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Abstract

This paper generalizes the Gan–Gross–Prasad (GGP) conjectures that were earlier formulated for tempered or more generally generic L-packets to Arthur packets, especially for the non-generic L-packets arising from Arthur parameters. The paper introduces the key notion of a *relevant pair* of Arthur parameters that governs the branching laws for GL_n and all classical groups over both local fields and global fields. It plays a role for all the branching problems studied in Gan *et al.* [*Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups. Sur les conjectures de Gross et Prasad. I*, Astérisque **346** (2012), 1–109] including Bessel models and Fourier–Jacobi models.

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1. Introduction

This paper is a sequel to our earlier work [GP92, GP94, GGP12a, GGP12b], which discussed several restriction (or branching) problems in the representation theory of classical groups. In the local case, the conjectural answer was given in terms of symplectic root numbers associated to the Langlands parameters (L-parameters). In the global case, for cuspidal automorphic representations, the conjectural answer was given in terms of the central value of automorphic L-functions. However, all of these predictions were for representations lying in local L-packets whose L-parameters are *generic* (in particular, for tempered representations). In this paper, we attempt to generalize these conjectures to certain non-generic L-packets.

We consider only those representations of a classical group G over a local field k that arise as local components of automorphic representations in the automorphic dual (a class of representations which was singled out by Arthur). They have Arthur parameters (A-parameters) of the form

$$\phi_{\mathrm{A}} : WD(k) \times \mathrm{SL}_2(\mathbb{C}) \to {}^{\mathrm{L}}G$$

where the restriction of ϕ_{A} to the Weil–Deligne group $WD(k)$ of k is an admissible homomorphism with bounded image and the restriction to SL_2 is algebraic. By the Jacobson–Morozov theorem, the conjugacy class of $\mathrm{SL}_2(\mathbb{C})$ in the dual group $\widehat{G}(\mathbb{C})$ corresponds to a unipotent conjugacy class in $\widehat{G}(\mathbb{C})$.

One can obtain an L-parameter from an A-parameter as follows. The abelianization of the Weil group is isomorphic to the multiplicative group k^*, by local class field theory. Let $|-|$ be the canonical absolute value on k^* and map the Weil–Deligne group to $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ by

$$w \to (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})).$$

Composing ϕ_{A} with this homomorphism gives an L-parameter ϕ. The map

$$\phi_{\mathrm{A}} \mapsto \phi$$

is an injection from the set of A-parameters to the set of L-parameters. We call L-parameters arising in this way the L-parameters of Arthur type. A particular example is when the restriction of ϕ_{A} to $\mathrm{SL}_2(\mathbb{C})$ is trivial, in which case ϕ is a tempered L-parameter. Hence, we have the containments

$$\{\text{tempered L-parameters}\} \subset \{\text{L-parameters of Arthur type}\} \subset \{\text{L-parameters}\}.$$

We consider the restriction problem only for those representations that are contained in an A-packet, to be called representations of Arthur type. Much of the paper, in fact, deals with a subclass of representations of Arthur type, contained in the L-packet associated to the L-parameter ϕ that is contained in the A-packet of ϕ_{A}.

When the L-parameter ϕ is tempered, the associated L-packet is generic (i.e. contains a generic member), so are covered by our original conjectures in [GGP12a]. In this case, the sum over the L-packet of a pair of classical groups of the multiplicity of restriction is always equal to one, and the adjoint L-function of ϕ is regular and non-zero at the point $s = 1$. On the other hand, when the restriction of ϕ_{A} to $\mathrm{SL}_2(\mathbb{C})$ is non-trivial, the L-parameter ϕ is neither tempered nor generic. In particular, the adjoint L-function of the parameter ϕ is not regular at the point $s = 1$. Indeed, the Lie algebra of the dual group contains the representation $\mathbb{C} \otimes \mathfrak{sl}_2(\mathbb{C})$ of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$, which contributes a simple pole to the adjoint L-function of ϕ at $s = 1$. For the general L-parameters of Arthur type, we conjecture that the sum of multiplicities of

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the restriction over an L-packet of Arthur type for a pair of classical groups is either zero or one. We give a precise conjectural criterion for which L-packets (of Arthur type) is the sum of the multiplicities equal to one. In this case, we further conjecture that the distinguished representation will be selected by the character of the component group of the L-parameter obtained from symplectic root numbers by the same recipe as in the tempered case.

One can consider the analogous restriction problem in the setting of unitary representations, where one works with the direct integral decomposition of a restriction. The work of Clozel [Clo04] in this direction suggests that an irreducible unitary representation of a reductive group with a specific unipotent conjugacy class in its A-parameter can weakly contain only those representations of a subgroup with a specific closely related unipotent class in their A-parameters. The work of Venkatesh [Ven05] makes this precise for the restriction of unitary representations of $\mathrm{GL}_n(k)$ to $\mathrm{GL}_{n-1}(k)$ (indeed, to $\mathrm{GL}_m(k)$ for any $m < n$), by explicating the map from unipotent classes of $\mathrm{GL}_n(\mathbb{C})$ to those of $\mathrm{GL}_{n-1}(\mathbb{C})$.

More precisely, let π be a representation of $\mathrm{GL}_n(k)$ of Arthur type for k a non-archimedean local field, for which the associated unipotent conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ corresponds to the partition

$$n_1 \geq n_2 \geq \cdots \geq n_r \geq 1.$$

Then according to Venkatesh [Ven05], the only unipotent conjugacy class of $\mathrm{GL}_{n-1}(\mathbb{C})$ involved in the restriction problem $\pi|_{\mathrm{GL}_{n-1}(k)}$ from $\mathrm{GL}_n(k)$ to $\mathrm{GL}_{n-1}(k)$, is that given by

$$n_1 - 1 \geq n_2 - 1 \geq \cdots \geq n_r - 1 \geq 0,$$

omitting those n_i which are one, and adding a few ones at the end if necessary. We should add that the work of Clozel and Venkatesh deals only with a crude question: that of determining the possible 'types' (i.e. the unipotent conjugacy class associated to the A-parameter) of representations of H which appear in the spectral decomposition of $\pi|_H$, and not the precise spectral decomposition or which representations of the correct type actually do appear in the spectral decomposition of $\pi|_H$. The extension of Venkatesh's results to the setting of classical groups has been carried out in the PhD thesis [Hen20] of Hendrickson (a student of the first author).

The work of Clozel and Venkatesh is, of course, in the context of unitary representations. In this paper, on the other hand, we work in the setting of smooth representation theory and formulate a conjecture for the restriction of irreducible smooth representations of classical groups in terms of the notion of a *relevant pair* of A-parameters. We find, in particular, that many more unipotent conjugacy classes of $\mathrm{GL}_{n-1}(\mathbb{C})$ are involved in the restriction problem from $\mathrm{GL}_n(k)$ to $\mathrm{GL}_{n-1}(k)$: these are, so to say, of distance one apart from the unipotent conjugacy class of the representation of $\mathrm{GL}_n(k)$ we are starting with.

The definition of a relevant pair of A-parameters is not too complicated but we defer its precise definition to § 3. An elegant reformulation of this notion was given by Zhiwei Yun and discussed in § 4; there, we also give a geometric interpretation in terms of a moment map (in the sense of symplectic geometry) arising in the theory of reductive dual pairs. In the rest of this introduction, we take this notion as a black box.

We first consider the restriction problem for GL_n in § 5. The case of $\mathrm{GL}_n(k)$ is simpler than the case of classical groups as A-packets and L-packets for $\mathrm{GL}_n(k)$ are singleton sets. Let π_M be an irreducible representation of $\mathrm{GL}_n(k)$ with A-parameter M_{A} and associated L-parameter M and let π_N be an irreducible representation of $\mathrm{GL}_{n-1}(k)$ with A-parameter N_{A} and associated L-parameter N. We conjecture that π_N appears as a quotient of the restriction of π_M to $\mathrm{GL}_{n-1}(k)$

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if and only if $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of A-parameters. We prove this in a number of cases for p-adic groups (such as when the Deligne $\mathrm{SL}_2(\mathbb{C})$ in $WD(k)$ acts trivially) using the theory of derivatives of Bernstein and Zelevinsky [BZ77]. Recently, Gurevich [Gur18] has extended this work to prove one direction of the conjecture in all cases. Namely, he showed that

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(k)}(\pi_M, \pi_N) \neq 0 \Longrightarrow (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is a relevant pair of A-parameters.}$$

Gurevich also showed the converse in some cases, such as when at least one of M_{A} or N_{A} is tempered.[1]

We also show that when the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, the ratio of the local L-functions

$$L(M, N, s) = \frac{L(M \otimes N^\vee, s+1/2) \cdot L(M^\vee \otimes N, s+1/2)}{L(M \otimes M^\vee, s+1) \cdot L(N \otimes N^\vee, s+1)} \tag{1.1}$$

does not vanish at the point $s = 0$ (but may have a pole). Note that the L-functions in the denominator are the adjoint L-functions for GL_n and GL_{n-1}, respectively. At least one of these L-functions has a pole at $s = 0$ in the non-tempered case, so the non-vanishing of $L(M, N, s)$ at $s = 0$ means that at least one of the L-functions in the numerator has a pole at $s = 0$. It is not clear to us whether this observation about the analytic behavior of $L(M, N, s)$ at $s = 0$ plays a role in the local restriction problem.

We now consider the problem of restriction from the split group $G = \mathrm{SO}(V) = \mathrm{SO}_{2n+1}(k)$ to the subgroup $H = \mathrm{SO}(W) = \mathrm{SO}_{2n}(k)$ fixing a non-isotropic line in the representation V (and acting on the orthogonal complement W). The representation

$$M_{\mathrm{A}} = \sum_{i=0}^{d} M_i \otimes \mathrm{Sym}^i(\mathbb{C}^2)$$

of dimension $2n$ gives an A-parameter for $G = \mathrm{SO}_{2n+1}$ if and only if the M_i are bounded selfdual representations of $WD(k)$ with M_i symplectic for i even and M_i orthogonal for i odd. Indeed, when these conditions are satisfied, M_{A} is a symplectic representation of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of dimension $2n$. Note that the action of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ on the Lie algebra of $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ is the representation $\mathrm{Sym}^2 M_{\mathrm{A}}$. Similarly, the representation

$$N_{\mathrm{A}} = \sum_{i=0}^{d} N_i \otimes \mathrm{Sym}^i(\mathbb{C}^2)$$

of dimension $2n$ is an A-parameter for $H = \mathrm{SO}_{2n}$ if and only if the N_i are bounded selfdual representations of $WD(k)$, with N_i orthogonal for i even, N_i symplectic for i odd and the quadratic character $\det N_{\mathrm{A}}$ is given by the discriminant of the even orthogonal space W. Indeed, in this case N_{A} is an orthogonal representation of the right dimension and determinant. Note that the action of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ on the Lie algebra of ${}^L H = O_{2n}(\mathbb{C})$ is the representation $\wedge^2 N_{\mathrm{A}}$.

We conjecture in § 6 that there is a representation $\pi_G \otimes \pi_H$ in the L-packet associated to M and N with

$$\dim \mathrm{Hom}_H(\pi_G \otimes \pi_H, \mathbb{C}) = 1$$

[1] As this paper was being revised, we became aware of the paper 'Restriction for general linear groups: the local non-tempered Gan-Gross-Prasad conjecture (non-Archimedean case)' by K.Y. Chan (arXiv:2006.02623), proving the full conjecture for GL_n in the non-archimedean case.

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if and only if the A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ form a relevant pair, and then the representation $\pi_G \otimes \pi_H$ in the L-packet associated to M and N with $\dim \mathrm{Hom}_H(\pi_G \otimes \pi_H, \mathbb{C}) = 1$ is unique. Furthermore, this representation is determined by the character of the component group of the L-parameter, which is given by the root numbers associated to the symplectic representation $M \otimes N$ using the recipe of [GGP12a]. When $(M_{\mathrm{A}}, N_{\mathrm{A}})$ form a relevant pair, we also show that the ratio of L-functions

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)} \tag{1.2}$$

does not vanish at the point $s = 0$, but may have a pole.

In § 7, we offer a conjecture (Conjecture 7.1) on which Arthur packets could have representations with $\mathrm{Hom}_H(\pi_G \otimes \pi_H, \mathbb{C}) \neq 0$, though the conjecture here is not as precise as that in the previous section for L-packets of Arthur type. The point here is that we consider all representations in the A-packet and not just those in the associated L-packet. The less definitive nature of the conjecture is due to the fact that A-packets for classical groups are not disjoint. Thus, for example, Conjecture 7.1 predicts that if $\pi_G \times \pi_H$ is a representation belonging to an A-packet (say for a pair of A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$) and satisfies $\mathrm{Hom}_H(\pi_G \otimes \pi_H, \mathbb{C}) \neq 0$, then $\pi_G \times \pi_H$ belongs to some A-packet for a relevant pair $(M'_{\mathrm{A}}, N'_{\mathrm{A}})$ of A-parameters. We prove certain theorems in § 7 to support this conjecture and give a general construction of Arthur packets where multiplicity greater than one is achieved. We also construct a counterexample to the naive expectation that if $\pi_G \times \pi_H$ is of Arthur type with A-parameter $(M_{\mathrm{A}}, N_{\mathrm{A}})$ and $\mathrm{Hom}_H(\pi_G \otimes \pi_H, \mathbb{C}) \neq 0$, then $(M_{\mathrm{A}}, N_{\mathrm{A}})$ must be relevant.

We note that the notion of a relevant pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of symplectic and orthogonal representations makes sense in general, and we use this to extend the conjecture on Bessel models for $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$ in [GGP12a] to the non-tempered case. It also works well for conjugate symplectic and conjugate orthogonal representations, which allows us to extend the conjectures on Bessel models for $U_{2n+1} \times U_{2m}$ in [GGP12a] to the non-tempered case. One can likewise formulate the analogous local conjecture in the setting of Fourier–Jacobi models for the symplectic/metaplectic groups and the unitary groups, following the procedure in [GGP12a]. We omit the details in this paper and leave the precise formulation to the reader, using [GGP12a] as a guide.

In § 9, we consider the global setting over a number field. Here, one is considering the automorphic period integral of the automorphic forms belonging to the global L-parameters of Arthur type. In the non-tempered setting, these automorphic forms are not necessarily cuspidal and so some regularization may be needed to make sense of the period integrals. A systematic equivariant regularization procedure for reductive periods has been developed in a recent paper of Zydor [Zyd19], following earlier works of others. Our local conjecture implies that there is at most one representation in the global L-packet (of Arthur type), which can have a non-vanishing period integral. Whether this distinguished global representation is automorphic or not is governed by Arthur's multiplicity formula in which a certain quadratic character of the global component group (of the A-parameter) plays a prominent role. In the tempered case, this character is trivial, but in general it is given in terms of certain symplectic global root numbers built out of the adjoint representation. The interaction of Arthur's character with the distinguished character arising from the restriction problem is quite interesting and will be discussed in § 10. In any case, we show that the global analog of the ratio of L-functions in (1.1) or (1.2) is holomorphic at $s = 0$ when $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of global A-parameters and its corresponding global root number is 1 when the distinguished representation in the global L-packet associated to (M, N)

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is automorphic. We then conjecture that its non-vanishing is equivalent to the non-vanishing of the (regularized) period integral.

In § 11, we discuss several families of examples in low-rank classical groups where the restriction problem has been addressed for non-tempered A-packets; these provide some additional support for our local conjecture.

It is interesting to note that in the ongoing work of Chaudouard and Zydor on the Jacquet–Rallis relative trace formula (which has been used to settle special cases of the global Gan–Gross–Prasad (GGP) conjectures for unitary groups), the notion of relevant pair of A-parameters seems to show up in the continuous spectrum on the spectral side. In another direction, we explain in § 12 how the automorphic descent method discovered by Ginzburg, Rallis and Soudry [GRS99a, GRS99b, GRS02, GRS11] and further extended in a recent work of Jiang and Zhang [JZ20] can be predicted using our global conjecture.

Thus, the main innovation of this paper is the introduction of the notion of a relevant pair of A-parameters, which governs the branching laws of all classical groups, at least for representations belonging to L-packets of Arthur type. It is meaningful for all local fields, and global fields, and makes sense for $\mathrm{GL}_n(k)$ as well as all classical groups, and plays a role for all the branching problems studied in [GGP12a] including Bessel models and Fourier–Jacobi models.

The notion of relevant pair of A-parameters appears in this paper from three relatively independent points of views.

(i) The condition was discovered in the course of studying branching laws from $\mathrm{GL}_{n+1}(k)$ to $\mathrm{GL}_n(k)$ via the Bernstein–Zelevinsky filtration of a representation of $\mathrm{GL}_{n+1}(k)$ restricted to its mirabolic subgroup.

(ii) After the work of Ichino and Ikeda, cf. [II10], it is natural to consider the ratio of L-functions $L(M, N, s)$ given in (1.2). From this L-function theoretic point of view, we prove in Theorem 14.1 that under some extra hypothesis, $L(M, N, s)$ has no zeros or poles at $s = 0$ if and only if (M, N) is a relevant pair of A-parameters.

(iii) From the point of view of epsilon factors and tempered GGP, it is Theorem 7.7 due to Waldspurger that brings out the notion of relevant pairs of A-parameters for those A-packets that contain a cuspidal representation.

Finally, from the global point of view, our local conjectures are still incomplete, as Conjecture 6.1 only deals with an L-packet of Arthur type that is a subset of the corresponding A-packet, and Conjecture 7.1 is not as precise as it could be. It would have been ideal to have a precise conjecture for the entire A-packet, because any representation in the A-packet could have been the local component of an automorphic representation and, thus, would intervene in the global period integral problem. It will thus be very interesting to have a precise prediction for the sum of multiplicities over the entire A-packet (over relevant pure inner forms) and a conjectural determination of the representations which have non-zero contribution. Unfortunately, despite trying for a few years, such a precise prediction continues to elude us.

We end the introduction with a general comment on the strategy of the proof of tempered GGP, where one exploits the transfer of the stable distribution on a classical group associated to a tempered L-packet to an invariant distribution on $\mathrm{GL}_n(k)$. The proof of tempered GGP uses such transfer to move the branching problem for classical groups to one on $\mathrm{GL}_n(k)$. For non-tempered A-packets, one still has such a transfer, but the stable distribution associated to an A-packet is not simply the sum of irreducible characters. Instead, it is a linear combination of such with coefficients $\eta(z)$, where z is the center of the Arthur SL_2, so that there are ± 1 in the coefficients.

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Thus one is led to wonder whether there is a nicer expression if, instead of summing over all multiplicities in the A-packet, one considers the signed sum of these multiplicities reflecting the signs in the stable characters.

As an example, one may consider a non-tempered A-packet of U_3 of the form $\{\pi_c, \pi_n\}$, with π_c cuspidal and π_n non-tempered. The cuspidal representation π_c belongs to a discrete L-packet $\{\pi_c, \pi_d\}$, where π_d is a non-cuspidal discrete series representation. There is a exact sequence arising from a principal series representation:

$$0 \to \pi_d \to \mathrm{Ind} \to \pi_n \to 0,$$

so that

$$\pi_c - \pi_n = (\pi_c + \pi_d) - (\pi_d + \pi_n) = (\pi_c + \pi_d) - \mathrm{Ind}$$

is stable. This lends some support to the expectation that for a tempered representation π of $U(2)$,

$$\dim \mathrm{Hom}_{U(2)}[\pi_c, \pi] - \dim \mathrm{Hom}_{U(2)}[\pi_n, \pi],$$

may have a nicer expression than

$$\dim \mathrm{Hom}_{U(2)}[\pi_c, \pi] + \dim \mathrm{Hom}_{U(2)}[\pi_n, \pi].$$

2. Notation and preliminaries

In this paper, unless otherwise specified, we use k to denote a local field. All the conjectures in this paper are formulated for all local fields, archimedean and non-archimedean, but no proofs are given for archimedean local fields. It is hoped that in due course, not only the few proofs we give in this paper, but also the conjectures that we formulate, will be considered by others for archimedean fields.

For a local field k, we let $W(k)$ be the Weil group of k and $WD(k)$ the Weil–Deligne group of k. Thus,

$$WD(k) = \begin{cases} W(k) & \text{if } k \text{ is archimedean;} \\ W(k) \times \mathrm{SL}_2(\mathbb{C}), & \text{if } k \text{ is non-archimedean.} \end{cases}$$

In the non-archimedean case, the $\mathrm{SL}_2(\mathbb{C})$ which arises here is called the Deligne SL_2.

For a reductive algebraic group G over k, let ${}^{\mathrm{L}}G = \widehat{G}(\mathbb{C}) \rtimes W(k)$ be the L-group of G with $\widehat{G}(\mathbb{C})$ the dual group over $\mathbb{C}$. L-parameters for G are admissible homomorphisms

$$\phi : WD(k) \to {}^{\mathrm{L}}G,$$

up to conjugacy by $\widehat{G}(\mathbb{C})$. A-parameters for G are admissible homomorphisms

$$\phi_{\mathrm{A}} : WD(k) \times \mathrm{SL}_2(\mathbb{C}) \to {}^{\mathrm{L}}G$$

up to conjugacy by $\widehat{G}(\mathbb{C})$, where the restriction of ϕ_{A} to the Weil–Deligne group $WD(k)$ is an admissible homomorphism with bounded image for $W(k)$ and the restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic. The extra $\mathrm{SL}_2(\mathbb{C})$ that enters here will be called the Arthur $\mathrm{SL}_2(\mathbb{C})$.

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One can obtain an L-parameter from an A-parameter as discussed in the introduction, using the map $WD(k) \to WD(k) \times \mathrm{SL}_2(\mathbb{C})$ given by

$$w \to (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})).$$

Composing ϕ_{A} with this homomorphism gives an L-parameter ϕ. The association

$$\phi_{\mathrm{A}} \mapsto \phi$$

is an injective map from the set of A-parameters to the set of L-parameters, and we call its image the set of L-parameters of Arthur type.

In the case of classical groups as discussed in [GGP12a], A-parameters are representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ on finite-dimensional complex vector spaces that may come equipped with a symmetric or skew-symmetric bilinear form. For an A-parameter M_{A}, with associated L-parameter M, we often denote the associated complex vector space (the representation space of M_{A} or M) as M; sometimes, for sake of clarity, we may use a different symbol V or W for the representation space of M_{A} or M.

By the work of Harris and Taylor [HT01] for $\mathrm{GL}_n(k)$, Arthur [Art13] for orthogonal and symplectic groups and Mok [Mok15] for unitary groups, the local Langlands conjecture for these quasi-split classical groups over local fields is now known. The extended version due to Vogan [Vog93], involving pure inner forms of these groups, is also known thanks to the work of Kaletha *et al.* [KMSW14] and Mœglin [Mœg07]. We assume this throughout the work, referring to [GGP12a] for precise assertions. For the notion of A-packets, and their relationship to A-parameters, we refer to [Art89, Art13], as well as to many works of Mœglin [Mœg06, Mœg09, Mœg11]. Both for L-packet and A-packet, we simultaneously consider all relevant pure inner forms of the pair of groups involved, and use what may be called Vogan L-packet and Vogan A-packet. We do not dwell further on these matters, as they are totally analogous to [GGP12a].

As is well-known, $\mathrm{Sym}^a(\mathbb{C}^2)$ is the unique $(a+1)$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We often denote this $(a+1)$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ as $[a+1]$. If we regard $[a+1]$ as a representation of the Arthur $\mathrm{SL}_2(\mathbb{C})$, then $[a+1]$ is an A-parameter for $\mathrm{GL}_{a+1}(k)$ whose corresponding A-packet is the singleton set containing only the trivial representation. Hence, by abuse of notation, we also denote by $[a+1]$ the trivial representation of $\mathrm{GL}_{a+1}(k)$. The context will make it clear whether $[a+1]$ denotes an A-parameter or the trivial representation of $\mathrm{GL}_{a+1}(k)$.

If π_1 is a representation of $\mathrm{GL}_m(k)$ and π_2 of $\mathrm{GL}_n(k)$, we let $\pi_1 \times \pi_2$ be the representation of $\mathrm{GL}_{m+n}(k)$ parabolically induced from the representation $\pi_1 \boxtimes \pi_2$ of the Levi subgroup $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ of $\mathrm{GL}_{m+n}(k)$. In particular, $[a] \times [b]$ denotes an irreducible (unitary) representation of $\mathrm{GL}_{a+b}(k)$ that has A-parameter $[a] \oplus [b]$. In § 5, we give a more precise description of the A-packets for $\mathrm{GL}_n(k)$.

For any integer $n \geq 1$, we set

$$\nu := |\det| : \mathrm{GL}_n(k) \to \mathbb{C}^\times.$$

When $n = 1$ so that $\mathrm{GL}_n(k) = k^\times$, we also view ν as the absolute value map on $WD(k)$.

An L-parameter for a classical group G (not $\mathrm{GL}_n(k)$)

$$\phi : WD(k) \to \mathrm{GL}(V),$$

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is called a discrete L-parameter if it does not factor through a proper Levi subgroup of the dual group $\hat{G}$; equivalently, if it is a multiplicity free sum of selfdual or conjugate-selfdual irreducible representations of $WD(k)$ of the correct parity. This terminology originates from the fact that an L-parameter of G is discrete if and only if the associated representations of $G(k)$ are discrete series representations. Similarly, an A-parameter for a classical group (not $\mathrm{GL}_n(k)$)

$$\phi_{\mathrm{A}} : WD(k) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V),$$

is called a discrete A-parameter if it is a multiplicity free sum of selfdual or conjugate-selfdual irreducible representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$.

For all the number theoretic background on L and ϵ factors, we refer to the article [Tat79], usually without explicit mention. All our global L-functions will be completed L-functions $L(s, \Pi)$ for an automorphic representation Π on $\mathrm{GL}_n(\mathbb{A}_F)$ for F a global field, or an automorphic representation on a classical group treated as an automorphic representation Π on $\mathrm{GL}_n(\mathbb{A}_F)$ after transfer.

3. Relevant pair of A-parameters

In this section, we formulate the notion of a relevant pair of A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ in the context of classical groups (including the case of GL). This notion plays a pivotal role in this paper. We also discuss some basic properties of this notion.

With k a local field, recall that a finite-dimensional complex representation M_{A} of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ which arises from an A-parameter (of a classical group) has a canonical decomposition of the form

$$M_{\mathrm{A}} = \sum_{i=0}^{d} M_i \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

where the M_i are bounded admissible representations of $WD(k)$. We say that two representations M_{A} and N_{A} form a relevant pair of A-parameters if we have a decomposition of the respective representations of $WD(k)$ as

$$M_i = M_i^+ + M_i^- \quad \text{and} \quad N_i = N_i^+ + N_i^-,$$

with the property that

$$M_i^+ = N_{i+1}^- \quad \text{for } i \geq 0 \quad \text{and} \quad M_i^- = N_{i-1}^+ \quad \text{for } i \geq 1.$$

Therefore, if we write the decomposition of M_{A} as

$$M_{\mathrm{A}} = M_0^+ + M_0^- + \sum_{i=1}^{d} (M_i^+ + M_i^-) \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

we have the following decomposition of N_{A}:

$$N_{\mathrm{A}} = N_0^- + \sum_{i=0}^{d} M_i^+ \otimes \mathrm{Sym}^{i+1}(\mathbb{C}^2) + \sum_{i=1}^{d} M_i^- \otimes \mathrm{Sym}^{i-1}(\mathbb{C}^2).$$

The notion of a relevant pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is symmetric, as we can also write the conditions on the summands as

$$N_i^+ = M_{i+1}^- \quad \text{for } i \geq 0 \quad \text{and} \quad N_i^- = M_{i-1}^+ \quad \text{for } i \geq 1.$$

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Therefore, if we write the decomposition of N_{A} as

$$N_{\mathrm{A}} = N_0^+ + N_0^- + \sum_{i=1}^{d} (N_i^+ + N_i^-) \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

we have the following decomposition of M_{A}:

$$M_{\mathrm{A}} = M_0^- + \sum_{i=0}^{d} N_i^+ \otimes \mathrm{Sym}^{i+1}(\mathbb{C}^2) + \sum_{i=1}^{d} N_i^- \otimes \mathrm{Sym}^{i-1}(\mathbb{C}^2).$$

Note that the two summands M_0^- and N_0^- in a relevant pair are not constrained by the other parameter. In particular, any two bounded representations (M_0, N_0) of $WD(k)$ with the trivial action of $\mathrm{SL}_2(\mathbb{C})$ form a relevant pair of A-parameters.

The decomposition of the representations M_i and N_i of $WD(k)$ in a relevant pair into components $M_i^{\pm}$ and $N_i^{\pm}$ is unique. Indeed, suppose that the highest (non-zero) summand in the decomposition of M_{A} is M_d. Then the highest summand in the decomposition of N_{A} is either N_{d-1}, N_d or N_{d+1}. We now consider the three cases separately.

