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NONERGODIC JACKSON NETWORKS WITH INFINITE SUPPLY—LOCAL STABILIZATION AND LOCAL EQUILIBRIUM ANALYSIS

JENNIFER SOMMER,* *HPC-Hamburg Port Consulting GmbH*
HANS DADUNA,** *University of Hamburg*
BERND HEIDERGOTT,*** *Tinbergen Institute and Vrije University Amsterdam*

Abstract

Classical Jackson networks are a well-established tool for the analysis of complex systems. In this paper we analyze Jackson networks with the additional features that (i) nodes may have an infinite supply of low priority work and (ii) nodes may be unstable in the sense that the queue length at these nodes grows beyond any bound. We provide the limiting distribution of the queue length distribution at stable nodes, which turns out to be of product form. A key step in establishing this result is the development of a new algorithm based on adjusted traffic equations for detecting unstable nodes. Our results complement the results known in the literature for the subcases of Jackson networks with either infinite supply nodes or unstable nodes by providing an analysis of the significantly more challenging case of networks with both types of nonstandard node present. Building on our product-form results, we provide closed-form solutions for common customer and system oriented performance measures.

Keywords: Jackson network; stability; instability; product-form solution; bottleneck analysis; shortest path

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Secondary 90B15

1. Introduction

Open Jackson networks and their successors, BCMP and Kelly networks, are a well-established class of models in, for example, production, telecommunication, and computer systems. Today's complex networks often meet the conditions of classical Jackson networks only locally. As Goodman and Massey [4] showed in a pioneering paper, even if a set of nodes in a network fails to be stable, i.e. queue lengths at these nodes builds up over time unboundedly, other parts of the network can operate in a stable manner and the asymptotic queue length distribution at stable nodes can be given in closed form. The analysis of locally unstable Jackson networks becomes more challenging when nodes in the network, although stable, i.e. building no infinite queues over time, are required to be fully utilized (working without intermediate idling); see [15]. This occurs in production control where a machine is monitored and by some external control additional raw material is supplied whenever there occurs the possibility that the queue

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* Postal address: HPC-Hamburg Port Consulting GmbH, PO Box 11 14 06, 20414 Hamburg, Germany. Email address: j.sommer@hpc-hamburg.de
** Postal address: Center of Mathematical Statistics and Stochastic Processes, Department of Mathematics, University of Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany. Email address: daduna@math.uni-hamburg.de
*** Postal address: Department of Econometrics and Operations Research, Vrije University Amsterdam, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands. Email address: b.f.heidergott@vu.nl

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empties. Such a system can be modeled by adding to the node which represents the fully utilized machine an infinite buffer (infinite supply or infinite virtual queue (IVQ)) of raw material from which the server takes material whenever the regular queue empties. Whenever a piece of such additional material is completely processed, it is sent out as standard item (customer, job) into the network's production process. Implementing an IVQ at some nodes has the aim to utilize capacities to the fullest, avoid idle times completely, and therefore enhance productivity. However, nodes with an IVQ act as additional sources to the network and thereby may lead to instability at downstream nodes.

The node set of our network is $\tilde{J} = \{1, \ldots, J\}$. We will distinguish specific subsets of $\tilde{J}$: nodes in $V \subseteq \tilde{J}$ have IVQs, while nodes in $W := \tilde{J} \setminus V$ operate without an IVQ. Nodes in S are stable, while nodes in $U = \tilde{J} \setminus S$ are unstable.

In this paper we provide an analysis of locally stable Jackson networks such that $V, U \neq \varnothing$ and possibly $V \cap U \neq \varnothing$. To illustrate this type of network, consider a large-scale network as in Figure 1 with two arrival streams with arrival intensity λ_1 and λ_2, respectively. The network contains three nonstandard subnetworks indicated by the shaded areas, where the term 'nonstandard' refers to the fact that the subnetworks contain nodes with an IVQ, unstable nodes, or both. Figure 1 illustrates possible cases. In the subnet shown at the top, a node is unstable due to the fact that it is preceded by a node with an IVQ, the service rate of which is larger than the service rate of this node. (The latter node might become stable if the infinite supply is removed.) In the subnetwork on the right-hand side, a node is unstable due to the load arriving from a stable and a node with an IVQ. Finally, in the subnetwork on the bottom, there are two nodes with an IVQ in a row, the second of which is unstable. Customers arriving via arrival stream λ_2 and traversing the net on the dotted path visit only stable nodes, some of which are fully utilized. Customers arriving via arrival stream λ_1 and traversing the net on the dashed path will pass through unstable nodes and will therefore experience a sequence of unboundedly growing passage times. Customers from arrival stream λ_1 traversing the network on the intermediate path visit only stable nodes without an IVQ. The difficulty with analyzing networks as in Figure 1 stems from the fact that we have two binary classifications of the nodes which can interact in any way: stable versus unstable and nodes with an IVQ versus standard nodes.

