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Distributional chaos in shrinking target problems

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Abstract. In this article, we prove that the set of well-approximable points $W_\varphi(z) = \{x \in X : d(f^n x, z) < \varphi(n) \text{ for infinite } n \in \mathbb{N}^+\}$ in the shrinking targets problems is distributional chaotic of type 1 for systems with a weak form of the exponential specification property. We apply it to transitive Anosov systems, β-shifts, etc.

Key words: distributional chaos, shrinking target problems, specification property
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1. *Introduction*

Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and denote $\mathcal{M}_{\text{erg}}(f, X)$ the set of f-ergodic probability measures. Let $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{R}$ denote the set of non-negative integers, positive integers, integers, and real numbers, respectively. We consider the set of *well-approximable points*

$$W_\varphi(z) = \{x \in X : d(f^n x, z) < \varphi(n) \text{ for infinite } n \in \mathbb{N}^+\}$$

for some fixed point $z \in X$ and $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$ defined by Hill and Velani in [13] in analogy to the classical theory of metric Diophantine approximations. These points' trajectories hit a shrinking target infinitely often. A simple case is that φ is a constant in the whole space and $z \in \text{supp}(\mu)$ where

$$\text{supp}(\mu) = \{x \in X : \mu(B(x, \varepsilon)) > 0 \text{ for any } \varepsilon > 0\}.$$

For any $\varepsilon > 0$, by Poincaré recurrence theorem, almost every $x \in X$ hits the shrinking target $B(z, \varepsilon)$ infinitely often. However, what happens when φ is not a constant? Does $W_\varphi(z)$ exhibit strong dynamical complexity?

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From the viewpoint of measures, a direct result by the Borel–Cantelli lemma is that $\mu(W_\varphi(z)) = 0$ when $\sum_{n=1}^{\infty} \mu(B(z, \varphi(n))) < +\infty$, and $\mu(W_\varphi(z)) = 1$ when $\sum_{n=1}^{\infty} \mu(B(z, \varphi(n))) = +\infty$ and $B(z, \varphi(n))$ are independent. Unfortunately, the hypothesis of independence for the sets $B(z, \varphi(n))$ is rarely satisfied in the field of dynamical systems. Therefore, researchers developed some weaker hypotheses to conclude $\mu(W_\varphi(z)) = 1$, called the 'dynamical Borel–Cantelli lemma'. Previous results include uniformly hyperbolic systems [5], partially hyperbolic systems [9], interval maps [20], irrational rotations [21, 43], geodesic flows [26], and systems with some abstract conditions [11, 12, 18].

Another viewpoint is the Hausdorff dimension, which has a strong connection to Diophantine approximations. The Hausdorff dimension of $W_\varphi(z)$ was studied in many systems, including expanding rational maps of the Riemann sphere [13, 14], actions on finite Kleinian groups [15], linear endomorphisms of the d-torus [16], parabolic rational maps [39], conformal iterated function systems [44], expanding Markov systems [12], irrational rotations on torus [3, 8, 10, 42], and a dynamical system of continued fractions [24].

The complexity of $W_\varphi(z)$ also can be measured from the viewpoint of chaos. Recall that *Li–Yorke chaos* focuses on the relatively complicated long-time behavior of pairs of two orbits, which first imported the concept of chaos to dynamical systems. Based on this, Schweizer and Smítal used ideas from the theory of probabilistic metric spaces to develop three versions of Li–Yorke chaos for maps of the interval in [37] from a statistical viewpoint, which are collectively called *distributional chaos* and numbered in the order of decreasing strength (DC1, DC2, DC3). Those concepts of distributional chaos were generalized to the case of maps on a metric space in [36, 37]. In this article, we focus on the complexity of $W_\varphi(z)$ from the viewpoint of distributional chaos.

1.1. *Main results.* In this article, we always assume a non-degenerate situation, which means that $|X| > 1$. Denote $\mathcal{M}(X)$ as the set of probability measures and denote $\mathcal{M}(f, X)$ as the set of f-invariant probability measures. We denote the set of all *transitive points* in X with respect to f by $\mathrm{Trans}(f) = \{x \in X : \overline{\{f^n x : n \in \mathbb{N}^+\}} = X\}$.

Let $Z = \{z_n\}_{n=1}^{\infty} \subset X$ and $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$. We call

$$W_\varphi(Z) = \{x \in X : d(f^n x, z_n) < \varphi(n) \text{ for infinite } n \in \mathbb{N}\}$$

the set of *well-approximable points* of Z with respect to φ. We say Z is the *target center*. When $z_n = z_m = z$ for any $n, m \in \mathbb{N}^+$, we also say that z is the target center and denote $W_\varphi(Z)$ by $W_\varphi(z)$. In addition, for infinite $T \subseteq \mathbb{N}^+$, we call

$$W_\varphi(Z, T) = \{x \in X : d(f^n x, z_n) < \varphi(n) \text{ for infinite } n \in T\}$$

the set of well-approximable points of Z with index T with respect to φ.

Next, we recall some concepts of distributional chaos. For convenience, we recall the definitions of the distributional functions first. For any pair (x, y) of points in X and any positive integer n, the *distributional function* $\Phi_{xy}^{(n)} : \mathbb{R} \to [0, 1]$ is defined by

$$\Phi_{xy}^{(n)}(t, f) = \frac{1}{n}|\{0 \leqslant i \leqslant n-1 : d(f^i x, f^i y) < t\}|.$$

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The *lower distributional function* and the *upper distributional function* are defined by

$$\Phi_{xy}(t, f) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t, f),$$

$$\Phi_{xy}^{*}(t, f) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t, f).$$

Now, we recall the definition of distributional chaos of type 1 (DC1), which is stronger than DC2 and DC3. A pair (x, y) of points in X is called *DC1-scrambled* (with respect to f) if

$$\Phi_{xy}^{*}(t, f) = 1 \quad \text{for any } t > 0,$$

$$\Phi_{xy}(t, f) = 0 \quad \text{for some } t > 0.$$

A set $K \subseteq X$ is called a *DC1-scrambled set* (with respect to f) if any pair (x, y) of points in K is DC1-scrambled (with respect to f). A subset $K \subseteq X$ is said to be *distributional chaotic of type 1*(*DC1*) (with respect to f) if it has an uncountable DC1-scrambled subset (with respect to f).

We want to investigate the distributional chaos property of the well-approximable points $W_\varphi(Z, T)$. Since the distributional chaos property has a strong connection to the specification property (cf. [28, 30]), we consider those systems with some form of the specification property. The main difficulty is to construct a set that is not only DC1, but also a subset of $W_\varphi(Z, T)$ and the specification property may not work for the decreasing $\varphi(n)$. Thus, we consider a strong form of the specification property, called the two-sided exponential specification property (see §2 for details), which appears in Axiom A systems and symbolic systems. It requires the shadowing to be exponential, which guarantees the existence of the well-approximable points. However, the uniformity of the gap is one of the most important properties in the (exponential) specification property. However, it does not play a significant role in the construction of chaos. Therefore, we consider new types of the specification property called the *two-sided (one-sided) exponential specification property with non-uniform gap* and conclude the following theorem.

THEOREM A. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and $T \subset \mathbb{N}^+$ be infinite. For any target center $Z = \{z_n\}_{n=1}^{+\infty} \subset X$:*

(1) *if f has the two-sided exponential specification property with a non-uniform gap with exponent $\lambda > 0$ and $\varphi : \mathbb{N} \to \mathbb{R}^+$ satisfies*

$$\liminf_{n\to+\infty, n\in T} \frac{e^{-n\lambda}}{\varphi(n)} = 0,$$

then, for any non-empty open set $U \subset X$, the set $W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ is DC1;

(2) *if f has the one-sided exponential non-uniformly specification property with exponent $\lambda > 0$ and $\varphi : \mathbb{N} \to \mathbb{R}^+$, then for any non-empty open set $U \subset X$, the set $W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ is DC1.*

This theorem implies that the well-approximable points in systems with exponential specification properties exhibit strong dynamical complexity. Applications of the theorem include Axiom A systems, mixing subshifts of finite type, etc (see §4).

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In particular, we can apply the theorem to the β-shifts for any $\beta > 1$. One of the key distinctions between β-shifts and 'good' shifts, such as topologically mixing subshifts of the finite type, is that β-shifts do not satisfy the specification property for every $\beta > 1$. By [4],

$$\{\beta \in (1, +\infty) : (\Sigma_\beta, \sigma) \text{ has the specification property}\}$$

is dense in $(1, +\infty)$ but has zero Lebesgue measure. In [32], Pfister and Sullivan proved that β-shifts satisfy a weak form of the specification property, called the *approximate product property*. This result was generalized to the g-almost product property in [33] and the almost specification property in [40]. In this article, we proved that the β-shifts have the one-sided exponential specification property with a non-uniform gap, which not only implies β-shifts satisfy Theorem A but also have independent interest of possibly playing important roles in other topics.

