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Abstract

If Y is a closed orientable graph manifold, we show that Y admits a coorientable taut foliation if and only if Y is not an L-space. Combined with previous work of Boyer and Clay, this implies that Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.

1. Introduction

An L-space is a rational homology sphere Y with simplest possible Heegaard Floer homology,[1] in the sense that $\operatorname{rank} \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. Ozsváth and Szabó have shown, by an argument analogous to one used by Kronheimer and Mrowka in the monopole setting [KM97], that the existence of a C^2 coorientable taut foliation ensures that $\operatorname{rank} \widehat{HF}(Y) > |H_1(Y;\mathbb{Z})|$ [OS04]. That is, L-spaces do not admit C^2 coorientable taut foliations.

For certain classes of manifolds the converse is known to hold. In particular, for Seifert fibered spaces with base orbifold S^2, Lisca and Stipsicz have shown that if Y is not an L-space then Y admits a coorientable taut foliation [LS07]. (In fact, it can be shown that the two conditions are equivalent for all Seifert fibered spaces; see [BGW13].) The main result of this note extends Lisca and Stipsicz's result to general graph manifolds. Recall that a graph manifold is a prime three-manifold admitting a JSJ decomposition into pieces admitting Seifert fibered structures.

Theorem 1. *Let Y be a closed, connected, orientable graph manifold. If Y is not an L-space then Y admits a C^0 coorientable taut foliation.*

An independent alternative proof of this result, together with an explicit classification of graph manifolds admitting cooriented taut foliations, appears in [Ras17], by the third author.

There is a third condition on three-manifolds that is relevant in this setting. Recall that a countable group is left-orderable if it admits an effective action on $\mathbb{R}$ by order-preserving homeomorphisms [BRW05]. There is a conjectured equivalence among prime three-manifolds between L-spaces and non-left-orderability of the fundamental group [BGW13]. Theorem 1 gives rise to an equivalence between all three conditions for graph manifolds.

Theorem 2. *If Y is a closed, connected, orientable graph manifold then the following are equivalent:*

(i) *Y is not an L-space;*
(ii) *Y admits a C^0 coorientable taut foliation;*
(iii) *Y has left-orderable fundamental group.*

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[1] We use Floer homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Other coefficient systems are discussed at the end of the paper.

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The equivalence (ii) $\Leftrightarrow$ (iii) is due to Boyer and Clay [BC17]. The implication (ii) $\Rightarrow$ (i) is established by Boyer and Clay in [BC15]. Alternatively, this implication follows from a theorem of Bowden [Bow16] and, independently, Kazez and Roberts [KR15, KR17] that taut C^0 foliations can be approximated by weakly semi-fillable contact structures, together with the earlier work of Ozsváth and Szabó [OS04]. Theorem 1 provides the final required implication (i) $\Rightarrow$ (ii). Among graph manifolds, the above equivalence was known for Seifert fibered spaces (see [BGW13] and references therein). The case of a graph manifold with a single JSJ torus was shown in [HW15, Theorem 1.1] of the first and fourth authors; this case also follows from the second and third authors' gluing theorem [RR17, Theorem 6.2]. In fact, Theorem 1 (and thus Theorem 2) also follows from results in [RR17], as we aim to show in this paper.

The equivalence (i) $\Leftrightarrow$ (iii) resolves [BGW13, Conjecture 1] in the affirmative for graph manifolds. We thank Tye Lidman for pointing out the following immediate consequence.

COROLLARY 3. *Suppose $f\colon Y_1 \to Y_2$ is a non-zero degree map between closed, connected, orientable graph manifolds. If Y_1 is an L-space then Y_2 is an L-space as well.*

Proof. This follows from Theorem 2 and [BRW05, Theorem 3.7]. Note that the existence of the non-zero degree map f induces a non-trivial homomorphism from $\pi_1(Y_1)$ to $\pi_1(Y_2)$ [BRW05, Lemma 3.8]. Hence if $\pi_1(Y_2)$ is left-orderable then so is $\pi_1(Y_1)$ [BRW05, Theorem 1.1]. □

Our work rests on a detailed study of the Heegaard Floer invariants of orientable three-manifolds M with torus boundary. Denote by $M(\alpha)$ the result of Dehn filling along a slope α in ∂M, that is, α represents a primitive class in $H_1(\partial M;\mathbb{Z})/\{\pm 1\}$. The set of slopes $Sl(M)$ may be identified with the extended rationals $\mathbb{Q}\cup\{\frac{1}{0}\}$, viewed as a subspace of $\mathbb{R}\mathrm{P}^1$. Consider the set of L-space slopes $\mathcal{L}_M = \{\alpha \,|\, M(\alpha) \text{ is an L-space}\}$; its interior $\mathcal{L}_M^\circ$ is the set of *strict L-space slopes*. The key tool used in the proof of Theorem 1 is the following non-L-space cutting theorem, which follows from results proved in [RR17].