Our main results are 'product-form results', which in the classical setting say that for a vector-valued Markov process the stationary distribution (at a fixed time point) is the product of the stationary marginal distributions. More specifically, we find either stationary, or limiting, distributions for subsets of nodes in the well-known form of Jackson's theorem [8], or limiting distributions in the sense of Goodman and Massey [4], respectively. In the infinite supply literature, see, for example, [15], the setting $V \neq \varnothing$ and $S = \tilde{J}$ was studied, i.e. no unstable

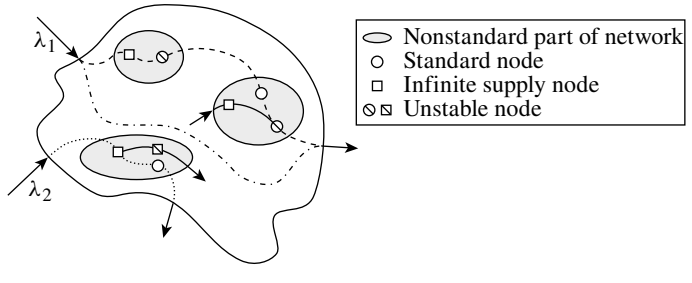


FIGURE 1: Example of a queueing network with nonstandard subnetworks.

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TABLE 1: Overview of results from this paper.

	$U = \varnothing, S = \tilde{J}$ all nodes stable (ergodic)	$U \neq \varnothing, S \subset \tilde{J}$ some nodes unstable (nonergodic)
$V = \varnothing, W = \tilde{J}$; no IVQ	classical theory	Goodman and Massey [4]
$V \neq \varnothing, W \subset \tilde{J}$; IVQ	Weiss [15], Theorem 2	Theorems 3 and 4

nodes; whereas in [4] the case $U \neq \varnothing$ and $V = \varnothing$ was analyzed. In this paper, we develop the missing theory for the case of networks with IVQs and unstable nodes, thereby combining the problem settings of [15] and [4]. For an overview and our contribution, see Table 1, where we refer to our results.

A key step in establishing the result on the limiting distribution is a new algorithm for detecting unstable nodes in the combined framework. We believe that this algorithm is of great practical value as it allows for a stability analysis of complex networks.

The paper is organized as follows. Analysis of Jackson networks with IVQs and unstable nodes is provided in Section 2. Explicit closed-form expressions for performance measures are provided in Section 3. We show in Section 3 how our results apply to identify bottlenecks in stable networks, and discuss throughputs and customer-related performance metrics, for example, mean shortest travel times along stable paths.

2. Jackson networks with infinite supply and unstable nodes

We introduce the following conventions: $A \subset B$ means that A is a strict subset of B, $A \subseteq B$ means $A \subset B \vee A = B$. Moreover, $\mathbb{N}_+ = \{1, 2, 3, \dots\}$, $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$, and $\mathbb{R}_+ = [0, \infty)$.

Definition 1. (*Jackson network [8].*) We consider a standard Jackson network with node set $\tilde{J} = \{1, \dots, J\}$. At node j jobs arrive as an external Poisson(λ_j)-arrival stream ($\lambda_j \geq 0$). We set $\lambda := \lambda_1 + \cdots + \lambda_J \geq 0$. Stations (nodes) are single servers with exponential(μ_j) distributed service times, and have infinite waiting room under a first-come–first-served (FCFS) regime. Jobs (customers) are indistinguishable. All interarrival and service times constitute a set of independent random variables.

Routing is Markovian: given the departure node i, selection of the next node is independent of the previous history. A customer departing from node i immediately proceeds to node j with probability $r(i, j) \geq 0$ and departs from the network with probability $r(i, 0)$. The artificial node 0 represents the outside, source, and sink, of the network, $r(0, 0) := 0$, $r(0, i) := \lambda_i/\lambda$. The routing matrix $\boldsymbol{R} = (r(i, j) : i, j \in \{0, 1, \dots, J\})$ is stochastic and irreducible.

Denote by $X_j(t)$ the queue length at node j at time t. The joint queue length process $X = ((X_1(t), \dots, X_J(t)) : t \geq 0)$ describing the network's evolution is Markovian on $\mathbb{N}^J$.

Definition 2. (*Jackson network with infinite supply [15].*) Consider a Jackson network from Definition 1. Adding an infinite supply of work (IVQ) at nodes in $\varnothing \neq V \subseteq \tilde{J}$ is done as follows.

- Customers arriving from the exterior are given high priority which they retain until departure from the network. Nodes in $W = \tilde{J} \setminus V$ (without an IVQ) function as in standard Jackson networks and are faced only with high priority customers.
- Low priority jobs are created only at nodes with an IVQ if these become idle: whenever

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all jobs of high priority queued at $i_0 \in V$ have departed and the node is idle, a job of low priority from the IVQ is taken to be served there. When service of a low priority job is completed, that job is converted into a high priority job, departs, and is routed according to routing matrix $\boldsymbol{R}$.