THEOREM B. *Let $\beta > 1$ and (Σ_β, σ) be a β-shift. Then, it has the one-sided exponential specification property with a non-uniform gap.*

1.2. *Organization.* In §2, we recall the definitions of some concepts to make precise statements of the theorems and their proofs and introduce some new forms of specification properties. In §3, we give some lemmas and prove Theorem A. In §4, we give some applications of Theorem A to symbolic systems and differential systems.

2. *The specification-like properties*

2.1. *Specification property and exponential specification property.* We first recall the definition of the specification property, which is weaker than the original definition given by Bowen. The latter is always called *Bowen's specification property* or the *periodic specification property* now.

Let $f : X \to X$ be a continuous map on a compact metric space (X, d).

Definition 2.1. We say that $f : X \to X$ has the *specification property* if for any $\varepsilon > 0$, there exists an integer $M = M(\varepsilon)$ (called the *gap*) such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k \quad \text{with } a_j - b_{j-1} \geqslant M \quad \text{for any } 2 \leqslant j \leqslant k,$$

there exists a point $y \in X$ such that

$$d(f^n y, f^n x_i) < \varepsilon$$

for any $a_i \leqslant n \leqslant b_i$ and any $1 \leqslant i \leqslant k$. We also say that y ε-shadows x_j on $[a_j, b_j]$ for any $1 \leqslant j \leqslant k$.

Definition 2.2. Let $\lambda > 0$. We say that $f : X \to X$ has the *two-sided exponential specification property with exponent* λ if for any $\varepsilon > 0$, there exists an integer $M = M(\varepsilon)$ such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k \quad \text{with } a_j - b_{j-1} \geqslant M \quad \text{for any } 2 \leqslant j \leqslant k,$$

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there exists a point $y \in X$ such that

$$d(f^n y,\, f^n x_i) < \varepsilon e^{-\lambda \min\{n-a_i, b_i-n\}}$$

for any $a_i \leqslant n \leqslant b_i$ and any $1 \leqslant i \leqslant k$. We also say that $y\varepsilon$-shadows x_j on $[a_j, b_j]$ with two-sided exponent λ for any $1 \leqslant j \leqslant k$.

Definition 2.3. Let $\lambda > 0$. We say that $f : X \to X$ has the *one-sided exponential specification property with exponent* λ if for any $\varepsilon > 0$, there exists an integer $M = M(\varepsilon)$ such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k \quad \text{with } a_j - b_{j-1} \geqslant M \quad \text{for any } 2 \leqslant j \leqslant k,$$

there exists a point $y \in X$ such that

$$d(f^n y,\, f^n x_i) < \varepsilon e^{-\lambda(b_i-n)}$$

for any $a_i \leqslant n \leqslant b_i$ and any $1 \leqslant i \leqslant k$. We also say that $y\varepsilon$-shadows x_j on $[a_j, b_j]$ with one-sided exponent λ for any $1 \leqslant j \leqslant k$.

By the definition of the specification property and exponential specification property, we can easily check the relationship between them and get the following remark.

Remark 2.4. Let $f : X \to X$ be a continuous map on a compact metric space (X, d).

- If f has the one-sided exponential specification property for some $\lambda > 0$, then f has the two-sided exponential specification property with exponent λ.
- If f has the two-sided exponential specification property for some $\lambda > 0$, then f has the specification property.

In other words, the one-sided exponential specification property is stronger than the two-sided exponential specification property. The two-sided exponential specification property is stronger than the specification property.

2.2. *Specification property and exponential specification property with a non-uniform gap.* In this subsection, we introduce some new forms of the specification property. In non-uniformly hyperbolic systems and non-uniformly expanding maps, a weaker form of the specification property called the non-uniform (exponential) specification property, is considered in [27, 41]. One important requirement of it is that the gap M of the non-uniformly (exponential) specification property is non-uniform to x and sublinear to n, where n is the length of the shadowing interval. However, it is difficult to estimate the gap for some systems like β-shifts. Therefore, we introduce a new form of the specification property, called the specification property with a non-uniform gap.

Definition 2.5. We say that f has the *specification property with a non-uniform gap* if for any $x \in X$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists a positive integer $M(x, n, \varepsilon)$ such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$0 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$$

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with

$$a_j - b_{j-1} \geqslant M(x_{j-1}, b_{j-1} - a_{j-1}, \varepsilon)$$

for any $2 \leqslant j \leqslant k$, there exists a point $y \in X$ such that

$$d(f^n y,\, f^{n-a_j} x_j) \leqslant \varepsilon$$

for any $a_j \leqslant n \leqslant b_j$ and any $1 \leqslant j \leqslant k$.

The main difference between the specification property and the specification property with a non-uniform gap is that the gap of the specification property with a non-uniform gap $M(x, n, \varepsilon)$ is non-uniform with respect to x and n.

Definition 2.6. Let $\lambda > 0$. We say that f has the *two-sided exponential specification property with a non-uniform gap with exponent* λ if for any $x \in X$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists an integer $M(x, n, \varepsilon)$ such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$0 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$$

with

$$a_j - b_{j-1} \geqslant M(x_{j-1}, b_{j-1} - a_{j-1}, \varepsilon)$$

for any $2 \leqslant j \leqslant k$, there exists a point $y \in X$ such that

$$d(f^n y,\, f^{n-a_j} x_j) \leqslant \varepsilon e^{-\lambda \min\{n-a_i, b_i-n\}}$$

for any $a_j \leqslant n \leqslant b_j$ and any $1 \leqslant j \leqslant k$.

Definition 2.7. Let $\lambda > 0$. We say that f has the *one-sided exponential specification property with a non-uniform gap with exponent* λ if for any $x \in X$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists an integer $M(x, n, \varepsilon)$ such that for any integer $k \geqslant 2$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $2k$ integers

$$0 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$$

with

$$a_j - b_{j-1} \geqslant M(x_{j-1}, b_{j-1} - a_{j-1}, \varepsilon)$$

for any $2 \leqslant j \leqslant k$, there exists a point $y \in X$ such that

$$d(f^n y,\, f^{n-a_j} x_j) \leqslant \varepsilon e^{-\lambda(b_i-n)}$$

for any $a_j \leqslant n \leqslant b_j$ and any $1 \leqslant j \leqslant k$.

Remark 2.8. Let $f : X \to X$ be a continuous map on a compact metric space (X, d).

- If f has the one-sided exponential specification property with a non-uniform gap for some $\lambda > 0$, then f has the two-sided exponential specification property with a non-uniform gap with exponent λ.
- If f has the two-sided exponential specification property with a non-uniform gap for some $\lambda > 0$, then f has the specification property with a non-uniform gap.

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In other words, the one-sided exponential specification property with a non-uniform gap is stronger than the two-sided exponential specification property with a non-uniform gap. The two-sided exponential specification property with a non-uniform gap is stronger than the specification property with a non-uniform gap.

One can easily check the following statement by [7, Proposition 6.11].

LEMMA 2.9. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d). If f has the specification property with a non-uniform gap, then f is topologically transitive and, hence,* Trans *is dense in X.*

The following proposition implies that the specification property with a non-uniform gap can be extended to the case of infinite shadowing intervals. Similar results hold for the specification property with a non-uniform gap and the one-sided exponential specification property with a uniform gap.

PROPOSITION 2.10. *Let $\lambda > 0$. If f has the two-sided exponential specification property with a non-uniform gap with exponent λ, then for any $x \in X$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists an integer $M(x, n, \varepsilon)$ such that for any sequence of points $\{x_j\}_{j=1}^{+\infty} \subseteq X$, and any two sequences of integers $\{a_j\}_{j=1}^{+\infty}$ and $\{b_j\}_{j=1}^{+\infty}$, if*

$$0 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_j \leqslant b_j < a_{j+1} \leqslant \cdots,$$

$$a_j - b_{j-1} \geqslant M(x_{j-1}, b_{j-1} - a_{j-1}, \varepsilon) \quad \text{for any } j \geqslant 2,$$

then, there exists a point $y \in X$ that ε-shadows x_j on $[a_j, b_j]$ with exponent λ for any $j \geqslant 1$.

Proof. Since f has the two-sided exponential specification property with a non-uniform gap, for any $l \geqslant 2$, there exists a point $y_l \in X$ that ε-shadows x_j on $[a_j, b_j]$ with exponent λ for any $1 \leqslant j \leqslant l-1$ and ε-shadows x_l on $[a_l, b_l]$ with exponent λ. Since f is a continuous map, any accumulation point of $\{y_l\}_{l=2}^{\infty}$ naturally ε-shadows x_j on $[a_j, b_j]$ with exponent λ for any $j \geqslant 1$. □

2.3. *g-almost product property.* In this section, we recall the definition of the g-almost product property introduced by Pfister and Sullivan in [33], and give some lemmas about the relationship between the g-almost product property and specification property.

Let $f : X \to X$ be a continuous map on a compact metric space (X, d). For any $n \in \mathbb{N}^+$, $x \in X$, and $\varepsilon > 0$, the *dynamical ball* $B_n(x, \varepsilon)$ is defined by

$$B_n(x, \varepsilon) = \{y \in X : d(f^j x, f^j y) < \varepsilon \text{ for any } 0 \leqslant j \leqslant n-1\}.$$

Definition 2.11. Let $g : \mathbb{N}^+ \to \mathbb{N}$ be a non-decreasing unbounded map such that

$$g(n) < n \text{ and } \lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

Such g is called a *blowup function*.