THEOREM 4. *Let N denote the twisted I-bundle over the Klein bottle, with rational longitude λ. Let M_1 and M_2 be compact, connected, orientable three-manifolds with torus boundary, and suppose that $Y \cong M_1 \cup_h M_2$ for some homeomorphism $h\colon \partial M_1 \to \partial M_2$. If Y is not an L-space, then*

(1) *there exists a slope α in ∂M_1 such that $\alpha \notin \mathcal{L}_{M_1}^\circ$ and $h(\alpha) \notin \mathcal{L}_{M_2}^\circ$; moreover,*
(2) *for any orientation-reversing homeomorphisms $\varphi_i : \partial N \to \partial M_i$ with $\varphi_1(\lambda) = \alpha$ and $\varphi_2(\lambda) = h(\alpha)$, the closed manifolds $N \cup_{\varphi_1} M_1$ and $N \cup_{\varphi_2} M_2$ are non-L-spaces.*

We note that statement (1) alternatively results from an enhanced gluing result introduced in Theorem 7 below, which is of independent interest.

2. Notions of simplicity

Before proving Theorem 4 we recall the main notions of [HW15] and [RR17] in order to highlight a key point of interaction between these two works. The subject of [RR17] is the class of *Floer simple* manifolds: a manifold with torus boundary M is Floer simple if and only if $\mathcal{L}_M$ contains more than one element [RR17, Proposition 1.3]. In [HW15], the main object of study is the class of *simple loop-type* manifolds. The bordered Floer homology [LOT18] of these manifolds has a particularly nice form. Below, we briefly summarize some relevant facts about bordered Floer homology. For a more detailed exposition we refer to [HW15, § 2].

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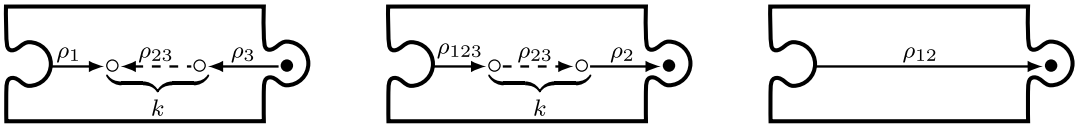


FIGURE 1. Puzzle pieces for simple loops; see [HW15]. The pieces represented by the letters $\bar{c}_k$, d_k and e are shown from left to right, where k is a positive integer determining the number of $\circ$ vertices. A simple loop may be expressed as a cyclic word in these letters.

The bordered Floer module $\widehat{CFD}$ is an invariant of a three-manifold with parametrized boundary. When ∂M is a torus we can specify a parametrization of ∂M by a pair of simple closed curves $\alpha, \beta \in H_1(\partial M; \mathbb{Z})$ with $\alpha \cdot \beta = 1$. In this case, the bordered Floer homology $\widehat{CFD}(M, \alpha, \beta)$ may be represented by a directed graph whose edges are labeled by elements of the set $\mathcal{A} = \{\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}\}$. The triple (M, α, β) is said to be of *loop type* if each vertex in the graph representing $\widehat{CFD}(M, \alpha, \beta)$ has valence 2 [HW15, Definition 3.2]. Such a graph can be decomposed into certain standard *puzzle pieces* as described in [HW15, § 3]. For our purposes, the relevant pieces are the ones shown in Figure 1. The property of being loop type is inherent in the underlying manifold M: if the triple (M, α, β) is of loop type for some choice of parametrizing curves α and β, then it is of loop type for any choice of α and β. In this case, we say the manifold M is of loop type.