(i) Suppose first that it is N_d. Then we conclude that $M_d = M_d^-$ and $N_d = N_d^-$. This implies that $N_{d-1}^+ = M_d^-$, which determines N_{d-1}^-. Similarly, $M_{d-1}^+ = N_d^-$, which determines M_{d-1}^-. Continuing to descend in this manner determines all of the decompositions.

(ii) Next assume that the highest summand in the decomposition of N_{A} is N_{d-1}. Again we conclude that $M_d = M_d^-$, but now we also have $M_{d-1} = M_{d-1}^-$ as $N_d = 0$. Then $N_{d-1}^+ = M_d^-$, which determines N_{d-1}^- and $N_{d-2}^+ = M_{d-1}^-$, which determines N_{d-2}^-. Continuing to descend in this manner gives the full decomposition.

(iii) Finally, if the highest summand in the decomposition of N_{A} is N_{d+1}, then one can switch the roles of M_{A} and N_{A} and use the previous argument.

We have occasion to use the following lemma about relevant pair of A-parameters whose simple proof is omitted.

Lemma 3.1. *Let*

$$M_{\mathrm{A}} = M_0^+ + M_0^- + \sum_{i=1}^{d} M_i \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

$$N_{\mathrm{A}} = N_0^+ + N_0^- + \sum_{i=1}^{d} N_i \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

be a relevant pair of A-parameters. Then we have the relations

$$\sum_{2i-1 \geq 1} M_{2i-1} = \sum_{2i \geq 0} N_{2i} - N_0^-,$$

$$\sum_{2i-1 \geq 1} N_{2i-1} = \sum_{2i \geq 0} M_{2i} - M_0^-.$$

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Suppose now that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a pair of representations with M_{A} symplectic and N_{A} even orthogonal. Then M_{A} is the A-parameter of an odd special orthogonal group whereas N_{A} is that of an even special orthogonal group. If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, the argument we gave for the unicity of the decomposition of M_i and N_i shows that the summands $M_i^{\pm}$ are each symplectic for i even and orthogonal for i odd, and the summands $N_i^{\pm}$ are each orthogonal for i even and symplectic for i odd.

Likewise, suppose that k/k_0 is a separable quadratic extension and $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a pair of representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ with M_{A} conjugate symplectic and N_{A} conjugate orthogonal, then M_{A} is an A-parameter for an even unitary group whereas N_{A} is an A-parameter of an odd unitary group. When $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, one sees that the summands $M_i^{\pm}$ and $N_i^{\pm}$ are conjugate symplectic and conjugate orthogonal, respectively.

On the other hand, suppose that F is a global field with conjectural Langlands group $L(F)$, so that global A-parameters of classical groups can be thought of as finite-dimensional complex representations of $L(F) \times \mathrm{SL}_2(\mathbb{C})$. Of course, because the existence of $L(F)$ is not known, one needs to interpret an irreducible n-dimensional representation of $L(F)$ as a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. In the context of classical groups, one needs to interpret an irreducible n-dimensional symplectic (respectively orthogonal) representation of $L(F)$ as a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for which the exterior square (respectively symmetric square L-function) has a pole at $s = 1$. With this caveat, one can similarly define the notion of a relevant pair of global A-parameters.

Since the work of Ichino and Ikeda, cf. [II10], it is natural to consider in the global setting the ratio of L-functions:

$$L(M, N, s) = \frac{L(M \otimes N^\vee, s + 1/2) \cdot L(M^\vee \otimes N, s + 1/2)}{L(M \otimes M^\vee, s + 1) \cdot L(N \otimes N^\vee, s + 1)}$$

if $M \times N$ is an L-parameter for $\mathrm{GL}_m \times \mathrm{GL}_n$, or the ratio

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)}$$

if $M \times N$ is an L-parameter for $\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}$. One may of course consider $L(M, N, s)$ in the local context as well. We conclude this section by highlighting some results about the analytic properties of $L(M, N, s)$ at $s = 0$. The proofs of these results, which proceed by explicit computation, will be given in §§ 13 and 14 at the end of the paper.

Theorem 3.2. *Let k be a non-archimedean local field and let $(M_{\mathrm{A}}, N_{\mathrm{A}})$ be a relevant pair of A-parameters for $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ with associated pair of L-parameters (M, N). Then the order of pole at $s = 0$ of*

$$L(M, N, s) = \frac{L(M \otimes N^\vee, s + 1/2) \cdot L(M^\vee \otimes N, s + 1/2)}{L(M \otimes M^\vee, s + 1) \cdot L(N \otimes N^\vee, s + 1)}$$

is greater than or equal to zero.

Theorem 3.3. *Let k be a non-archimedean local field and let $(M_{\mathrm{A}}, N_{\mathrm{A}})$ be a pair of A-parameters for $\mathrm{SO}_{2m+1}(k) \times \mathrm{SO}_{2n}(k)$ with associated pair of L-parameters (M, N).*

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(i) *If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of A-parameters, then the order of pole at $s = 0$ of the function*

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)}$$

is greater than or equal to zero.

(ii) *Suppose that M_{A} and N_{A} are multiplicity-free representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ on which the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially. Then, at $s = 0$, the function $L(M, N, s)$ has a zero of order at least zero. It has neither a zero nor a pole at $s = 0$ if and only if $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of A-parameters.*

4. Correlator

In this section, we describe an elegant formulation of the notion of a relevant pair of A-parameters which is due to Zhiwei Yun (personal correspondence).

Consider a pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of selfdual finite-dimensional representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ with M_{A} symplectic and N_{A} orthogonal, realized on vector spaces V and W, respectively. The action of the diagonal torus of $\mathrm{SL}_2(\mathbb{C})$ induces a $\mathbb{Z}$-grading on V and W. More precisely, if we identify $\mathbb{G}_m$ with the diagonal torus, taking $t \in \mathbb{G}_m$ to the diagonal matrix (t, t^{-1}), then the degree n part of V is the eigenspace for the character $t \mapsto t^n$.

Let e (respectively, e') be the nilpotent endomorphism of V (respectively, W) given by the image of the usual upper triangular element in the Lie algebra $\mathfrak{sl}_2$ of $\mathrm{SL}_2(\mathbb{C})$. Then the action of e and e' shifts degree by 2 on V and W. In other words, e and e' are degree 2 elements in $\mathrm{End}\,(V)$ and $\mathrm{End}\,(W)$ (equipped with the induced grading).

The key definition of Zhiwei Yun is as follows.

Definition 4.1. For a pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of finite-dimensional representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ realized on vector spaces (V, W), together with invariant non-degenerate bilinear forms, a correlator for $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a $WD(k)$-equivariant linear map

$$T : V \to W,$$

such that:

(i) T shifts degree by 1, that is, T is an element of degree 1 in $\mathrm{Hom}(V, W)$;
(ii) $T^*T = e$ and $TT^* = e'$, where $T^* : W \to V$ is the adjoint of T.

Remark 4.2. Observe that condition (ii) forces the parity of the forms on V and W to be opposite, unless e and e' are zero.

Now we have the main observation.

Lemma 4.3. *The pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of local A-parameters is relevant if and only if there exists a correlator $T : V \to W$.*

Proof. Given a relevant pair of A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$, we construct a correlator as follows. Writing

$$V = \bigoplus_n V_n \otimes \mathrm{Sym}^n(\mathbb{C}^2),$$

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the action of the diagonal torus of $\mathrm{SL}_2(\mathbb{C})$ gives a further decomposition of the form

$$V = \sum_n \sum_a V_n \otimes t_n^a,$$

where the inner sum is taken over integers $|a| \leq n$ with $a \equiv n (\mathrm{mod} 2)$. Moreover, we have written t_n^a for a non-zero eigenvector in $\mathrm{Sym}^n(\mathbb{C}^2)$ for the character $t \mapsto t^a$. We normalize the choice of t_n^a so that the nilpotent element e maps $V_n \otimes t_n^a$ to $V_n \otimes t_n^{a+2}$. In particular, the kernel of e on V is $\sum_n V_n \otimes t_n^n$.

Now we define a correlator T by giving $WD(k)$-equivariant surjective maps

$$\begin{cases} V_n^+ \otimes t_n^a \longrightarrow W_{n+1}^- \otimes t_{n+1}^{a+1} \\ V_n^- \otimes t_n^a \longrightarrow W_{n-1}^+ \otimes t_{n-1}^{a+1} \end{cases}$$

for each n. In particular,

$$\mathrm{Ker}(T) = \bigoplus_n V_n^- \otimes t_n^n,$$

and one can easily check that T is a correlator.

Conversely, one argues that a correlator T must send $V_n \otimes t_n^a$ to $W_{n+1}^- \otimes t_{n+1}^{a+1} \oplus W_{n-1}^+ \otimes t_{n-1}^{a+1}$ for any $|a| \leq n$. Then $\mathrm{Ker}(T)$ determines V_n^- for each n and, hence, the representation V_n^+ (because V_n is semisimple). Hence, the existence of T implies that the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant. □

Remark 4.4. In the trivial case, when $e = 0$ and $e' = 0$, one can take $T = 0$ and $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is indeed always relevant in this case.

For $(M_{\mathrm{A}}, N_{\mathrm{A}})$ a pair of local A-parameters for $\mathrm{GL}_n \times \mathrm{GL}_m$, we can apply the same reformulation above to the selfdual A-parameters $M_{\mathrm{A}} + M_{\mathrm{A}}^\vee$ and $N_{\mathrm{A}} + N_{\mathrm{A}}^\vee$ (on vector spaces V, W of dimensions $2n, 2m$). A correlator in this situation becomes a pair of maps of $WD(k)$-representations $T : V + V^\vee \to W + W^\vee$ and $T^* : W + W^\vee \to V + V^\vee$, each of degree 1, such that $T^*T = e$ and $TT^* = e'$. One can then check by a similar argument as given previously that the existence of such a correlator is equivalent to the relevance of $(M_{\mathrm{A}}, N_{\mathrm{A}})$.

The notion of a correlator T defined above should remind the reader familiar with the theory of reductive dual pairs (in the sense of Howe) of the moment map that arises there. Let us explicate this connection here. Given a pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of A-parameters where M_{A} (respectively, N_{A}) is a symplectic (respectively, orthogonal) representation of $WD(k)$ on a vector space V (respectively, W), one has a symplectic representation $V \otimes W$ of $WD(k)$:

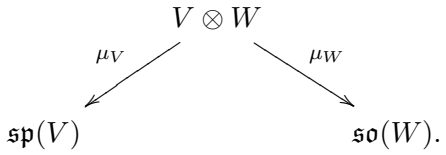
$$WD(k) \longrightarrow \mathrm{Sp}(V) \times O(W) \longrightarrow \mathrm{Sp}(V \otimes W).$$

The $\mathrm{Sp}(V) \times O(W)$-symplectic variety $V \otimes W$ is Hamiltonian and possesses a $WD(k)$-equivariant moment map

$$\mu = \mu_V \times \mu_W : V \otimes W \longrightarrow \mathfrak{sp}(V) \times \mathfrak{so}(W),$$

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where $\mathfrak{sp}(V)$ (respectively, $\mathfrak{so}(W)$) is the Lie algebra of $\mathrm{Sp}(V)$ (respectively, $\mathrm{SO}(W)$), which we prefer to write as a double fibration:



As V and W are equipped with non-degenerate bilinear forms, we have isomorphisms

$$V \otimes W \cong \mathrm{Hom}(V, W) \cong \mathrm{Hom}(W, V).$$

The maps μ_V and μ_W are then given by

$$\mu_V(T) = T^*T \quad \text{and} \quad \mu_W(T) = TT^* \quad \text{for } T \in \mathrm{Hom}(V, W).$$

The double fibration diagram gives one a correspondence between the set of $\mathrm{Sp}(V)$-adjoint orbits and $O(W)$-adjoint orbits: say that a $\mathrm{Sp}(V)$-adjoint orbit $\mathcal{O}_V$ and an $O(W)$-adjoint orbit $\mathcal{O}_W$ correspond if there exists $T \in V \otimes W$ such that

$$\mu_V(T) \in \mathcal{O}_W \quad \text{and} \quad \mu_W(T) \in \mathcal{O}_W.$$

Now we may reformulate the notion of a correlator in terms of this moment map. Given a pair of A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$, recall that one has a pair of nilpotent elements

$$e \in \mathfrak{sp}(V) \quad \text{and} \quad e' \in \mathfrak{so}(W)$$

which are fixed by $WD(k)$. Recalling that $V \otimes W \cong \mathrm{Hom}(V, W)$ has a $\mathbb{Z}$-grading, we see that a correlator for $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a degree 1 element $T \in \mathrm{Hom}(V, W)$ that is fixed by $WD(k)$ and such that

$$\mu_V(T) = e \quad \text{and} \quad \mu_W(T) = e'.$$

Hence, we have a nice geometric formulation of the notion of relevance: $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant if and only if the nilpotent orbits of e and e' are in the moment map correspondence via a $WD(k)$-fixed degree 1 element T.

We note that such correlator maps T have also appeared in the work of Gomez and Zhu [Zhu19] where they studied the transfer of generalized Gelfand–Graev models (attached to nilpotent orbits) under the local theta correspondence. In their work, the correlator maps were considered on the side of representation theory of the real or p-adic groups, whereas in our work, they appeared on the side of the Langlands dual groups.

5. Local conjecture for GL_n

Having introduced the key notion of relevant pairs of A-parameters, we can begin the consideration of restriction problems. In this section, we discuss the restriction problem for $\mathrm{GL}_n(k) \subset \mathrm{GL}_{n+1}(k)$ in the context of L-parameters of Arthur type. We first explicate the representations of $\mathrm{GL}_n(k)$ which belong to such L-packets.

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An A-parameter of $\mathrm{GL}_n(k)$ is an n-dimensional representation of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of the form

$$M_{\mathrm{A}} = \bigoplus_{i=1}^{r} M_i \boxtimes \mathrm{Sym}^{d_i}(\mathbb{C}^2),$$

where M_i is an irreducible m_i-dimensional representation of $WD(k)$. As discussed in the introduction, M_{A} gives rise to an L-parameter M. Moreover, the local A-packet associated to M_{A} is equal to the local L-packet associated to M that is a singleton set. We can describe this unique representation π_M as follows. By the local Langlands correspondence, each M_i corresponds to a discrete series representation π_{M_i} of $\mathrm{GL}_{m_i}(k)$. The representation in the A-packet associated to the A-parameter $M_i \otimes \mathrm{Sym}^{d_i}(\mathbb{C}^2)$ is a Speh representation $\mathrm{Speh}(M_i, d_i)$, which is the unique irreducible quotient of the standard module

$$\pi_{M_i}|\det|^{d_i/2} \times \pi_{M_i}|\det|^{d_i/2-1} \times \cdots \times \pi_{M_i}|\det|^{-d_i/2},$$

where we have used the standard notation for parabolic induction. Then the representation π_M is given by the irreducible parabolic induction

$$\pi_M = \mathrm{Speh}(M_1, d_1) \times \mathrm{Speh}(M_2, d_2) \times \cdots \times \mathrm{Speh}(M_r, d_r).$$

As an example, when $M = \chi \boxtimes \mathrm{Sym}^{n-1}(\mathbb{C}^2)$ with χ a one-dimensional character of $WD(k)$, the associated representation π_M is the character $\chi \circ \det$ of $\mathrm{GL}_n(k)$.

We can now consider the restriction problem. Suppose that M_{A} is an A-parameter of $\mathrm{GL}_{n+1}(k)$ and N_{A} one for $\mathrm{GL}_n(k)$. Let π_M and π_N be the irreducible representations in the respective L-packets (of Arthur type), which we have described previously. By [AGRS10], it is known that

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \leq 1.$$

When M_{A} and N_{A} are tempered A-parameters, so that π_M and π_N are both tempered (and generic), it is well-known that the Hom space given in this equation is one-dimensional. In general, we make the following conjecture.

Conjecture 5.1. In the context given here (k any local field, archimedean or non-archimedean),

$$\mathrm{Hom}_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \neq 0 \Longleftrightarrow \text{the pair } (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ of A-parameters is relevant.}$$

Of course, one could have formulated a restriction problem for an arbitrary $\mathrm{GL}_n \times \mathrm{GL}_m$, that is, including the case of Bessel models and Fourier–Jacobi models. We highlight this particular case here because of its intrinsic simplicity, and defer the general case to the next section where we deal with classical groups.

Before proceeding further, it is good to do a reality check. Suppose that

$$M_{\mathrm{A}} = \mathrm{Sym}^n(\mathbb{C}^2)$$

is the A-parameter of the trivial representation of $\mathrm{GL}_{n+1}(k)$, so that π_M is the trivial representation $\mathrm{GL}_{n+1}(k)$. Then its restriction to $\mathrm{GL}_n(k)$ is certainly the trivial representation, that is,

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the only A-parameter N_{A} of $\mathrm{GL}_n(k)$ for which $\mathrm{Hom}_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \neq 0$ is

$$N_{\mathrm{A}} = \mathrm{Sym}^{n-1}(\mathbb{C}^2).$$

The reader can easily verify that the only N_{A} (of dimension n) such that $(\mathrm{Sym}^n(\mathbb{C}^2), N_{\mathrm{A}})$ is relevant is $N_{\mathrm{A}} = \mathrm{Sym}^{n-1}(\mathbb{C}^2)$.

On the other hand, if we start with $N_{\mathrm{A}} = \mathrm{Sym}^{n-1}(\mathbb{C}^2)$, then the A-parameters M_{A} (of dimension $(n+1)$) of $\mathrm{GL}_{n+1}(k)$ such that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant are:

(i) $M_{\mathrm{A}} = \mathrm{Sym}^n(\mathbb{C}^2)$; or
(ii) $M_{\mathrm{A}} = \mathrm{Sym}^{n-2}(\mathbb{C}^2) \oplus M_0$ where M_0 is a bounded two-dimensional representation of $WD(k)$.

This reflects the known fact that the irreducible admissible representations of $\mathrm{GL}_{n+1}(k)$ which are of Arthur type and which are $\mathrm{GL}_n(k)$-distinguished are either trivial or theta lifts of tempered representations of $\mathrm{GL}_2(k)$ (see [Ven13] for the precise results for *all* irreducible representations of $\mathrm{GL}_{n+1}(k)$ for k a non-archimedean local field).

The rest of this section is devoted to proving a special case of Conjecture 5.1.

Theorem 5.2. *Let k be a non-archimedean local field. Consider a pair of local A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ for $\mathrm{GL}_{n+1}(k) \times \mathrm{GL}_n(k)$ satisfying the following extra properties:*

(a) *on every irreducible summand of M_{A} or N_{A}, at least one of the two $\mathrm{SL}_2(\mathbb{C})$ in*

$$WD(k) \times \mathrm{SL}_2(\mathbb{C}) = W(k) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$$

acts trivially;
(b) *for any irreducible representation ρ of $W(k)$ and $a \geq 1$ an integer,*

$$\rho \otimes \mathbb{C} \otimes \mathrm{Sym}^a(\mathbb{C}^2) \oplus \rho \otimes \mathrm{Sym}^a(\mathbb{C}^2) \otimes \mathbb{C} \not\subseteq M_{\mathrm{A}},$$

and likewise for N_{A}.

Then

$$\mathrm{Hom}_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \neq 0 \Longleftrightarrow (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is relevant.}$$

In particular, the two properties are satisfied and, hence, the result holds in the case when the Deligne $\mathrm{SL}_2(\mathbb{C})$ (i.e. the $\mathrm{SL}_2(\mathbb{C})$ in $WD(k)$) acts trivially on both M_{A} and N_{A}.

Proof. The proof proceeds by analyzing the restriction of the irreducible representation π_M of $\mathrm{GL}_{n+1}(k)$ to its mirabolic subgroup. This restriction was described in a classic theorem of Bernstein and Zelevinsky in [BZ77], in terms of certain induced representations arising from the Bernstein–Zelevinsky derivatives π_M^i of the representation π_M. As the mirabolic subgroup contains $\mathrm{GL}_n(k)$, this gives one a way to understand the restriction of π_M to $\mathrm{GL}_n(k)$.

By this theorem of Bernstein and Zelevinsky, if $\mathrm{Hom}_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \neq 0$, then for some $i \geq 0$, there exists:

(i) an irreducible composition factor A of $\nu^{1/2}(\pi_M)^{i+1}$;
(ii) an irreducible composition factor B of $((\pi_N^\vee)^i)^\vee$;

such that

$$\mathrm{Hom}_{\mathrm{GL}_{n-i}(k)}(A, B) \neq 0.$$

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For the converse, given some (i, A, B) as described previously, suppose that we are lucky enough to have the further property:

(i) for any $j \leq i$ and any $(C, D) \neq (A, B)$, where C is an irreducible composition factor of $\nu^{1/2}(\pi_M)^{j+1}$ and D is an irreducible composition factor of $((\pi_N^\vee)^j)^\vee$,

$$\mathrm{Ext}^1_{\mathrm{GL}_{n-j}(k)}[C, D] = 0.$$

Then the non-zero homomorphism from a certain submodule of π_M to π_N represented by a homomorphism in $\mathrm{Hom}_{\mathrm{GL}_{n-i}(k)}[A, B] \neq 0$, extends to all of π_M.

To have this vanishing of $\mathrm{Ext}^1[C, D]$, the simplest reason would be when the cuspidal support of any irreducible composition factor of $\nu^{1/2}(\pi_N^\vee)^j$ and of $(\pi_M^{j+1})^\vee$ are different (except for a unique choice (A, B) among all pairs (C, D)). The conditions (a) and (b) in the theorem are imposed to ensure this.

Denote by $[d]$ the trivial representation of $\mathrm{GL}_d(k)$, and by $[d]^{\times m}$ the irreducible admissible representation of $\mathrm{GL}_{d\cdot m}(k)$, which is $[d] \times \cdots \times [d]$. If σ is a cuspidal representation of $\mathrm{GL}_r(k)$, let $\sigma[d]$ be the Speh representation of $\mathrm{GL}_{r\cdot d}(k)$ constructed from σ, that is, in the Zelevinsky notation $\sigma[d] = Z[\sigma\nu^{-(d-1)/2}, \ldots, \sigma\nu^{(d-1)/2}]$. It is known that the only non-zero derivative of $\sigma[d]$ is $\sigma[d]^i$ for $i = 0, r$, and $\sigma[d]^r = \nu^{-1/2}\sigma[d-1] = Z[\sigma\nu^{-(d-1)/2}, \ldots, \sigma\nu^{(d-3)/2}]$.

Denote by $\mathrm{St}_d[\rho]$, the generalized Steinberg representation of $\mathrm{GL}_{dd'}(k)$ where ρ is a cuspidal unitary representation of the general linear group $\mathrm{GL}_{d'}(k)$. The derivatives of $\mathrm{St}_d[\rho]^i$ are known to be non-zero only for $i = jd'$ for some $0 \leq j \leq d$ with $\mathrm{St}_d[\rho]^{jd'} = \nu^{j/2}\mathrm{St}_{d-j}[\rho]$. A tempered representation of $\mathrm{GL}_m(k)$ is built as a product of the generalized Steinberg representations $\mathrm{St}_d[\rho]$.

For a cuspidal representation ρ of $\mathrm{GL}_m(k)$ (or the corresponding irreducible representation of $W(k)$), we call the set of representations $\{\rho\nu^i \mid i \in \mathbb{Z}\}$, the cuspidal line passing through ρ. By Leibnitz rule, if a representation has cuspidal support contained in the cuspidal line passing through ρ, all its derivatives have the same property. Therefore, the following analysis which mostly involves dealing with the non-tempered parts of π_M and π_N, to find an irreducible composition factor A of $\nu^{1/2}(\pi_M)^{i+1}$, and an irreducible composition factor B of $((\pi_N^\vee)^i)^\vee$, such that $\mathrm{Hom}_{\mathrm{GL}_{n-i}(k)}[A, B] \neq 0$, we can focus attention on a cuspidal line passing through a fixed cuspidal representation ρ, that is, we can assume that the non-tempered parts of M_{A} and N_{A} when restricted to $W(k)$ are multiples of ρ, without any constraint on the tempered parts of M_{A} and N_{A}. Replacing ρ (an irreducible bounded representation of $W(k)$) by the trivial representation of $W(k)$ has no effect on the following analysis, and this is what we assume in the rest of the proof, that is, we assume in the rest of the proof that the non-tempered parts of M_{A} and N_{A} when restricted to $W(k)$ are multiples of the trivial representation.

Let

$$\pi_M = [a_1] \times \cdots \times [a_r] \times [b_1] \times \cdots \times [b_s] \times \mathrm{St}[c_1] \times \cdots \times \mathrm{St}[c_t] \times \mathrm{St}[d_1] \times \cdots \times \mathrm{St}[d_u],$$
$$\pi_N = [a'_1] \times \cdots \times [a'_{r'}] \times [b'_1] \times \cdots \times [b'_{s'}] \times \mathrm{St}[c'_1] \times \cdots \times \mathrm{St}[c'_{t'}] \times \mathrm{St}[d'_1] \times \cdots \times \mathrm{St}[d'_{u'}],$$

where $\mathrm{St}[a]$ denotes the Steinberg representation of $\mathrm{GL}_a(k)$. There could be a further non-Steinberg tempered part in π_M, and in π_N, which we do not write here out for brevity of notation and which will not play any role in the arguments given in the following. Note also that $\mathrm{St}[1] = [1]$, the trivial representation of $\mathrm{GL}_1(k)$; in fact, we use the notation $\mathrm{St}[a]$ in the expressions given previously for π_M, π_N only for $a > 1$.

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This notation is so made that for calculating $(j+1)$th derivative of π_M, and jth derivative of π_N, we differentiate all the terms $[a_i], [a_i'], \mathrm{St}[d_i], \mathrm{St}[d_i']$ at least once, and no more for $[a_i], [a_i']$; whereas none of the terms involving $[b_i], [b_i'], \mathrm{St}[c_i], \mathrm{St}[c_i']$ are to be differentiated. If the cuspidal support of an irreducible composition factor C of $\nu^{1/2}\pi_M^{j+1}$ and D of $(\pi_N^\vee)^{j^\vee}$ are to be the same, we must have equality of cuspidal supports of $\tilde{C}$, and $\tilde{D}$ defined as follows:

$$\begin{aligned}\tilde{C} &= \nu^{1/2}\{\nu^{-1/2}[a_1-1]\times\cdots\times\nu^{-1/2}[a_r-1]\times[b_1]\times\cdots\times[b_s]\\ &\quad\times\mathrm{St}[c_1]\times\cdots\times\mathrm{St}[c_t]\times\nu^{e_1/2}\mathrm{St}[d_1-e_1]\times\cdots\times\nu^{e_u/2}\mathrm{St}[d_u-e_u]\},\\ \tilde{D} &= \nu^{1/2}[a_1'-1]\times\cdots\times\nu^{1/2}[a_{r'}'-1]\times[b_1']\times\cdots\times[b_{s'}']\times\\ &\quad\times\mathrm{St}[c_1']\times\cdots\times\mathrm{St}[c_{t'}']\times\nu^{-f_1/2}\mathrm{St}[d_1'-f_1]\times\cdots\times\nu^{-f_{u'}/2}\mathrm{St}[d_{u'}'-f_{u'}].\end{aligned}$$

If the cuspidal supports of the representations $\tilde{C}$ and $\tilde{D}$ are to be the same, we draw the following conclusions from Lemma 5.3. Let us recall that the cuspidal support of $\mathrm{St}[a]$ and $[a]$ is the same, so when using Lemma 5.3, we replace all $\mathrm{St}[a]$ in the previous expressions for $\tilde{C}$ and $\tilde{D}$ by the corresponding $[a]$. Further, the hypothesis in Lemma 5.3 is easily seen to be the same as the hypothesis in the Theorem 5.2 that for any integer $a\geq 1$, the representation $\mathbb{C}\otimes\mathrm{Sym}^a(\mathbb{C}^2)\oplus\mathrm{Sym}^a(\mathbb{C}^2)\otimes\mathbb{C}$ does not appear in M_{A} or in N_{A}.

(i) There is no differentiation for $\mathrm{St}[d_i]$ and $\mathrm{St}[d_i']$.
(ii) For each $[a_i]$ in π_M (which get differentiated once), there is exactly one of $[a_i-1]$ or $\mathrm{St}[a_i-1]$ amongst $[b_i']$ or $\mathrm{St}[c_i']$ for π_N (these are not differentiated in achieving π_N^j).
(iii) For each $[a_i']$ in π_N (which get differentiated once), there is exactly one of $[a_i'-1]$ or $\mathrm{St}[a_i'-1]$ amongst $[b_i]$ or $\mathrm{St}[c_i]$ for π_M (these are not differentiated in achieving π_M^{j+1}).
(iv) Each of the terms in $\tilde{C}$ and $\tilde{D}$ (in expressing these representations as a sum, in the Grothendieck group of representations, of products of representations of the form $[a]$ and $\mathrm{St}[b]$) is matched through conclusions (i), (ii), and (iii).