Denote $\{0, 1, 2, \ldots, n-1\}$ by Λ_n.

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Definition 2.12. Let $g : \mathbb{N}^+ \to \mathbb{N}$ be a blowup function. We say that (X, f) has the *g-almost product property* if there exists a function $M : \mathbb{R}^+ \to \mathbb{N}$ such that for any integer $k \geqslant 2$, any k positive real numbers $\{\varepsilon_j\}_{j=1}^k$, any k points $\{x_j\}_{j=1}^k \subseteq X$, and any $k+1$ integers

$$0 = a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k \quad \text{with } a_j - a_{j-1} \geqslant M(\varepsilon_j) \quad \text{for any } 1 \leqslant j \leqslant k,$$

there exists a point $y \in X$ such that

$$\#\{n \in [a_{j-1}, a_j) : d(f^n y, f^n x_j) \geqslant \varepsilon_j\} \leqslant g(a_j - a_{j-1})$$

for any $1 \leqslant j \leqslant k$.

Intuitively, the g-almost product property means that for any k points $x_1, \ldots, x_k \in X$ and k positive scales $\varepsilon_1, \ldots, \varepsilon_k$, there exists a point $y \in X$ that ε_j-shadows x_j on $[a_{j-1}, a_j - 1]$ with some 'bad times' if $a_j - a_{j-1}$ is large enough. In addition, those 'bad times' account for up to the proportion of $g(a_j - a_{j-1})/(a_j - a_{j-1})$ on $[a_{j-1}, a_j - 1]$.

Remark 2.13. There are three main differences between the g-almost product property and the specification property.

- The g-almost product property allows different scales through shadowing different points, while the specification property requires the scales to be the same.
- The proportion of the 'bad times' of the g-almost product property is sublinear, while the proportion of the 'bad times' of the specification property is a constant.
- The distribution of 'bad times' of the g-almost product property is unknown, while the 'bad times' of the specification property is at the end of every shadowing interval.

There are three main differences between the g-almost product property and the specification property with a non-uniform gap.

- The g-almost product property allows different scales through shadowing different points, while the specification property with a non-uniform gap requires the scales to be the same.
- The proportion of the 'bad times' of the g-almost product property is sublinear, while the proportion of the 'bad times' of the specification property with a non-uniform gap is unknown.
- The distribution of 'bad times' of the g-almost product property is unknown, while the 'bad times' of the specification property with a non-uniform gap is at the end of every shadowing interval.

Figure 1 shows the relationship between those specification-like properties.

3. *Proof of Theorem A*

In this section, we prove Theorem A. Before our proof, we recall some definitions and lemmas related to distributional chaos.

3.1. *Strongly distributional chaos.*

In this subsection, we recall the definition of a new form of the distributional chaos property called strongly distributional chaos introduced in [17], which is stronger than DC1.

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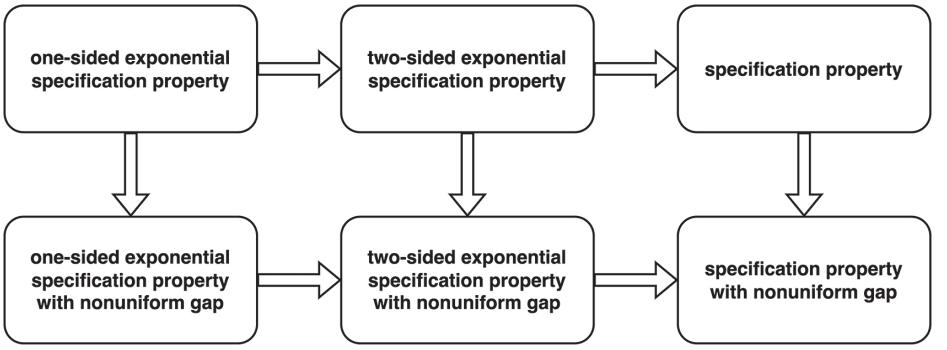


FIGURE 1. Relationship between different forms of specification-like properties.

Definition 3.1. Let $\mathcal{A}$ be a set of maps defined by

$$\mathcal{A} = \left\{\alpha : \mathbb{N} \to [0, +\infty],\ \alpha \text{ is non-decreasing, } \lim_{n\to\infty} \alpha(n) = +\infty \text{ and } \lim_{n\to\infty} \frac{\alpha(n)}{n} = 0\right\}.$$

For any $\alpha \in \mathcal{A}$, any pair (x, y) of points in X, and any positive integer n, the *α-distributional function* $\Phi_{xy}^{(n)}(\cdot, \alpha) : \mathbb{R} \to [0, 1]$ is defined by

$$\Phi_{xy}^{(n)}(t, \alpha, f) = \frac{1}{n}\left|\left\{1 \leqslant i \leqslant n : \sum_{j=0}^{i-1} d(f^j x, f^j y) < \alpha(i)t\right\}\right|.$$

The *upper α-distributional function* is defined by

$$\Phi_{xy}^{*}(t, \alpha, f) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t, \alpha, f).$$

A pair (x, y) of points in X is called *α-DC1-scrambled* (with respect to f) if

$$\Phi_{xy}^{*}(t, \alpha, f) = 1 \quad \text{for any } t > 0,$$

$$\Phi_{xy}(t, f) = 0 \quad \text{for some } t > 0.$$

A set $K \subseteq X$ is called a *α-DC1-scrambled set* (with respect to f) if any pair (x, y) of points in K is α-DC1-scrambled (with respect to f). A subset $K \subseteq X$ is said to be *strongly distributional chaotic* (with respect to f) if it has an uncountable α-DC1-scrambled subset (with respect to f) for any $\alpha \in \mathcal{A}$.

It is easy to check the relationship between strong distributional chaotic sets and DC1 sets.

Remark 3.2. [17, Proposition 2.5] Any strongly distributional chaotic subset $K \subseteq X$ is DC1, which means that strongly distributional chaos is stronger than DC1.

3.2. *Existence of distal pairs.* A pair (x, y) of points in X is called a *distal pair* if

$$\liminf_{n\to\infty} d(f^n x, f^n y) > 0.$$

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Otherwise, we call it a *proximal pair*. We say a continuous map $f : X \to X$ is *proximal* if there is no distal pair in X for f. Additionally, we say a subset $B \subseteq X$ is *minimal* if B is compact, f-invariant, and has no proper subset that is non-empty, compact, and f-invariant.

Oprocha gave an equivalent condition of systems to be proximal. The existence of distal pairs in systems with the two-sided exponential specification property is a direct consequence.

LEMMA 3.3. [29, Lemma 18] *A continuous map $f : X \to X$ is proximal if and only if it has a fixed point which is the unique minimal subset of X.*

LEMMA 3.4. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) with two-sided exponential specification property with a non-uniform gap. Then, f has a distal pair.*

Proof. Since $|X| > 1$, fix $x, y \in X$ with $d(x, y) > 0$. Let $\varepsilon = \frac{1}{2}d(x, y)$. By the two-sided exponential specification property with a non-uniform gap and Proposition 2.10, there exists $l \geqslant M(x, 0, \varepsilon)$ and points that ε-shadows x on $[il, il]$ for any $i \in \mathbb{N}$. Let $S(x)$ be the set of all points that satisfy the above condition. Then, $\bigcup_{i=0}^{l-1} f^{-i}S(x)$ is proper, non-empty, compact, and f-invariant. Hence, it has a proper minimal subset $U(x)$. Similarly, there exists a minimal set $U(y) \neq U(x)$, which implies the conclusion by Lemma 3.3. □

3.3. *The case of the two-sided exponential specification property with a non-uniform gap.* Term (1) of Theorem A is a direct corollary of the following theorem.

THEOREM 3.5. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and $T \subset \mathbb{N}$ be infinite. For any target center $Z = \{z_n\}_{n=1}^{+\infty} \subset X$, if f has the two-sided exponential specification property with a non-uniform gap with exponent $\lambda > 0$ and $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$ satisfies*

$$\liminf_{n\to+\infty, n\in T} \frac{e^{-n\lambda}}{\varphi(n)} = 0,$$

then for any non-empty open set $U \subset X$ and any $\alpha \in \mathcal{A}$, there exists an uncountable subset $S_\alpha \subseteq W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ such that there exists $t_0 > 0$ such that for any $x, y \in S_\alpha$ with $x \neq y$,

$$\limsup_{n\to\infty} \frac{1}{n}\left|\left\{1 \leqslant i \leqslant n : \sum_{j=0}^{i-1} d(f^j x, f^j y) < \alpha(i)t\right\}\right| = 1 \quad \text{for any } t > 0,$$

$$\liminf_{n\to\infty} \frac{1}{n}|\{0 \leqslant i \leqslant n-1 : d(f^i x, f^i y) < t_0\}| = 0.$$

Therefore, $W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ is strongly distributional chaotic and, hence, distributional chaotic of type 1.