DEFINITION 5 [HW15, Definition 4.19]. A manifold with torus boundary M is of simple loop type if it is of loop type, the number of connected components of the graph is equal to the number of spin^c structures on M, and, for some choice of parametrizing curves α and β, $\widehat{CFD}(M, \alpha, \beta)$ is expressible in the letters $\bar{c}_k$, d_k, and e.

PROPOSITION 6. *M is Floer simple if and only if M is of simple loop type.*

Proof. The bordered Floer homology of a Floer simple manifold M was explicitly computed in [RR17, Proposition 3.9] for an appropriate choice of parametrization (α, β). In the course of the proof, it is shown that $\widehat{CFD}(M, \alpha, \beta)$ is composed of puzzle pieces of type $\bar{c}_k$. Thus to see that M is of simple loop type, we need only check that the number of loops is equal to the number of spin^c structures on M, which is $|H^2(M; \mathbb{Z})| = |H_1(M, \partial M; \mathbb{Z})| = |H_1(M; \mathbb{Z})/\langle \alpha, \beta \rangle|$.

Each vertex v of $\widehat{CFD}(M, \alpha, \beta)$ is labeled by a relative spin^c structure $\mathfrak{s}(v)$, which we can view as an element of $H^2(M, \partial M; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$. By [RR17, Lemma 3.8], edges of the graph labeled by ρ_1 preserve this labeling, edges labeled by ρ_{23} shift the labeling by α and edges labeled by ρ_3 shift the labeling by $\alpha + \beta$.

Given a puzzle piece in $\widehat{CFD}(M, \alpha, \beta)$, let v be its unique black vertex, and label the piece by the image of $\mathfrak{s}(v)$ in $H_1(M; \mathbb{Z})/\langle \alpha \rangle$. This labeling defines a bijection between the set of puzzle pieces in $\widehat{CFD}(M, \alpha, \beta)$ and $H_1(M; \mathbb{Z})/\langle \alpha \rangle$. Moreover, if the label on a given piece is a, the label on the next piece in the loop is $a + \beta$. It follows that the set of loops is in bijection with $(H_1(M; \mathbb{Z})/\langle \alpha \rangle)/\langle \beta \rangle \cong H_1(M; \mathbb{Z})/\langle \alpha, \beta \rangle$. Thus a Floer simple manifold is of simple loop type.

Conversely, given a simple loop-type manifold M, fix parametrizing curves α and β such that $\widehat{CFD}(M, \alpha, \beta)$ consists only of segments of type $\bar{c}_k$, d_k and e. The slope ∞ is a strict L-space slope for (M, α, β) by [HW15, Proposition 4.18]; that is, $\alpha \in \mathcal{L}_M^\circ$. This implies that $|\mathcal{L}_M| > 1$ and therefore, by [RR17, Proposition 1.3], that M is Floer simple. □

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3. Cutting and gluing

Combining Proposition 6 with the gluing theorem [HW15, Theorem 1.3] for simple loop-type manifolds, we obtain a gluing result for Floer simple manifolds.

THEOREM 7. *Suppose that M_1 and M_2 are Floer simple manifolds, and consider the closed manifold $M_1 \cup_h M_2$ for some homeomorphism $h\colon \partial M_1 \to \partial M_2$.*

(1) *If neither M_1 nor M_2 is solid torus-like, then $M_1 \cup_h M_2$ is not an L-space if and only if there is a slope α in ∂M_1 such that $\alpha \notin \mathcal{L}^\circ_{M_1}$ and $h(\alpha) \notin \mathcal{L}^\circ_{M_2}$.*
(2) *If either M_1 or M_2 is solid-torus like, then $M_1 \cup_h M_2$ is not an L-space if and only if there is a slope α in ∂M_1 such that $\alpha \notin \mathcal{L}_{M_1}$ and $h(\alpha) \notin \mathcal{L}_{M_2}$.*

The two cases arising in this statement are expected: the second accounts for Dehn filling (that is, when one of the M_i is a solid torus) and simply verifies the definition of an L-space slope. More generally, we must appeal to a larger class of manifolds which are called *solid torus-like* [HW15, Definition 3.23], as they are characterized by having bordered Floer homology which resembles that of a solid torus in every spin^c structure [HW15], or equivalently, by having empty $\mathcal{D}^\tau$ in the sense of [RR17]. It was proved in [Gil16] that a solid torus-like manifold must be a solid torus connected sum with an L-space. In particular, if we assume that M_1 and M_2 are boundary incompressible, then the conclusion in case (1) holds.