It follows that if the cuspidal supports of $\tilde{C}$ and $\tilde{D}$ are to be the same, we must have the following structure for π_M and π_N (up to multiplication by a tempered representation π_M^t in π_M and up to multiplication by a tempered representation π_N^t in π_N; in using the Leibnitz rule to calculate π_M^{j+1} and π_N^j, one takes the highest possible derivatives for π_M^t and π_N^t, which has the effect that the terms π_M^t and π_N^t do not show up in π_M^{j+1} and π_N^j):

$$\begin{aligned}\pi_M &\overset{\text{(i)}}{=} [b_1'+1]\times\cdots\times[b_{s'}'+1]\times[c_1'+1]\times\cdots\times[c_{t'}'+1]\times[b_1]\times\cdots\times[b_s]\\ &\quad\times\mathrm{St}[c_1]\times\cdots\times\mathrm{St}[c_t],\\ \pi_N &\overset{\text{(ii)}}{=} [b_1+1]\times\cdots\times[b_s+1]\times[c_1+1]\times\cdots\times[c_t+1]\times[b_1']\times\cdots\times[b_{s'}']\\ &\quad\times\mathrm{St}[c_1']\times\cdots\times\mathrm{St}[c_{t'}'].\end{aligned}$$

For these representations π_M, π_N, if there is a composition factor C of $\nu^{1/2}\pi_M^{j+1}$, and D of $(\pi_N^\vee)^{j^\vee}$, for which C, D have the same cuspidal support as representations of $\mathrm{GL}_{n-j}(k)$, the representations $\tilde{C}$ and $\tilde{D}$ must be

$$\begin{aligned}\tilde{C} &\overset{\text{(iii)}}{=} [b_1']\times\cdots\times[b_{s'}']\times[c_1']\times\cdots\times[c_{t'}']\times\nu^{1/2}\{[b_1]\times\cdots\times[b_s]\times\mathrm{St}[c_1]\times\cdots\times\mathrm{St}[c_t]\},\\ \tilde{D} &\overset{\text{(iv)}}{=} \nu^{1/2}\{[b_1]\times\cdots\times[b_s]\times[c_1]\times\cdots\times[c_t]\}\times[b_1']\times\cdots\times[b_{s'}']\times\mathrm{St}[c_1']\times\cdots\times\mathrm{St}[c_{t'}'].\end{aligned}$$

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By Lemma 5.5, such representations $\tilde{C}, \tilde{D}$ are irreducible, and for $\mathrm{Hom}_{\mathrm{GL}_{n-j}(k)}[\tilde{C}, \tilde{D}]$ to be non-zero, we must have

$$c_i = 1 \quad \text{for all } i,$$
$$c'_i = 1 \quad \text{for all } i.$$

Thus, as a consequence, we find that if any of the spaces $\mathrm{Hom}_{\mathrm{GL}_{n-j}(k)}[C, D]$ is non-zero, then $M_{\mathrm{A}}, N_{\mathrm{A}}$ are a relevant pair of A-parameters.

Next, we prove that $\mathrm{Hom}_{\mathrm{GL}_n(k)}[\pi_M, \pi_N] \neq 0$ when the A-parameter M_{A} for an irreducible representation π_M of $\mathrm{GL}_{n+1}(k)$ and N_{A} for an irreducible representation π_N of $\mathrm{GL}_n(k)$ satisfies all conditions in the statement of this theorem with $M_{\mathrm{A}}, N_{\mathrm{A}}$, a relevant pair of A-parameters. With the structure of π_M, π_N given in conclusions (i) and (ii), the only way to obtain an irreducible composition factor $\tilde{C}$ of $\nu^{1/2}\pi_M^{j+1}$ and $\tilde{D}$ of $(\pi_N^\vee)^{j^\vee}$ to have same cuspidal supports is that none of the terms $\mathrm{St}(c_i), \mathrm{St}(c'_i)$ should be present in the expressions for π_M and π_N given in conclusions (i) and (ii), and also in $\tilde{C}, \tilde{D}$ given in conclusions (iii) and (iv), that is, we must take full derivative of all tempered factors of π_M that do not match-up with presence of a [2] in π_N and we must take full derivative on all tempered factors of π_N which do not match-up with presence of a [2] in π_M.

Thus, the strategy outlined in the beginning of the proof of the theorem that there is a unique j, and a unique irreducible composition factor C of $\nu^{1/2}\pi_M^{j+1}$ and D of $(\pi_N^\vee)^{j^\vee}$ with the same cuspidal supports holds; we elaborate more on this now.

Write

$$\pi_M = [b'_1 + 1] \times \cdots \times [b'_{s'} + 1] \times [b_1] \times \cdots \times [b_s] \times \tau_1,$$
$$\pi_N = [b_1 + 1] \times \cdots \times [b_s + 1] \times [b'_1] \times \cdots \times [b'_{s'}] \times \tau_2,$$

where τ_1, τ_2 are tempered representations, not generalized Steinberg, of $\mathrm{GL}_{d_1}(k)$ and $\mathrm{GL}_{d_2}(k)$, for certain integers d_1, d_2 with

$$s' + d_1 + \sum_{i=1}^{s} b_i + \sum_{i=1}^{s'} b'_i = n + 1,$$
$$s + d_2 + \sum_{i=1}^{s} b_i + \sum_{i=1}^{s'} b'_i = n.$$

We see that there is exactly one integer j, which, by the previous equations, has the property that

$$j + 1 = s' + d_1,$$
$$j = s + d_2,$$

and for this integer j, there is only one composition factor, say A, of $\nu^{1/2}\pi_M^{j+1}$ and one of $(\pi_N^\vee)^{j^\vee}$, say B, for which $\mathrm{Hom}_{\mathrm{GL}_{n-j}(k)}[A, B]$ is non-zero, and all the other composition factors of $\nu^{1/2}\pi_M^{j+1}$ and of $(\pi_N^\vee)^{j^\vee}$ have different cuspidal supports, and this is also the case for any composition factors of $\nu^{1/2}\pi_M^{i+1}$ and of $(\pi_N^\vee)^{i^\vee}$ for $i \neq j$. (A subtlety in this argument may be pointed out, which is that a particular composition factor may appear with higher (Jordan–Hölder) multiplicity, for example, the first derivative of the representation 1×1 of $\mathrm{GL}_2(k)$ is the trivial representation of

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$\mathrm{GL}_1(k)$ with multiplicity two, one which appears as a submodule, and the other which appears as a quotient. This subtlety will show up as soon as $b_i = b'_j + 1$ for some i, j. As we are not trying to prove multiplicity one theorem for $\mathrm{Hom}(\pi_M, \pi_N)$, but rather only the existence of a non-zero homomorphism, this subtlety does not create a problem for us.)

Therefore,

$$\mathrm{Ext}^{\ell}_{\mathrm{GL}_{n-i}(k)}[C, D] = 0, \quad \text{for } (A, B) \neq (C, D), \quad \text{and for all } \ell \geq 0,$$

completing the proof of the 'existence' part of the theorem, that is, when the parameters $M_{\mathrm{A}}, N_{\mathrm{A}}$ are a relevant pair of A-parameters, and are representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ as in the statement of the theorem, $\mathrm{Hom}_{\mathrm{GL}_n(k)}[\pi_M, \pi_N] \neq 0$. The other part of the theorem was already proved earlier, thus the proof of the theorem is complete. □

LEMMA 5.3. *Suppose that the cuspidal supports of*

$$V = [a_1] \times \cdots \times [a_r] \times \nu^{e_1/2}[b_1] \times \cdots \times \nu^{e_s/2}[b_s],$$
$$W = [c_1] \times \cdots \times [c_t] \times \nu^{1/2}\{[d_1] \times \cdots \times [d_u]\} \times \nu^{-f_1/2}[g_1] \times \cdots \times \nu^{-f_v/2}[g_v],$$

are the same (where $e_i > 0$, $f_i > 0$ are integers). Assume that the following two conditions (V), (W) are satisfied:

(V) $a_i + 1 \neq b_j + e_j - 1$ *for any pair* (i, j) *with* $e_j > 1$;
(W) $d_i + 1 \neq g_j + f_j$ *for any pair* (i, j).

Then:

(i) $e_i = 1$ *for all* i;
(ii) $f_j = 0$ *for all* j;
(iii) *the set of irreducible representations appearing in writing V as a product and W as a product is unique up to permutation of the representations involved.*

Proof. We prove the lemma by choosing a component $[a_i]$ or $\nu^{e_j/2}[b_j]$ from V with the property that V contains ν^x, with x the maximum half integer such that ν^x is contained in the cuspidal support of V, and proving that this component ($[a_i]$ or $\nu^{e_j/2}[b_j]$) lies in W too. Removing this common component from V and W proves the lemma by an inductive process. (Recall that the cuspidal support of V is the union with multiplicities of the support of individual factors in it, and that the cuspidal support of $[a]$ is $\{\nu^{-(a-1)/2}, \ldots, \nu^{(a-1)/2}\}$.

The proof of the lemma will be carried out in the following two cases.

Case 1. Suppose V contains ν^x, with x the maximum half integer such that ν^x is contained in the cuspidal support of V, and that $\nu^x \in [a_i] \subset V$.

In this case, ν^{-x} also belongs to the support of V, $\{-x, x\}$ are the extreme points of the support of $[a_i]$ as well as V (because $e_i > 0$). We would like to prove that $[a_i] = [c_j]$ for some $[c_j] \subset W$. Assume the contrary, that is, $x \neq c_j$ for any $[c_j] \subset W$.

As $f_j > 0$, if $\nu^x \in \nu^{-f_j/2}[g_j] \subset W$, then $\nu^{-y} \in W$ for some $y > x$, a contradiction to the fact that ν^{-y}, $y > x$ does not belong to the support of V.

If $\nu^x \in \nu^{1/2}[d_j] \subset W$, then $x = d_j/2$. As $\nu^{-x} \in V$, ν^{-x} also belongs to W, but ν^{-x} does not belong to terms of the form $[c_j], \nu^{1/2}[d_j] \subset W$, so the only option is that ν^{-x} belongs to $\nu^{-f_j/2}[g_j] \subset W$, that is, $2x = f_j + g_j - 1$. On the other hand, $x = d_j/2$.

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Thus, W contains both $\nu^{1/2}[d_j]$ and $\nu^{-f_j/2}[g_j]$ with

$$2x = d_j = f_j + g_j - 1,$$

which is not allowed by the condition (W) in the statement of the lemma. Thus, the only option left is that $\nu^x \in [c_j] \subset W$.

Case 2. Suppose V contains ν^x, with x the maximum half integer such that ν^x is contained in the cuspidal support of V, and that $\nu^x \in \nu^{e_i/2}[b_i] \subset V$ with e_i minimal positive integer, and with $\nu^x \notin [a_i]$ for any $[a_i] \subset V$.

In this case, ν^{-x} does not belong to the support of V, hence does not belong to the support of W either. It follows that ν^x, which is supposed to belong to W, must belong to either $\nu^{1/2}[d_j]$ or $\nu^{-f_j/2}[g_j]$. Clearly, ν^x cannot belong to $\nu^{-f_j/2}[g_j]$ because otherwise ν^{-x} will belong to the support of W.

If ν^x belongs to $\nu^{1/2}[d_j] \subset W$, x being the largest exponent in W, $x = d_j/2$ and $\nu^{-x+1} \in \nu^{1/2}[d_j]$, hence $\nu^{-x+1} \in V$. If $e_i > 1$, the only option is that $\nu^{-x+1} \in [2x-1] \subset V$. However, we already have $\nu^x \in \nu^{e_i/2}[b_i] \subset V$, so V contains $[2x-1]$ as well as $\nu^{e_i/2}[b_i]$ with

$$2x = e_i + b_i - 1,$$

once again a contradiction to our condition (V) in the statement of the lemma. Thus, $e_i = 1$, $\nu^x \in \nu^{1/2}[b_i] \subset V$ and $\nu^x \in \nu^{1/2}[d_j] \subset W$, completing the proof of the lemma. □

Example 5.4. We present two examples to show that the conditions in Lemma 5.3 are necessary:

(i) $[a] \times \nu[a]$ and $\nu^{1/2}[a+1] \times \nu^{1/2}[a-1]$ have the same cuspidal support; said another way, $\nu^{1/2}[a] \times \nu^{-1/2}[a]$ and $[a+1] \times [a-1]$ have the same cuspidal support;
(ii) $\nu^{1/2}[n-1] \times \nu^{-(n-1)/2}$ and $[n]$ have the same cuspidal support.

We thank A. Minguez for the proof of the following lemma.

Lemma 5.5. *The representations*

$$V = [a_1] \times \cdots \times [a_r] \times \nu^{1/2}\{[b_1] \times \cdots \times [b_s] \times \mathrm{St}[d_1] \times \cdots \times \mathrm{St}[d_u]\},$$
$$W = \nu^{1/2}\{[a'_1] \times \cdots \times [a'_{r'}]\} \times [b'_1] \times \cdots \times [b'_{s'}] \times \mathrm{St}[d'_1] \times \cdots \times \mathrm{St}[d'_{u'}],$$

are irreducible, and are isomorphic if and only if the expressions for V, W as products are the same up to a permutation, in particular, if $V \cong W$, then $d_i, d'_i \leq 1$.

Proof. Irreducibility is part of Theorem 3.9 of the paper [BLM13] due to Badulescu *et al.*, which gives a sufficient condition for a product $Z(\mathfrak{m}) \times L(\mathfrak{m}')$ to be irreducible: when no segment of $\mathfrak{m}$ is juxtaposed (in the obvious sense) to a segment in $\mathfrak{m}'$.

To prove that $V \cong W$ if and only if the expressions for V, W as products are the same up to a permutation, define

$$R = \sum_{n \geq 0} R_n,$$

where $R_0 = \mathbb{Z}$, and R_d is the Grothendieck group of finite length representations of $\mathrm{GL}_d(k)$. Then R is a ring under the product $R_n \times R_m \to R_{n+m}$ given by parabolic induction.

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The ring R is known to be a polynomial ring $\mathbb{Z}[S(C)]$ on indeterminates $S(C)$, where $S(C)$ is the set of all segments $\{\rho, \rho\nu, \dots, \rho\nu^{n-1}\}$ where ρ is a cuspidal representation of some $\mathrm{GL}_d(k)$. There is a natural map from the polynomial ring $\mathbb{Z}[S(C)]$ to R taking $S(C)$ to the unique irreducible submodule of the full parabolic induction: $\rho \times \rho\nu \times \cdots \times \rho\nu^{n-1}$, which is an isomorphism of rings. In particular, R is a unique factorization domain (UFD). Further, it is known (by Tadic [Tad86, Section 3]) that the representations $[a]$ and $\mathrm{St}[a]$ are prime elements of the ring R, proving the second part of the lemma. □

Remark 5.6. An explicit example of a representation π_M of $\mathrm{GL}_5(k)$ and π_N of $\mathrm{GL}_4(k)$ was provided by Chan [Cha19] where the previous proof cannot be carried out to prove that $\mathrm{Hom}_{\mathrm{GL}_4(k)}(\pi_M, \pi_N) \neq 0$. For this, take the representation π_M of $\mathrm{GL}_5(k)$ and π_N of $\mathrm{GL}_4(k)$ which in the previous notation are

$$\pi_M = [3] \times [1] \times [1],$$
$$\pi_N = \mathrm{St}[2] \times [2].$$

In this case, it is easy to see $\nu^{1/2}(\pi_M)^{i+1}$, and $((\pi_N^\vee)^i)^\vee$ have the same cuspidal support for $i = 1, 2$, and the strategy of the proof of the theorem to conclude $\mathrm{Hom}_{\mathrm{GL}_4(k)}(\pi_M, \pi_N) \neq 0$ fails. In fact, we do not know at this point whether $\mathrm{Hom}_{\mathrm{GL}_4(k)}(\pi_M, \pi_N) = 0$ or is non-zero although the representations have a pair of relevant A-parameters.

The proof of the following proposition follows from the proof of Theorem 5.2.

Proposition 5.7. *Let π_M be an irreducible admissible representation of $\mathrm{GL}_{n+1}(k)$ with A-parameter M_A of dimension $(n+1)$, and π_N be an irreducible admissible representation of $\mathrm{GL}_n(k)$ with A-parameter N_A of dimension n. Assume that the representations $M_\mathrm{A}, N_\mathrm{A}$ of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ are, in fact, representations of $W(k) \times \mathrm{SL}_2(\mathbb{C})$. Then, if*

$$\mathrm{Ext}^i_{\mathrm{GL}_n(k)}(\pi_M, \pi_N) \neq 0, \quad \text{for some } i \geq 0,$$

$M_\mathrm{A}, N_\mathrm{A}$ are a relevant pair of A-parameters.

Remark 5.8. It appears to be an interesting problem to understand when $\mathrm{Ext}^i_{\mathrm{GL}_n(k)}(\pi_M, \pi_N)$ is non-zero for some $i \geq 0$, and then to calculate them, among irreducible representations π_M, π_N of $\mathrm{GL}_n(k)$ with A-parameters $M_\mathrm{A}, N_\mathrm{A}$ (where there is the necessary condition that $M_\mathrm{A}, N_\mathrm{A}$ restricted to $W(k) \times \Delta\mathrm{SL}_2(\mathbb{C}) \subset WD(k) \times \mathrm{SL}_2(\mathbb{C})$ are isomorphic), and also for the restriction problem when π_M is an irreducible representation of $\mathrm{GL}_{n+1}(k)$ and π_N of $\mathrm{GL}_n(k)$, both with an A-parameter, where the previous proposition provides the answer in the first non-trivial case. See a recent work of Chan [Cha19] for some results in both the cases.

6. Local conjecture for classical groups

In this section, we consider the restriction problem for classical groups. Hence, we shall work in the context of [GGP12a]. More precisely, we fix local fields $k_0 \subset k$ with $[k : k_0] \leq 2$. Then, with $\epsilon = \pm$, we consider a pair of non-degenerate ϵ-Hermitian spaces $W \subset V$ over k such that:

(i) $\epsilon \cdot (-1)^{\dim W^\perp} = -1$;
(ii) $W^\perp$ is a split space.

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Setting $G(V)$ to be the identity component of the automorphism group of the space V, we assume further that $G(V)$ is quasi-split. We thus have a diagonal embedding

$$G(W)^{\Delta} \hookrightarrow G(V) \times G(W).$$

Indeed, as was explained in [GGP12a], one has a subgroup

$$G(W)^{\Delta} \subset H = G(W)^{\Delta} \cdot N \subset G(V) \times G(W)$$

where N is a certain unipotent subgroup of $G(V)$ normalized by $G(W)^{\Delta}$. Moreover, a certain representation ν of $H(k)$ was defined in [GGP12a, § 14]: it is a one-dimensional character if $\dim W^{\perp}$ is odd and is essentially a Weil representation when $\dim W^{\perp}$ is even. When $\dim W^{\perp} = 1$, for example, ν is simply the trivial representation.

Given an irreducible representation $\pi_V \otimes \pi_W$ of $G(V) \times G(W)$, we defined in [GGP12a, § 14] a multiplicity $d(\pi_V, \pi_W)$, given by

$$d(\pi_V, \pi_W) = \dim \mathrm{Hom}_{G(W)}(\pi_V \otimes \pi_W, \nu).$$

It is known that

$$d(\pi_V, \pi_W) \leq 1.$$

We have also introduced in [GGP12a] a notion of relevant pure inner forms $G(V') \times G(W')$ of $G(V) \times G(W)$, and one can likewise consider the multiplicity $d(\pi_{V'}, \pi_{W'})$ for representations of the pure inner form. Our goal is to determine the multiplicity $d(\pi_V, \pi_W)$ when $\pi_V \otimes \pi_W$ belongs to local L-packets of Arthur type. Before formulating the conjecture, we recall some basic facts about A-packets of classical groups.

Suppose that M_{A} is a local A-parameter for $G(V)$, with associated L-parameter M. By the local Langlands correspondence, M gives rise to a Vogan L-packet Π_M consisting of irreducible representations of pure inner forms $G(V')$ of $G(V)$. Moreover, fixing generic data as in [GGP12a], there is a bijection

$$\Pi_M \longleftrightarrow \mathrm{Irr}(A_M),$$

where A_M is the component group of the L-parameter M and is an elementary abelian 2-group with a canonical basis. Hence,

$$\Pi_M = \{\pi_\eta \in \mathrm{Irr}(G(V')) : \eta \in \mathrm{Irr}(A_M)\}.$$

On the other hand, the A-parameter M_{A} gives rise to a local A-packet $\Pi_{M_{\mathrm{A}}}$, which is *a priori* a multi-set of unitary representations of the pure inner forms of $G(V)$ in correspondence with the irreducible characters of the component group $A_{M_{\mathrm{A}}}$ (which is also an elementary abelian 2-group with a canonical basis). In other words, one has

$$\Pi_{M_{\mathrm{A}}} = \{\pi_\eta : \eta \in \mathrm{Irr}(A_{M_{\mathrm{A}}})\}$$

where now π_η is a (possibly zero and possibly reducible) finite length unitary representation of some pure inner form $G(V')$ of $G(V)$. The work of Mœglin [Mœg06, Mœg09, Mœg11] and Mœglin and Renard [MR18] has shown that (except possibly over $\mathbb{R}$) the local A-packets $\Pi_{M_{\mathrm{A}}}$ are sets rather than multi-sets. This means that the representations π_η are multiplicity-free and π_η and $\pi_{\eta'}$ have no common constituents if $\eta \neq \eta'$. One expects similar results over $\mathbb{R}$ for which special cases have been shown by Mœglin and Renard [MR17, MR19].

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Regarding $\Pi_{M_{\mathrm{A}}}$ as a set of irreducible representations (by considering the irreducible summands of π_η), one thus has a map

$$\Pi_{M_{\mathrm{A}}} \longrightarrow \mathrm{Irr}(A_{M_{\mathrm{A}}}),$$

which is neither surjective nor injective in general. There is a natural surjective map

$$A_{M_{\mathrm{A}}} \longrightarrow A_M,$$

giving rise to a natural injective map

$$\mathrm{Irr}(A_M) \hookrightarrow \mathrm{Irr}(A_{M_{\mathrm{A}}}),$$

and a commutative diagram

$$\begin{array}{ccc} \Pi_M & \xrightarrow{\text{inj.}} & \Pi_{M_{\mathrm{A}}} \\ {\scriptstyle\text{bij.}}\downarrow & & \downarrow \\ \mathrm{Irr}(A_M) & \xrightarrow[\text{inj.}]{} & \mathrm{Irr}(A_{M_{\mathrm{A}}}). \end{array}$$

In other words, Π_M is a subset of $\Pi_{M_{\mathrm{A}}}$ (this is actually part of the defining property of an A-packet) and the labelling of its elements by characters of the component groups A_M or $A_{M_{\mathrm{A}}}$ is consistent, cf. Conjecture 6.1(iv) of [Art89].

After this short preparation, we can now formulate the conjecture.

CONJECTURE 6.1. Let M_{A} and N_{A} be local A-parameters for $G(V)$ and $G(W)$, respectively, with associated A-packets $\Pi_{M_{\mathrm{A}}}$ and $\Pi_{N_{\mathrm{A}}}$. Let M and N be the L-parameters associated to M_{A} and N_{A} with corresponding L-packets Π_M and Π_N.

(a) If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is not relevant, then

$$d(M,N) := \sum_{(\pi,\pi') \in (\Pi_M \times \Pi_N)^{\mathrm{rel}}} d(\pi,\pi') = 0.$$

Here, $(\Pi_M \times \Pi_N)^{\mathrm{rel}}$ denotes the subset of representations of relevant pure inner forms of $G(V) \times G(W)$.

(b) If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, then there is a unique relevant pair of representations $(\pi_M, \pi_N) \in \Pi_M \times \Pi_N$ such that the multiplicity $d(\pi_M, \pi_N) = 1$. In other words,

$$d(M,N) = \sum_{(\pi,\pi') \in (\Pi_M \times \Pi_N)^{\mathrm{rel}}} d(\pi,\pi') = 1.$$

(c) In the setting of part (b), the distinguished representation $\pi_M \boxtimes \pi_N$ corresponds to the distinguished character χ of the component group $A_M \times A_N$ given by the same recipe as in [GGP12a].

The recipe for χ is sufficiently intricate that we have no desire to repeat it here; we refer the reader to [GGP12a] for its precise definition. Let us make a few remarks concerning this conjecture.

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Remark 6.2. (i) Modulo the notion of 'relevant A-parameters', the previous conjecture is a direct generalization of our original conjectures for tempered L-packets in [GGP12a] to the setting of L-packets of Arthur type.

(ii) Unlike the case of $G = \mathrm{GL}_n$, we cannot treat the special case of this conjecture for those A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ whose associated L-parameters (M, N) are unramified (in particular, trivial on the Deligne $\mathrm{SL}_2(\mathbb{C})$ in $WD(k)$). The work of Hendrickson [Hen20] offers a possible strategy for establishing this unramified case, using a combination of Mackey theory and theta correspondence.

7. A conjecture for A-packets

In the previous section, Conjecture 6.1 addressed the restriction problem for the local L-packet $\Pi_M \times \Pi_N$ contained in a local A-packet $\Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}}$. In this section, we attempt to address the restriction problem for all members of A-packets, but our conjecture in this section is not as precise. Ideally, one would like to know the multiplicities for all relevant pairs of representations in the A-packet. Namely, one would like to have a prediction for the sum

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) := \sum_{(\pi,\pi') \in (\Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}})^{\mathrm{rel}}} d(\pi, \pi')$$

and a prediction of which relevant pairs (π, π') give non-zero contribution. Examples of restriction problems for low-rank groups (some of which we discuss later) indicate that the sum can be greater than one; see Remark 7.8.

Recall that unlike the case of GL_n where $\Pi_{M_{\mathrm{A}}} = \Pi_M$ (i.e. an A-packet is equal to its associated L-packet) are singletons and where A-packets corresponding to distinct A-parameters are disjoint, the A-packets of classical groups are more complicated and more interesting.

(i) First, it is rarely the case that the containment $\Pi_M \subset \Pi_{M_{\mathrm{A}}}$ is an equality.
(ii) Second, the sets Π_M and $\Pi_{M_{\mathrm{A}}}$ are typically not singletons.
(iii) Third, A-packets are not necessarily disjoint.

These complications make the problem of predicting the multiplicities in an A-packet rather tricky. At this moment, we can only offer the following (optimistic) conjecture, which may be treated more as a question.

Conjecture 7.1.

(i) If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a pair of A-parameters (for a pair of classical groups $G(V) \times G(W)$) such that the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially, then

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) \neq 0 \Longleftrightarrow (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is relevant.}$$

(ii) For a representation $\pi_1 \times \pi_2$, an irreducible representation of $G(V) \times G(W)$, and belonging to an Arthur packet,

$$d(\pi_1, \pi_2) \neq 0 \Longrightarrow \pi_1 \times \pi_2 \in \Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}} \text{ for some relevant pair } (M_{\mathrm{A}}, N_{\mathrm{A}}).$$

We make a few remarks about this conjecture.

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Remark 7.2. We outline a heuristic global justification, without in the least pretending to be complete, of part (ii) of Conjecture 7.1. Assume that $d(\pi_1, \pi_2) \neq 0$. Since π_1 and π_2 are of Arthur type, we have automorphic representations $\Pi_1 = \otimes \Pi_{1,v}$, and $\Pi_2 = \otimes \Pi_{2,v}$ with $\pi_1 = \Pi_{1,v_0}$ and $\pi_2 = \Pi_{2,v_0}$ for some place v_0 of a global field F. Now we can assume by a form of the Burger–Sarnak principle that Π_1 and Π_2 are so chosen that $\Pi_1 \otimes \Pi_2$ has a non-zero global period integral. By Arthur, $\Pi_1 \otimes \Pi_2$ has a (uniquely determined) global A-parameter, such that all the local components of $\Pi_1 \otimes \Pi_2$ (in particular, $\pi_1 \times \pi_2$) have the corresponding local A-parameter. As the period integral on $\Pi_1 \otimes \Pi_2$ is non-zero, it follows from the global Conjecture 9.1 (see later) that the global A-parameter of (Π_1, Π_2) is relevant. Hence, (π_1, π_2) has a relevant pair of local A-parameters.

Remark 7.3. It appears to us that higher multiplicities in a given A-packet is the result of other A-packets (with relevant A-parameters) intersecting this A-packet. To be more precise, we feel that for $(M_{\mathrm{A}}, N_{\mathrm{A}})$, a pair of A-parameter for $G(V) \times G(W)$,

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) \leq |X|,$$

where X is a set of maximal cardinality, consisting of pairs $(\pi_1' \times \pi_2', M_{\mathrm{A}}' \times N_{\mathrm{A}}')$ such that:

(i) $M_{\mathrm{A}}' \times N_{\mathrm{A}}'$ is any relevant pair of A-parameter for $G(V) \times G(W)$;
(ii) $\pi_1' \times \pi_2'$ is a representation belonging to both the A-packets $\Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}}$ and $\Pi_{M_{\mathrm{A}}'} \times \Pi_{N_{\mathrm{A}}'}$
(iii) the projections from X to both the first and the second factor are injective.