Proof. Given $\alpha \in \mathcal{A}$, non-empty open set $U \subseteq X$. By Lemma 2.9, Trans is dense in X. Fix some $z \in \mathrm{Trans} \cap U$ with $B(z, \varepsilon_0) \subseteq U$ for some $0 < \varepsilon_0 < 1$. Let $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$ for any $k \in \mathbb{N}$.

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By Lemma 3.4, there exists a distal pair (q_1, q_2) in X. Since (q_1, q_2) is a distal pair, then by definition, there exists $\zeta > 0$ such that for any $n \in \mathbb{N}^+$,

$$d(f^n q_1, f^n q_2) > \zeta.$$

Let $\{c_k\}_{k=1}^{+\infty} = \{c_k(\varepsilon_k, \lambda)\}_{k=1}^{+\infty}$ be a sequence of positive integers where

$$c_k = \min\{n \in \mathbb{N} : \varepsilon_0 e^{-n\lambda} < \varepsilon_k\}$$

for any $k \in \mathbb{N}^+$.

We construct $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ inductively. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, define $M^*(n, \varepsilon) := \max\{M(q_1, n, \varepsilon), M(q_2, n, \varepsilon)\}$.

Step (1): construction of $\{a_n\}_{n=1}^4$ and $\{b_n\}_{n=1}^4$. Let $a_1 = 0$, $b_1 = 0$, $a_2 = b_1 + M(z, b_1 - a_1, \varepsilon_0)$.

Since $\liminf_{n\to+\infty, n\in T}(e^{-n\lambda})/(\varphi(n)) = 0$ and $\lim_{n\to\infty} \alpha(n) = +\infty$, there exists $n_1 \in T$ large enough such that

$$\varepsilon_0 e^{-\lambda(n_1 - a_2)} < \varphi(n_1), \quad a_2 \mathrm{diam}(X) + \frac{4\varepsilon_0}{1 - e^{-\lambda}} < \alpha(a_2 + n_1)\varepsilon_1,$$

where $\mathrm{diam}(X) := \sup_{x_1 \neq x_2 \in X} d(x_1, x_2)$. Let b_2 be large enough such that

$$\frac{a_2 + 2n_1 + 2c_1}{b_2} < \varepsilon_1$$

and $a_3 = b_2 + M(f^{-(n_1 - a_2)} z_{n_1}, b_2 - a_2, \varepsilon_0)$, and b_3 be large enough such that

$$\frac{a_3 + 2c_1}{b_3} < \varepsilon_1.$$

Let $a_4 = b_3 + M^*(b_3 - a_3, \varepsilon_0)$ and $b_4 = a_4 + 1$.

. . .

Step (k): construction of $\{a_n\}_{n=k(k+3)/2}^{k(k+3)/2+k+1}$ and $\{b_n\}_{n=k(k+3)/2}^{k(k+3)/2+k+1}$. Now, $\{a_n\}_{n=1}^{k(k+3)/2-1}$ and $\{b_n\}_{n=1}^{k(k+3)/2-1}$ have been defined. Let

$$a_{k(k+3)/2} = b_{k(k+3)/2-1} + M(f^{-k+1} z, b_{k(k+3)/2-1} - a_{k(k+3)/2-1}, \varepsilon_0).$$

Since $\liminf_{n\to+\infty, n\in T} e^{-n\lambda})/(\varphi(n)) = 0$ and $\lim_{n\to\infty} \alpha(n) = +\infty$, there exists $n_k \in T$ large enough such that

$$\varepsilon_0 e^{-\lambda(n_k - a_{k(k+3)/2})} < \varphi(n_k), \tag{3.1}$$

$$a_{k(k+3)/2} \mathrm{diam}(X) + \frac{4\varepsilon_0}{1 - e^{-\lambda}} < \alpha(a_{k(k+3)/2} + n_k)\varepsilon_k. \tag{3.2}$$

Let $b_{k(k+3)/2}$ be large enough such that

$$\frac{a_{k(k+3)/2} + 2n_k + 2c_k}{b_{k(k+3)/2}} < \varepsilon_k \tag{3.3}$$

and $a_{k(k+3)/2+1} = b_{k(k+3)/2} + M(f^{-n_k + a_{k(k+3)/2}} z_{n_k}, b_{k(k+3)/2} - a_{k(k+3)/2}, \varepsilon_0)$.

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For any $1 \leqslant i \leqslant k$, we define $\{b_n\}_{n=k(k+3)/2+1}^{k(k+3)/2+k}$ and $\{a_n\}_{n=k(k+3)/2+2}^{k(k+3)/2+k+1}$ in the following way inductively. Let $b_{k(k+3)/2+i}$ be large enough such that

$$\frac{a_{k(k+3)/2+i} + 2c_k}{b_{k(k+3)/2+i}} < \varepsilon_k \tag{3.4}$$

and let $a_{k(k+3)/2+i+1} = b_{k(k+3)/2+i} + M^*(b_{k(k+3)/2+i} - a_{k(k+3)/2+i}, \varepsilon_0)$. Finally, let $b_{k(k+3)/2+k+1} = a_{k(k+3)/2+k+1} + 2k - 1$.

...

Now, $\{a_i\}_{i=1}^{+\infty}$ and $\{b_i\}_{i=1}^{+\infty}$ are already defined. By the two-sided exponential specification property with a non-uniform gap and Proposition 2.10, for any $\xi = (\xi_1\xi_2\cdots) \in \{1, 2\}^{+\infty}$, there exists $x_\xi \in X$ that ε_0-shadows

$$\begin{array}{lll}
z & \text{on} & [a_1, b_1], \\
f^{-n_1+a_2}(z_{n_1}) & \text{on} & [a_2, b_2], \\
q_{\xi_1} & \text{on} & [a_3, b_3], \\
 & \cdots & \\
f^{-k+1}z & \text{on} & [a_{k(k+3)/2-1}, b_{k(k+3)/2-1}], \\
f^{-n_k+a_{k(k+3)/2}}(z_{n_k}) & \text{on} & [a_{k(k+3)/2}, b_{k(k+3)/2}], \\
q_{\xi_1} & \text{on} & [a_{k(k+3)/2+1}, b_{k(k+3)/2+1}], \\
q_{\xi_2} & \text{on} & [a_{k(k+3)/2+2}, b_{k(k+3)/2+2}], \\
 & \cdots & \\
q_{\xi_k} & \text{on} & [a_{k(k+3)/2+k}, b_{k(k+3)/2+k}], \\
f^{-k}z & \text{on} & [a_{k(k+3)/2+k+1}, b_{k(k+3)/2+k+1}], \\
 & \cdots &
\end{array}$$

with two-sided exponent λ, which implies that $x_\xi \varepsilon_k$-shadows

$$\begin{array}{lll}
f^{-n_k+c_k+a_{k(k+3)/2}}(z_{n_k}) & \text{on} & [a_{k(k+3)/2} + c_k, b_{k(k+3)/2} - c_k], \\
f^{c_k}(q_{\xi_1}) & \text{on} & [a_{k(k+3)/2+1} + c_k, b_{k(k+3)/2+1} - c_k], \\
f^{c_k}(q_{\xi_2}) & \text{on} & [a_{k(k+3)/2+2} + c_k, b_{k(k+3)/2+2} - c_k], \\
 & \cdots & \\
f^{c_k}(q_{\xi_k}) & \text{on} & [a_{k(k+3)/2+k} + c_k, b_{k(k+3)/2+k} - c_k]
\end{array}$$

with two-sided exponent λ for any $k \in \mathbb{N}^+$.

First, we prove that $\{x_\xi\}_{\xi\in\{1,2\}^{+\infty}}$ is an uncountable α-DC1-scrambled set. Fixing $\xi \neq \eta \in \{1, 2\}^{+\infty}$, we claim that x_ξ and x_η are an α-DC1-scrambled pair. Suppose $\xi_s \neq \eta_s$ for some $s \in \mathbb{N}^+$, then $x_\xi\varepsilon_k$-shadows $f^{c_k}(q_{\xi_s})$ on $[a_{k(k+3)/2+s} + c_k, b_{k(k+3)/2+s} - c_k]$ and $x_\eta\varepsilon_k$-shadows $f^{c_k}(q_{\eta_s})$ on $[a_{k(k+3)/2+s} + c_k, b_{k(k+3)/2+s} - c_k]$ for any $k \geqslant s$. For any $\kappa < \zeta$, there exists an $M_\kappa \geqslant s$ such that $\zeta - \kappa > 2\varepsilon_k$ for any $k \geqslant M_\kappa$. By (3.4), we have

