## 102.19 A black hole conundrum

As a potential source of energy, Stephen Hawking suggested dropping small black holes through the Earth, [1, pp. 108-109]. Presumably these black holes would be so small that they would fall through the Earth without resistance and without absorbing appreciable portions of the Earth's mass.

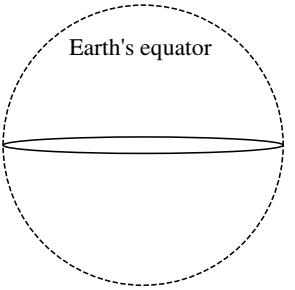



FIGURE 1: The elliptical orbit of a black hole through a stationary Earth

As a fun problem when introducing parametric equations (and in particular epicycloids and hypocycloids) and under our black hole science fiction assumptions, suppose that an international research lab in Borneo (on the Earth's equator) plans to release a black hole. If the Earth is stationary, the black hole yo-yos through the Earth in simple harmonic motion along an Earth diameter. Yet because the Earth rotates, the black hole has initial tangential velocity with respect to the Earth's surface. To model the motion of the black hole we shall artificially assume that the Earth's radius $R$ is exactly 6400 km and that the acceleration $g$ of gravity at the Earth's surface is exactly 98 decimetres/sec$^2$. Assuming that the Earth is a sphere and is uniformly dense, Isaac Newton in the *Principia* (Book I, Proposition 10, Corollary (i)) showed that the black hole's orbit is an ellipse, whose centre is the Earth's centre at the origin in the Earth's equatorial plane, and whose parametric equations are

$$\left(R\cos\sqrt{k}t,\ \frac{h}{R\sqrt{k}}\sin\sqrt{k}t\right), \tag{1}$$

where $h = \dfrac{2\pi R^2}{q}$ is the constant of angular momentum of the black hole, $q = 86164.0916$ seconds is the mean sidereal period of the Earth's rotation about its axis with respect to the fixed stars (a value almost four minutes shorter than the standard Earth day of $p = 60 \times 60 \times 24 = 86400$ seconds), $k \approx 1.53125 \times 10^{-6}$ / sec$^2$ is a constant (where $kR = g$), and at time $t = 0$, the black hole is dropped at $(R,\ 0)$. Thus the period $T$ of the black hole in its elliptical orbit is

$$T = \frac{2\pi}{\sqrt{k}} \approx 5077.5805 \text{ seconds} \approx 84.6263 \text{ minutes}.$$

https://doi.org/10.1017/mag.2018.28 Published online by Cambridge University Press

With respect to the fixed stars, at time $t$, the Borneo lab is at

$$R\left(\cos\frac{2\pi t}{q},\ \sin\frac{2\pi t}{q}\right). \tag{2}$$

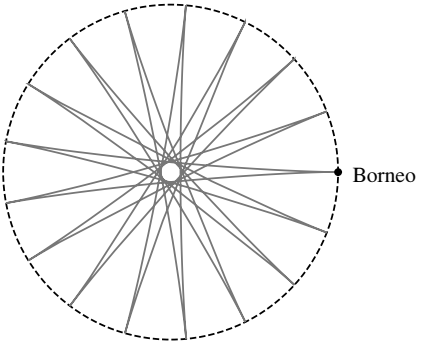



FIGURE 2: Black hole trajectory through a rotating Earth

Once the lab releases the black hole at noon on day $X$, the team hopes to recapture it as a safety drill just to convince themselves that if events go awry they can prevent the black hole from gobbling the entire Earth. Figure 1 shows the orbit of the black hole through the Earth with respect to the fixed stars. Since the Earth rotates, the orbit of the black hole through the Earth with respect to the Earth's centre and Borneo is a hypocycloid, Figure 2, as shown for example in [2] or [3, pp. 198-203].

"Aha," said Otto, a member of the team, after seeing the graph. "Let's recapture our black hole after seventeen oscillations."

"Why is that?" asked Marie, another member of the team.

"Hmm," he hesitated. "One oscillation takes $T \approx 84.6263$ minutes. Dividing a sidereal day by $T$ gives $\omega \approx 16.969\,517\,588$. After seventeen oscillations," he paused again to check his calculations, "the difference between $17T$ and one standard Earth day is about 81 seconds $(17T - p \approx -81.1)$. So at 11:58:39 am on day $X + 1$ we can recover the black hole about 72.2 km west of the lab (about $R \sin\left(\frac{2\pi}{q} \times 17T\right) \approx 72.2$ from the second component of (2))."

"No," replied Marie, "That's in the ocean. Let's wait a bit longer." Seeing Otto's blank look, she elaborated. "Look at the nearest integer continued fraction for $\omega$, an algorithm introduced by Adolf Hurwitz in the late nineteenth century [4]:"

$$\omega \approx \frac{1 \text{ sidereal day}}{5077.5805 \text{ sec}} \approx 16.969\,517\,588$$

$$\approx [17;\ -33,\ -5,\ 7,\ -3] \to \left\{17,\ \frac{560}{33},\ \frac{2783}{164},\ \frac{20041}{1181}\right\}.$$

https://doi.org/10.1017/mag.2018.28 Published online by Cambridge University Press

where

$$[17;-33] = 17 + \frac{-1}{33} = \frac{560}{33}, [17;-33,-5] = 17 + \frac{-1}{33 + \dfrac{-1}{5}} = \frac{2783}{164},$$

and so on.

She continued, "When we plug these denominators 1, 33, 164, 1181 as time in sidereal days into (1), we see that the black hole will surface nearer and nearer to us," directing attention to Table 1.

| periods | 17 | 560 | 2783 | 20041 |
|---|---|---|---|---|
| standard Earth days | 0.999 | 32.91 | 163.55 | 1177.78 |
| time | 11:58:39 am | 9:50:45 am | 1:15:07 pm | 6:36:31 pm |
| kilometres | 72.2 west | 14.0 west | 2.1 east | 0.64 east |

TABLE 1: Black hole recapture dates and distances.

Marie continued, "In particular, take this first convergent, $\frac{560}{33}$."

Otto grunted an assent.

"In about thirty-three sidereal days, the black hole will make about 560 oscillations. In particular, a time lapse of $560T$ is approximately 32.91 standard Earth days (or about 33.0003 sidereal days). A lapse of 0.91 Earth days is about 21 hours, 50 minutes, and 45 seconds. So at 9:50:45 am on day $X + 33$, the black hole surfaces about 14 km west of us. Personally, I like this last convergent," whereupon she pointed to the convergent $\frac{20041}{1181}$.

"But that's over three years from now," Otto protested.

"Yet the resurface site is an easy walking distance east of here," Marie smiled, "and maybe by then we can determine how to recapture black holes."

*References*

1. S. Hawking, *A brief history of time*, Bantam Books (1988).
2. A. Simoson, Black holes through the Mirrour, *Math. Mag.* **82** (2009) pp. 372-381.
3. A. Simoson, *Voltaire's riddle: micromégas and the measure of all things*, Mathematical Association of America (2010).
4. A. Hurwitz, Über eine besondere Art der Kettenbruch entwicklung reeller Grøßen, *Acta Math.* **12** (1889) pp. 367-405.

10.1017/mag.2018.28

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https://doi.org/10.1017/mag.2018.28 Published online by Cambridge University Press