We now offer some evidence for Conjecture 7.1, but before doing so, we need to recall some results of Mœglin. Given a discrete L-parameter M of a classical group, Mœglin determined precisely which elements of its L-packet Π_M can be a supercuspidal representation. To formulate the result, we introduce the following two notions.

(i) Say that a discrete L-parameter M is *without gaps* (or holes) if the following holds: for any selfdual irreducible representation ρ of $W(k)$ and $a \geq 1$,

$$\rho \boxtimes [a+2] \subset M \Longrightarrow \rho \boxtimes [a] \subset M.$$

Equivalently, let

$$\rho \boxtimes [1] + \rho \boxtimes [3] + \cdots \quad \text{or} \quad \rho \boxtimes [2] + \rho \boxtimes [4] + \cdots$$

be a maximal chain appearing as a direct summand in M. Then M is without gaps if these chains (as ρ varies) span M.

(ii) Let M be a discrete L-parameter without gaps and let A_M be its component group, so that Π_M is in bijection with $\mathrm{Irr}(A_M)$. Recall that A_M is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with a canonical basis indexed by the irreducible summands of M. Say that $\alpha \in \mathrm{Irr}(A_M)$ is *alternating* if the following holds: for any $\rho \boxtimes [a] + \rho \boxtimes [a+2] \subset M$,

$$\alpha(\rho \boxtimes [a]) = -\alpha(\rho \boxtimes [a+2]),$$

with the convention that if $a = 0$, then $\rho \boxtimes [a] = 0$, and $\alpha(\rho \boxtimes [0]) = 1$, so that we must have $\alpha(\rho \boxtimes [2]) = -1$.

Here then is Mœglin's first result [Mœg09] that we use.

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Theorem 7.4. *Let M be a discrete L-parameter for a classical group over a p-adic field and let $\alpha \in Irr(A_M)$ be a character of its component group. Then the corresponding representation $\pi(M, \alpha) \in \Pi_M$ is supercuspidal if and only if M is without gaps and α is alternating.*

The second result of Mœglin [Mœg06, Mœg09] we need concerns the possible overlaps of different local A-packets. If M_{A} is the local A-parameter of a (classical) group G over a non-archimedean local field k, let M_{A}^{Δ} be the tempered L-parameter defined by

$$M_{\mathrm{A}}^{\Delta} : W(k) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Id}\times\Delta} W(k) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{M_{\mathrm{A}}} {}^L G$$

where Δ is the diagonal embedding of $\mathrm{SL}_2(\mathbb{C})$.

Theorem 7.5. *Let M_{A} and M'_{A} be two local A-parameters for a classical group over a p-adic field with associated A-packets $\Pi_{M_{\mathrm{A}}}$ and $\Pi_{M'_{\mathrm{A}}}$. Then*

$$\Pi_{M_{\mathrm{A}}} \cap \Pi_{M'_{\mathrm{A}}} \neq \emptyset \Longrightarrow M_{\mathrm{A}}^{\Delta} \cong M_{\mathrm{A}}'^{\Delta}.$$

Further, if $\Pi^{\mathrm{sc}}_{M_{\mathrm{A}}^{\Delta}}$ denotes the set of supercuspidal representations in $\Pi_{M_{\mathrm{A}}^{\Delta}}$, then

$$\Pi^{\mathrm{sc}}_{M_{\mathrm{A}}^{\Delta}} \subset \Pi_{M_{\mathrm{A}}}.$$

The first part of this theorem of Mœglin is eventually related to the observation that two A-parameters M_{A} and M'_{A} of $\mathrm{GL}_n(k)$ have the same cuspidal support if and only if M_{A}^{Δ} and $M_{\mathrm{A}}'^{\Delta}$ are equivalent.

At this point, let us take note of the following consequence of tempered GGP, which was pointed out to us by Waldspurger.

Proposition 7.6. *Let $M_0 \times N_0$ be a tempered L-parameter for $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$ over a non-archimedean local field k with associated L-packets $\Pi_{M_0} \times \Pi_{N_0}$. Let (π, σ) be the unique member of $\Pi_{M_0} \times \Pi_{N_0}$ such that $d(\pi, \sigma) \neq 0$, so that (π, σ) corresponds to the distinguished character $\chi := \chi_{M_0,N_0}$ of the component group by [GGP12a].*

(i) *Suppose that ρ is an irreducible selfdual parameter for $\mathrm{GL}_d(k)$ such that for some $a \geq 3$, both the representations $\rho \otimes [a]$ and $\rho \otimes [a-2]$ of $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C})$ appear with multiplicity 1 in M_0. In this case, these two summands give rise to two basis elements of the component group associated to M_0. Let*

$$\chi(\rho, a), \chi(\rho, a-2) \in \{\pm 1\}$$

be the value of the distinguished character χ on these two elements respectively. Then

$$\chi(\rho, a) = -\chi(\rho, a-2)$$

if and only if the parameter N_0 contains $\rho \otimes [a-1]$ with odd multiplicity.

(ii) *Further, if $\rho \otimes [2]$ appears with multiplicity 1 in M_0, then*

$$\chi(\rho, 2) = -1$$

if and only if the parameter N_0 contains ρ with odd multiplicity.

Furthermore, there are similar assertions when both the representations $\rho \otimes [a]$ and $\rho \otimes [a-2]$ of $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C})$ appear with multiplicity 1 in N_0.

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Proof. Let us begin by recalling Conjecture 20.1 of [GGP12a], giving a recipe for $\chi(\rho, a)$:

$$\chi(\rho, a) = \epsilon(\rho \otimes [a] \otimes N_0) \cdot (\det \rho)(-1)^{(a \dim N_0)/2} \cdot (\det N_0)(-1)^{(a \dim \rho)/2}.$$

Define more generally for any selfdual representation μ of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ with even parity (i.e. the same as that of N_0),

$$\chi_\mu(\rho, a) = \epsilon(\rho \otimes [a] \otimes \mu) \cdot (\det \rho)(-1)^{(a \dim N_0)/2} \cdot (\det \mu)(-1)^{(a \dim \rho)/2},$$

so that if $N_0 = \oplus_i \mu_i$, a sum of irreducible selfdual representations μ_i of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ with even parity (i.e. the same as that of N_0), then

$$\chi(\rho, a) = \prod_i \chi_{\mu_i}(\rho, a).$$

Thus, we need to compare $\chi_{\mu_i}(\rho, a)$ and $\chi_{\mu_i}(\rho, a-2)$ for any μ_i as before.

Recall from [Tat79] that for an irreducible representation $\lambda \otimes [n]$ of $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C})$, one has

$$\epsilon(\lambda \otimes [n]) = \epsilon(\lambda)^n \cdot \det(-F, \lambda^I)^{n-1},$$

where λ^I denotes the subspace of λ fixed by the inertia group I and F denotes the Frobenius element of $W(k)/I$. In particular, if λ is an irreducible selfdual representation of $W(k)$, then

$$\epsilon(\lambda \otimes [n]) = \begin{cases} \epsilon(\lambda)^n & \text{if } \lambda \neq 1; \\ (-1)^{n-1} & \text{if } \lambda = 1. \end{cases}$$

Using the Clebsch–Gordon theorem,

$$[a] \otimes [b] = [a+b-1] \oplus [a+b-3] \oplus \cdots \oplus [|a-b|+1],$$

it follows that if ρ, τ are irreducible selfdual representations of $W(k)$, then

$$\epsilon(\rho \otimes [a] \otimes \tau \otimes [b]) = \begin{cases} \epsilon(\rho \otimes \tau)^{ab} & \text{if } \rho \ncong \tau, \\ \epsilon(\rho \otimes \tau)^{ab}(-1)^{n(a,b)} & \text{if } \rho \cong \tau, \end{cases}$$

where $n(a, b) = \min\{a, b\} \cdot [\max\{a, b\} - 1]$. In particular, observe that for a, b positive integers of different parity,

$$\epsilon([a] \otimes [b]) \cdot \epsilon([a-2] \otimes [b]) = \begin{cases} 1 & \text{if } b \neq a-1, \\ -1 & \text{if } b = a-1. \end{cases}$$

More generally, it is easily seen that for any irreducible representation μ of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ as previously and $a \geq 3$,

$$\chi_\mu(\rho, a) = \begin{cases} \chi_\mu(\rho, a-2), & \text{if } \mu \neq \rho \otimes [a-1], \\ -\chi_\mu(\rho, a-2), & \text{if } \mu = \rho \otimes [a-1]. \end{cases}$$

It follows that $\chi(\rho, a) = -\chi(\rho, a-2)$ if and only if $\rho \otimes [a-1]$ occurs in N_0 with odd multiplicity, thus completing the proof of part (i). The proof of part (ii), where $a = 2$, follows along the same lines; we omit the details. □

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After this preparation, we can prove the following theorem. Part (b) of the theorem is a contribution to Conjecture 7.1(i) and (ii).

Theorem 7.7. (a) *Let $M_0 \times N_0$ be a discrete L-parameter for $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$ without gaps with associated L-packet $\Pi_{M_0} \times \Pi_{N_0}$. Let (π, σ) be the unique member of $\Pi_{M_0} \times \Pi_{N_0}$ such that $d(\pi, \sigma) \neq 0$. Let $D(M_0) \times D(N_0)$ be the A-parameter obtained from the L-parameter $M_0 \times N_0$ by interchanging the Deligne $\mathrm{SL}_2(\mathbb{C})$ by Arthur $\mathrm{SL}_2(\mathbb{C})$.*

Then (π, σ) is supercuspidal if and only if $(D(M_0), D(N_0))$ is relevant.

(b) *Let $M_{\mathrm{A}} \times N_{\mathrm{A}}$ be a discrete A-parameter for $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$ over a non-archimedean local field k, on which the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially, with associated A-packet $\Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}}$. Assume that the A-parameter $M_{\mathrm{A}} \times N_{\mathrm{A}}$ has no gaps (relative to the Arthur $\mathrm{SL}_2(\mathbb{C})$). Then there exists supercuspidal $(\pi, \sigma) \in \Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}}$ such that $d(\pi, \sigma) \neq 0$ if and only if $M_{\mathrm{A}} \times N_{\mathrm{A}}$ is relevant.*

Proof. (a) This is a consequence of the tempered GGP [Wal12], Theorems 7.4 and 7.5, and Proposition 7.6.

(b) This follows from part (a) on noting that the Aubert–Zelevinsky involution

$$\pi \to D(\pi)$$

on the category of admissible representations of a group $G(k)$ takes the tempered L-packet $\Pi_{M_0} \times \Pi_{N_0}$ to $D(\Pi_{M_0}) \times D(\Pi_{N_0})$, which is the A-packet associated to the A-parameter $D(M_0) \times D(N_0)$. Now $\pi = D(\pi)$ (up to a sign) for supercuspidal representations π. Hence, if (π, σ) is a supercuspidal member of $\Pi_{M_0} \times \Pi_{N_0}$ such that $d(\pi, \sigma) \neq 0$, then (π, σ) is also a member of the A-packet $D(\Pi_{M_0}) \times D(\Pi_{N_0})$ with $d(\pi, \sigma) \neq 0$. □

Remark 7.8. In Theorem 7.7(b), the representation $\pi \times \sigma$ we produced in the A-packet $\Pi_{M_{\mathrm{A}}} \times \Pi_{N_{\mathrm{A}}}$ with $d(\pi, \sigma) \neq 0$ is supercuspidal and, hence, lies outside of the associated L-packet $\Pi_M \times \Pi_N$ (because the L-packet underlying an A-parameter with non-trivial restriction to the Arthur $\mathrm{SL}_2(\mathbb{C})$ consists only of non-tempered representations). Thus, by Conjecture 6.1, the sum of multiplicities in such A-packets is at least two. This remark could be considered to be a contribution to Remark 7.3.

In the rest of this section, we construct examples of non-relevant A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ for which $d(M_{\mathrm{A}}, N_{\mathrm{A}}) > 0$, a subtlety that Conjecture 7.1(ii) takes into account.

For any subset $J \subset \{1, 2, \ldots, n\}$ (for n fixed), consider the representation of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ given by

$$M_{\mathrm{A},J} = \bigoplus_{i \notin J} [2i] \otimes [1] \oplus \bigoplus_{j \in J} [1] \otimes [2j].$$

Then $M_{\mathrm{A},J}$ is a multiplicity-free sum of symplectic representation of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ with $W(k)$ acting trivially. Hence, $M_{\mathrm{A},J}$ is a discrete A-parameter for an odd special orthogonal group of the appropriate size.

Now observe that $M_{\mathrm{A},J}^{\Delta}$ is independent of J and is a discrete L-parameter without gaps:

$$M_{\mathrm{A},J}^{\Delta} = M_{A,\emptyset} = \bigoplus_{j=1}^{n} 1_{W(k)} \otimes [2j].$$

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By Theorem 7.4, the L-packet of $M_{\mathrm{A},J}^{\Delta}$ contains a unique supercuspidal representation π_{sc} for an odd special orthogonal group of the appropriate size, attached to the unique alternating character α_{sc} of the component group:

$$\alpha_{\mathrm{sc}}([2j]) = (-1)^j.$$

By Theorem 7.5, we see that

$$\pi_{\mathrm{sc}} \in \Pi_{M_{\mathrm{A},J}} \quad \text{for any } J.$$

We now consider the restriction problem for the pair $(M_{\mathrm{A},J}, N)$ where $M_{\mathrm{A},J}$ is as previously and N is the discrete L-parameter

$$N := [1] \oplus [3] \oplus \cdots \oplus [2n-1]$$

on which $W(k)$ acts trivially. One may apply Proposition 7.6 (together with Theorem 7.4) to deduce that

$$d(\pi_{\mathrm{sc}}, N) = 1,$$

where we recall that π_{sc} is the unique supercuspidal representation with L-parameter $M_{A,\emptyset}$. Since $\pi_{\mathrm{sc}} \in \Pi_{M_{\mathrm{A},J}}$ for any J, we thus conclude that

$$d(M_{\mathrm{A},J}, N) \neq 0 \quad \text{for any } J.$$

It is easy to see that

$$(M_{\mathrm{A},J}, N) \text{ is relevant} \iff J = \emptyset, \text{ or } J = \{1\}.$$

Hence, if $n > 1$, we have produced non-relevant pairs $(M_{\mathrm{A},J}, N)$ such that $d(M_{\mathrm{A},J}, N) > 0$. In other words, 'relevance' is not a necessary condition for branching in A-packets.

The simplest example one can take is $n = 2$. In this case, we have the A-parameters

$$M_{\mathrm{A}} = [1] \boxtimes [4] + [2] \boxtimes [1] \quad \text{or} \quad [1] \boxtimes [4] + [1] \boxtimes [2],$$

and

$$N = [3] \boxtimes [1] + [1] \boxtimes [1].$$

The pair (M_{A}, N) is not relevant, yet $d(M_{\mathrm{A}}, N) > 0$. (We note that the A-packets of SO_7 corresponding to the two choices of M_{A} given previously can be constructed by theta lifting from an appropriate L-packet and an A-packet on Mp_2, respectively, and the unique supercuspidal representation in them is simply the theta lift (with respect to ψ) of the odd Weil representation ω_ψ).

8. A special case of the conjecture

The results of Mœglin and Waldspurger that were used in the previous section to provide a counterexample to the necessity of relevance for the restriction problem for A-packets can be used in other circumstances to verify special cases of our Conjecture 6.1. In this section, we consider one such special case of Conjecture 6.1, the verification of which is due to Mœglin. As a general reference for A-packets on not necessarily quasi-split classical groups, we refer to [MR18]. In what follows, we use $\mathrm{SO}(V)^+$ or $\mathrm{SO}^+_{2\ell+1}$ (respectively, $\mathrm{SO}(V)^-$, or $\mathrm{SO}^-_{2\ell+1}$) to denote the split (respectively, non-split) orthogonal group in odd number of variables of discriminant 1; the superscript $\pm$ will be omitted when we consider either of the two possibilities.

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To describe the special case, we introduce the following notation. Let ρ be an irreducible d-dimensional representation of $W(k)$ and write $[b]$ for the b-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Then an irreducible representation of $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ has the form $\rho \otimes [a] \otimes [b]$ for integers $a, b > 0$.

Assume now that ρ is an irreducible d-dimensional orthogonal representation of $W(k)$. We consider the following A-parameters

$$M_{\mathrm{A}} = \rho \otimes [2] \otimes [1] + \rho \otimes [1] \otimes [2],$$

and

$$N_{\mathrm{A}} = \rho \otimes [1] \otimes [3] + N_0,$$

where N_0 is a d-dimensional tempered orthogonal L-parameter. Then M_{A} is an A-parameter for SO_{4d+1} whereas N_{A} is an A-parameter for SO_{4d} (orthogonal group of a quadratic space whose discriminant is dictated by the A-parameter N_{A}). Observe that the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant. Thus, according to our conjecture, we expect that there should be a pair of representations (π, σ) in the L-packet $\Pi_M \times \Pi_N$ associated to M_{A} and N_{A} such that $\mathrm{Hom}_{\mathrm{SO}_{4d}}(\pi, \sigma) \neq 0$. To verify this, we first describe the elements in these L-packets more concretely.

The L-parameter associated to M_{A} is given by

$$M = \rho \otimes [2] + \rho\nu^{1/2} + \rho\nu^{-1/2},$$

where ν is the character of $W(k)$ corresponding to the absolute value of $k^\times$ under the local class field theory. The associated L-packet Π_M consists of various Langlands quotients of the split group SO^+_{4d+1} and the non-split SO^-_{4d+1} defined as follows.

Let P_d be the maximal parabolic subgroup of SO_{4d+1} stabilizing a d-dimensional isotropic space, so that its Levi factor is isomorphic to $\mathrm{GL}_d \times \mathrm{SO}_{2d+1}$. Then Π_M consists of the unique irreducible quotients of the standard modules

$$\mathrm{Ind}_{P_d}^{\mathrm{SO}_{4d+1}} \pi_\rho |\det|^{1/2} \otimes \tau \twoheadrightarrow J(\rho, \tau),$$

where π_ρ is the irreducible cuspidal representation of GL_d with L-parameter ρ and τ runs over the Vogan L-packet of SO_{2d+1} associated to the L-parameter $\rho \otimes [2]$.

To further explicate the L-packet Π_M, we need to describe the L-packet of SO_{2d+1} with L-parameter $\rho \otimes [2]$. The component group of the discrete L-parameter $\rho \otimes [2]$ is $\mathbb{Z}/2\mathbb{Z}$. Hence, its associated L-packet has the form

$$\Pi_{\rho \otimes [2]} = \{\tau^+, \tau^-\},$$

with τ^+ a discrete series representation of the split group SO^+_{2d+1} and τ^- that of the non-split inner form SO^-_{2d+1}. More precisely, τ^+ is the unique irreducible submodule of the induced representation $\mathrm{Ind}_P^{\mathrm{SO}^+(V)} \rho |\det|^{1/2}$, where P has Levi factor GL_d, and τ^- is a supercuspidal representation of SO^-_{2d+1}.

To summarize, we have

$$\Pi_M = \{\pi_L^+, \pi_L^-\},$$

where

$$\pi_L^+ = J(\rho, \tau^+) \in \mathrm{Irr}\big(\mathrm{SO}^+_{4d+1}\big) \quad \text{and} \quad \pi_L^- = J(\rho, \tau^-) \in \mathrm{Irr}\big(\mathrm{SO}^-_{4d+1}\big).$$

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In fact, we can explicate not just the L-packet Π_M but also the entire A-packet $\Pi_{M_{\mathrm{A}}}$. For the split group SO^+_{4d+1}, the A-packet $\Pi^+_{M_{\mathrm{A}}}$ has two elements:

$$\Pi^+_{M_{\mathrm{A}}} = \{\pi^+_L, \pi^+_T\},$$

where π^+_L lies in the L-packet Π_M as described before, and π^+_T is a tempered representation. The standard module with π^+_L as its Langlands quotient has composition series given by

$$0 \to \pi^+_g \to \mathrm{Ind}_{P_d}^{\mathrm{SO}^+_{4d+1}}(\rho\nu^{1/2} \otimes \tau^+) \to \pi^+_L \to 0,$$

with π^+_g an irreducible generic tempered representation of SO^+_{4d+1}. The representation π^+_T, on the other hand, is the non-generic component of the following induced representation whose other irreducible constituent is π^+_g,

$$\pi^+_T + \pi^+_g = \mathrm{Ind}_{P_{2d}}^{\mathrm{SO}^+_{4d+1}}\mathrm{St}(\rho, 2).$$

Here $\mathrm{St}(\rho, 2)$ is the generalized Steinberg representation of GL_{2d} (the Levi factor of P_{2d}). Observe that the character of $\pi^+_L - \pi^+_T$ is a stable distribution, because

$$\pi^+_L - \pi^+_T = \mathrm{Ind}_{P_d}^{\mathrm{SO}^+_{4d+1}}(\rho\nu^{1/2} \otimes \tau^+) - \mathrm{Ind}_{P_{2d}}^{\mathrm{SO}^+_{4d+1}}\mathrm{St}(\rho, 2). \tag{A}$$

From (A), we obtain the packet $\Pi^-_{M_{\mathrm{A}}}$ on the non-split orthogonal group SO^-_{4d+1} by transfer of the stable distribution (A). As the second induced representation on the right-hand side of (A) does not transfer to SO^-_{4d+1} (as the parabolic subgroup P_{2d} is irrelevant for SO^-_{4d+1}), this gives

$$\Pi^-_{M_{\mathrm{A}}} = \{\pi^-_L\} = \left\{\mathrm{Ind}_{P^-_d}^{\mathrm{SO}^-_{4d+1}}(\rho\nu^{1/2} \otimes \tau^-)\right\}. \tag{B}$$

The standard module $\mathrm{Ind}_{P^-_d}^{\mathrm{SO}^-_{4d+1}}(\rho\nu^{1/2} \otimes \tau^-)$ is known to be irreducible, thus is equal to its Langlands quotient π^-_L. This fact is crucially used in the restriction problem considered later.

Now we consider the orthogonal A-parameter

$$N_{\mathrm{A}} = \rho \otimes [1] \otimes [3] + N_0.$$

Its associated L-parameter is

$$N = (N_0 + \rho) + \rho\nu + \rho\nu^{-1}.$$

If Q_d denotes the maximal parabolic subgroup of SO_{4d} with Levi factor $\mathrm{GL}_d \times \mathrm{SO}_{2d}$, the elements of the L-packet Π_N consists of Langlands quotient of the standard modules

$$\mathrm{Ind}_{Q_d}^{\mathrm{SO}_{4d}}\pi_\rho|\det| \otimes \tau' \twoheadrightarrow J(\rho, \tau'),$$

as τ' runs over the Vogan L-packet associated to the L-parameter $N_0 + \rho$.

To understand the restriction problem for the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ on $\mathrm{SO}_{4d+1} \times \mathrm{SO}_{4d}$, we first need to understand the restriction problem for the tempered pair $(\rho \otimes [2], \rho + N_0)$ of $\mathrm{SO}_{2d+1} \times \mathrm{SO}_{2d}$. This latter problem has been understood because Waldspurger has proven the tempered GGP conjecture.

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More precisely, for any tempered L-parameter Σ of SO_{2d}, with associated L-packet Π_Σ, let us set

$$d(\tau^\pm, \Sigma) = \sum_{\sigma \in \Pi_\Sigma} d(\tau^\pm, \sigma),$$

where we recall that

$$\Pi_{\rho \otimes [2]} = \{\tau^+, \tau^-\}.$$

Then one knows by tempered GGP that

$$d(\tau^+, \Sigma) + d(\tau^-, \Sigma) = 1.$$

One may ask: for which Σ is $d(\tau^-, \Sigma) = 1$? Applying Proposition 7.6, we deduce the following.

Corollary 8.1. *Let ρ be an irreducible orthogonal representation of $W(k)$ of dimension d. Let $\{\tau^+, \tau^-\}$ be the L-packet of SO_{2d+1} associated to the discrete L-parameter $\rho \otimes [2]$ and let Σ be a tempered L-parameter of SO_{2d}. Then*

$$d(\tau^-, \Sigma) = 1 \Longleftrightarrow \Sigma = \rho + N_0 \quad \textit{for some tempered } N_0.$$

The following proposition, which is the main result of this section, lends some support to Conjecture 6.1.

Proposition 8.2. *Let*

$$M_{\mathrm{A}} = \rho \boxtimes [2] \boxtimes [1] + \rho \boxtimes [1] \boxtimes [2],$$
$$N_{\mathrm{A}} = \rho \boxtimes [1] \boxtimes [3] + N_0,$$

be a relevant pair of A-parameters for $\mathrm{SO}_{4d+1} \times \mathrm{SO}_{4d}$ where ρ is an irreducible orthogonal representation of $W(k)$ of dimension d. Let

$$(\pi, \sigma) \in \Pi_M \times \Pi_N$$

be the representations in the associated L-packets indexed by the distinguished character of the relevant component group as given in Conjecture 6.1(c). Then

$$d(\pi, \sigma) \neq 0.$$

Proof. In the context of Corollary 8.1, let

$$(\tau, \tau') \in \Pi^-_{\rho \boxtimes [2]} \times \Pi^-_{\rho + N_0}$$

be such that

$$\mathrm{Hom}_{\mathrm{SO}^-_{2d}}(\tau, \tau') \neq 0,$$

so that the pair (τ, τ') corresponds to the distinguished character of the component group for $(\rho \boxtimes [2]) \times (\rho + N_0)$

Let P^-_d be a parabolic subgroup in $G = \mathrm{SO}^-_{4d+1}$ stabilizing an isotropic subspace of dimension d. The subgroup $\mathrm{SO}^-_{4d} \subset \mathrm{SO}^-_{4d+1}$ operates on the corresponding flag variety $\mathrm{SO}^-_{4d+1}/P^-_d$ with two orbits: an open dense orbit and a closed orbit that is given by SO^-_{4d}/Q^-_d where Q^-_d is the parabolic in SO^-_{4d} with Levi subgroup $\mathrm{GL}_d \times \mathrm{SO}^-_{2d}$.

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Via restriction to the closed orbit SO^-_{4d}/Q_d, one has a surjective SO^-_{4d}-equivariant homomorphism

$$\pi_L^- = \mathrm{Ind}_{P_d^-}^{\mathrm{SO}^-_{4d+1}} \rho\nu^{1/2} \otimes \tau \to \mathrm{Ind}_{Q_d^-}^{\mathrm{SO}^-_{4d}} \rho\nu \otimes \tau|_{\mathrm{SO}^-_{2d}} \to \mathrm{Ind}_{Q_d^-}^{\mathrm{SO}^-_{4d}} \rho\nu \otimes \tau'.$$

Composing this map with the projection from the last standard module to its Langlands quotient σ^- (which is an element of Π_N^-), we have thus shown that $d(\pi_L^-, \sigma^-) \neq 0$. Moreover, by the properties of the local Langlands correspondence, the characters of component groups associated to the representations π_L^- and σ^- are inherited from those of τ and τ' and is equal to the distinguished character for (M, N). This completes the proof of the proposition. □

9. Global conjecture

In this section, we formulate a conjecture for the global analog of the restriction problem, which concerns the non-vanishing of automorphic period integrals. Thus, let F be a global field with ring of adeles $\mathbb{A}$ and F/F_0 a separable extension with $[F : F_0] \leq 2$.

For a connected reductive group G over F, let $\mathcal{A}(G)$ denote the space of automorphic forms on G (with a fixed unitary central character if G is not semisimple), which is a $G(\mathbb{A})$-module; thus, $\mathcal{A}(G)$ is the space of smooth functions on $G(F)\backslash G(\mathbb{A})$ with a fixed unitary central character that are $Z(U(\mathfrak{g}))$-finite under the center of the enveloping algebra of $G(F \otimes \mathbb{R})$, and are of uniform moderate growth. Any irreducible subquotient of $\mathcal{A}(G)$ is called an automorphic representation of G. One has the following natural submodules

$$\mathcal{A}_{\mathrm{cusp}}(G) \subset \mathcal{A}_{\mathrm{disc}}(G) \subset \mathcal{A}(G)$$

consisting of the cusp forms and the square-integrable (modulo center) automorphic forms, respectively. The submodules $\mathcal{A}_{\mathrm{cusp}}(G)$ and $\mathcal{A}_{\mathrm{disc}}(G)$ are semisimple and any irreducible summand of $\mathcal{A}_{\mathrm{cusp}}(G)$ (respectively, $\mathcal{A}_{\mathrm{disc}}(G)$) is called a cuspidal automorphic representation (respectively, a discrete automorphic representation). For quasi-split classical groups G, the discrete automorphic representations are classified by Arthur [Art13] (for symplectic and orthogonal groups) and Mok [Mok15] (for unitary groups) in terms of discrete global A-parameters. In particular, the local components of discrete automorphic representations are among those considered by our local conjectures in §§ 5 and 6.