Proof of Theorem 4. According to [RR17, Theorem 1.6], the set $\mathcal{L}^\circ_{M_i}$ (for $i = 1, 2$) is either empty or it is (the restriction to $\mathbb{Q} \cup \{\frac{1}{0}\}$ of) a connected interval with rational endpoints. In the case $\mathcal{L}^\circ_{M_2} = \emptyset$, let α be the rational longitude of M_1, which is not an L-space slope. Similarly, if $\mathcal{L}^\circ_{M_1} = \emptyset$ we choose α such that $h(\alpha)$ is the rational longitude of M_2. If $\mathcal{L}^\circ_{M_1}$ and $\mathcal{L}^\circ_{M_2}$ are both non-empty, then M_1 and M_2 are both Floer simple [RR17, Proposition 1.3]; since Y is not an L-space it follows from Theorem 7 that there is a slope $\alpha \notin \mathcal{L}^\circ_{M_1} \subset \mathcal{L}_{M_1}$ such that $h(\alpha) \notin \mathcal{L}^\circ_{M_2} \subset \mathcal{L}_{M_2}$ as required.

Part (2) of Theorem 4 is subsumed as a special case of [RR17, Proposition 7.9] by the second and third authors, but the result still merits some explanation. Again, and henceforth in this paper, N denotes the twisted I-bundle over the Klein bottle, with rational longitude λ. It is straightforward to compute, for example from [RR17, Theorem 5.1], that $\mathcal{L}_N = \mathcal{L}^\circ_N = Sl(N)\backslash\{\lambda\}$. Thus, for M Floer simple, Theorem 7 implies that for any gluing map $\varphi\colon \partial N \to \partial M$,

$$N \cup_\varphi M \text{ is a non-L-space} \iff \varphi(\lambda) \notin \mathcal{L}^\circ_M.$$

Note that, similar to a Dehn filling, the above non-L-space criterion for $N \cup_\varphi M$ depends only on the slope $\varphi(\lambda) \in Sl(M)$, and is independent of the choice of framing, relative to λ, of φ. In fact, even the $\mathbb{Z}/2\mathbb{Z}$-graded groups $\widehat{HF}(N \cup_\varphi M)$ are independent of this choice of framing [BGW13, Proposition 21], although we will not need this stronger statement. We therefore call any $N \cup_\varphi M$ with $\alpha = \varphi(\lambda)$ an *N-filling* of M along the slope α. The above non-L-space criterion, that an N-filling is a non-L-space if and only the filling is along a non-strict-L-space slope, also holds for N-fillings of an arbitrary rational homology solid torus M, but the argument is more subtle in the non-Floer-simple case. □

4. The main theorem

Decomposing along tori

Since Theorem 4 generically produces lower-complexity closed non-L-spaces from a closed non-L-space, it provides an *iterative* decomposition tool for the proof of Theorem 1. Moreover,

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at each decomposition step, the new N-filling slopes record one non-strict-L-space slope for each boundary component, with these slopes pairwise identified under the original gluing maps. In particular, by iteratively decomposing a non-L-space graph manifold along a suitable JSJ decomposition, we can produce a collection of non-L-space N-filled Seifert fibered spaces, with N-filling slopes specifying gluing-compatible non-strict-L-space slopes on all boundary components of the Seifert fibered spaces. Appealing to [RR17, Proposition 7.9] translates these data into the language of non-L-space-detected (NLS-detected) slopes in the sense of Boyer and Clay [BC17, Definition 7.16], whose machinery then automatically produces a cooriented taut foliation on the original graph manifold.

To describe this process in more detail, we first need to set up some notation for cutting along tori. Note that we may assume without loss of generality that $b_1(Y) = 0$: in the case where $b_1(Y) > 0$, work by Gabai guarantees the existence of a coorientable taut foliation [Gab83]. Thus, every torus $T \hookrightarrow Y$ separates Y into two rational homology solid tori.

Given Y and a collection of disjoint embedded tori $T_1, \dots, T_n$ in Y, let $Y \setminus\setminus \{T_i\}$ denote the result of cutting Y along each torus T_i. Since every torus is separating, this process produces $n+1$ pieces; that is, $Y \setminus\setminus \{T_i\} \cong M_1 \amalg \cdots \amalg M_{n+1}$, where each M_j is a three-manifold with ∂M_j a disjoint union of tori.