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$$\begin{aligned}&\frac{1}{b_{k(k+3)/2+s}}|\{i\in[0,b_{k(k+3)/2+s}-1]:d(f^ix_\xi,f^ix_\eta)>\kappa\}|\\&\geqslant\frac{b_{k(k+3)/2+s}-a_{k(k+3)/2+s}-2c_k}{b_{k(k+3)/2+s}}\\&>1-\varepsilon_k\end{aligned}$$

for any $k\geqslant M_\kappa$, which implies $x_\xi\neq x_\eta$ since $\lim_{n\to+\infty}\varepsilon_k=0$. Hence, $\{x_\xi\}_{\xi\in\{1,2\}^\infty}$ is an uncountable set. Meanwhile,

$$\begin{aligned}&\liminf_{n\to+\infty}\frac{1}{n}|\{j\in[0,n-1]:d(f^jx_\xi,f^jx_\eta)<\kappa\}|\\&\leqslant\liminf_{k\geqslant M_\kappa\to+\infty}\frac{1}{b_{k(k+3)/2+s}}|\{j\in[0,b_{k(k+3)/2+s}-1]:d(f^jx_\xi,f^jx_\eta)<\kappa\}|\\&\leqslant\liminf_{k\geqslant M_\kappa\to+\infty}\frac{a_{k(k+3)/2+s}+2c_k}{b_{k(k+3)/2+s}}\\&\leqslant\liminf_{k\geqslant M_\kappa\to+\infty}\varepsilon_k=0.\end{aligned}$$

However, for any $t>0$, we can choose $K_t\in\mathbb{N}$ large enough such that $\varepsilon_k<t$ for any $k\geqslant K_t$. Note that x_ξ and x_η both ε_0-shadow $f^{-n_k+a_{k(k+3)/2}}(z_k)$ on $[a_{k(k+3)/2},b_{k(k+3)/2}]$ with two-sided exponent λ for any $k\geqslant1$. For any $i\in[a_{k(k+3)/2}+n_k,b_{k(k+3)/2}]$ and any $k\in\mathbb{N}^+$, by (3.2),

$$\begin{aligned}\sum_{j=0}^{i-1}d(f^jx_\xi,f^jx_\eta)&\leqslant\sum_{j=0}^{a_{k(k+3)/2}-1}d(f^jx_\xi,f^jx_\eta)+\sum_{j=a_{k(k+3)/2}}^{b_{k(k+3)/2}-1}d(f^jx_\xi,f^jx_\eta)\\&\leqslant a_{k(k+3)/2}\mathrm{diam}X+2\cdot2\varepsilon_0\cdot\frac{1}{1-e^{-\lambda}}<\varepsilon_k\alpha(i).\end{aligned}$$

Therefore, by (3.3), we have

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{n}\left|\left\{i\in[1,n]:\sum_{j=0}^{i-1}d(f^jx_\xi,f^jx_\eta)<\alpha(i)t\right\}\right|\\&\geqslant\limsup_{n\to+\infty}\frac{1}{n}\left|\left\{i\in[1,n]:\sum_{j=0}^{i-1}d(f^jx_\xi,f^jx_\eta)<\alpha(i)\varepsilon_k\right\}\right|\\&\geqslant\limsup_{k\geqslant K_t\to+\infty}\frac{1}{b_{k(k+3)/2}}\left|\left\{i\in[1,b_{k(k+3)/2}]:\sum_{j=0}^{i-1}d(f^jx_\xi,f^jx_\eta)<\alpha(i)\varepsilon_k\right\}\right|\\&\geqslant\limsup_{k\geqslant K_t\to+\infty}\left(1-\frac{a_{k(k+3)/2}+n_k}{b_{k(k+3)/2}}\right)\\&\geqslant\limsup_{k\geqslant K_t\to+\infty}(1-\varepsilon_k)=1.\end{aligned}$$

So far, we have proved that $\{x_\xi\}_{\xi\in\{1,2\}^{+\infty}}$ is an uncountable α-DC1-scrambled set. To complete the proof, we need to check that $x_\xi\in W_\varphi(Z,T)\cap\mathrm{Trans}$ for any $\xi\in\{1,2\}^{+\infty}$.

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Since $x_\xi \varepsilon_0$-shadows $f^{-k+1}z$ on $[a_{k(k+3)/2-1}, b_{k(k+3)/2-1}]$ with two-sided exponent λ for any $k \in \mathbb{N}^+$, the orbit of x_ξ shadows the orbit of z increasingly more closely. Then, $\omega_f(z) \subseteq \omega_f(x_\xi)$, which implies that x_ξ is a transitive point.

Finally, we prove that $x_\xi \in W_\varphi(Z, T)$. For any $k \in \mathbb{N}^+$, one has that $x_\xi \varepsilon_0$-shadows $f^{-n_k+a_{k(k+3)/2}}(z_{n_k})$ on $[a_{k(k+3)/2}, b_{k(k+3)/2}]$ with two-sided exponent λ, which implies that

$$d(f^{n_k}x_\xi, z_{n_k}) < \varepsilon_0 e^{-\lambda(n_k - a_{k(k+3)/2})} < \varphi(n_k)$$

by (3.1). This completes our proof. □

3.4. *The case of the one-sided exponential specification property with a non-uniform gap.* Term (2) of Theorem A is a direct corollary of the following theorem.

THEOREM 3.6. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and $T \subset \mathbb{N}$ be infinite. For any target center $Z = \{z_n\}_{n=1}^{+\infty} \subset X$, if f has the one-sided exponential specification property with a non-uniform gap with exponent $\lambda > 0$ and $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$, then for any non-empty open set $U \subset X$ and any $\alpha \in \mathcal{A}$, there exists an uncountable subset $S_\alpha \subseteq W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ such that there exists $t_0 > 0$ such that for any $x, y \in S_\alpha$ with $x \neq y$,*

$$\limsup_{n\to\infty} \frac{1}{n}\left|\left\{1 \leqslant i \leqslant n : \sum_{j=0}^{i-1} d(f^j(x), f^j(y)) < \alpha(i)t\right\}\right| = 1 \quad \text{for any } t > 0,$$

$$\liminf_{n\to\infty} \frac{1}{n}|\{0 \leqslant i \leqslant n-1 : d(f^i(x), f^i(y)) < t_0\}| = 0.$$

Therefore, $W_\varphi(Z, T) \cap U \cap \mathrm{Trans}$ is strongly distributional chaotic and, hence, distributional chaotic of type 1.

Proof. Given $\alpha \in \mathcal{A}$, non-empty open set $U \subseteq X$. By Lemma 2.9, Trans is dense in X. Fix some $z \in \mathrm{Trans} \cap U$ with $B(z, \varepsilon_0) \subseteq U$ for some $0 < \varepsilon_0 < 1$. Let $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$ for any $k \in \mathbb{N}^+$.

By Lemma 3.4, there exists a distal pair (q_1, q_2) in X. Since (q_1, q_2) is a distal pair, then by definition, there exists $\zeta > 0$ such that for any $n \in \mathbb{N}^+$,

$$d(f^n q_1, f^n q_2) > \zeta.$$

Let $\{c_k\}_{k=1}^{+\infty} = \{c_k(\varepsilon_k, \lambda)\}_{k=1}^{+\infty}$ be a sequence of positive integers where

$$c_k = \min\{n \in \mathbb{N} : \varepsilon_0 e^{-n\lambda} < \varepsilon_k\}$$

for any $k \in \mathbb{N}^+$.

We construct $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ inductively. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, define $M^*(n, \varepsilon) := \max\{M(q_1, n, \varepsilon), M(q_2, n, \varepsilon)\}$.

Step (1): construction of $\{a_n\}_{n=1}^4$ and $\{b_n\}_{n=1}^4$. Let $a_1 = 0$, $b_1 = 0$, and $a_2 \geqslant b_1 + M(z, b_1 - a_1, \varepsilon_0)$ such that $a_2 \in T$. Since $\lim_{n\to\infty} \alpha(n) = +\infty$, there exists n_1 large enough such that

$$a_2 \mathrm{diam}(X) + \frac{2\varepsilon_0}{1 - e^{-\lambda}} < \alpha(a_2 + n_1)\varepsilon_1.$$

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Let b_2 be large enough such that

$$\frac{a_2+n_1+c_1}{b_2}<\varepsilon_1,\quad \varepsilon_0 e^{-\lambda(b_2-a_2)}<\varphi(a_2),$$

and $a_3=b_2+M(z_{a_2},b_2-a_2,\varepsilon_0)$, and b_3 be large enough such that

$$\frac{a_3+c_1}{b_3}<\varepsilon_1.$$

Let $a_4=b_3+M^*(b_3-a_3,\varepsilon_0)$ and $b_4=a_4+1$.

...

Step (k): construction of $\{a_n\}_{n=k(k+3)/2}^{k(k+3)/2+k+1}$ and $\{b_n\}_{n=k(k+3)/2}^{k(k+3)/2+k+1}$. Now, $\{a_n\}_{n=1}^{k(k+3)/2-1}$ and $\{b_n\}_{n=1}^{k(k+3)/2-1}$ have been defined. Let $a_{k(k+3)/2}\geqslant b_{k(k+3)/2-1}+M(z,b_{k(k+3)/2-1}-a_{k(k+3)/2-1},\varepsilon_0)$ such that $a_{k(k+3)/2}\in T$. Since $\lim_{n\to\infty}\alpha(n)=+\infty$, there exists n_k large enough such that

$$a_{k(k+3)/2}\mathrm{diam}(X)+\frac{2\varepsilon_0}{1-e^{-\lambda}}<\alpha(a_{k(k+3)/2}+n_k)\varepsilon_k. \tag{3.5}$$

Let $b_{k(k+3)/2}$ be large enough such that

$$\frac{a_{k(k+3)/2}+n_k+c_k}{b_{k(k+3)/2}}<\varepsilon_k, \tag{3.6}$$

$$\varepsilon_0 e^{-\lambda(b_{k(k+3)/2}-a_{k(k+3)/2})}<\varphi(a_{k(k+3)/2}), \tag{3.7}$$

and $a_{k(k+3)/2+1}=b_{k(k+3)/2}+M(z_{a_{k(k+3)/2}},b_{k(k+3)/2}-a_{k(k+3)/2},\varepsilon_0)$.