One may also consider the unitary representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$ (with a fixed central character), which possesses a direct integral decomposition. The isomorphism classes of irreducible representations which are weakly contained in this direct integral decomposition give a subset of the unitary dual of $G(\mathbb{A})$, the closure of which (with respect to the Fell topology) is called the automorphic dual $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$. The spectral decomposition theorem of Langlands describes the elements of $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$ as unitary principal series representations induced from the discrete automorphic representations (twisted by unitary automorphic characters) of Levi subgroups of G (including G itself). Moreover, the theory of Eisenstein series provides concrete equivariant embeddings of the elements of the automorphic dual $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$ into $\mathcal{A}(G)$. Via the theory of Eisenstein series, one can thus consider elements of $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$ as irreducible submodules of $\mathcal{A}(G)$, and let $\mathcal{A}_{\mathrm{aut}}(G)$ denote the submodule generated by these realizations of the elements of $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$. In particular, one has

$$\mathcal{A}_{\mathrm{cusp}}(G) \subset \mathcal{A}_{\mathrm{disc}}(G) \subset \mathcal{A}_{\mathrm{aut}}(G) \subset \mathcal{A}(G).$$

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For classical groups, the results of Arthur [Art13] and Mok [Mok15] alluded to previously allow one to assign (not-necessarily discrete) global A-parameters to the elements of the automorphic dual $\widehat{G(\mathbb{A})}_{\mathrm{aut}}$. In other words, the local components of the constituents of $\mathcal{A}_{\mathrm{aut}}(G)$ are precisely the representations considered in our local conjecture, so that $\mathcal{A}_{\mathrm{aut}}(G)$ is the natural setting on which to consider the global branching problem and to formulate the global version of our conjectures.

Now we place ourselves in the global setting of [GGP12a], so that we have a pair $W \subset V$ of non-degenerate ϵ-Hermitian spaces over F satisfying the conditions highlighted in § 6. As in the local case, we have a pair of groups

$$H = G(W) \cdot N \hookrightarrow G = G(V) \times G(W).$$

As explained in [GGP12a, § 23], there is a automorphic representation ν of $H(\mathbb{A})$ which is a character when $\epsilon = +$ and is essentially a Weil representation when $\epsilon = -$. Thus, one may consider the global period integral

$$F(\nu) : \mathcal{A}_{\mathrm{cusp}}(G) \longrightarrow \mathbb{C}$$

defined by an absolutely convergent integral

$$F(\nu)(f) = \int_{H(\mathbb{A})\backslash H(\mathbb{A})} f(h) \cdot \overline{\nu(h)}\, dh$$

when $\epsilon = +$ and by an analogous integral involving theta functions when $\epsilon = -$.

The above definition of $F(\nu)$ on the cuspidal spectrum is sufficient for the restriction problem considered in [GGP12a] because we only dealt with tempered global A-packets (or, equivalently, tempered L-packets) there: the automorphic representations in these tempered L-packets are necessarily cuspidal. In this paper, we are dealing with possibly non-tempered A-parameters. The automorphic representations in global A-packets are no longer necessarily cuspidal. This means that, for a meaningful consideration of the global period problem, one needs to extend the definition of $F(\nu)$ to the larger space $\mathcal{A}_{\mathrm{disc}}(G)$ or even the still larger space $\mathcal{A}_{\mathrm{aut}}(G)$.

The definition of a regularized period integral on the non-cuspidal part of the spectrum is a basic problem in the analysis of the spectral side of the relative trace formula. Recently, a general definition of such a regularized period integral over reductive subgroups of G was given by Zydor [Zyd19], following earlier work of Jacquet *et al.* [JLR99] and Ichino and Yamana [IY15]. In particular, the work of Ichino and Yamana [IY15] provides a regularized period integral on the space of automorphic forms on $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ and $U_n \times U_{n+1}$.

For the purpose of this section, we assume the working hypothesis that one has a canonical equivariant extension of $F(\nu)$ *to* $\mathcal{A}_{\mathrm{aut}}(G)$.

With this background and caveat, we can now formulate our global conjecture.

Conjecture 9.1. Let $\pi \otimes \pi'$ be an irreducible representation of $G(\mathbb{A}) = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ which occurs as a sub-representation of $\mathcal{A}_{\mathrm{aut}}(G)$. Then the restriction of the regularized period integral $F(\nu)$ to $\pi \otimes \pi'$ is non-zero if and only if the following conditions hold:

(i) the pair of global A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ associated to $\pi \otimes \pi'$ is relevant;
(ii) the local multiplicity $d(\pi_v, \pi'_v)$ is non-zero for all places v of F;
(iii) the ratio of L-functions $L(M, N, s)$ defined by (1.1) in the case of GL_n and (1.2) in the case of classical groups is non-zero at $s = 0$.

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Further, if conditions (i) and (iii) hold, then there exists a globally relevant pure inner form $G' = G(V') \times G(W')$ of G and an automorphic representation $\tilde{\pi} \otimes \tilde{\pi}'$ of G' with the same A-parameter $(M_{\mathrm{A}}, N_{\mathrm{A}})$ such that $F(\nu)$ is non-zero on $\tilde{\pi} \otimes \tilde{\pi}'$.

Let us make a few remarks about this global conjecture.

• As we noted in [GGP12a, Pg. 88], the formulation of the previous conjecture essentially decouples the global restriction problem from the local ones. Unfortunately, as we noted before, our local conjecture is somewhat deficient, and does not allow one to completely address the condition (ii). In the next section, we shall consider the global restriction problem starting from our local conjecture and examine its interaction with the Arthur multiplicity formula analogous to what we did in [GGP12a, § 26].

• It is reasonable to ask what one knows *a priori* about the analytic behavior of the function $L(M, N, s)$ in condition (iii) at $s = 0$. Condition (iii) seems only reasonable if one knows *a priori* that $L(M, N, s)$ is holomorphic at $s = 0$. If M_{A} and N_{A} are tempered, this is automatic because the adjoint L-functions in the denominator of $L(M, N, s)$ do not vanish at $s = 0$. We show later that when $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair, then $L(M, N, s)$ is holomorphic at $s = 0$. Moreover, the function $L(M, N, s)$ decomposes into a product of various automorphic L-functions and the curious reader will wonder which factors in $L(M, N, s)$ have the potential to contribute a zero at $s = 0$. We address this question at the end of this section when we compute the function $L(M, N, s)$ more explicitly.

• As mentioned earlier, Ichino and Yamana have defined a regularized global period integral in [IY15] on automorphic forms of $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$. On the cuspidal spectrum, the global period integral is known to be non-zero on any irreducible cuspidal $\pi \otimes \pi'$ if and only if

$$L\left(\tfrac{1}{2}, \pi \otimes \pi'\right) \neq 0.$$

Ichino and Yamana showed further that when $\pi \otimes \pi'$ is in the part of the discrete spectrum orthogonal to the cuspidal spectrum and the one-dimensional summands, their regularized period integral $F(\nu)$ vanishes on $\pi \otimes \pi'$. As we now explain, this is consistent with the previous conjecture and, in fact, follows from our local Theorem 5.2.

Indeed, by a result of Mœglin and Waldspurger, a representation $\pi \subset \mathcal{A}_{\mathrm{disc}}(\mathrm{GL}_n)$ has an irreducible A-parameter, that is, one of the form $M \otimes \mathrm{Sym}^d(\mathbb{C}^2)$ where M is a cuspidal representation of GL_r with $r \cdot (d+1) = n$. Moreover, π is cuspidal if and only if $d = 0$. Now given $\pi \otimes \pi'$ in the discrete spectrum of $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$, we have a pair of A-parameters

$$M \otimes \mathrm{Sym}^d(\mathbb{C}^2) \quad \text{and} \quad N \otimes \mathrm{Sym}^e(\mathbb{C}^2).$$

It is easy to see that for this pair of A-parameters to be relevant, we must have either:

(i) $d = e = 0$, so that $\pi \otimes \pi'$ is cuspidal; or
(ii) $d + 1 = n = e$ and $M = N$ is a one-dimensional character χ of GL_1, in which case $\pi = \chi \circ \det$ and $\pi' = \chi \circ \det$.

As at all but finitely many places of F, the local representations π_v and π'_v are unramified (hence, parametrized by representations of $WD(F_v) \times \mathrm{SL}_2(\mathbb{C})$ on which $WD(F_v)$ acts through $W(F_v)$), Theorem 5.2 can be applied to give a local proof of the theorem of Ichino and Yamana from [IY15] about the vanishing of the regularized period integral on the (non-one-dimensional and non-cuspidal part of) the discrete spectrum of $\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_{n+1}(\mathbb{A})$.

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• On the other hand, still in the setting of $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, an ongoing work of Chaudouard and Zydor on the spectral side of the Jacquet–Rallis relative trace formula indicates that the regularized periods of some representations in the automorphic dual do intervene in the analysis. Moreover, only relevant pairs of A-parameters can contribute in this analysis. These results, together with those of Ichino and Yamana recalled previously, indicate that in the global setting, it is natural to consider the restriction problem for $\mathcal{A}_{\mathrm{aut}}(G)$ and not just for the discrete spectrum $\mathcal{A}_{\mathrm{disc}}(G)$. However, for classical groups, where discrete A-parameters are simply multiplicity-free sum of selfdual representations of the appropriate sign and are not necessarily irreducible, the consideration of the discrete spectrum should already provide many interesting examples and checks on our global conjecture.

Remark 9.2. It can happen that two global A-parameters are not relevant but they are locally relevant at all places. For example, for a non-CM automorphic representation Π on $\mathrm{PGL}_2(\mathbb{A}_F)$, which is a principal series at all places v of F, then at each local place v, the L-parameter of $\mathrm{Sym}^2(\Pi_v)$ will contain the trivial representation of $WD(F_v)$. Hence, the A-parameter $\mathrm{Sym}^1(\mathbb{C}^2)$ of $L(F) \times \mathrm{SL}_2(\mathbb{C})$ on which $L(F)$ acts trivially, and the L-parameter of $\mathrm{Sym}^2(\Pi)$ are locally relevant at all places of F, but not globally relevant. Thus, the local–global principle does not hold for *relevance* in general. However, in many practical situations, it poses no problem to conclude the relevance of global parameters by local arguments (if one knew local relevance at unramified places for example) as we saw above for the local proof of the result of Ichino and Yamana in the case of $(\mathrm{GL}_n(\mathbb{A}_F), \mathrm{GL}_{n-1}(\mathbb{A}_F))$.

Remark 9.3. Just as understanding multiplicities in a local A-packet is of interest (see Remark 7.3), so is a global analog of it: for a given group $G(\mathbb{A}) = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ and a global A-packet of automorphic representations on $G(\mathbb{A})$ (thus, a collection of nearly equivalent automorphic representations on $G(\mathbb{A})$), one may ask how many representations in this global A-packet support a non-vanishing global period integral. In the case of tempered A-parameters, the answer is at most one. For general A-parameters, considering the restriction problem for the trivial representation of $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ to $\mathrm{PGL}(2)$ (respectively, the trivial representation of $\mathrm{PD}^\times \times \mathrm{PD}^\times$ to $\mathrm{PD}^\times$ where D is any quaternion division algebra over the global field F), we find that global period integral is non-zero on $\mathrm{PGL}(2)$ as well as $\mathrm{PD}^\times$ for any D, so global multiplicity 1 for non-vanishing of period integrals fails, at least when we allow the group $G(\mathbb{A}) = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ to vary. Can one formulate a reasonable answer to this question?

We conclude this section with a result about the function $L(M, N, s)$ for a relevant pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of discrete A-parameters. To describe it, we need to introduce some more notation. Assume for simplicity that we are dealing with the Bessel case, so that M_{A} and N_{A} are selfdual or conjugate-selfdual representations of opposite sign; the Fourier–Jacobi case can be similarly treated. To fix ideas, we assume that the sign of M_{A} is $b(M_{\mathrm{A}}) = -1$ whereas the sign of N_{A} is $b(N_{\mathrm{A}}) = +1$. In the following, we use the symbol M_i to denote a selfdual (or conjugate-selfdual) representation of sign -1 whereas N_j denotes one with sign $+1$. With this convention and understanding, one can decompose the relevant pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ of discrete A-parameters as follows:

$$M_{\mathrm{A}} = \bigoplus_{i \in I} M_i \boxtimes \mathrm{Sym}^{m_i - 1}(\mathbb{C}^2) \oplus \bigoplus_{j \in J} N_j \boxtimes \mathrm{Sym}^{n_j - 1}(\mathbb{C}^2), \tag{9.4}$$

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and

$$N_{\mathrm{A}} = \bigoplus_{i\in I} M_i \boxtimes \mathrm{Sym}^{m_i'-1}(\mathbb{C}^2) \oplus \bigoplus_{j\in J} N_j \boxtimes \mathrm{Sym}^{n_j'-1}(\mathbb{C}^2), \tag{9.5}$$

where the numbers m_i, n_j, m_i', and n_j' are positive integers satisfying

$$m_i - 1 \equiv n_j \equiv 0 \mod 2,$$

and

$$|m_i - m_i'| = 1 = |n_j - n_j'|.$$

Here, we understand $\mathrm{Sym}^{-1}(\mathbb{C}^2)$ to be 0. Moreover, we assume that M_i (for $i \in I$) and N_j (for $j \in J$) are irreducible. In other words, each irreducible summand in M_{A} has a partner in N_{A} and vice versa. As M_{A} and N_{A} are discrete A-parameters, these are multiplicity-free decompositions.

Now consider a pair of indices $(i, j) \in I \times J$. Assume without loss of generality that $m_i > n_j$ (recalling that they are of different parity). As $m_i' = m_i \pm 1$ and $n_j' = n_j \pm 1$, we see that, typically, one would expect $m_i' > n_j'$. This is the case, for example, if $m_i - n_j > 2$. We now make the following definition.

Definition 9.6 (Special pairs). Let $M_{\mathrm{A}}, N_{\mathrm{A}}$ be a pair of relevant discrete global A-parameters, which are a sum of distinct irreducible representations as in (9.4) and (9.5). We call a pair $(i, j) \in I \times J$ special if $m_i - n_j$ and $m_i' - n_j'$ are of opposite sign and denote the subset of special (i, j) by $(I \times J)^{\heartsuit}$, thus

$$(I \times J)^{\heartsuit} = \{(i, j) \in (I \times J) \,|\, (m_i, m_i') = (n_j', n_j)\}.$$

Equivalently, special pairs are direct summands in M_{A} and N_{A} of the form

$$V \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus W \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2) \subset M_{\mathrm{A}},$$
$$V \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2) \oplus W \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \subset N_{\mathrm{A}},$$

for some $i \geq 0$, with V, W irreducible, one orthogonal and the other symplectic.

We now state the main result of this section.

Theorem 9.7. *Let $M_{\mathrm{A}}, N_{\mathrm{A}}$ be a pair of relevant discrete global A-parameters, which are a sum of distinct irreducible representations as in (9.4) and (9.5), with M_{A} symplectic and N_{A} orthogonal. Then the function*

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)}$$

is holomorphic at $s = 0$. Moreover, with the notation as given previously, and with $(I \times J)^{\heartsuit}$ the set of special pairs, one has

$$L(M, N, 0) = C \cdot \prod_{(i,j)\in(I\times J)^{\heartsuit}} L(M_i \otimes N_j, 1/2)$$

for some $C \in \mathbb{C}^{\times}$; more precisely, the order of zero at $s = 0$ of the L-function on the left-hand side is the same as that of the L-function on the right-hand side.

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Proof. To simplify notation, for any global L-parameter Π, let us set

$$L(\Pi \boxtimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s) := \prod_{i=0}^{d-1} L\Big(\Pi, s + \frac{d-1-2i}{2}\Big).$$

Before we begin the proof of the theorem, let us observe that for any irreducible cuspidal automorphic representation Π_1 of $\mathrm{GL}_m(\mathbb{A})$, and Π_2 of $\mathrm{GL}_n(\mathbb{A})$, we have the following assertions due to Jacquet *et al.* [JPS83] for the completed, that is, including the factors at infinity, Rankin product L-function $L(s, \Pi_1 \otimes \Pi_2)$:

(a) $L((\Pi_1 \otimes \Pi_2) \boxtimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + \frac{1}{2})$ has a zero at $s = 0$ if and only if d is odd and $L(\Pi_1 \otimes \Pi_2, \frac{1}{2}) = 0$;
(b) $L((\Pi_1 \otimes \Pi_2) \boxtimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + \frac{1}{2})$ has a pole at $s = 0$ if and only if $d \neq 0$ is even, $m = n$, and $\Pi_1 \cong \Pi_2^\vee$, in which case there is a double pole at $s = 0$ arising from poles of $L(\Pi_1 \otimes \Pi_2, s)$ at both $s = 0, 1$;
(c) $L((\Pi_1 \otimes \Pi_2) \boxtimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + 1)$ has a zero at $s = 0$ if and only if $d \neq 0$ is even, and $L(\Pi_1 \otimes \Pi_2, \frac{1}{2}) = 0$;
(d) $L((\Pi_1 \otimes \Pi_2) \boxtimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + 1)$ has a pole at $s = 0$ if and only if d is odd, $m = n$, and $\Pi_1 \cong \Pi_2^\vee$, in which case there is a simple pole at $s = 0$ if $d = 1$, and a double pole at $s = 0$ if d is odd and $d > 1$.

We first prove the theorem in the case when $M_{\mathrm{A}}, N_{\mathrm{A}}$ are the global A-parameters,

$$M_{\mathrm{A}} = M_1 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus N_1 \boxtimes \mathrm{Sym}^{j-1}(\mathbb{C}^2), \tag{9.8}$$

$$N_{\mathrm{A}} = M_1 \boxtimes \mathrm{Sym}^{i-2}(\mathbb{C}^2) \oplus N_1 \boxtimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2), \tag{9.9}$$

where $\epsilon = \pm 1$, M_1 is symplectic and N_1 is orthogonal. It follows that

$$M_{\mathrm{A}} \otimes N_{\mathrm{A}} = C + D,$$

where

$$C = (M_1 \otimes M_1) \boxtimes (\mathrm{Sym}^{i-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{i-2}(\mathbb{C}^2)) + (N_1 \otimes N_1) \boxtimes (\mathrm{Sym}^{j-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2)),$$
$$D = (M_1 \otimes N_1) \boxtimes [\mathrm{Sym}^{i-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2) + \mathrm{Sym}^{i-2}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j-1}(\mathbb{C}^2)].$$

Similarly, we write

$$\mathrm{Sym}^2(M_{\mathrm{A}}) \oplus \Lambda^2(N_{\mathrm{A}}) = C' + D',$$

where

$$\begin{aligned} C' = {} & \mathrm{Sym}^2(M_1) \boxtimes [\mathrm{Sym}^2(\mathrm{Sym}^{i-1}(\mathbb{C}^2)) + \Lambda^2(\mathrm{Sym}^{i-2}(\mathbb{C}^2))] \\ & + \Lambda^2(M_1) \boxtimes [\Lambda^2(\mathrm{Sym}^{i-1}(\mathbb{C}^2)) + \mathrm{Sym}^2(\mathrm{Sym}^{i-2}(\mathbb{C}^2))] \\ & + \mathrm{Sym}^2(N_1) \boxtimes [\mathrm{Sym}^2(\mathrm{Sym}^{j-1}(\mathbb{C}^2)) + \Lambda^2(\mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2))] \\ & + \Lambda^2(N_1) \boxtimes [\Lambda^2(\mathrm{Sym}^{j-1}(\mathbb{C}^2)) + \mathrm{Sym}^2(\mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2))], \\ D' = {} & (M_1 \otimes N_1) \boxtimes [\mathrm{Sym}^{i-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{i-2}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2)]. \end{aligned}$$

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With this notation, we have

$$L(M,N,s)=\frac{L(C,s+\ 1/2)}{L(C',s+1)}\cdot\frac{L(D,s+1/2)}{L(D',s+1)}.$$

Note that the representations of $\mathrm{SL}_2(\mathbb{C})$ appearing in the definition of C given previously have non-trivial central characters, and therefore only irreducible non-trivial representations of $\mathrm{SL}_2(\mathbb{C})$ appear in C. Similarly, we analyze the terms in C' that contribute a pole to $L(C',s)$ at $s=1$. As we are assuming that M_1 is symplectic and N_1 orthogonal, these are

$$\begin{aligned}&\Lambda^2(M_1)\boxtimes[\Lambda^2(\mathrm{Sym}^{i-1}(\mathbb{C}^2))+\mathrm{Sym}^2(\mathrm{Sym}^{i-2}(\mathbb{C}^2))]\\&\quad+\mathrm{Sym}^2(N_1)\boxtimes[\mathrm{Sym}^2(\mathrm{Sym}^{j-1}(\mathbb{C}^2))+\Lambda^2(\mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2))].\end{aligned}$$

However, the representation

$$\Lambda^2(\mathrm{Sym}^{i-1}(\mathbb{C}^2))+\mathrm{Sym}^2(\mathrm{Sym}^{i-2}(\mathbb{C}^2))+\mathrm{Sym}^2(\mathrm{Sym}^{j-1}(\mathbb{C}^2))+\Lambda^2(\mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2))$$

of $\mathrm{SL}_2(\mathbb{C})$ does not contain the trivial representation as we are assuming M_{A},M_1 are symplectic and N_{A},N_1 are orthogonal, therefore i is odd, and j even.

From the assertions (a)–(d) at the beginning of the proof, we therefore find that:

(i) the terms in C can only give rise to a (double) pole at $s=0$ for $L(C,s+\frac{1}{2})$;
(ii) the terms in C' can only give rise to a (double) pole at $s=0$ for $L(C',s+1)$;
(iii) the terms in D can only give rise to a zero at $s=0$ for $L(D,s+\frac{1}{2})$;
(iv) the terms in D' can only give rise to a zero at $s=0$ for $L(D',s+1)$.

Thus,

$$\begin{aligned}\mathrm{ord}_{s=0}\frac{L\left(C,s+\frac{1}{2}\right)}{L(C',s+1)}&=\mathrm{ord}_{s=0}L\left(C,s+\frac{1}{2}\right)-\mathrm{ord}_{s=0}L(C',s+1)\\&=-2[i-1+\min\{j,j+\epsilon\}]+[(i-1)+j+(i-1)+(j+\epsilon-1)]\\&=0\quad\text{for either choice of }\epsilon=\pm1.\end{aligned}$$

Hence, $L(C,s+\frac{1}{2})/L(C',s+1)$ has neither a zero nor a pole at $s=0$.

Similarly, we have

$$\mathrm{ord}_{s=0}\frac{L(D,s+\ 1/2)}{L(D',s+1)}=\mathrm{ord}_{s=0}L(M_1\otimes N_1,s+1/2)^{\min\{i,j+\epsilon\}+\min\{i-1,j\}-\min\{i-1,j+\epsilon\}-\min\{i,j\}}.$$

It can be easily checked that for i and j of different parity, the exponent is zero in all cases, except when

$$\begin{aligned}M_{\mathrm{A}}&=M_1\boxtimes\mathrm{Sym}^{i-1}(\mathbb{C}^2)\oplus N_1\boxtimes\mathrm{Sym}^{i-2}(\mathbb{C}^2),\\N_{\mathrm{A}}&=M_1\boxtimes\mathrm{Sym}^{i-2}(\mathbb{C}^2)\oplus N_1\boxtimes\mathrm{Sym}^{i-1}(\mathbb{C}^2),\end{aligned}$$

in which case

$$\mathrm{ord}_{s=0}\frac{L(D,s+\ 1/2)}{L(D',s+1)}=\mathrm{ord}_{s=0}L(M_1\otimes N_1,s+1/2).$$

Thus, the theorem is proved in the case when $(M_{\mathrm{A}},N_{\mathrm{A}})$ are given by (9.8) and (9.9).

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There is the analogous case when

$$M_{\mathrm{A}} = M_2 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus N_2 \boxtimes \mathrm{Sym}^{j-1}(\mathbb{C}^2),$$
$$N_{\mathrm{A}} = M_2 \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2) \oplus N_2 \boxtimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2),$$

where $\epsilon = \pm 1$, M_2 is symplectic, and N_2 is orthogonal. This can be analyzed in the same way, and we find that if for some $i \geq 0$,

$$M_{\mathrm{A}} = M_2 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus N_2 \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2),$$
$$N_{\mathrm{A}} = M_2 \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2) \oplus N_2 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2),$$

we have

$$\mathrm{ord}_{s=0} L(M, N, s) = \mathrm{ord}_{s=0} L(M_2 \otimes N_2, s + 1/2),$$

and in all other cases (with $M_{\mathrm{A}}, N_{\mathrm{A}}$ as here)

$$\mathrm{ord}_{s=0} L(M, N, s) = 0.$$

Finally, there is the case when

$$M_{\mathrm{A}} = M_3 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus N_3 \boxtimes \mathrm{Sym}^{j-1}(\mathbb{C}^2),$$
$$N_{\mathrm{A}} = M_3 \boxtimes \mathrm{Sym}^{i}(\mathbb{C}^2) \oplus N_3 \boxtimes \mathrm{Sym}^{j-1+\epsilon}(\mathbb{C}^2),$$

where $\epsilon = \pm 1$, M_3, N_3 are either both symplectic or both orthogonal. This can be analyzed in the same way, and we find that

$$\mathrm{ord}_{s=0} L(M, N, s) = 0.$$

Now we prove the theorem in the general case. Assume that

$$M_{\mathrm{A}} = \bigoplus_{\alpha} V_\alpha,$$
$$N_{\mathrm{A}} = \bigoplus_{\alpha} W_\alpha,$$

is a relevant pair of discrete global A-parameters with M_{A} symplectic and N_{A} orthogonal and where (V_α, W_α) are relevant pairs of irreducible global A-parameters.

Then

$$\begin{aligned} M_{\mathrm{A}} \otimes N_{\mathrm{A}} &= \sum_{\alpha} V_\alpha \otimes W_\alpha + \sum_{\alpha \neq \beta} (V_\alpha \otimes W_\beta + V_\beta \otimes W_\alpha), \\ \mathrm{Sym}^2 M_{\mathrm{A}} + \Lambda^2 N_{\mathrm{A}} &= \sum_{\alpha} (\mathrm{Sym}^2 V_\alpha + \Lambda^2 W_\alpha) + \sum_{\alpha \neq \beta} (V_\alpha \otimes V_\beta + W_\alpha \otimes W_\beta), \end{aligned} \tag{9.10}$$

where in both of the sums, the sum $\sum_{\alpha \neq \beta}$ is over unordered distinct pair of indices α, β.

We now analyze $L(M, N, s)$ in two steps.

Step 1. We prove that the diagonal terms

$$L(V_\alpha, W_\alpha, s) = \frac{L(V_\alpha \otimes W_\alpha, s + 1/2)}{L(\mathrm{Sym}^2 V_\alpha \oplus \wedge^2 W_\alpha, s + 1)},$$

have neither a zero nor a pole at $s = 0$.

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Let

$$V_\alpha = V \boxtimes \mathrm{Sym}^i(\mathbb{C}^2) \quad \text{and} \quad W_\alpha = V \boxtimes \mathrm{Sym}^{i+\epsilon}(\mathbb{C}^2),$$

with V irreducible and $\epsilon = \pm 1$. In this case it is easy to see that the numerator and denominators of $L(s, V_\alpha, W_\alpha)$ can only have poles at $s = 0$. Assume first that V is symplectic so that i is even (because V_α is symplectic). Then we have

$$-\mathrm{ord}_{s=0} L\big(V_\alpha \otimes W_\alpha, s + \tfrac{1}{2}\big) = \min\{i+1, i+1+\epsilon\}$$

and

$$-\mathrm{ord}_{s=0} L(\mathrm{Sym}^2 V_\alpha \oplus \wedge^2 W_\alpha, s+1) = \begin{cases} i, & \text{if } \epsilon = -1, \\ i+1, & \text{if } \epsilon = 1. \end{cases}$$

Therefore, we find that $L(V_\alpha, W_\alpha, s)$ has neither a zero nor a pole at $s = 0$. An analogous argument works in the case when V is orthogonal.

Step 2. Observe that

$$\begin{aligned} L(V_\alpha + V_\beta, W_\alpha + W_\beta, s) &= \frac{L([V_\alpha + V_\beta] \otimes [W_\alpha + W_\beta], s + 1/2)}{L(\mathrm{Sym}^2[V_\alpha + V_\beta] \oplus \wedge^2[W_\alpha + W_\beta], s+1)}, \\ &= L(V_\alpha, W_\alpha, s) L(V_\beta, W_\beta, s) \frac{L(V_\alpha \otimes W_\beta + V_\beta \otimes W_\alpha, s + 1/2)}{L(V_\alpha \otimes V_\beta + W_\alpha \otimes W_\beta, s+1)}. \end{aligned}$$

As, by step 1, $L(V_\alpha, W_\alpha, s)$ and $L(V_\beta, W_\beta, s)$ have neither a zero nor a pole at $s = 0$, the order of zero or pole at $s = 0$ of $L(V_\alpha + V_\beta, W_\alpha + W_\beta, s)$ is the same as that of

$$\frac{L(V_\alpha \otimes W_\beta + V_\beta \otimes W_\alpha, s + 1/2)}{L(V_\alpha \otimes V_\beta + W_\alpha \otimes W_\beta, s+1)}.$$

Now, by the special case of the theorem treated previously, we understand the order of zero or pole at $s = 0$ of $L(V_\alpha + V_\beta, W_\alpha + W_\beta, s)$ (in terms of special pairs), and therefore by the two equalities in (9.10), the proof of the theorem is complete. □

Using Theorem 9.7, we can reformulate Conjecture 9.1 for the discrete spectrum as follows.