If we further specify a collection of slopes $\alpha_* = (\alpha_1, \dots, \alpha_n)$ on each of the tori in $\{T_i\}$, we can extend each M_j (for $1 \leqslant j \leqslant n+1$) to a closed manifold $Y_j^{\alpha_*}$ in the following way. The collection of slopes α_* induces a collection of slopes on the boundary tori of each M_j, since each boundary component of each M_j is identified with one of the T_i. A closed manifold $Y_j^{\alpha_*}$ is obtained from M_j by gluing a copy of N (the twisted I-bundle over the Klein bottle) to each boundary of M_j such that, for each gluing, the slope λ in ∂N is identified with the slope in the relevant component of ∂M_j induced by α_*. We say that $Y_j^{\alpha_*}$ is an *N-filling* of M_j along the slopes induced by α_*.

Note that the manifold $Y_j^{\alpha_*}$ described above is not uniquely determined by M_j and α_* since each time a copy of N is glued to M_j there is an infinite family of gluing maps which take λ in ∂N to the desired slope in ∂M_j. A particular gluing map is specified by choosing slopes dual to λ in ∂M and dual to each slope induced by α_* in ∂M_j; the manifold $Y_j^{\alpha_*}$ depends on the particular choice of dual slopes. However, Theorem 4 again tells us that the question of whether $Y_j^{\alpha_*}$ is an L-space is determined solely by α_*, thus is independent of these choices. Incidentally, one again has the stronger result that the $\mathbb{Z}/2\mathbb{Z}$-graded Heegaard Floer groups $\widehat{HF}(Y_j^{\alpha_*})$ are independent of these choices [BGW13, Proposition 21].

Slope detection

Given a three-manifold Y, a collection of tori $\{T_i\}_{i=1}^n$ and a collection of slopes α_*, we have explained how to construct manifolds $\{Y_j^{\alpha_*}\}_{j=1}^{n+1}$, and observed that the $\mathbb{Z}/2\mathbb{Z}$-graded groups $\widehat{HF}(Y_j^{\alpha_*})$ do not depend on the choices made in the construction. In particular, whether or not each $Y_j^{\alpha_*}$ is an L-space is a well-defined question. The key step in proving Theorem 1 is the following proposition.

Proposition 8. *Let Y be an irreducible three-manifold and fix a collection of disjoint embedded tori $\{T_1, \dots, T_n\}$ in Y such that each torus is separating. If Y is a non-L-space, then there is some collection of slopes α_* on these tori with the property that each of the manifolds $Y_j^{\alpha_*}$ defined above is a non-L-space.*

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Proof. First observe that if $n = 1$ (that is, the collection of tori consists of just one torus), this is equivalent to Theorem 4. In this case, $Y \cong M_1 \cup_h M_2$ for some gluing map h. By Theorem 4, there is a slope α in ∂M_1 such that N-filling M_1 along α gives a non-L-space and N-filling M_2 along $h(\alpha)$ gives a non-L-space. Let α_1 be the slope in T_1 that corresponds to the slopes $\alpha \in \partial M_1$ and $h(\alpha) \in \partial M_2$. Then $\alpha_* = (\alpha_1)$ gives the desired collection of slopes.

For the general case, we proceed by induction on n. Assume $n > 1$ and the result holds for collections of fewer than n tori. First cut Y along the torus T_1 to produce two manifolds M_1 and M_2. By the $n = 1$ case, there is a slope α_1 in T_1 such that N-filling M_1 and M_2 along the slopes corresponding to α_1 produces non-L-spaces. We denote the resulting closed manifolds by $Y_1^{(\alpha_1)}$ and $Y_2^{(\alpha_1)}$.

Having cut along T_1, the remaining collection of tori $\{T_2, \dots, T_n\}$ splits into two subsets depending on whether each torus is contained in M_1 or M_2. Up to relabeling the tori, we may assume that $\{T_2, \dots, T_m\}$ is the subset of tori contained in M_1 and $\{T_{m+1}, \dots, T_n\}$ is the subset of tori contained in M_2, for some $1 \leqslant m \leqslant n$ (note that if $m = 1$ the first subset is empty, and if $m = n$ the second subset is empty). We consider these subsets as collections of tori on $Y_1^{(\alpha_1)}$ and $Y_2^{(\alpha_1)}$, respectively. Note that each collection has at most $n - 1$ tori.