For any $1\leqslant i\leqslant k$, we define $\{b_n\}_{n=k(k+3)/2+1}^{k(k+3)/2+k}$ and $\{a_n\}_{n=k(k+3)/2+2}^{k(k+3)/2+k+1}$ in the following way inductively. Let $b_{k(k+3)/2+i}$ be large enough such that

$$\frac{a_{k(k+3)/2+i}+c_k}{b_{k(k+3)/2+i}}<\varepsilon_k \tag{3.8}$$

and let $a_{k(k+3)/2+i+1}=b_{k(k+3)/2+i}+M^*(b_{k(k+3)/2+i}-a_{k(k+3)/2+i},\varepsilon_0)$. Finally, let $b_{k(k+3)/2+k+1}=a_{k(k+3)/2+k+1}+k$.

...

Now, $\{a_i\}_{i=1}^{+\infty}$ and $\{b_i\}_{i=1}^{+\infty}$ are already defined. By the one-sided exponential specification property with a non-uniform gap and Proposition 2.10, for any $\xi=(\xi_1\xi_2\cdots)\in\{1,2\}^{+\infty}$, there exists $x_\xi\in X$ that ε_0-shadows

$$\begin{array}{lll}
z & \text{on} & [a_1,b_1],\\
z_{a_2} & \text{on} & [a_2,b_2],\\
q_{\xi_1} & \text{on} & [a_3,b_3],\\
 & \cdots & \\
z & \text{on} & [a_{k(k+3)/2-1},b_{k(k+3)/2-1}],\\
z_{a_{k(k+3)/2}} & \text{on} & [a_{k(k+3)/2},b_{k(k+3)/2}],\\
q_{\xi_1} & \text{on} & [a_{k(k+3)/2+1},b_{k(k+3)/2+1}],\\
q_{\xi_2} & \text{on} & [a_{k(k+3)/2+2},b_{k(k+3)/2+2}],\\
 & \cdots &
\end{array}$$

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$$\begin{aligned} &q_{\xi_k} && \text{on} && [a_{k(k+3)/2+k}, b_{k(k+3)/2+k}],\\ &z && \text{on} && [a_{k(k+3)/2+k+1}, b_{k(k+3)/2+k+1}],\\ & && \cdots \end{aligned}$$

with one-sided exponent λ, which implies that $x_\xi \varepsilon_k$-shadows

$$\begin{aligned} &z_{a_{k(k+3)/2}} && \text{on} && [a_{k(k+3)/2}, b_{k(k+3)/2} - c_k],\\ &q_{\xi_1} && \text{on} && [a_{k(k+3)/2+1}, b_{k(k+3)/2+1} - c_k],\\ &q_{\xi_2} && \text{on} && [a_{k(k+3)/2+2}, b_{k(k+3)/2+2} - c_k],\\ & && \cdots \\ &q_{\xi_k} && \text{on} && [a_{k(k+3)/2+k}, b_{k(k+3)/2+k} - c_k] \end{aligned}$$

with one-sided exponent λ for any $k \in \mathbb{N}^+$.

First, we prove that $\{x_\xi\}_{\xi\in\{1,2\}^{+\infty}}$ is an uncountable α-DC1-scrambled set. Fix $\xi \neq \eta \in \{1,2\}^{+\infty}$. We claim that x_ξ and x_η is an α-DC1-scrambled pair. Suppose $\xi_s \neq \eta_s$ for some $s \in \mathbb{N}^+$, then $x_\xi \varepsilon_k$-shadows q_{ξ_s} on $[a_{k(k+3)/2+s}, b_{k(k+3)/2+s} - c_k]$ and $x_\eta \varepsilon_k$-shadows q_{η_s} on $[a_{k(k+3)/2+s}, b_{k(k+3)/2+s} - c_k]$ for any $k \geqslant s$. For any $\kappa < \zeta$, there exists an $M_\kappa \geqslant s$ such that $\zeta - \kappa > 2\varepsilon_k$ for any $k \geqslant M_\kappa$. By (3.8), we have

$$\begin{aligned} &\frac{1}{b_{k(k+3)/2+s}} |\{i \in [0, b_{k(k+3)/2+s} - 1] : d(f^i(x_\xi), f^i(x_\eta)) > \kappa\}| \\ &\quad \geqslant \frac{b_{k(k+3)/2+s} - a_{k(k+3)/2+s} - c_k}{b_{k(k+3)/2+s}} \\ &\quad > 1 - \varepsilon_k \end{aligned}$$

for any $k \geqslant M_\kappa$, which implies $x_\xi \neq x_\eta$ since $\lim_{n\to+\infty} \varepsilon_k = 0$. Hence, $\{x_\xi\}_{\xi\in\{1,2\}^\infty}$ is an uncountable set. Meanwhile,

$$\begin{aligned} &\liminf_{n\to+\infty} \frac{1}{n} |\{j \in [0, n-1] : d(f^j x_\xi, f^j x_\eta) < \kappa\}| \\ &\quad \leqslant \liminf_{k\geqslant M_\kappa \to+\infty} \frac{1}{b_{k(k+3)/2+s}} |\{j \in [0, b_{k(k+3)/2+s} - 1] : d(f^j x_\xi, f^j x_\eta) < \kappa\}| \\ &\quad \leqslant \liminf_{k\geqslant M_\kappa \to+\infty} \frac{a_{k(k+3)/2+s} + c_k}{b_{k(k+3)/2+s}} \\ &\quad \leqslant \liminf_{k\geqslant M_\kappa \to+\infty} \varepsilon_k = 0. \end{aligned}$$

However, for any $t > 0$, we can choose $K_t \in \mathbb{N}$ large enough such that $\varepsilon_k < t$ for any $k \geqslant K_t$. Note that x_ξ and x_η both ε_0-shadow $z_{a_{k(k+3)/2}}$ on $[a_{k(k+3)/2}, b_{k(k+3)/2}]$ with one-sided exponent λ for any $k \geqslant 1$. For any $i \in [a_{k(k+3)/2} + n_k, b_{k(k+3)/2}]$ and any $k \in \mathbb{N}^+$, by (3.5),

$$\begin{aligned} \sum_{j=0}^{i-1} d(f^j x_\xi, f^j x_\eta) &\leqslant \sum_{j=0}^{a_{k(k+3)/2}-1} d(f^j x_\xi, f^j x_\eta) + \sum_{j=a_{k(k+3)/2}}^{b_{k(k+3)/2}-1} d(f^j x_\xi, f^j x_\eta) \\ &\leqslant a_{k(k+3)/2} \mathrm{diam} X + 2 \cdot \varepsilon_0 \cdot \frac{1}{1 - e^{-\lambda}} < \varepsilon_k \alpha(i). \end{aligned}$$

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Therefore, by (3.6), we have

$$\begin{aligned}
&\limsup_{n\to+\infty}\frac{1}{n}\left|\left\{i\in[1,n]:\sum_{j=0}^{i-1}d(f^j(x_\xi),f^j(x_\eta))<\alpha(i)t\right\}\right|\\
&\geqslant\limsup_{n\to+\infty}\frac{1}{n}\left|\left\{i\in[1,n]:\sum_{j=0}^{i-1}d(f^j(x_\xi),f^j(x_\eta))<\alpha(i)\varepsilon_k\right\}\right|\\
&\geqslant\limsup_{k\geqslant K_t\to+\infty}\frac{1}{b_{k(k+3)/2}}\left|\left\{i\in[1,b_{k(k+3)/2}]:\sum_{j=0}^{i-1}d(f^j(x_\xi),f^j(x_\eta))<\alpha(i)\varepsilon_k\right\}\right|\\
&\geqslant\limsup_{k\geqslant K_t\to+\infty}\left(1-\frac{a_{k(k+3)/2}+n_k}{b_{k(k+3)/2}}\right)\\
&\geqslant\limsup_{k\geqslant K_t\to+\infty}(1-\varepsilon_k)=1.
\end{aligned}$$

So far, we have proved that $\{x_\xi\}_{\xi\in\{1,2\}^{+\infty}}$ is an uncountable α-DC1-scrambled set. To complete the proof, we need to check that $x_\xi\in W_\varphi(Z,T)\cap\mathrm{Trans}$ for any $\xi\in\{1,2\}^{+\infty}$.