Conjecture 9.11. Let $M_{\mathrm{A}} \times N_{\mathrm{A}}$ be a discrete A-parameter of the quasi-split group $G(\mathbb{A}) = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ with endoscopic decomposition as given in (9.4) and (9.5):

$$M_{\mathrm{A}} = \bigoplus_{i \in I} M_i \boxtimes \mathrm{Sym}^{m_i - 1}(\mathbb{C}^2) \oplus \bigoplus_{j \in J} N_j \boxtimes \mathrm{Sym}^{n_j - 1}(\mathbb{C}^2),$$

and

$$N_{\mathrm{A}} = \bigoplus_{i \in I} M_i \boxtimes \mathrm{Sym}^{m'_i - 1}(\mathbb{C}^2) \oplus \bigoplus_{j \in J} N_j \boxtimes \mathrm{Sym}^{n'_j - 1}(\mathbb{C}^2).$$

Then the regularized period integral is non-zero on the submodule of the automorphic discrete spectrum associated to $M_{\mathrm{A}} \times N_{\mathrm{A}}$ on some relevant pure inner form $G'(\mathbb{A}) = G(V')(\mathbb{A}) \times G(W')(\mathbb{A})$ of $G(\mathbb{A}) = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ if and only if $L(1/2, M_i \otimes N_j)$ is non-zero for all special pairs $(i, j) \in (I \times J)^\heartsuit$ (as in Definition 9.6).

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Finally, it is natural to ask whether the refined conjecture of Ichino and Ikeda [II10], which gives a precise formula relating the period integral and the relevant L-value in the case of tempered L-parameters, can be formulated in the non-tempered setting. An important issue in the formulation of the Ichino–Ikeda conjecture is the definition of the local period functionals that intervene in their conjectural formula. In the tempered case, these local functionals are given by (absolutely convergent) integrals of matrix coefficients and can be characterized using the spectral decomposition in the L^2-theory. We do not know how to formulate an extension to the non-tempered case, but hope that this paper will stimulate some work in this direction.

10. Revisiting the global conjecture

We now revisit the global Conjecture 9.1 from the viewpoint of the local Conjecture 6.1.

Fix a pair of spaces $W_0 \subset V_0$ and a pair of global A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$ for $G(V_0) \times G(W_0)$. This gives rise to local A-parameters $(M_{A,v}, N_{A,v})$ at all places v and their associated local A-packets. Because our local conjecture does not address the restriction problem for the whole A-packet, we restrict our attention only to the associated Vogan L-packet $\Pi_{M_v} \times \Pi_{N_v}$ contained in the A-packet. Based on this, we can draw the following conclusions from our local conjecture.

(i) The pair $(M_{A,v}, N_{A,v})$ of local A-parameters should be relevant if we want a non-zero local multiplicity in $\Pi_{M_v} \times \Pi_{N_v}$. This is implied by the relevance of the pair of global A-parameters $(M_{\mathrm{A}}, N_{\mathrm{A}})$. Hence, we assume that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant in the ensuing discussion.

(ii) When $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, there is a unique representation in $\Pi_{M_v} \times \Pi_{N_v}$ with non-zero multiplicity. This unique representation is indexed by a character χ_v of the local component group $A_{M,v} \times A_{N,v}$ and is a representation of a pure inner form of $G_v := G(V)(F_v) \times G(W)(F_v)$. We denote it by π_{χ_v}.

(iii) For almost all v, χ_v is trivial and the representation π_{χ_v} is an unramified representation of $G_{0,v} = G(V_0)(F_v) \times G(W_0)(F_v)$.

In view of this, we can form the restricted direct product groups

$$G_{\mathbb{A}} = \prod_v{}' G_v \supset H_{\mathbb{A}} = \prod_v{}' H_v$$

and the abstract representations

$$\pi_\chi := \otimes_v{}' \pi_{\chi_v} \text{ of } G_{\mathbb{A}} \quad \text{and} \quad \nu = \otimes_v \nu_v \text{ of } H_{\mathbb{A}}.$$

The abstract representation π_χ is the only one in the global L-packet $\Pi_M \times \Pi_N$ that could possibly have a non-zero global period integral. However, before considering the global restriction problem, we need to ensure that it is actually an automorphic representation. For this, we need to address the following questions in turn.

(i) Is $H_{\mathbb{A}} \hookrightarrow G_{\mathbb{A}}$ the group of adelic points of algebraic groups

$$H = G(W) \cdot N \hookrightarrow G = G(V) \times G(W)$$

associated to a relevant pair of spaces $W \subset V$?

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This question can be answered by the same consideration as in [GGP12a, Pg 99]. We see that the answer is Yes if and only if

$$\epsilon(1/2, M \otimes N) = 1.$$

Assuming this, we are led to the next questions.

(ii) Is the abstract representation π_χ of $G_{\mathbb{A}} = G(V)(\mathbb{A}) \times G(W)(\mathbb{A})$ automorphic?

Finally, if the answer to question (ii) is affirmative, one can consider the restriction of the regularized period integral $F(\nu)$ to π_χ and ask the last question.

(iii) Is this restriction non-zero?

We now consider question (ii) in some detail. To address this question, we can reduce to the case of discrete A-parameters, so we assume that M_{A} and N_{A} are discrete. Then the automorphy of π_χ is determined by the Arthur multiplicity formula, where a certain quadratic character ϵ_{Art} of the global component group of the A-parameter intervenes. More precisely, given the global A-parameter $M_{\mathrm{A}} \times N_{\mathrm{A}}$ with associated L-parameter $M \times N$, one has the following commutative diagram of component groups.

$$\begin{array}{ccc} A_{M_{\mathrm{A}}} \times A_{N_{\mathrm{A}}} & \xrightarrow{\Delta} & \prod_v (A_{M_{A,v}} \times A_{N_{A,v}}) \\ \downarrow & & \downarrow \\ A_M \times A_N & \xrightarrow{\Delta} & \prod_v (A_{M_v} \times A_{N_v}) \end{array}$$

The representation $\pi_\chi = \otimes'_v \pi_{\chi_v}$ corresponds to a character $\otimes_v \chi_v$ of the group $\prod_v (A_{M_v} \times A_{N_v})$. Pulling back by the previous diagram, one obtains a character χ of $A_{M_{\mathrm{A}}} \times A_{N_{\mathrm{A}}}$. Now

$$\pi_\chi \text{ is automorphic if and only if } \chi = \epsilon_{\mathrm{Art}}.$$

The difference between [GGP12a] and the situation here is that for tempered A-parameters considered in [GGP12a], this quadratic character ϵ_{Art} is trivial, whereas here it is not necessarily trivial.

To explicate the condition $\chi = \epsilon_{\mathrm{Art}}$, we need to determine ϵ_{Art}. For this, let us return to the decomposition of M_{A} and N_{A} given in (9.4) and (9.5). Then we may write

$$A_{M_{\mathrm{A}}} = \prod_{i \in I, m_i > 0} \mathbb{Z}/2\mathbb{Z} \cdot a_i \times \prod_{j \in J, n_j > 0} \mathbb{Z}/2\mathbb{Z} \cdot b_j$$

and

$$A_{N_{\mathrm{A}}} = \prod_{i \in I : m'_i > 0} \mathbb{Z}/2\mathbb{Z} \cdot a'_i \times \prod_{j \in J : n'_j > 0} \mathbb{Z}/2\mathbb{Z} \cdot b'_j.$$

In other words, these component groups are vector spaces over $\mathbb{Z}/2\mathbb{Z}$ equipped with canonical bases. To specify ϵ_{Art}, it suffices to evaluate the signs $\epsilon_{\mathrm{Art}}(a_i)$, $\epsilon_{\mathrm{Art}}(b_j)$, $\epsilon_{\mathrm{Art}}(a'_i)$, and $\epsilon_{\mathrm{Art}}(b'_i)$.

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Lemma 10.1. *With the above notation, one has*

$$\epsilon_{\mathrm{Art}}(a_i) = \prod_{j\in J} \epsilon(1/2, M_i \otimes N_j)^{\min(m_i,n_j)} = \prod_{j\in J: m_i<n_j} \epsilon(1/2, M_i \otimes N_j),$$

$$\epsilon_{\mathrm{Art}}(b_j) = \prod_{i\in I} \epsilon(1/2, M_i \otimes N_j)^{\min(m_i,n_j)} = \prod_{i\in I: m_i<n_j} \epsilon(1/2, M_i \otimes N_j),$$

$$\epsilon_{\mathrm{Art}}(a'_i) = \prod_{j\in J} \epsilon(1/2, M_i \otimes N_j)^{\min(m'_i,n'_j)} = \prod_{j\in J: m'_i>n'_j} \epsilon(1/2, M_i \otimes N_j),$$

$$\epsilon_{\mathrm{Art}}(b'_j) = \prod_{i\in I} \epsilon(1/2, M_i \otimes N_j)^{\min(m'_i,n'_j)} = \prod_{i\in I: m'_i>n'_j} \epsilon(1/2, M_i \otimes N_j).$$

On the other hand, we have the following result.

Lemma 10.2. *The character χ is given by*

$$\chi(a_i) = \epsilon(1/2, M_i \otimes N) = \prod_{j\in J} \epsilon(1/2, M_i \otimes N_j),$$

and

$$\chi(b'_j) = \epsilon(1/2, M \otimes N_j) = \prod_{i\in I} \epsilon(1/2, M_i \otimes N_j).$$

Moreover, $\chi(b_j) = 1 = \chi(a'_i)$.

As a consequence of Lemmas 10.1 and 10.2, we deduce the following.

Corollary 10.3. *The representation π_χ in the L-packet associated to $M \times N$ is automorphic if and only if the following hold:*

$$\prod_{j\in J: m_i>n_j} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for all } i \in I \text{ with } m_i > 0,$$

$$\prod_{j\in J: m'_i>n'_j} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for all } i \in I \text{ with } m'_i > 0,$$

$$\prod_{i\in I: m_i<n_j} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for all } j \in J \text{ with } n_j > 0,$$

$$\prod_{i\in I: m'_i<n'_j} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for all } j \in J \text{ with } n'_j > 0.$$

In particular, when these conditions hold, one has

$$\prod_{j:(i,j)\in(I\times J)^\heartsuit} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for any fixed } i \in I. \tag{10.4}$$

Likewise,

$$\prod_{i:(i,j)\in(I\times J)^\heartsuit} \epsilon(1/2, M_i \otimes N_j) = 1 \quad \text{for any fixed } j \in J. \tag{10.5}$$

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Proof. The first four identities simply follow by equating χ with ϵ_{Art}. Equation (10.4) follows by dividing the first and second identities, whereas (10.5) follows by dividing the third and fourth identities. □

At this point, we have explicated the answer to question (ii), and it is instructive to recall Theorem 9.7, which states that

$$L(M,N,0) = C \cdot \prod_{(i,j)\in(I\times J)^{\heartsuit}} L(M_i \times N_j, 1/2) \quad \text{for some } C \in \mathbb{C}^{\times}.$$

One can rewrite

$$L(M,N,0) = C \cdot \prod_{i\in I} \Big(\prod_{j:(i,j)\in(I\times J)^{\heartsuit}} L(M_i \otimes N_j, 1/2) \Big) \tag{10.6}$$

or

$$L(M,N,0) = C \cdot \prod_{j\in J} \Big(\prod_{i:(i,j)\in(I\times J)^{\heartsuit}} L(M_i \otimes N_j, 1/2) \Big). \tag{10.7}$$

Now observe that (10.4) is saying that the global root number of the interior product of L-functions in (10.6) is $+1$. Likewise, (10.5) is saying that the global root number of the interior product of L-functions in (10.7) is $+1$.

Given this, it is not unreasonable to conjecture that, under the hypothesis that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant and in the context of question (iii), the regularized period integral $F(\nu)$ is non-zero on π_χ if and only if $L(M,N,0) \neq 0$, given that the conditions of automorphy of π_χ expressed in Corollary 10.3 hold.

11. Low-rank examples

In this section, we consider a few examples of our conjectures in low-rank groups. The restriction problem considered in this paper has been studied in several low-rank examples by various people and we examine their results in light of our conjectures.

Example 11.1: $\mathrm{SO}_2 \times \mathrm{SO}_3$. We begin with this simple example where $G = \mathrm{SO}_3(k) = \mathrm{PGL}_2(k)$ and $\widehat{G} = \mathrm{SL}_2(\mathbb{C})$. Consider the A-parameter

$$M_{\mathrm{A}} = M_\alpha \otimes \mathrm{Sym}^1(\mathbb{C}^2)$$

where M_α is a one-dimensional orthogonal representation given by a quadratic character

$$\alpha : k^*/k^{*2} \to \langle \pm 1 \rangle.$$

The L-packet of the corresponding L-parameter M contains a single representation $\pi(\alpha)$ of dimension one, where the group G acts through a composition of α with the determinant character

$$\det : \mathrm{PGL}_2(k) \to k^*/k^{*2}.$$

Let $H = \mathrm{SO}_2$ be the subgroup of G fixing a non-isotropic line in the standard three dimensional representation of G. Then there is an étale quadratic algebra K over k such that $H = K^*/k^*$ and the irreducible representations χ of H all have dimension one. In this case,

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the A-parameters and L-parameters of H are both equal to the two-dimensional orthogonal representation

$$N = \mathrm{Ind}\,(\chi)$$

of $W(k)$. When $K = k + k$, $N = \chi + \chi^{-1}$. When K is a field corresponding to a non-trivial quadratic character β of k^*, one has $\det(N) = \beta$. In this case, the representation N is irreducible unless χ is the composition of a quadratic character χ_k of k^* with the norm, when $N = \chi_k + \chi_k\beta$. There are two representations in the Vogan L-packet, for the groups of the two orthogonal spaces of dimension 2 with the correct discriminant, but only one of these spaces is relevant.

The restriction of the one-dimensional representation $\pi(\alpha)$ of $G = \mathrm{SO}_3$ to the subgroup $H = \mathrm{SO}_2$ is the character χ_α given by the composition of the quadratic character α of k^* with the norm from K^* to k^*. (As the norms from K^* form a subgroup of index two in k^* when K is a field, this character is also the composition of the quadratic character $\alpha.\beta$ with the norm.) As $\chi_\alpha = \chi_\alpha^{-1}$, we have

$$\mathrm{Hom}_H(\pi(\alpha) \otimes \chi_\alpha, \mathbb{C}) = \mathbb{C}.$$

In this case, the A-parameters (M, N) form a relevant pair: $N = \alpha.\beta + \alpha = N_0 + M_\alpha$. This is the only possible N of dimension 2 and the correct determinant that combines with M to give a relevant pair. For characters χ of H that are not equal to χ_α, we have

$$\mathrm{Hom}_H(\pi(\alpha) \otimes \chi, \mathbb{C}) = 0.$$

In particular, the results of this elementary example is in accordance with our local Conjecture 6.1.

The L-function of the tensor product representation is given by

$$L(M \otimes N, s) = L(\alpha \otimes \mathrm{Ind}\,(\chi), s + 1/2) \cdot L(\alpha \otimes \mathrm{Ind}\,(\chi), s - 1/2).$$

This has a pole at $s = 1/2$ if and only if the quadratic character α appears in the representation $\mathrm{Ind}\,(\chi)$, in which case the order of the pole is the multiplicity of the character α in this induced representation. The L-function of the adjoint representation is given by

$$L(\mathrm{Sym}^2 M, s) \cdot L(\wedge^2, s) = L(\mathbb{C}, s + 1) \cdot L(\mathbb{C}, s) \cdot L(\mathbb{C}, s - 1) \cdot L(\beta, s).$$

This has a simple pole at $s = 1$. We conclude that the ratio

$$L(M, N, s) = L(M \otimes N, s + 1/2)/L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)$$

has order less than or equal to zero at $s = 0$ if and only if α appears in $\mathrm{Ind}\,(\chi)$ or, equivalently, if and only if

$$\mathrm{Hom}_H(\pi(\alpha) \otimes \chi, \mathbb{C}) = \mathbb{C}.$$

When K is a field, the parameter is discrete, and we see that $L(M, N, s)$ is regular and non-zero at $s = 0$, When $K = k + k$, the parameter for χ_α is not discrete, and the character α appears with multiplicity 2 in the induced representation. In this case, the ratio $L(M, N, s)$ has a simple pole at $s = 0$.

Example 11.2: *restriction of one-dimensional characters.* One can easily extend the previous example to cover the restriction of one-dimensional representations of the split group

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$\mathrm{SO}_{2n+1} = \mathrm{SO}(V)$ to the special orthogonal group $\mathrm{SO}_{2n} = \mathrm{SO}(W)$ of a codimension-one subspace W. The one-dimensional representations of SO_{2n+1} have symplectic A-parameters

$$M = M_\alpha \otimes \mathrm{Sym}^{2n-1}(\mathbb{C}^2),$$

where M_α is a one-dimensional orthogonal representation given by a quadratic character

$$\alpha : k^*/k^{*2} \to \langle \pm 1 \rangle.$$

The associated representation of $\mathrm{SO}(V)$ is the composition of α with the spinor norm : $\mathrm{SO}(V) \to \langle \pm 1 \rangle$. Let β be the quadratic character $\beta : k^*/k^{*2} \to \langle \pm 1 \rangle$, given by $\beta(x) = (x, d)$ where d is the (normalized) discriminant of the even-dimensional subspace W, and $(-,-)$ is the Hilbert symbol of k. Then the only A-parameter for the subgroup $\mathrm{SO}_{2n} = \mathrm{SO}(W)$ that gives a relevant pair (M, N) is

$$N = N_0^- + N_{2n-2}^+ \otimes \mathrm{Sym}^{2n-2}(\mathbb{C}^2)$$

with $N_0^- = \alpha.\beta$ and $N_{2n-2}^+ = \alpha$. This is indeed the A-parameter for the restricted one-dimensional representation of SO_{2n}.

We can easily calculate the order of $L(M, N, s)$ at $s = 0$ in this case. A short computation gives

$$\begin{aligned}
\mathrm{Sym}^2(M) &= \mathrm{Sym}^{4n-2}(\mathbb{C}^2) + \mathrm{Sym}^{4n-6}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^2(\mathbb{C}^2),\\
\wedge^2(N) &= \mathrm{Sym}^{4n-6}(\mathbb{C}^2) + \mathrm{Sym}^{4n-10}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^2(\mathbb{C}^2) + \mathrm{Sym}^{2n-2}(\mathbb{C}^2) \otimes \beta,\\
M \otimes N &= \mathrm{Sym}^{4n-3}(\mathbb{C}^2) + \mathrm{Sym}^{4n-7}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^1(\mathbb{C}^2) + \mathrm{Sym}^{2n-1}(\mathbb{C}^2) \otimes \beta.
\end{aligned}$$

Therefore, the L-function of the numerator $L(M \otimes N, s + 1/2)$ has a pole of order $2n - 1$ at $s = 0$ if $\beta \neq 1$, and a pole of order $2n$ at $s = 0$ if $\beta = 1$. The same is true for the L-function of the denominator $L(\mathrm{Sym}^2 M, s + 1) \cdot L(\wedge^2 N, s + 1)$, provided that $n > 1$. Hence, when $n > 1$, the ratio $L(M, N, s)$ is regular and non-zero at the point $s = 0$. In the special case where $n = 1$ and $\beta = 1$ (so $\mathrm{SO}(W) \cong \mathbb{G}_m$), this ratio has a simple pole at $s = 0$ as we noted already at the end of the previous example.

Example 11.3: $\mathrm{SO}_3 \subset \mathrm{SO}_4$. We consider the case of $\mathrm{SO}_3 \subset \mathrm{SO}_4$ with SO_4 quasi-split. This corresponds to

$$\mathrm{PGL}_2(k) \longrightarrow \mathrm{GL}_2^+(K)/k^\times,$$

where K is a quadratic algebra over a p-adic field k and $\mathrm{GL}_2^+(K)$ is the subgroup of $\mathrm{GL}_2(K)$ with determinant in $k^\times$. We can, in fact, consider slightly more generally the embedding $\mathrm{GL}_2(k) \hookrightarrow \mathrm{GL}_2^+(K)$. Thus, for irreducible representations π_1 of $\mathrm{GL}_2^+(K)$ and π_2 of $\mathrm{GL}_2(k)$, we would like to compute $d(\pi_1, \pi_2) = \dim \mathrm{Hom}_{\mathrm{GL}_2(k)}(\pi_1, \pi_2)$.

We consider the case when π_1 is a generic (thus, infinite-dimensional) representation. In this case, the four-dimensional L-parameter associated to π_1 is more usually called the Asai representation, or tensor induction, denoted as $As(\pi_1)$. This notion is typically defined for representations of $\mathrm{GL}_2(K)$, but one can also define it for representations π_1 of $\mathrm{GL}_2^+(K)$ by choosing an arbitrary irreducible representation of $\mathrm{GL}_2(K)$ containing π_1 and doing the Asai construction for this. The result is independent of the choice of the representation of $\mathrm{GL}_2(K)$, so it is legitimate to call it $As(\pi_1)$ for π_1 on $\mathrm{GL}_2^+(K)$.

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If the representation π_2 of $\mathrm{GL}_2(k)$ is infinite dimensional, this amounts to questions already considered in [Pra92], which is in the setting of tempered GGP. So we only analyze the situation when the representation π_2 of $\mathrm{GL}_2(k)$ is one dimensional, which to simplify notation, we take to be the trivial representation.

Thus, we want to understand which generic representations π_1 of $\mathrm{GL}_2^+(K)$ have an invariant form for $\mathrm{GL}_2(k)$. By our conjecture, one would therefore expect that a representation of $\mathrm{GL}_2^+(K)$ has a $\mathrm{GL}_2(k)$ invariant form if and only if the four-dimensional representation of W_k' contains the trivial representation, or equivalently the Asai L-function has a pole at 0.

This is exactly what was proved globally by Harder *et al.* [HLR86], and later also locally by others, see [Fli91, Kab04] for results for GL_2, and Lemma 4.1 of [AP18] for SL_2. The nice fact is that this criterion, which is usually stated for discrete series representations, is valid for all generic representations π_1 of $\mathrm{GL}_2^+(K)$.

Theorem 11.4. *For K a quadratic extension of a p-adic field k, an infinite-dimensional irreducible representation π of $\mathrm{GL}_2(K)$ whose central character is trivial when restricted to $k^\times$ has a non-zero $\mathrm{GL}_2(k)$ invariant linear form if and only if its Asai L-function has a pole at $s = 0$.*

Assuming that the representation π of $\mathrm{GL}_2(K)$ has a $\mathrm{GL}_2(k)$ invariant form, an irreducible sub-representation π' of π restricted to $\mathrm{GL}_2^+(K)$ has a $\mathrm{GL}_2(k)$ invariant linear form if and only if π' is generic for a character $\psi : K \to \mathbb{C}^\times$ that is trivial on k.

In this case, where the A-parameter of π_2 is [2], the distinguished character predicted by our local conjecture is the trivial character, which indexes the representation of $\mathrm{GL}_2^+(K)$ in the L-packet with Whittaker model with respect to a character of K/k. This is indeed what the previous theorem proves. Our may also consider the case when $\mathrm{GL}_2(k)$ is replaced by $D^\times$, where D is the unique quaternion k-algebra. In this case, the restriction problem is addressed by the tempered GGP.

Example 11.5: $U_2 \subset U_3$. For a quadratic extension K/k of local fields, we consider a pair of unitary group $U_2 \subset U_3$ and the restriction of representations from U_3 to U_2. In what follows, we are interested in understanding which representations of U_3 contain the trivial character of U_2 as a quotient; such representations of U_3 are said to be U_2-distinguished. This has been considered by Gelbart *et al.*, culminating in their work [GRS97]. We recall their results briefly, casting them in the language of our local and global conjectures.

Theorem 11.6. *Let $U_2 \subset U_3$ be a pair of quasi-split unitary groups defined for a quadratic extension K/k of non-archimedean local fields. Suppose that M_{A} is an A-parameter for U_3 (regarded as a conjugate orthogonal representation of $W(K)$) with associated A-packet $\Pi_{M_{\mathrm{A}}}$. A representation $\pi \in \Pi_{M_{\mathrm{A}}}$ is U_2-distinguished if and only if one of the following holds:*

(i) *π is the trivial representation of U_3, i.e. $M_{\mathrm{A}} = [3]$;*
(ii) *π is a tempered generic representation such that $L(s, BC(\pi))$ has a pole at $s = 0$ (where $BC(\pi)$ is the base change of π to K); in this case, $M_{\mathrm{A}} = 1 + M_0$ with M_0 tempered.*

In particular, the pair $(M_{\mathrm{A}}, [2])$ is relevant and the unique representation π which is U_2-distinguished lies in Π_M and corresponds to the distinguished character of Conjecture 6.1(c).

We now state the corresponding global theorem from [GRS97].

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Theorem 11.7. *Let E/F be a quadratic extension of number fields with corresponding adele rings $\mathbb{A}_E$ and $\mathbb{A}_F$ and consider a pair $U_2 \subset U_3$ of unitary groups over F. Let $\pi = \otimes \pi_v$ be an irreducible infinite-dimensional cuspidal automorphic representation of $U_3(\mathbb{A}_F)$ with base change $BC(\pi)$ on $\mathrm{GL}_3(E)$. Then the period integral over U_2 is non-zero on π if and only if:*

(i) *for each place v of F, π_v is $U_2(F_v)$-distinguished;*
(ii) *$L(s, BC(\pi))$ has a pole at $s = 1$.*

In particular, π belongs to a tempered L-packet with L-parameter $M = 1 + M_0$, so that (M, N_{A}) is relevant (with $N_{\mathrm{A}} = [2]$). Moreover, the L-function

$$L(M, N, s) = \frac{L_E(M, s) \cdot L_E(M, s+1)}{\zeta_F(s) \cdot \zeta_F(s+1) \cdot \zeta_F(s+2) \cdot L(\omega_{E/F}, s+1) \cdot L(M, Ad, s+1)}$$

is finite non-zero at $s = 0$.

One may also start with a non-tempered A-parameter M_{A} of U_3 and consider the restriction of the representations in the associated A-packets to U_2. Disregarding one-dimensional representations of U_3, these A-parameters are of the form $M_{\mathrm{A}} = \mu + \nu \boxtimes [2]$, with μ, ν one-dimensional characters that are conjugate dual of appropriate sign. This restriction problem has been studied in two papers [Haa16, Haa17] of Haan. His results are in accordance with the expectations of Conjecture 6.1. We refer the reader to his papers for the precise results.

Example 11.8: $\mathrm{SO}_4 \subset \mathrm{SO}_5$. We now consider the restriction problem for some non-tempered A-packets of $\mathrm{SO}_5 = \mathrm{PGSp}_4$, namely the Saito–Kurokawa packets and the A-packets of Soudry type. The restriction of representations in these packets to SO_4 has been determined in [GG09, GS15].

Consider first the Saito–Kurokawa case over a non-archimedean local field k. Here the A-parameter M_{A} of SO_5 is of the form

$$M_{\mathrm{A}} = M_0 + [2],$$

where M_0 is a tempered L-parameter of SO_3. The A-packet of M_{A} can be explicated as follows. The component group of M_{A} is

$$A_{M_{\mathrm{A}}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } M_0 \text{ irreducible;} \\ \{1\} \times \mathbb{Z}/2\mathbb{Z} & \text{if } M_0 \text{ reducible.} \end{cases}$$

Here, we think of the first component in the direct product as associated to M_0, whereas the second $\mathbb{Z}/2\mathbb{Z}$ is associated to $[2]$. The A-packet $\Pi_{M_{\mathrm{A}}}$ thus consists of four or two representations in the respective cases and we may label them by the irreducible characters of $A_{M_{\mathrm{A}}}$ as

$$\Pi_{M_{\mathrm{A}}} = \{\pi^{\epsilon_1, \epsilon_2} : \epsilon_1, \epsilon_2 = \pm\}$$

where we understand that when M_0 is reducible, $\pi^{-,\epsilon} = 0$. Moreover, a representation $\pi^{\epsilon_1, \epsilon_2}$ is a representation of the split group SO_5^+ if and only if $\epsilon_1 = \epsilon_2$; otherwise, it is a representation of the non-split inner form SO_5^-. The associated L-packet Π_M is the subset

$$\Pi_M = \{\pi^{\epsilon_1, +} : \epsilon_1 = \pm\} \subset \Pi_{M_{\mathrm{A}}}.$$

The A-parameters N_{A} of SO_4 for which $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant are those of the form:

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(a) $N_{\mathrm{A}} = N_0 + 1$, where N_0 is a three-dimensional orthogonal L-parameter;
(b) $N_{\mathrm{A}} = \chi_0 + [3]$, where χ_0 is a one-dimensional character.