By the inductive hypothesis applied to $Y_1^{(\alpha_1)}$ with the collection of tori $\{T_2, \dots, T_m\}$, there is a collection of slopes $(\alpha_2, \dots, \alpha_m)$ such that cutting along each torus and N-filling along the corresponding slopes produces only non-L-spaces. Similarly, there is collection of slopes $(\alpha_{m+1}, \dots, \alpha_n)$ on the tori $\{T_{m+1}, \dots, T_n\}$ in $Y_2^{(\alpha_1)}$ with this property. Finally, observe that the non-L-space manifolds obtained from $Y_1^{(\alpha_1)}$ by this process together with the non-L-space manifolds obtained from $Y_2^{(\alpha_1)}$ are exactly the same as the manifolds obtained by cutting Y along the tori $\{T_1, \dots, T_n\}$ and N-filling along the slopes induced by $\alpha_* := (\alpha_1, \dots, \alpha_n)$. □

The proposition above can be restated using the notion of NLS-detected slopes defined in [BC17]. Let M be a manifold with ∂M a disjoint union of n tori, and let $\alpha_* = (\alpha_1, \dots, \alpha_n)$ be a collection of slopes on the boundary tori. Following [BC17, Definition 7.2], let $\mathcal{M}_t(\emptyset; [\alpha_*])$ denote the collection of manifolds obtained by filling each boundary component of M by a copy of N_t where the rational longitude of the ith copy of N_t is sent to α_i. The manifold N_t is the Seifert fibered space over the disk with two cone points of order t and Seifert invariants $(1/t, (t-1)/t)$. Note that $N_2 = N$. In particular, in the notation introduced above, the set $\mathcal{M}_2(\emptyset; [\alpha_*])$ is the set of all possible N-fillings Y^{α_*} of M along the slopes α_*. Recall that all manifolds in this set have the same Heegaard Floer homology. According to [BC17, Definition 7.16], the collection of slopes $\alpha_* = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is NLS detected if for every $t > 1$, the set $\mathcal{M}_t(\emptyset; [\alpha_*])$ contains no L-spaces. We can now restate Proposition 8 as follows.

PROPOSITION 9. *Let Y be an irreducible three-manifold with $b_1 = 0$ and fix a collection of disjoint tori $\{T_1, \dots, T_n\}$ in Y. If Y is a non-L-space, then there is some collection of slopes α_* on these tori with the property that the restriction of α_* to each M_j in $Y \backslash\backslash \{T_i\}$ is NLS detected.*

Proof. In [RR17, Proposition 7.9], the second and third authors show that for any rational homology solid torus M, 'generalized solid torus' N' with rational longitude λ', and gluing map $\varphi : \partial N' \to \partial M$, the closed manifold $N' \cup_\varphi M$ is a non-L-space if and only if $\varphi(\lambda') \notin \mathcal{L}^\circ_M$. In the discussion preceding that proposition, they also prove that N_t is a generalized solid torus for any $t > 1$. Thus, for a manifold with slope α_*, the set $\mathcal{M}_2(\emptyset, [\alpha_*])$ contains a non-L-space if and only if for all $t > 1$, the set $\mathcal{M}_t(\emptyset, [\alpha_*])$ contains no L-spaces, hence if and only if α_* is NLS detected.

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By Proposition 8, there is a collection of slopes α_* such that each $Y_j^{\alpha_*}$ is a non-L-space. For each M_j, where $1 \leqslant j \leqslant n+1$, $Y_j^{\alpha_*}$ is by construction a non-L-space element of $\mathcal{M}_2(\emptyset; [\alpha_*^j])$, where α_*^j denotes the restriction of α_* to ∂M_j. Thus each α_*^j is NLS detected on M_j. □

Remark 10. Notice that we have yet to restrict to graph manifolds, or even to incompressible tori. Indeed, given a rational homology sphere Y that is not an L-space, a collection of disjoint tori $\{T_i\}$ always gives rise to an NLS detected collection of slopes on the boundary of each component of $Y \setminus\setminus \{T_i\}$. This suggests that the same behavior for taut foliations and/or for left-orders on the fundamental group should be explored for more general prime three-manifolds.