Since

$$d(f^{a_{k(k+3)/2-1}}x_\xi,z)\leqslant\varepsilon_0e^{-\lambda(b_{k(k+3)/2-1}-a_{k(k+3)/2-1})}=\varepsilon_0e^{-(k-1)\lambda}$$

for any $k\in\mathbb{N}^+$, the orbit of x_ξ shadows the orbit of z increasingly more closely. Then, $\omega_f(z)\subseteq\omega_f(x_\xi)$, which implies that x_ξ is a transitive point.

Finally, we prove that $x_\xi\in W_\varphi(Z,T)$. For any $k\in\mathbb{N}^+$, one has that

$$d(f^{a_{k(k+3)/2}}x_\xi,z)\leqslant\varepsilon_0e^{-\lambda(b_{k(k+3)/2}-a_{k(k+3)/2})}<\varphi(a_{k(k+3)/2})$$

by (3.7). This completes our proof. □

4. *Applications*

In this section, we give some applications of Theorem A.

4.1. *Shifts.*

Before applying our theorem, we recall some basic concepts and results of symbolic systems. For any finite alphabet A, the *full symbolic space* is the set

$$A^{\mathbb{Z}}=\{\ldots,x_{-1}x_0x_1\ldots:x_i\in A\},$$

which is viewed as a compact topological space with the discrete product topology. The shift action on full symbolic space is defined by

$$\sigma:A^{\mathbb{Z}}\to A^{\mathbb{Z}},\quad \ldots x_{-1}x_0x_1\ldots\mapsto\ldots x_0x_1x_2\ldots.$$

Here, $(A^{\mathbb{Z}},\sigma)$ forms a dynamical system under the discrete product topology and is called a two-sided shift. We equip $A^{\mathbb{Z}}$ with the following metric. Let $n=|A|$. For $x=(\ldots x_{-1}x_0x_1\ldots)$ and $y=(\ldots y_{-1}y_0y_1\ldots)$ with $x\neq y$, let

$$d(x,y)=n^{-\min\{|i|:i\in\mathbb{Z},x_i\neq y_i\}}.$$

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A closed subset $X \subseteq A^{\mathbb{Z}}$ is called a *two-sided subshift* if it is invariant under the shift action σ. Define $A^n = \{x_1x_2 \dots x_n : x_i \in A\}$. Then, $w \in A^n$ is a *word* of the subshift X if there is an $x \in X$ and $k \in \mathbb{N}^+$ such that $w = x_k x_{k+1} \cdots x_{k+n-1}$. We say n is the *length* of w. The *language* of a subshift X, denoted by $\mathcal{L}(X)$, is the set of all words of the subshift X. Denote $\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n$, all words of X with length n.

Similarly, one can define the one-sided shifts. The set $A^{\mathbb{N}^+} = \{x_1 \cdots : x_i \in A\}$ is called the *one-sided full symbolic space*. The shift action on one-sided full symbolic space is defined by

$$\sigma : A^{\mathbb{N}^+} \to A^{\mathbb{N}^+}, \quad x_1x_2 \cdots \mapsto x_2x_3 \cdots .$$

Here, $(A^{\mathbb{N}^+}, \sigma)$ forms a dynamical system under the discrete product topology and is called a one-sided shift. We equip $A^{\mathbb{N}^+}$ with the following metric. For $x = (x_1x_2 \cdots)$ and $y = (y_1y_2 \cdots)$ with $x \neq y$, let

$$d(x, y) = n^{-\min\{i \in \mathbb{N}^+ : x_i \neq y_i\}}.$$

Hou and Tian proved in [17, Lemma 6.3] that the 'ε-shadow' in a one-sided subshift must be a 'ε-shadow with exponent ln n'. It directly implies the following proposition.

PROPOSITION 4.1. *If a one-sided (two-sided) subshift (X, σ) with $X \subseteq \{0, 1, \dots, n-1\}^{\mathbb{N}^+}$ ($X \subseteq \{0, 1, \dots, n-1\}^{\mathbb{Z}}$) has the specification property with a non-uniform gap, then it has the one-sided (two-sided) exponential specification property with a non-uniform gap.*

Next, we give the equivalent definition of the specification with a non-uniform gap in symbolic systems, described by words and letters. Note this lemma is for one-sided subshifts and can be easily generalized to two-sided subshifts.

LEMMA 4.2. *Let (X, σ) be a one-sided subshift. It has the specification with a non-uniform gap if and only if for any $w^1, w^2, \dots, w^m \in \mathcal{L}(X)$, there exists a function $L : \mathcal{L}(X) \to \mathbb{N}$ such that for any $k_j \geqslant L(w^j)$ with $1 \leqslant j \leqslant m-1$, there exist $u^1, u^2, \dots, u^{m-1} \in \mathcal{L}(X)$ such that $w^1u^1w^2u^2 \cdots w^{m-1}u^{m-1}w^m \in \mathcal{L}(X)$ and $|u^j| = k_j$ for any $1 \leqslant j \leqslant m-1$.*

Proof. Let $N = |A|$, where A is the alphabet of the subshift (X, σ).

First, we assume that (X, σ) has the specification with a non-uniform gap. Let $L(w) = \min\{M(x, |w|, 1/(N+1)) : x \in X, w = x_1 \cdots x_{|w|}\}$ in Definition 2.5. Fix $w^1, w^2, \dots, w^m \in \mathcal{L}(X)$. There exists $x^j \in X$ such that $w^j = x_1^j \cdots x_{|w|}^j$ and $L(w) = M(x^j, |w^j|, 1/(N+1))$ for any $1 \leqslant j \leqslant m$. Next, we prove that for any $k_j \geqslant L(w^j)$ with $1 \leqslant j \leqslant m$, there exist $u^1, u^2, \dots, u^{m-1} \in \mathcal{L}(X)$ such that $w^1u^1w^2u^2 \cdots w^{m-1}u^{m-1}w^m \in \mathcal{L}(X)$ and $|u^j| = k_j$ for any $1 \leqslant j \leqslant m-1$. Fix $k_1, \dots, k_m$. Let $a_1 = 0$ and $a_j = \sum_{i=1}^{j-1}(k_i + |w^i|)$ for any $2 \leqslant j \leqslant m$. Let $b_j = \sum_{i=1}^{j-1} k_i + \sum_{i=1}^{j}|w^i|$ for any $1 \leqslant j \leqslant m$. Since (X, σ) has the specification with a non-uniform gap, there exists a point $y \in X$ such that

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$$d(\sigma^n y, \sigma^{n-a_j} x^j) \leqslant \frac{1}{N+1}$$

for any $a_j \leqslant n \leqslant b_j$ and any $1 \leqslant j \leqslant m$. Then, $y_n = x^j_{n-a_j}$ for any $a_j \leqslant n \leqslant b_j$ and any $1 \leqslant j \leqslant m$, which implies that $y = w^1u^1w^2u^2 \cdots w^{m-1}u^{m-1}w^m y^0$ for some $u^1, u^2, \ldots, u^{m-1} \in \mathcal{L}(X)$ with $|u^j| = k_j$ for any $1 \leqslant j \leqslant m$ and some $y^0 \in X$. Therefore, $w^1u^1w^2u^2 \cdots w^{m-1}u^{m-1}w^m \in \mathcal{L}(X)$.

Now, we assume the second half of this lemma holds. Let

$$M(x, n, \varepsilon) = L(x_1 \cdots x_{n+\min\{k\in\mathbb{N}:N^{k+1}\varepsilon\geqslant 1\}}).$$

Next, we prove that (X, σ) has the specification with a non-uniform gap. Fix $k \in \mathbb{N}$, $x^1, \ldots, x^k \in X$, and $2k$ integers

$$0 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$$

with

$$a_{j+1} - b_j \geqslant M(x^j, b_j - a_j, \varepsilon)$$

for any $1 \leqslant j \leqslant k-1$. Let $w^j = x^j_1 \cdots x^j_{b_j-a_j+\min\{k\in\mathbb{N}:N^{k+1}\varepsilon\geqslant 1\}}$ for any $1 \leqslant j \leqslant k$. By the second half of this lemma, there exists $u^1, \ldots, u^k \in \mathcal{L}(X)$ such that $w^1u^1w^2u^2 \cdots w^{k-1}u^{k-1}w^k \in \mathcal{L}(X)$ and $|u^j| = a_{j+1} - b_j$ for any $1 \leqslant j \leqslant k-1$. Then, there exists $y \in X$ with prefix $w^1u^1w^2u^2 \cdots w^{k-1}u^{k-1}w^k$. By the definition of the metric on $A^{\mathbb{N}^+}$, such y satisfies the requirement of the specification with a non-uniform gap. □

4.1.1. *Shifts satisfying the specification-like properties.* Since the specification property with a non-uniform gap is weaker than the specification property (and even weaker than the non-uniform specification property [25] and the weak specification property [23, Definition 14]), shifts with the (non-uniform or weak) specification property satisfy the hypothesis of Theorem A. Here, we list some of them.

- One-sided (two-sided) full shifts, by [38, Example (A)].
- One-sided (two-sided) mixing subshifts of finite type (including mixing sofic shifts), by [38, Example (C)].
- Mixing cocyclic subshifts [22], by [23, Theorem 47].
- Mixing S-gap shifts for $S = \{n_1, n_2, \ldots\}$ with $\sup_{i\to\infty}(n_{i+1} - n_i) < +\infty$, by [19, Example 3.4].
- β-shifts for a dense set of $\beta \in (1, +\infty)$, by [35].