For simplicity, we assume that the SO_4 in question has trivial discriminant, in which case N_0 and χ_0 have trivial determinant. Thus, in part (b), N_{A} is the A-parameter of the trivial representation. As this case is treated in some generality in the last example, we focus on part (a) here.

In this case, the results of [GG09] can be summarized as follows.

Theorem 11.9. *Fix a Saito–Kurokawa A-parameter $M_{\mathrm{A}} = M_0 + [2]$ of SO_5 and let N_{A} be a tempered A-parameter of SO_4 with $\det(N_{\mathrm{A}})$ trivial. For $\pi^{\epsilon_1,\epsilon_2} \in \Pi_{M_{\mathrm{A}}}$, set*

$$d(\pi^{\epsilon_1,\epsilon_2}, N_{\mathrm{A}}) = \sum_{\sigma \in \Pi_{N_{\mathrm{A}}}} d(\pi^{\epsilon_1,\epsilon_2}, \sigma).$$

In addition, set

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) = \sum_{\epsilon_1,\epsilon_2} d(\pi^{\epsilon_1,\epsilon_2}, N_{\mathrm{A}}).$$

Then

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) \neq 0 \Longleftrightarrow (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is relevant.}$$

When $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, with $N_{\mathrm{A}} = N_0 + 1$ tempered, we have

$$d(\pi^{\epsilon_1,\epsilon_2}, N_{\mathrm{A}}) \neq 0 \Longleftrightarrow \epsilon_1 = \epsilon(1/2, M_0 \otimes N_{\mathrm{A}}) = \epsilon(1/2, M_0) \cdot \epsilon(1/2, M_0 \otimes N_0).$$

Further, if $d(\pi^{\epsilon_1,\epsilon_2}, N_{\mathrm{A}}) \neq 0$, then $d(\pi^{\epsilon_1,\epsilon_2}, N_{\mathrm{A}}) = 1$.

Corollary 11.10. *In the context of the previous theorem, if one considers the L-packet Π_M, then we have*

$$d(M, N) = \begin{cases} 0, & \text{if } (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is not relevant;} \\ 1, & \text{if } (M_{\mathrm{A}}, N_{\mathrm{A}}) \text{ is relevant.} \end{cases}$$

When $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, the unique representation $\pi^{\epsilon_1,+} \in \Pi_M$ for which $d(\pi^{\epsilon_1,+} N) \neq 0$ is given by

$$\epsilon_1 = \epsilon(1/2, M_0 \otimes N_{\mathrm{A}}).$$

This is the distinguished character predicted by Conjecture 6.1.

Observe that the above theorem goes beyond our local Conjecture 6.1 as it addresses the restriction problem for the whole A-packet $\Pi_{M_{\mathrm{A}}}$ and not just the L-packet Π_M. Note also that when M_0 is irreducible, one has $d(M_{\mathrm{A}}, N_{\mathrm{A}}) = 2$.

There is also an analogous global theorem, which we state here.

Theorem 11.11. *Consider the non-tempered global A-parameter $M_{\mathrm{A}} = M_0 \oplus [2]$ over a number field F, where M_0 corresponds to a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A}_F)$. Consider an element of the global A-packet $\Pi_{M_{\mathrm{A}}}$ for $G(\mathbb{A}_F) = \mathrm{SO}_5(\mathbb{A}_F)$:*

$$\pi^{\epsilon_1,\epsilon_2} = \otimes_v \pi^{\epsilon_{1,v},\epsilon_{2,v}}.$$

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(i) *This representation occurs in the automorphic discrete spectrum if and only if*

$$\prod_v \epsilon_{1,v} = \epsilon(1/2, M_0) = \prod_v \epsilon_{2,v}.$$

(ii) *When condition (i) holds, the global period integral of* $\pi^{\epsilon_1,\epsilon_2}$ *against the cuspidal representations with tempered L-parameter* N_{A} *is zero unless* $(M_{\mathrm{A}}, N_{\mathrm{A}})$ *is relevant, that is, when* $N_{\mathrm{A}} = N_0 + 1$.

(iii) *When condition (i) holds and* $(M_{\mathrm{A}}, N_{\mathrm{A}})$ *is relevant, the period integral in condition (ii) is non-zero if and only if*

$$L(1/2, M_0 \otimes N_0) \neq 0.$$

Noting that

$$L(s, M_{\mathrm{A}}, N_{\mathrm{A}}) = \frac{L(s+1/2, M_0 \otimes N_0)}{\zeta_F(s+2) \cdot L(s+3/2, M_0) \cdot L(s+1, M_0, Ad)},$$

we see that condition (iii) is equivalent to saying that $L(0, M_{\mathrm{A}}, N_{\mathrm{A}}) \neq 0$.

Now we consider the case of A-packets of Soudry type, that is, an A-parameter of SO_5 of the form

$$M_{\mathrm{A}} = M_0 \otimes [2]$$

where M_0 is an orthogonal L-parameter. Over a local field k, we enumerate the possible component group $A_{M_{\mathrm{A}}}$ according to the following four disjoint scenarios:

(a) if $M_0 = \chi + \chi^{-1}$ with χ unitary but non-quadratic, then $\det(M_0)$ is trivial and $A_{M_{\mathrm{A}}}$ is trivial;
(b) if $M_0 = \chi + \chi$, with χ quadratic, then $\det(M_0)$ is trivial and $A_{M_{\mathrm{A}}} = \mathbb{Z}/2\mathbb{Z}$;
(c) if M_0 is irreducible, then $\det(M_0)$ is non-trivial and $A_{M_{\mathrm{A}}} = \mathbb{Z}/2\mathbb{Z}$;
(d) if $M_0 = \chi_1 + \chi_2$ with $\chi_1 \neq \chi_2$ quadratic, then $\det(M_0)$ is non-trivial and $A_{M_{\mathrm{A}}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Accordingly, the local A-packet $\Pi_{M_{\mathrm{A}}}$ has one, two, two, or four representations in the respective cases. We may denote the A-packet by

$$\Pi_{M_{\mathrm{A}}} = \{\pi^{\epsilon_1,\epsilon_2} : \epsilon_1, \epsilon_2 = \pm\}$$

with the understanding that:

- in case (a), all representations are 0 except for π^{++};
- in case (b), the representations $\pi^{+,-}$ and $\pi^{-,+}$ are zero;
- in case (c), the representations $\pi^{-,-}$ and $\pi^{+,-}$ are zero.

Moreover, the representation $\pi^{\epsilon_1,\epsilon_2}$ is a representation of the split SO_5^+ if and only if $\epsilon_1 = \epsilon_2$. Thus, the A-packet $\Pi_{M_{\mathrm{A}}}$ contains some representations of the non-split SO_5^- only in cases (c) and (d) (that is, when $\det(M_0)$ is non-trivial). The associated L-packet Π_M is the subset

$$\Pi_M = \{\pi^{+,+}\} \subset \Pi_{M_{\mathrm{A}}}.$$

The A-parameters N_{A} of SO_4 for which the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant are those of the form:

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(a) $N_{\mathrm{A}} = M_0 + N_0$ with N_0 tempered orthogonal of dimension 2;
(b) $N_{\mathrm{A}} = \chi_1 \otimes [3] + \chi_2$, if $M_0 = \chi_1 + \chi_2$ (χ_1, χ_2 not necessarily distinct).

Here is a sample of the local results of [GS15].

Theorem 11.12. *Consider the non-tempered local A-parameter $M_{\mathrm{A}} = M_0 \boxtimes [2]$ for SO_5 and let N_{A} be a tempered A-parameter for SO_4. Assume for simplicity that $\det(N_{\mathrm{A}})$ is trivial.*

(i) *If $d(M_{\mathrm{A}}, N_{\mathrm{A}}) \neq 0 \Longrightarrow (M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant.*
(ii) *Suppose that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant and $\det(M_0)$ is trivial (i.e. in cases (a) and (b)). Then $d(\pi, N_{\mathrm{A}}) = 1$ for any $\pi \in \Pi_{M_{\mathrm{A}}}$. Thus,*

$$d(M_{\mathrm{A}}, N_{\mathrm{A}}) = \begin{cases} 1, & \text{if } M_0 = \chi + \chi^{-1},\ \chi \text{ non-quadratic;} \\ 2, & \text{if } M_0 = \chi + \chi^{-1},\ \text{with } \chi \text{ quadratic.} \end{cases}$$

(iii) *Suppose that $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant and $\det(M_0)$ is non-trivial (i.e. in cases (c) and (d)). Then we have:*
- *in case (c), where M_0 is irreducible, we have $d(\pi, N_{\mathrm{A}}) = 1$ for any $\pi \in \Pi_{M_{\mathrm{A}}}$, so that $d(M_{\mathrm{A}}, N_{\mathrm{A}}) = 2$;*
- *in case (d), where M_0 is reducible, we have $d(\pi, N_{\mathrm{A}}) = 1$ for any $\pi \in \Pi_{M_{\mathrm{A}}}$, so that $d(M_{\mathrm{A}}, N_{\mathrm{A}}) = 4$, unless $N_{\mathrm{A}} = M_0 + N_0$ with N_0 reducible; in this latter case, we have*

$$d(\pi^{\epsilon_1, \epsilon_2}, N_{\mathrm{A}}) = 1 \Longleftrightarrow \epsilon_1 = +,$$

so that $d(M_{\mathrm{A}}, N_{\mathrm{A}}) = 2$.

(iv) *In all cases where $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is relevant, we have*

$$d(\pi^{++}, N) = 1.$$

This theorem verifies Conjecture 6.1 in this particular case; indeed it goes further by determining $d(\pi, N_{\mathrm{A}})$ for all $\pi \in \Pi_{M_{\mathrm{A}}}$. There is also a companion global theorem for which we refer the reader to [GS15].

Example 11.13: $\mathrm{Sp}_4(k) \subset \mathrm{GL}_4(k)$. Representations of $\mathrm{GL}_{2n}(k)$ distinguished by $\mathrm{Sp}_{2n}(k)$ are classified by the work of Offen and Sayag, cf. [OS07]. A low-rank case of their work, $\mathrm{Sp}_4(k) \subset \mathrm{GL}_4(k)$, is closely related to the pair $\mathrm{SO}_5(k) \subset \mathrm{SO}_6(k)$ and can be used to verify our conjectures, which we do now.

In the relationship of $\mathrm{GL}_4(k)$ with $\mathrm{SO}_6(k)$, if the L-parameter of a representation π of $\mathrm{GL}_4(k)$ is ϕ, then the corresponding parameter of the representation of $\mathrm{SO}_6(k)$ with values in $\mathrm{SO}_6(\mathbb{C})$ is $M = \det^{-1/2}(\phi)\Lambda^2(\phi)$ where $\det^{1/2}(\phi)$ is a square root of $\det(\phi)$ (which must exist for a representation π of $\mathrm{GL}_4(k)$ to be related to a representation of $\mathrm{SO}_6(k)$).

As the A-parameter of the trivial representation of $\mathrm{SO}_5(k)$ is $[4] = \mathrm{Sym}^3(\mathbb{C}^2)$, it follows by our conjecture that the only option for the six-dimensional orthogonal representation $M = \det^{-1/2}(\phi)\Lambda^2(\phi)$ are:

(i) $[5] + [1]$;
(ii) $[3] + M_0$ with M_0 tempered and three dimensional.

Here, case (i) corresponds to characters of $\mathrm{GL}_4(k)$, whereas in case (ii), the M_0 must be orthogonal. It can be seen that $M = \det^{-1/2}(\phi)\Lambda^2(\phi)$ has this shape for a four-dimensional L-parameter

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ϕ of Arthur type with A-parameter ϕ_{A} if and only if ϕ_{A} is of the form $\tau \otimes [2]$, where τ is a two-dimensional tempered parameter. Thus, M is of Arthur type with associated A-parameter

$$M_{\mathrm{A}} = \Lambda^2(\tau \otimes [2]) = \mathrm{Sym}^2(\tau) \oplus \Lambda^2(\tau)[3].$$

Thus, our conjecture is in conformity with the results of Offen and Sayag that the only representations of $\mathrm{GL}_4(k)$ which are distinguished by $\mathrm{Sp}_4(k)$ are either one dimensional, or have A-parameter of the form $\tau \otimes [2]$.

Example 11.14: *distinction of* SO_n *by* SO_{n-1}. We have considered several low-rank instances of the problem of determining the representations (of Arthur type) of SO_n (respectively U_n) distinguished by SO_{n-1} (respectively U_{n-1}). This problem can be studied in general using the theory of theta correspondence. More precisely, one can show the following.

Proposition 11.15. *Let k be a non-archimedean local field and consider a non-degenerate quadratic space V over k. For $a \in k$, set $V_a = \{x \in V : q(x) = a\}$. Let*

$$G = \begin{cases} \mathrm{SL}_2(k) & \text{if } \dim V \text{ is even;} \\ \mathrm{Mp}_2(k), & \text{(the two fold metaplectic cover of } \mathrm{SL}_2(k)\text{), if } \dim V \text{ is odd.} \end{cases}$$

For a fixed non-trivial additive character ψ of k, one may consider the ψ-theta correspondence for $G \times O(V)$. Given an irreducible representation π of $O(V)$, let $\Theta_\psi(\pi)$ be the big theta lift of π to $G(k)$. Then, for $a \in k^\times$,

$$\Theta_\psi(\pi)_{N,\psi_a} \cong \begin{cases} 0, & \text{if } V_a(k) \text{ is empty;} \\ \mathrm{Hom}_{O(U_a)}(\pi, \mathbb{C}), & \text{if } V_a(k) \text{ is non-empty;} \end{cases}$$

where in the latter case, $U_a = x_a^\perp$ for some $x_a \in V_a(k)$. Here, N is a maximal unipotent subgroup of G and $\psi_a(x) = \psi(ax)$ is regarded as a generic character of $N(k) = k$.

Corollary 11.16. *Let π be an irreducible admissible representation of $O(V)$. Then π is distinguished by $O(W)$ with $V/W = \langle a \rangle$ if and only if the big ψ-theta lift of π to G is ψ_a-generic. In particular, one has the following.*

(i) *If π is distinguished by $O(W)$ where W is a codimension-one non-degenerate subspace of V, then π has a non-zero theta lift to G.*
(ii) *If π is the ψ-theta lift of a ψ_a-generic representation of G, then π is distinguished by $O(W)$ for a codimension-one non-degenerate subspace W of V with $V/W = \langle a \rangle$.*

When $\dim V \geq 3$, as is well-known, the theta correspondence for $G \times O(V)$ reduces to one for $G \times \mathrm{SO}(V)$. If $\sigma \in \mathrm{Irr}(G)$ has tempered A-parameter M_0, one knows precisely how to describe its small theta lift $\theta_\psi(\sigma)$ to $\mathrm{SO}(V)$ in terms of the local Langlands correspondence (see [Mui04, Mui08a, Mui08b, AG17]). From this description, one sees that $\theta_\psi(\sigma)$ is of Arthur type with A-parameter of the form

$$\begin{cases} M_0 + [2n-3], & \text{if } \dim V = 2n; \\ M_0 + [2n-2], & \text{if } \dim V = 2n+1. \end{cases}$$

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Moreover, one knows from results of Mœglin [Mœg11, GI18] that all representations with such an A-parameter are obtained as theta lifts from the L-packet of G with L-parameter M_0. From this and Corollary 11.16, we see that the following holds.

Theorem 11.17. *Let π be an irreducible admissible representation of $\mathrm{SO}(V)$ of Arthur type with A-parameter M_{A} and suppose that π is distinguished by $\mathrm{SO}(W)$ for some codimension-one non-degenerate subspace W of V. Then M_{A} must have one of the following forms:*

- *if $\dim V = 2n+1$ is odd, then $M_{\mathrm{A}} = [2n]$ (which corresponds to the trivial representation) or $M_{\mathrm{A}} = [2n-2] + M_0$, with M_0 tempered two dimensional (which corresponds to theta lifts of tempered representations of $\mathrm{Mp}_2(k)$);*
- *if $\dim V = 2n$ is even, then $M_{\mathrm{A}} = [2n-1] + \chi$ (which corresponds to the trivial representation) or $M_{\mathrm{A}} = [2n-3] + M_0$ with M_0 tempered three dimensional (which corresponds to theta lifts of tempered representations of $\mathrm{SL}_2(k)$).*

Conversely, if M_{A} is an A-parameter of $\mathrm{SO}(V)$ of the form described previously, then any representation of $\mathrm{SO}(V)$ in the associated A-packet $\Pi_{M_{\mathrm{A}}}$ is distinguished by $\mathrm{SO}(W)$ for some codimension-one non-degenerate subspace $W \subset V$.

This theorem is in accordance with Conjecture 6.1. There is of course an entirely analogous theorem (and proof) in the unitary case, with the quasi-split U_2 playing the role of G.

12. Automorphic descent

In a series of papers [GRS99a, GRS99b, GRS02, GRS11], Ginzburg *et al.* pioneered the automorphic descent or backward lifting technique, which allowed them to construct the backward functorial lifting from general linear groups to classical groups. The input of their construction is an discrete tempered global L-parameter M for the classical group in question, thought of as an isobaric automorphic representation of the appropriate GL_N, and the output is the unique globally generic cuspidal representation in global L-packet determined by M.

In a recent paper [JZ20], Jiang and Zhang extended this construction so that it has the potential to construct all the automorphic members of the global L-packet associated to M. By their very construction, the automorphic descent method is closely connected to the conjectures of [GGP12a] because it involves the consideration of Bessel and Fourier–Jacobi models. In this section, we explain how these automorphic descent constructions fit with our conjectures.

Let us consider the example of SO_{2n+1} and give a brief description of the descent construction. Suppose we are given a tempered L-parameter $M = M_1 \oplus \cdots \oplus M_r$ of SO_{2n+1}, with M_i distinct cuspidal representations of some GL of symplectic type. We may regard M as an L-parameter of GL_{2n}, giving rise to an automorphic representation

$$\pi_M = \pi_{M_1} \times \cdots \times \pi_{M_r}$$

of GL_{2n}. Now consider the non-tempered A-parameter

$$\tilde{M}_{\mathrm{A}} = M \boxtimes [2].$$

It is an A-parameter of SO_{4n} whose associated L-parameter is

$$\tilde{M} = M \cdot |-|^{1/2} \oplus M \cdot |-|^{-1/2}.$$

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One sees that the local and global component groups of $\tilde{M}$ are trivial, so that the local L-packets and global L-packet of SO_{4n} associated to the L-parameter $\tilde{M}$ are singleton sets. This unique global representation is the Langlands quotient $J(\pi_M, 1/2)$ of the standard module induced from the representation $\pi_M|-|^{1/2}$ on the Siegel parabolic subgroup of SO_{4n}. The embedding of $J(\pi_M, 1/2)$ into the automorphic discrete spectrum can be constructed as an iterated residue of the corresponding Eisenstein series.

Consider now the Bessel coefficient of $J(\pi_M, 1/2)$ with respect to SO_{2n+1}. This gives rise to an automorphic representation of SO_{2n+1} that Ginzburg *et al.* showed to be an irreducible globally generic cuspidal representation belonging to the L-packet of M. In fact, because it is globally generic, it is the cuspidal representation σ_M whose local components correspond to the trivial character of the local component group at each place.

To explain this construction from the viewpoint of our conjectures in this paper, observe that we are considering the pair $\mathrm{SO}_{4n} \times \mathrm{SO}_{2n+1}$ and we have a non-tempered A-parameter $\tilde{M}_{\mathrm{A}}$ of SO_{4n}. One now reasons as follows.

- For which A-parameter N_{A} of SO_{2n+1} can the pair $(\tilde{M}_{\mathrm{A}}, N_{\mathrm{A}})$ be relevant? It is clear that the only possibility is $N_{\mathrm{A}} = M$. Hence, by our global Conjecture 9.1(i), the Bessel descent of $J(\pi_M, 1/2)$ down to SO_{2n+1} (which can be shown to be cuspidal) can only contain representations in the L-packet of M. At this point, however, it is still possible that this Bessel descent is 0.
- Is there any reason to hope that the Bessel descent is non-zero? To address this, we first consider the local setting. As $J(\pi_M, 1/2)$ belongs to the L-packet $\Pi_{\tilde{M}}$, which is a singleton (both locally and globally), our local Conjecture 6.1 can be applied. One can compute the distinguished character of the local component group $A_{\tilde{M}_v} \times A_M$ given by Conjecture 6.1(c); in this case, it turns out to be the trivial character. Hence, our local Conjecture 6.1(c) implies that at each place v, the multiplicity

$$d(J(\pi_{M,v}, 1/2), \sigma_{M_v}) = 1.$$

  Because of the uniqueness part in Conjecture 6.1(b), we see that σ_M is the only element σ in the global L-packet of M such that $J(\pi_M, 1/2) \otimes \sigma$ supports an abstract Bessel period.
- Finally, is the global Bessel period integral non-zero on the automorphic representation $J(\pi_M, 1/2) \otimes \sigma_M$? We can appeal to our global Conjecture 9.1. We have shown that conditions (i) and (ii) in Conjecture 9.1 already hold, so it remains to verify condition (iii). The function $L(\tilde{M}, M, s)$ is given by

$$\frac{L(s+1, M \times M) \cdot L(s, M \times M)}{L(s+1, \mathrm{Sym}^2 M) \cdot L(s+1, \mathrm{Sym}^2 M) \cdot L(s, \wedge^2 M) \cdot L(s+1, \wedge^2 M) \cdot L(s+2, \wedge^2 M)}.$$

  At $s = 0$, this is holomorphic and non-zero (alternatively, one can appeal directly to Theorem 9.7). Hence, our global conjecture implies that the global Bessel period integral is non-zero on $J(\pi_M, 1/2) \times \sigma_M$. In particular, the Bessel descent of $J(\pi_M, 1/2)$ to SO_{2n+1} is non-zero and equal to σ_M.

We mention that there is a local analog of the construction: the local descent map. This was considered in, for example, [GRS99a, GRS02, GRS11, ST15]. While the local descent can be defined starting from any L-parameter of a classical group, the results obtained in these works are most complete in the following two situations.

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(a) When the L-parameter is supercuspidal, that is, a multiplicity-free sum of selfdual representations of the Weil group (instead of the Weil–Deligne group) of the relevant sign. In this case, it was shown that the local descent is an irreducible generic supercuspidal representation of the classical group with the initially given L-parameter.

(b) When the L-parameter involved is unramified. In this case, it was shown that the only unramified representation which could occur as a quotient in the local descent is that with the given L-parameter. Such results in the unramified case is necessary for the proof of the global results described previously.

Thus, situation (a) is in complete accordance with our local Conjecture 6.1 whereas situation (b) provides supporting evidence for the same conjecture.

The twisted automorphic descent of Jiang and Zhang offers the possibility of constructing the other members of the global L-packet M of SO_{2n+1} beyond the representation σ_M. In the above construction, Ginzburg *et al.* showed that the automorphic descent of $J(\pi_M, 1/2)$ is equal to σ_M by showing that this descent is cuspidal and all of its irreducible summands are globally generic. Suppose that $\sigma_\eta = \otimes'_v \sigma_{\eta_v}$ is a cuspidal representation of the global L-packet of M. How can one construct σ_η by an analogous process as given previously?

The idea of Jiang and Zhang is as follows. The cuspidal representation σ_η must possess some non-zero Bessel coefficient with respect to a smaller even orthogonal group SO_{2m}. Suppose that the Bessel coefficient of σ_η down to SO_{2m} contains a cuspidal representation $\tau_\chi = \otimes_v \tau_{\chi_v}$ belonging to a tempered A-parameter N of SO_{2m}. By the uniqueness part of the global conjecture in [GGP12a] (in the tempered case), one knows that σ_η is characterized as the unique representation in the L-packet of M such that $\sigma_\eta \otimes \tau_\chi$ supports the relevant global Bessel period for some τ_χ in the L-packet of N. This means that we have a way of distinguishing the representation σ_η from the other members of the L-packet of M (just as one uses the Whittaker–Fourier coefficient to detect the representation σ_M). If this is how one is hoping to detect the representation σ_η, then it is only reasonable that one should somehow involve the auxiliary τ_χ in the descent construction of σ_η.

Here then is the construction of Jiang and Zhang. We first pick a tempered L-parameter N of a smaller group SO_{2m} such that the global Bessel period of $\sigma_\eta \otimes \tau_\chi$ is non-zero for some τ_χ with L-parameter N. The existence of the data $(\mathrm{SO}_{2m}, N, \tau_\chi)$ is not obvious, especially with the condition that N is tempered. This is, in fact, taken as a hypothesis in [JZ20]. Hence, the construction in [JZ20] is so far conditional, though one would like to think that this hypothesis is not unreasonable. In any case, under this hypothesis, one considers the standard module

$$I(\pi_M \otimes \tau, 1/2) = \pi_M \cdot |-|^{1/2} \rtimes \tau_\chi$$

of SO_{4n+2m} defined by induction from the parabolic subgroup with Levi factor $\mathrm{GL}_{2n} \times \mathrm{SO}_{2m}$. Let $J(\pi_M \otimes \tau_\chi, 1/2)$ be the Langlands quotient of this standard module. It can be embedded into the automorphic discrete spectrum as an iterated residue of the corresponding Eisenstein series and one can then consider the Bessel descent of $J(\pi_M \otimes \tau_\chi, 1/2)$ down to SO_{2n+1}. Jiang and Zhang verified that one obtains a cuspidal automorphic representation of SO_{2n+1} all of whose irreducible components σ' belong to the global L-packet of M. They then computed the Bessel period of $\sigma' \otimes \tau_\chi$ and showed that it is non-zero, thus showing that $\sigma' \cong \sigma_\eta$.

Again, we can understand this construction of Jiang and Zhang from the viewpoint of the conjectures of this paper.

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- The choice of $(\mathrm{SO}_{2m}, N, \tau_\chi)$ so that $\sigma_\eta \otimes \tau_\chi$ has non-zero global Bessel period implies that for each place v, the local character $\eta_v \times \chi_v$ is the distinguished character of the local component group $A_{M_v} \times A_{N_v}$ from the local conjecture of [GGP12a] for the pair $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$.
- The discrete automorphic representation $J(\pi_M \otimes \tau_\chi, 1/2)$ belongs to the A-packet of SO_{4n+2m} with non-tempered A-parameter

$$\tilde{M}_{\mathrm{A}} = M \boxtimes [2] \oplus N.$$

  Indeed, it lies in the L-packet associated to the corresponding L-parameter

$$\tilde{M} = M \cdot |-|^{1/2} \oplus M \cdot |-|^{-1/2} \oplus N.$$

  One has a natural identification of the local component group

$$A_{\tilde{M}_v} \cong A_{N_v}$$

  for each place v. Under this identification, the character of $A_{\tilde{M}_v}$ that corresponds to the representation $J(\pi_M \otimes \tau_\chi, 1/2)$ is the character χ_v (which indexes the representation τ_{χ_v} of SO_{2m}).
- Now if one considers the Bessel period in the context of $\mathrm{SO}_{4n+2m} \times \mathrm{SO}_{2n+1}$, one sees that the pair of A-parameters $(\tilde{M}_{\mathrm{A}}, M)$ is relevant. Indeed, M is the only A-parameter of SO_{2n+1} that can form a relevant pair with $\tilde{M}_{\mathrm{A}}$. Hence, by our global Conjecture 9.1, the only cuspidal representations of SO_{2n+1} that can potentially be contained in the Bessel descent of $J(\pi_M \otimes \tau_\chi, 1/2)$ are those with L-parameter M.
- To determine which cuspidal representations in the L-packet of M are contained in the Bessel descent, we appeal to our local Conjecture 6.1 for the relevant pair $(\tilde{M}_{\mathrm{A}}, M)$ on $\mathrm{SO}_{4n+2m} \times \mathrm{SO}_{2n+1}$. The conjectures gives a distinguished representation with non-zero multiplicity in the local L-packet of $\tilde{M}_v \times M_v$ corresponding to a distinguished character on the local component group

$$A_{\tilde{M}_v} \times A_{M_v} \cong A_{N_v} \times A_{M_v}.$$

  A short computation shows that this distinguished character of $A_{\tilde{M}_v} \times A_{M_v}$ is equal to the distinguished character of $A_{N_v} \times A_{M_v}$ under the natural identification of component groups given previously, and thus is none other than $\chi_v \times \eta_v$. Hence, the global representation $J(\pi_M \otimes \tau_\chi, 1/2) \otimes \sigma_\eta$ is the unique representation in the global L-packet of $\tilde{M} \times M$ that supports a non-zero abstract Bessel period. This implies that the only cuspidal representation that could be contained in the Bessel descent of $J(\pi_M \otimes \tau_\chi, 1/2)$ to SO_{2n+1} is σ_η.
- It remains to show that the Bessel descent is non-zero or, equivalently, that the global Bessel period of $J(\pi_M \otimes \tau_\chi, 1/2) \otimes \sigma_\eta$ is non-zero. For this, our global Conjecture 9.1 implies that it suffices to check the non-vanishing of $L(\tilde{M}, M, s)$ at $s = 0$. However, Theorem 9.7 implies that the desired non-vanishing. This shows that the cuspidal part of the Bessel descent is precisely σ_η, as desired.