Proof of Theorem 1. When Y is a graph manifold and $\{T_i\}$ is the collection of JSJ tori, note that Proposition 9 verifies one direction of Boyer and Clay's Conjecture 1.10 in [BC17] about cutting and gluing along NLS-detected slopes, namely that a non-L-space graph manifold can be cut into Seifert fibered pieces with gluing-compatible NLS-detected slopes on all boundary components. This allows us to complete the proof of Theorem 1.

Suppose that Y is a non-L-space graph manifold with $b_1(Y) = 0$. If Y has a trivial JSJ decomposition, then Y is Seifert fibered and therefore already known to admit a cooriented taut foliation. Next suppose Y has a non-trivial JSJ decomposition. Since $b_1(Y) = 0$, every JSJ torus separates; we take $\{T_i\}$ to be JSJ tori such that the components of $Y \setminus\setminus \{T_i\}$ are Seifert fibered. By Proposition 9 there is a collection of slopes $\alpha_* = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in T_i$, such that the restrictions of α_* to each component of $Y \setminus\setminus \{T_i\}$ are NLS detected. In [BC17, Theorem 8.1], Boyer and Clay show that on Seifert fibered spaces, NLS-detected slopes are equivalent to what they call 'foliation detected' slopes. Thus, we have finally produced foliation-detected slopes on all the JSJ tori decomposing Y into Seifert fibered pieces, and that is precisely what Boyer and Clay's foliation gluing theorem [BC17, Theorem 1.7] requires, in order to guarantee the existence of a cooriented taut foliation on Y. □

Coefficients

Up until this point, we have used Floer homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. This choice was imposed by our use of bordered Floer homology, which is only defined over $\mathbb{Z}/2\mathbb{Z}$. We now briefly discuss what happens for other coefficient systems. If G is an abelian group, we say Y is a G L-space if $\widehat{HF}(Y, \mathfrak{s}; G) \cong G$ for all $\mathfrak{s} \in \mathrm{spin}^c(Y)$. For a closed orientable graph manifold Y, we consider the following conditions:

(i) Y is a $\mathbb{Z}$ L-space;
(ii) Y is a $\mathbb{Z}/2\mathbb{Z}$ L-space;
(iii) Y does not admit a C^0 coorientable taut foliation.

In fact, if Y is a closed graph manifold, these three conditions are equivalent. To see this, we briefly sketch the points at which our argument used $\mathbb{Z}/2\mathbb{Z}$ coefficients. First, the proof of our enhanced gluing result Theorem 7 depends on bordered Floer homology, hence requires $\mathbb{Z}/2\mathbb{Z}$ coefficients. For the proof of Theorem 4, however, Theorem 7 can be replaced with [RR17, Theorem 1.1], which works over $\mathbb{Z}$ coefficients. The theorem of Lisca and Stipsicz uses $\mathbb{Z}/2\mathbb{Z}$ coefficients; however, for any manifold obtained by Dehn filling a Floer simple manifold, the properties of being a $\mathbb{Z}$ L-space and a $\mathbb{Z}/p\mathbb{Z}$ L-space are equivalent [RR17, Proof of Proposition 3.6]. Moreover, [RR17, Theorem 5.1] re-proves Lisca and Stipsicz's result over $\mathbb{Z}$ by performing a direct computation of the L-space slope interval for any Seifert fibered space over the disk. Thus, Theorems 1, 2 and 4 hold over $\mathbb{Z}$.

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Closing remarks

The first author has applied bordered Floer homology to give an algorithm for computing the Heegaard Floer homology of an arbitrary graph manifold [Han16]. This has been implemented on computer, with considerable savings in computation time if the combinatorics developed in [HW15] are incorporated (see [HW15, Remark 6.10] in particular). As a consequence of Theorem 2 two questions are now algorithmically decidable:

- does a graph manifold Y admit a coorientable taut foliation?
- does a graph manifold Y have a left-orderable fundamental group?

The answer to either question is *yes* if and only if $\operatorname{rank}\widehat{HF}(Y) > \chi(\widehat{HF}(Y))$ (recall that $\chi(\widehat{HF}(Y)) = |H_1(Y;\mathbb{Z})|$). This gives a direct, in fact combinatorial, verification of two conditions on a three-manifold that seem quite difficult to certify in general.

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