4.1.2. *β-shifts.* Here, we present one type of one-sided subshifts, β-shifts, basically referring to [6, 33–35]. We denote by $[x]$ and $\{x\}$ the integer and fractional part of the real number x. For any $\beta > 1$, considering the β-transformation $f_\beta : [0, 1) \to [0, 1)$ given by

$$f_\beta(x) = \beta x \bmod 1,$$

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let $b = [\beta]$ for $\beta \notin \mathbb{N}$ and $b = \beta - 1$ for $\beta \in \mathbb{N}^+$. Then, we split the interval $[0, 1)$ into $b + 1$ parts as follows:

$$J_0 = \left[0, \frac{1}{\beta}\right); \quad J_1 = \left[\frac{1}{\beta}, \frac{2}{\beta}\right); \quad \ldots; \quad J_b = \left[\frac{b}{\beta}, 1\right).$$

For $x \in [0, 1)$, let $w(x, \beta) = (w_j(x, \beta))_{j=1}^{\infty}$ be the sequence given by $w_j(x, \beta) = i$ when $f_\beta^{j-1}(x) \in J_i$. We call $i(x, \beta)$ the *greedy β-expansion* of x and we have

$$x = \sum_{j=1}^{\infty} w_j(x, \beta)\beta^{-j}.$$

We call (Σ_β, σ) the β-shift, where σ is the shift map and Σ_β is the closure of $\{w(x, \beta)\}_{x\in[0,1)}$ in $\prod_{i=1}^{\infty}\{0, 1, \ldots, b\}$. From the previous discussion, we can define the greedy β-expansion of 1, denoted by $w(1, \beta)$, as the lexicographic supremum over all β-expansions. It satisfies

$$1 = \sum_{j=1}^{\infty} w_j(1, \beta)\beta^{-j}.$$

Then, we can compute $w(1, \beta)$ by the algorithm in [32, 5.2] as follows:

$$r_0 := 1, \quad w_{j+1}(1, \beta) := \lceil \beta r_j \rceil - 1, \quad r_{j+1} := \beta r_j - w_{j+1}(1, \beta),$$

which ensures that $r_j > 0$ for all $j \in \mathbb{N}^+$. Therefore, the sequence $w(1, \beta) = \{w_j(1, \beta)\}_{j=1}^{\infty}$ cannot end with zeros only. Parry showed that the set of sequences that belong to Σ_β can be characterized as

$$x \in \Sigma_\beta \Leftrightarrow f^n x \leqslant w(1, \beta) \quad \text{for all } n \geqslant 1,$$

where '$\leqslant$' is taken in the lexicographic ordering [31]. By [4],

$$\{\beta \in (1, +\infty) : (\Sigma_\beta, \sigma) \text{ has the specification property}\}$$

is dense in $(1, +\infty)$, but has zero Lebesgue measure. Those β terms are considered in §4.1.1. Here, we deal with other β terms. By [1], a β-shift has the specification property if and only if the $w(1, \beta)$ does not contain arbitrarily long strings of zeroes. It implies that $w(1, \beta)$ is not eventually periodic for those β terms considered in this section.

For those β terms, Pfister and Sullivan proved that β-shifts have a weaker form of the specification, called the g-almost product property (see §2.3) for any $\beta > 1$ in [33, Example on p. 6]. The main observation of [33, Example on p. 6] and [32, Proposition 5.1] is that, given $w \in \mathcal{L}(\Sigma_\beta)$, one can find a word $\hat{w} \in \mathcal{L}(\Sigma_\beta)$, which differs from w by at most one character, and such that for any $w' \in \Sigma_\beta$, the concatenated word $\hat{w}w' \in \mathcal{L}(\Sigma_\beta)$. More precisely, Thompson gave an explanation in [40, §5.1]. The modified word w' is given by replacing the last non-zero entry of the word w by a 0. With the same observation, we can prove Theorem B.

We use the presentation of Σ_β by a labeled graph $\mathcal{G}_\beta$ introduced by Blanchard and Hansel in [2]. Here, we discuss the case where the β-expansion of 1 is not eventually

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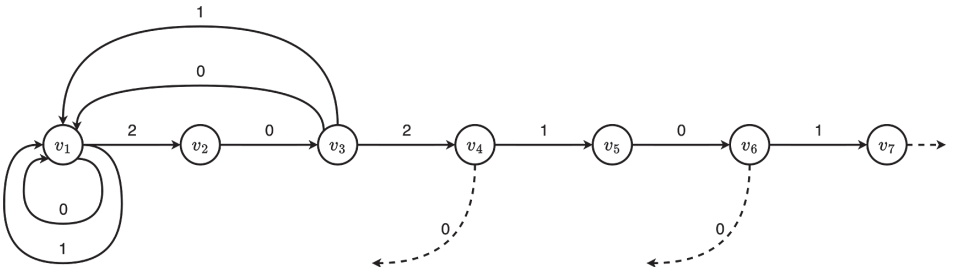


FIGURE 2. The graph $\mathcal{G}_\beta$ for $(w_j(1,\beta))_{j=1}^6 = (2, 0, 2, 1, 0, 1)$.

periodic. Let $v_1, v_2, \ldots$ be a countable set of vertices. For any $j \geqslant 1$, we draw some labeled edges with the following rules.

(1) Draw a directed edge from v_j to v_{j+1} and label it with the value $w_j(1,\beta)$.
(2) If $b > 1$ and $w_j(1,\beta) > 0$, for any $0 \leqslant i < w_j(1,\beta)$, draw a directed edge from v_j to v_1 and label it with the value i.

We have $x \in \Sigma_\beta$ if and only if x labels an infinite path of directed edges of $\mathcal{G}_\beta$ which starts at the vertex v_1. It follows that $w \in \mathcal{L}(\Sigma_\beta)$ corresponds to a finite path of directed edges of $\mathcal{G}_\beta$ that starts at the vertex v_1. Figure 2 depicts part of the graph $\mathcal{G}_\beta$ for a value of β satisfying $(w_j(1,\beta))_{j=1}^6 = (2, 0, 2, 1, 0, 1)$.

Proof of Theorem B. By Proposition 4.1, it remains to prove that (Σ_β, σ) has the specification property with a non-uniform gap. Here, we only prove the case that $m = 2$ in Lemma 4.2 for convenience. Other cases are similar.

The function $L : \mathcal{L}(X) \to \mathbb{N}$ is defined in the following way. Given w, let $l = |w|$. There exists $x \in \Sigma_\beta$ such that $x = (w, (x_{l+1}x_{l+2}\ldots))$, which means that $x_j = w_j$ for any $1 \leqslant j \leqslant l$.

(1) If $x_{l+j} = 0$ for any $j \in \mathbb{N}^+$, then there exists $p \in \mathbb{N}^+$ such that $(w, x_{l+1}, x_{l+2}, \ldots, x_{l+p})$ can be regarded as a path of directed edges in G_β that ends with the vertex v_1 since the β-expansion of 1 is not eventually periodic.
(2) If there exists $q \in \mathbb{N}^+$ such that $x_{l+q} > 0$, then $(w, x_{l+1}, x_{l+2}, \ldots, x_{l+q-1}, 0)$ can be regarded as a path of directed edges in G_β that ends with the vertex v_1.

Therefore, there exists $L(w, x) \in \mathbb{N}^+$ such that $(w, x_{l+1}, x_{l+2}, \ldots, x_{l+L(w,x)-1}, 0)$ can be regarded as a path of directed edges in G_β that ends with the vertex v_1. Let

$$L(w) = \min\{L(w, x) : x \in \Sigma_\beta, x_j = w_j \text{ for any } 1 \leqslant j \leqslant l\}.$$

Fix $w^1, w^2 \in \mathcal{L}(\Sigma_\beta)$ and $k_1 \geqslant L(w^1)$. There exists $x \in \Sigma_\beta$ with prefix w^1 such that $(w^1, x_{l+1}, x_{l+2}, \ldots, x_{l+L(w^1)-1}, 0)$ can be regarded as a path of directed edges in G_β that ends with the vertex v_1. By adding zeroes at the end of $(w, x_{l+1}, x_{l+2}, \ldots, x_{l+L(w)-1}, 0)$, there exists a word u^1 with $|u^1| = k_1$ such that w^1u^1 can be regarded as a path of directed edges in G_β that ends with the vertex v_1. Since w^2 can be regarded as a path of directed edges in G_β that starts with the vertex v_1, one has that $w^1u^1w^2 \in \mathcal{L}(\Sigma_\beta)$, which completes the proof. □

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4.2. *Applications to other systems.* Finally, we recall other examples in dynamic systems that exhibit the exponential specification property.

- Uniformly hyperbolic systems: elementary parts of Axiom A systems, by [38, Proposition 23.20]; transitive Anosov diffeomorphisms, by [17, Proposition 2.11 and Theorem 6.2].
- Mixing expanding map on a compact manifold, by [17, Propositions 2.11 and 6.1].

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