This concludes our explanation of the automorphic descent method from the viewpoint of our conjectures.

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13. L-functions: GL case

The purpose of this section is to give the proof of Theorem 3.2. For the convenience of the reader, we restate Theorem 3.2 here.

Theorem 13.1. *Let k be a non-archimedean local field and let $(M_{\mathrm{A}}, N_{\mathrm{A}})$ be a relevant pair of A-parameters for $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ with associated pair of L-parameters (M, N). Then the order of pole at $s = 0$ of*

$$L(M, N, s) = \frac{L(M \otimes N^\vee, s + 1/2) \cdot L(M^\vee \otimes N, s + 1/2)}{L(M \otimes M^\vee, s + 1) \cdot L(N \otimes N^\vee, s + 1)}$$

is greater than or equal to zero.

Before we start making the calculations on the order of the pole of $L(s, M, N)$, we need to recall some generalities on L-functions, referring the reader to [Tat79] for L-functions $L(s, V_{\mathrm{A}}) := L(s, V)$ attached to A-parameters V_{A} of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ with associated L-parameter V of $WD(k)$.

For any irreducible representation $V \otimes \mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^j(\mathbb{C}^2)$ of $WD(k) \times \mathrm{SL}_2(\mathbb{C}) = W(k) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, the L-function $L(s, V \otimes \mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^j(\mathbb{C}^2))$ has a pole at $s = 1/2$ if and only if:

(i) V is the trivial representation of $W(k)$;
(ii) $j > i$, $j \equiv (i + 1) \bmod 2$.

Similarly, the L-function $L(s, V \otimes \mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^j(\mathbb{C}^2))$ has a pole at $s = 1$ if and only if

(i) V is the trivial representation of $W(k)$;
(ii) $j > i$, $j \equiv i \bmod 2$.

In either of the two cases, the pole is simple if it exists. As a consequence, we note the following lemma.

Lemma 13.2. *For any representation M of $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C})$, the order of the pole of $L(s, M \otimes \mathrm{Sym}^j(\mathbb{C}^2))$ at $s = 1/2$ is the same as the order of pole of $L(s, M \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^2))$ at $s = 1$, and these are the same as the corresponding orders of pole for M replaced by $M^\vee$.*

This lemma suggests the introduction of a map on the set of representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ to itself, which we denote by $M \to M^+$, and defined as follows. For M_{A} a representation of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of the form

$$M_{\mathrm{A}} = \sum_{i \geq 0} M_i \otimes \mathrm{Sym}^i(\mathbb{C}^2),$$

define

$$M_{\mathrm{A}}^+ = \sum_{i \geq 0} M_i \otimes \mathrm{Sym}^{i+1}(\mathbb{C}^2).$$

From Lemma 13.2, the order of pole of M_{A} at $s = 1/2$ is the same as the order of pole of M_{A}^+ at $s = 1$.

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We are now ready to begin the proof of Theorem 13.1. We write the parameters M_{A} and N_{A} as

$$M_{\mathrm{A}} = A_1 + A_2 + \cdots + A_n + A_0,$$
$$N_{\mathrm{A}} = B_1 + B_2 + \cdots + B_n + B_0,$$

where, for $i \geq 1$, A_i and B_i are irreducible representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of the form,

$$A_i = M_i \boxtimes \mathrm{Sym}^{a_i - 1}(\mathbb{C}^2),$$
$$B_i = M_i \boxtimes \mathrm{Sym}^{b_i - 1}(\mathbb{C}^2),$$

with

$$a_i - b_i = \pm 1,$$

and with A_0, B_0 tempered representations of $WD(k)$. Clearly, the following proposition proves the theorem.

PROPOSITION 13.3. *Let C_1, C_2, D_1, D_2 be irreducible representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of the form:*

$$C_i = M_i \boxtimes \mathrm{Sym}^{a_i - 1}(\mathbb{C}^2),$$
$$D_i = M_i \boxtimes \mathrm{Sym}^{b_i - 1}(\mathbb{C}^2),$$

for irreducible representations σ_i of $WD(k)$ with $a_i - b_i = \pm 1$ for $i = 1, 2$. Then,

$$L(s, C_1, C_2, D_1, D_2) = \frac{L(s + 1/2, C_1 \otimes D_2 + C_2 \otimes D_1)}{L(s + 1, C_1 \otimes C_2) L(s + 1, D_1 \otimes D_2)}, \tag{a}$$

has a pole of order at least zero at $s = 0$.

Further,

$$L(s, C_1, D_1) = \frac{L(s + 1/2, C_1 \otimes D_1)^2}{L(s + 1, C_1 \otimes C_1) L(s + 1, D_1 \otimes D_1)}, \tag{b}$$

has a pole of order at least zero at $s = 0$.

Finally, for any tempered representation A_0 of $WD(k)$,

$$L(s, C_1, D_1, A_0) = \frac{L(s + 1/2, C_1 \otimes A_0)}{L(s + 1, D_1 \otimes A_0)}, \tag{c}$$

has a pole of order at least zero at $s = 0$.

Proof. We only prove part (a) of the proposition, the other parts are proved analogously. It suffices to prove part (a) of the proposition in the following two cases:

(i) $b_1 = a_1 + 1$, $b_2 = a_2 + 1$;
(ii) $b_1 = a_1 - 1$, $b_2 = a_2 + 1$.

We deal with these two cases separately by a direct calculation, starting with case (i).

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Let

$$C_1 = M_1 \boxtimes \mathrm{Sym}^i(\mathbb{C}^2), \qquad C_2 = M_2 \boxtimes \mathrm{Sym}^j(\mathbb{C}^2),$$
$$D_1 = M_1 \boxtimes \mathrm{Sym}^{i+1}(\mathbb{C}^2), \quad D_2 = M_2 \boxtimes \mathrm{Sym}^{j+1}(\mathbb{C}^2),$$

be irreducible representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ for irreducible representations M_1, M_2 of $WD(k)$ where we assume without loss of generality that $i \geq j$. It follows that

$$C_1 \otimes D_2 + C_2 \otimes D_1 = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^2) + \mathrm{Sym}^j(\mathbb{C}^2) \otimes \mathrm{Sym}^{i+1}(\mathbb{C}^2)],$$
$$C_1 \otimes C_2 + D_1 \otimes D_2 = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^j(\mathbb{C}^2) + \mathrm{Sym}^{i+1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^2)].$$

Hence, $C_1 \otimes D_2 + C_2 \otimes D_1$ is equal to (assuming $i > j$),

$$(M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j+1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-(j+1)}(\mathbb{C}^2) + \mathrm{Sym}^{j+i+1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i+1-j}(\mathbb{C}^2)]$$

and $C_1 \otimes C_2 + D_1 \otimes D_2$ is equal to

$$(M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-j}(\mathbb{C}^2) + \mathrm{Sym}^{i+j+2}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-j}(\mathbb{C}^2)].$$

For any representation M_{A} of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$, we have defined the representation M_{A}^+ earlier. With this notation, we have

$$(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2)$$
$$= (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j+2}(\mathbb{C}^2) - \mathrm{Sym}^{i-j}(\mathbb{C}^2)].$$

By Lemma 13.2, the order of pole of

$$\frac{L(s+1/2, C_1 \otimes D_2 + C_2 \otimes D_1)}{L(s+1, C_1 \otimes C_2) L(s+1, D_1 \otimes D_2)},$$

at $s = 0$ is the same as the order of pole of the virtual representation

$$(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2),$$

of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ at $s = 1$.

On the other hand, for any irreducible representation M of $WD(k)$, the order of pole at $s = 1$ of the virtual representation

$$M \boxtimes [\mathrm{Sym}^{i+j+2}(\mathbb{C}^2) - \mathrm{Sym}^{i-j}(\mathbb{C}^2)]$$

is at least zero, because the order of pole at $s = 1$ of the representation $M \boxtimes \mathrm{Sym}^a(\mathbb{C}^2)$ is always greater than or equal to the order of pole at $s = 1$ of the representation $M \boxtimes \mathrm{Sym}^{a-2b}(\mathbb{C}^2)$ for any integer $b \geq 0$. Hence, so $L(s, C_1, C_2, D_1, D_2)$ has a pole of order at least zero at $s = 0$, proving the proposition in this case (assuming $i > j$ here).

If $i = j$, it can be seen that

$$(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2)$$
$$= (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{2i+2}(\mathbb{C}^2) - 2\mathbb{C}].$$

As tempered representations have no poles in the region $Re(s) > 0$, the term $2\mathbb{C}$ contributes to no poles for the virtual representation $M_1 \otimes M_2 \boxtimes [\mathrm{Sym}^{2i+2}(\mathbb{C}^2) - 2\mathbb{C}]$, leaving us with the true representation $M_1 \otimes M_2 \boxtimes \mathrm{Sym}^{2i+2}(\mathbb{C}^2)$, which can only contribute a non-negative number of poles at $s = 1$, completing the proof of the proposition for $i = j$ case of case (i).

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Next we consider case (ii), where

$$C_1 = M_1 \boxtimes \mathrm{Sym}^i(\mathbb{C}^2), \qquad C_2 = M_2 \boxtimes \mathrm{Sym}^j(\mathbb{C}^2),$$
$$D_1 = M_1 \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2), \quad D_2 = M_2 \boxtimes \mathrm{Sym}^{j+1}(\mathbb{C}^2),$$

for irreducible representations M_1, M_2 of $WD(k)$, and assume that $i \geq j$. It follows that

$$C_1 \otimes D_2 + C_2 \otimes D_1 = (M_1 \otimes M_2) \otimes [\mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^2) + \mathrm{Sym}^j(\mathbb{C}^2) \otimes \mathrm{Sym}^{i-1}(\mathbb{C}^2)],$$
$$C_1 \otimes C_2 + D_1 \otimes D_2 = (M_1 \otimes M_2) \otimes [\mathrm{Sym}^i(\mathbb{C}^2) \otimes \mathrm{Sym}^j(\mathbb{C}^2) + \mathrm{Sym}^{i-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^2)].$$

Assuming $i > j+1$, we deduce that $C_1 \otimes D_2 + C_2 \otimes D_1$ is equal to

$$(M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j+1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-(j+1)}(\mathbb{C}^2) + \mathrm{Sym}^{j+i-1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-1-j}(\mathbb{C}^2)],$$

and $C_1 \otimes C_2 + D_1 \otimes D_2$ is equal to

$$(M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-j}(\mathbb{C}^2) + \mathrm{Sym}^{i+j}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{i-j-2}(\mathbb{C}^2)].$$

Therefore,

$$\begin{aligned}(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2)\\ = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j+2}(\mathbb{C}^2) - \mathrm{Sym}^{i-j-2}(\mathbb{C}^2)].\end{aligned}$$

As in case (i), the order of pole of

$$\frac{L(s+1/2, C_1 \otimes D_2 + C_2 \otimes D_1)}{L(s+1, C_1 \otimes C_2) L(s+1, D_1 \otimes D_2)},$$

at $s = 0$ is the same as the order of pole for the L-function of the virtual representation

$$\begin{aligned}(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2)\\ = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{i+j+2}(\mathbb{C}^2) - \mathrm{Sym}^{i-j-2}(\mathbb{C}^2)]\end{aligned}$$

at $s = 1$ which is at least zero.

If $i = j+1$, then

$$(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2) = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{2i+1}(\mathbb{C}^2)],$$

and we find that the order of pole of

$$\frac{L(s+1/2, C_1 \otimes D_2 + C_2 \otimes D_1)}{L(s+1, C_1 \otimes C_2) L(s+1, D_1 \otimes D_2)},$$

at $s = 0$ is at least zero.

If $i = j$, then

$$(C_1 \otimes D_2 + C_2 \otimes D_1)^+ - (C_1 \otimes C_2 + D_1 \otimes D_2) = (M_1 \otimes M_2) \boxtimes [\mathrm{Sym}^{2i+2}(\mathbb{C}^2) - \mathbb{C}],$$

and, once again, we find that the order of pole of

$$\frac{L(s+1/2, C_1 \otimes D_2 + C_2 \otimes D_1)}{L(s+1, C_1 \otimes C_2) L(s+1, D_1 \otimes D_2)},$$

at $s = 0$ is at least zero. □

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We have thus completed the proof of Theorem 13.1 (i.e.Theorem 3.2). In fact, the order of pole in Theorem 13.1 can be explicitly determined if the action of the Deligne $\mathrm{SL}_2(\mathbb{C})$ is trivial on the representations M_{A} and N_{A}. More precisely, we have the following result.

Proposition 13.4. *Let $(M_{\mathrm{A}}, N_{\mathrm{A}})$ be a pair of relevant A-parameters for $(\mathrm{GL}_m(k), \mathrm{GL}_n(k))$ on which the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially. Write*

$$M_{\mathrm{A}} = \sum_{i\geq 1} M_i \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) = \sum_{i\geq 1} (M_i^+ + M_i^-) \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2),$$
$$N_{\mathrm{A}} = \sum_{i\geq 1} W_i \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) = \sum_{i\geq 1} (N_i^+ + N_i^-) \boxtimes \mathrm{Sym}^{i-1}(\mathbb{C}^2),$$

with M_i, N_i representations of $W(k)$, such that $M_i^+ = N_{i+1}^-$ for $i \geq 1$, and $M_i^- = N_{i-1}^+$ for $i \geq 2$.

Then the order of pole at $s = 0$ of $L(s, M, N)$ is given by the dimension of $W(k)$-invariants in

$$\sum_{i\geq 1} \big\{ \mathrm{Hom}[M_i, N_{i-1}] + \mathrm{Hom}[M_i, N_{i+1}] - \mathrm{Hom}[M_i^+, N_{i-1}^-] - \mathrm{Hom}[M_i^-, N_{i+1}^+] \big\},$$

with the understanding that $N_0 = N_0^- = 0$.

Proof. The proof follows by examining the arguments made in Proposition 13.3 carefully, a task which we leave to the reader. We only point out here that in the course of the proof of this proposition, say in case 1 of case (a), we had to deal with L-function of the virtual representation:

$$M \boxtimes [\mathrm{Sym}^{i+j+2}(\mathbb{C}^2) - \mathrm{Sym}^{i-j}(\mathbb{C}^2)],$$

at the point $s = 1$. Clearly, if M is a representation of $WD(k)$ which factors through $W(k)$, such a virtual representation has neither a zero nor a pole at $s = 1$ for $i > j$, allowing us to calculate all the poles appearing in Proposition 13.3. □

14. L-functions: classical groups

The goal of this section is to give the proof of Theorem 3.3. For the convenience of the reader, we restate the theorem here.

Theorem 14.1. *Let k be a non-archimedean local field and let $(M_{\mathrm{A}}, N_{\mathrm{A}})$ be a pair of A-parameters for $\mathrm{SO}_{2m+1}(k) \times \mathrm{SO}_{2n}(k)$ with associated pair of L-parameters (M, N).*

(i) *If $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of A-parameters, then the order of pole at $s = 0$ of the function*

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M \oplus \wedge^2 N, s + 1)}$$

is greater than or equal to zero.

(ii) *Suppose that M_{A} and N_{A} are multiplicity-free representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ on which the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially. Then, at $s = 0$, the function $L(M, N, s)$ has a zero of order at least zero. It has neither a zero nor a pole at $s = 0$ if and only if $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is a relevant pair of A-parameters.*

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We begin with the proof of part (i), which is analogous to that of Theorem 13.1. Write the parameters M_{A} and N_{A} as

$$M_{\mathrm{A}} = A_1 + A_2 + \cdots + A_n + A_0,$$
$$N_{\mathrm{A}} = B_1 + B_2 + \cdots + B_n + B_0,$$

where for $i \geq 1$, A_i and B_i are irreducible representations of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$ of the form,

$$A_i = M_i \otimes \mathrm{Sym}^{a_i}(\mathbb{C}^2),$$
$$B_i = M_i \otimes \mathrm{Sym}^{b_i}(\mathbb{C}^2),$$

with

$$a_i - b_i = \pm 1,$$

and with A_0, B_0 tempered parameters of $WD(k)$.

Observe that

$$\mathrm{Sym}^2 M_{\mathrm{A}} = \sum_{i>j} A_i \otimes A_j + \sum_i \mathrm{Sym}^2(A_i)$$

and, similarly,

$$\Lambda^2 N_{\mathrm{A}} = \sum_{i>j} B_i \otimes B_j + \sum_i \Lambda^2(B_i).$$

Therefore, the *non-diagonal* contributions to the order of pole at $s = 1$ of the L-function $L(\mathrm{Sym}^2 M_{\mathrm{A}}, s)L(\Lambda^2 N_{\mathrm{A}}, s)$ is as in cases (a) and (c) in the statement of Proposition 13.3.

It suffices to prove the following analog of case (b) in the statement of Proposition 13.3, asserting that

$$L(C, D, s) = \frac{L(C \otimes D, s + 1/2)}{L(\mathrm{Sym}^2(C), s + 1)L(\Lambda^2(D), s + 1)}$$

has a pole of order at least zero at $s = 0$ whenever

$$C = M \otimes \mathrm{Sym}^a(\mathbb{C}^2),$$
$$D = M \otimes \mathrm{Sym}^b(\mathbb{C}^2),$$

for M an irreducible tempered representation of $WD(k)$ with

$$a - b = \pm 1.$$

To calculate symmetric and exterior square of representations C and D of $WD(k) \times \mathrm{SL}_2(\mathbb{C})$, note that for any two representations V, W of any group G, we have the identity of representations

$$\mathrm{Sym}^2(V \otimes W) = \mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(W) \oplus \Lambda^2(V) \otimes \Lambda^2(W)$$

and

$$\Lambda^2(V \otimes W) = \mathrm{Sym}^2(V) \otimes \Lambda^2(W) \oplus \Lambda^2(V) \otimes \mathrm{Sym}^2(W).$$

Using the well-known structure of $\mathrm{Sym}^2(\mathrm{Sym}^i(\mathbb{C}^2))$ and $\Lambda^2(\mathrm{Sym}^i(\mathbb{C}^2))$ given by

$$\mathrm{Sym}^2(\mathrm{Sym}^i(\mathbb{C}^2)) = \mathrm{Sym}^{2i}(\mathbb{C}^2) + \mathrm{Sym}^{2i-4}(\mathbb{C}^2) + \cdots$$

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and

$$\Lambda^2(\mathrm{Sym}^i(\mathbb{C}^2)) = \mathrm{Sym}^{2i-2}(\mathbb{C}^2) + \mathrm{Sym}^{2i-6}(\mathbb{C}^2) + \cdots,$$

one easily concludes that $L(s, C, D)$ has a pole of order at least zero at $s = 1$, concluding the proof of part (i) of Theorem 14.1.

We come now to the proof of part (ii). Thus, let M_{A} and N_{A} be A-parameters for which the Deligne $\mathrm{SL}_2(\mathbb{C})$ acts trivially.

For any irreducible representation ρ of $W(k)$, let $M_{\mathrm{A}}[\rho], N_{\mathrm{A}}[\rho]$ be the ρ-isotypic part of $M_{\mathrm{A}}, N_{\mathrm{A}}$ (as a $W(k)$-module). As the representations $M_{\mathrm{A}}, N_{\mathrm{A}}$ of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ are multiplicity free and are selfdual, if $M_{\mathrm{A}}[\rho] \neq 0$, or $N_a[\rho] \neq 0$, ρ must be a selfdual representation of $W(k)$.

It is easy to see that the order of zero of $L(M, N, s)$ at $s = 0$ is the sum of the order of zeros of $L(M_{\mathrm{A}}[\rho], N_{\mathrm{A}}[\rho], s)$ at $s = 0$ for various distinct irreducible representations ρ of $W(k)$. For ρ, an irreducible selfdual representation of $W(k)$, let us write

$$M_{\mathrm{A}}[\rho] = \rho \boxtimes V \quad \text{and} \quad N_{\mathrm{A}}[\rho] = \rho \boxtimes W$$

for representations V and W of (the Arthur) $\mathrm{SL}_2(\mathbb{C})$. As M_{A} is supposed to be a symplectic representation and N_{A} an orthogonal representation, if ρ is orthogonal, V will be a symplectic representation of (the Arthur) $\mathrm{SL}_2(\mathbb{C})$, and W will be an orthogonal representation of (the Arthur) $\mathrm{SL}_2(\mathbb{C})$. Now using the identities

$$\begin{aligned}\mathrm{Sym}^2(\rho \boxtimes V) &= \mathrm{Sym}^2(\rho) \boxtimes \mathrm{Sym}^2(V) + \Lambda^2(\rho) \boxtimes \Lambda^2(V),\\ \Lambda^2(\rho \boxtimes V) &= \mathrm{Sym}^2(\rho) \boxtimes \Lambda^2(V) + \Lambda^2(\rho) \boxtimes \mathrm{Sym}^2(V),\end{aligned}$$

we find that if ρ is irreducible and orthogonal representation of $W(k)$,

$$\mathrm{ord}_{s=0}\big(L(M_{\mathrm{A}}[\rho], N_{\mathrm{A}}[\rho], s)\big) = \mathrm{ord}_{s=0}\big(L(V, W, s)\big).$$

On the other hand, if ρ is an irreducible symplectic representation of $W(k)$, W will be a symplectic representation of (the Arthur) $\mathrm{SL}_2(\mathbb{C})$, V will be an orthogonal representation of (the Arthur) $\mathrm{SL}_2(\mathbb{C})$, and

$$\mathrm{ord}_{s=0}\big(L(M_{\mathrm{A}}[\rho], N_{\mathrm{A}}[\rho], s)\big) = \mathrm{ord}_{s=0}\big(L(W, V, s)\big).$$

Thus, it suffices to prove the theorem assuming that $WD(k)$ acts trivially on M_{A} and N_{A}. In other words, we have multiplicity-free representations

$$\begin{aligned}M_{\mathrm{A}} = V &= \mathrm{Sym}^{a_1-1}(\mathbb{C}^2) + \mathrm{Sym}^{a_2-1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{a_r-1}(\mathbb{C}^2),\\ N_{\mathrm{A}} = W &= \mathrm{Sym}^{b_1-1}(\mathbb{C}^2) + \mathrm{Sym}^{b_2-1}(\mathbb{C}^2) + \cdots + \mathrm{Sym}^{b_s-1}(\mathbb{C}^2),\end{aligned}$$

with V symplectic and W orthogonal, thus with all a_i even, and b_i odd.

We prove the theorem using an inductive argument from the 'top'. Suppose d is the largest integer a such that $\mathrm{Sym}^{a-1}(\mathbb{C}^2)$ is contained in either V or W. We assume that $\mathrm{Sym}^{d-1}(\mathbb{C}^2)$ appears in V, a similar argument can be given if $\mathrm{Sym}^{d-1}(\mathbb{C}^2)$ appears in W. Suppose e is the largest integer such that $\mathrm{Sym}^{e-1}(\mathbb{C}^2)$ appears in W. By hypothesis, all the integers a_i for which $\mathrm{Sym}^{a_i-1}(\mathbb{C}^2) \subset V$ is even, and all the integers b_i for which $\mathrm{Sym}^{b_i-1}(\mathbb{C}^2) \subset W$ is odd, in particular, d is even, e is odd, and $d > e$.

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Define

$$V' = V - \mathrm{Sym}^{d-1}(\mathbb{C}^2),$$
$$W' = W - \mathrm{Sym}^{e-1}(\mathbb{C}^2).$$

We prove the following proposition, which then allows us to complete the proof of Theorem 14.1(ii) by an inductive argument.

PROPOSITION 14.2. *With the notation and assumptions given previously, in particular V symplectic and W orthogonal multiplicity-free representation of the Arthur $\mathrm{SL}_2(\mathbb{C})$, the order of pole at $s = 0$ of $L(V, W, s)$ defined by*

$$L(V, W, s) = \frac{L(V \otimes W, s + 1/2)}{L(\mathrm{Sym}^2 V \oplus \wedge^2 W, s + 1)},$$

is less than or equal to that of $L(V', W', s)$. Equality holds if and only if $d = (e + 1)$.

Proof. Clearly,

$$V \otimes W = V' \otimes W' + W' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) + V' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2) + \mathrm{Sym}^{d-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2),$$
$$\mathrm{Sym}^2(V) = \mathrm{Sym}^2(V') + \mathrm{Sym}^2(\mathrm{Sym}^{d-1}(\mathbb{C}^2)) + V' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2),$$
$$\Lambda^2(W) = \Lambda^2(W') + \Lambda^2(\mathrm{Sym}^{e-1}(\mathbb{C}^2)) + W' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2).$$

Therefore,

$$\frac{L(V, W, s)}{L(V', W', s)} = \frac{L\big(W' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) + V' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2) + \mathrm{Sym}^{d-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2), s + \tfrac{1}{2},\big)}{L(W' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2) + V' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) + \mathrm{Sym}^2\mathrm{Sym}^{d-1}(\mathbb{C}^2) + \Lambda^2\mathrm{Sym}^{e-1}(\mathbb{C}^2), s + 1)}.$$

We analyze the order of pole at $s = 0$ of the following L-functions,

$$A(s) = \frac{L\big(\mathrm{Sym}^{d-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2), s + \tfrac{1}{2}\big)}{L(\mathrm{Sym}^2\mathrm{Sym}^{d-1}(\mathbb{C}^2) + \Lambda^2\mathrm{Sym}^{e-1}(\mathbb{C}^2), s + 1)},$$
$$B(s) = \frac{L\big(W' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + \tfrac{1}{2}\big)}{L(W' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2), s + 1)},$$
$$C(s) = \frac{L\big(V' \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2), s + \tfrac{1}{2}\big)}{L(V' \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2), s + 1)}.$$

- *Analyzing* $A(s)$. As d is even, e is odd, and $d > e$, $A(s)$ has a zero at $s = 0$ of order

$$a = -e + (d + e - 1)/2 = (d - e - 1)/2.$$

- *Analyzing* $B(s)$. Again, as d is even, e is odd, and $d > e$, and e is bigger than all integers b for which $\mathrm{Sym}^{b-1}(\mathbb{C}^2)$ is contained in W', it follows that, in writing W' as a sum of irreducible pieces $\mathrm{Sym}^{b-1}(\mathbb{C}^2)$'s, the $\mathrm{Sym}^{b-1}(\mathbb{C}^2)$'s contribute the same number of poles in the denominator as in the numerator of $B(s)$. Therefore, $B(s)$ has neither a zero nor a pole at $s = 0$.

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- *Analyzing* $C(s)$. Again, as d is even, e is odd, and $d > e$, the number poles at $s = 0$ in the denominator of $C(s)$ is greater than or equal to the number of poles at $s = 0$ in the numerator of $C(s)$, the difference is contributed by those $\mathrm{Sym}^{t-1}(\mathbb{C}^2)$, with $d > t > e$ and t even, which may belong to V'. Thus, we obtain

$$0 \leq c \leq (d - e - 1)/2,$$

for the number of zeros for the factor $C(s)$ at $s = 0$.

In conclusion, we find that

$$\mathrm{ord}_{s=0}\big(A(s)B(s)C(s)\big) \geq 0.$$

Moreover, if $A(s)B(s)C(s)$ has no zero at $s = 0$, then we must have $d = e + 1$. Conversely, if $d = e + 1$, then there are no zeros or poles for $A(s)B(s)C(s)$ at $s = 0$, completing the proof of the proposition. □

There seems no simple generalization of Theorem 14.1 to all discrete A-parameters of classical groups, although it would be highly desirable. We give two examples of its failure, beginning with one in which we allow general tempered parts for M_{A} and N_{A}, keeping other conditions in Theorem 14.1 intact. Let

$$M_{\mathrm{A}} = [10] \times [1],$$
$$N_{\mathrm{A}} = [5] \times [1] + [1] \times [7] + [1] \times [9].$$

It can be seen that

$$L(M, N, s) = \frac{L(M \otimes N, s + 1/2)}{L(\mathrm{Sym}^2 M, s + 1)L(\Lambda^2 N, s + 1)},$$

has neither a zero nor a pole at $s = 0$, even though the pair $(M_{\mathrm{A}}, N_{\mathrm{A}})$ is irrelevant.

As second example, consider the relevant pair of A-parameters

$$M_{\mathrm{A}} = [3] \times [4] + [5] \times [4],$$
$$N_{\mathrm{A}} = [3] \times [3] + [5] \times [5].$$

These satisfy all the conditions in Theorem 14.1 except that the Deligne $\mathrm{SL}_2(\mathbb{C})$ does not act trivially. It can be seen that in this case, $L(M \otimes N, s + 1/2)$ has a pole of order 25, whereas $L(\mathrm{Sym}^2 M, s + 1)L(\Lambda^2 N, s + 1)$ has a pole of order 20, so in this case $L(M, N, s)$ has a pole at $s = 0$ of order 5.

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