- If at node $i_0 \in V$, servicing a low priority job from the IVQ, a high priority job from the outside, or from a different node arrives, this new job has preemptive priority and the job from the IVQ is sent back to the IVQ immediately.
- As long as a job has low priority, it is not counted in the state space as a queued job, so the state description of the node does not change with its entrance into service at i_0.
- The joint queue length process $X = ((X_1(t), \dots, X_J(t)) \colon t \in \mathbb{R}_+)$ is Markovian on $\mathbb{N}^J$. The strictly positive entries of its rate matrix $\boldsymbol{Q} = (q(z, z') \colon z, z' \in \mathbb{N}^J)$ are for $z = (n_1, \dots, n_J)$ and all $i, j \in \tilde{J}$, $i \neq j$,

$$
\begin{aligned}
q(n_1, \dots, n_i, \dots, n_J; n_1, \dots, n_i + 1, \dots, n_J) &= \lambda_i + \sum_{j \in V} \mu_j r(j, i)\, \mathbf{1}_{\{0\}}(n_j), \\
q(n_1, \dots, n_i, \dots, n_J; n_1, \dots, n_i - 1, \dots, n_J) &= \mu_i r(i, 0)\, \mathbf{1}_{\mathbb{N}_+}(n_i), \\
q(n_1, \dots, n_i, \dots, n_j, \dots, n_J; n_1, \dots, n_i - 1, \dots, n_j + 1, \dots, n_J) & \\
= \mu_i r(i, j)\, \mathbf{1}_{\mathbb{N}_+}(n_i), & \\
q(n_1, \dots, n_J; n_1, \dots, n_J) & \\
= -\sum_{i \in \tilde{J}} \lambda_i - \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j, i)\, \mathbf{1}_{\{0\}}(n_j) - \sum_{i \in \tilde{J}} \mu_i (1 - r(i, i))\, \mathbf{1}_{\mathbb{N}_+}(n_i), &
\end{aligned}
$$

where $\mathbf{1}$ is the indicator function.

Remarks. Nodes with an IVQ are busy all the time; hence, their service capacity is fully utilized. An ongoing service at time $t \geq 0$ of a low priority job at node $i_0 \in V$ is detected by $X_{i_0}(t) = 0$. In [15] no immediate feedback was required, i.e. $r(j, j) = 0$, for all $i \in \tilde{J}$. We do not impose this in general.

The following property of the networks with IVQs will be fundamental.

Theorem 1. ([15, Proposition 1(iii)].) *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have IVQs. Then the departure streams from nodes $j \in V$ are independent* Poisson(μ_j) *streams and the departure stream from $j \in V$ to $i \in \tilde{J}$ is* Poisson$(\mu_j r(j, i))$.

Example 1. Open Jackson networks with an IVQ may have external total arrival rate $\lambda = 0$. For example (see [1] and [15]), two nodes with IVQs, service rates μ_i, $i = 1, 2$, and routing for $i = 1, 2$ fulfills $r(i, i) = 0$, $r(i, 0) > 0$, and $r(i, j) > 0$, $i, j = 1, 2$, $i \neq j$. So $W = \varnothing$ and $V = \tilde{J}$. The steady state distribution (which is not of product form) is computed in [1] whenever it exists.

2.1. Literature review

Investigation of generalized Jackson networks with infinite supply (IVQs) has recently found much interest in the literature and it turns out that the feature of infinite supply makes analysis of the network considerably harder than that of classical product-form networks. Weiss [15] considered the case $V \neq \varnothing$ and $S = \tilde{J}$ with the additional condition that for $j \in V$ a customer finishing service at j is *not* directly rerouted to j. For this class of networks, product-form

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steady-state results are provided in [15] and compared with results on a special class of multiclass queueing networks with virtual infinite buffers, introduced in [9] and [1]. Specifically, [15] considered Jackson networks with IVQs with jobs of two priority classes, and with only one server at each station which can serve both classes of jobs. Jobs moving between the stations are of high priority, the IVQ at some stations is filled only with jobs of low priority. Once completely served at their first station, the low priority jobs turn into higher priority jobs on their subsequent path through the network.

Infinite supply of lower priority work is used frequently. For some early work using this concept of infinite supply see [11], where IVQ attached to an M/G/1 queueing system was used to utilize idle times. For more recent examples see [5] where generalized Jackson networks were considered and [10] where a push–pull network with infinite supply was investigated.

The work of Guo [5] and Guo *et al.* [6] was on general multiclass queueing networks with IVQs under different scheduling policies for the servers. These policies guide the nodes' decisions on how to dedicate their activities to either the regular standard queues or the infinite virtual queues. The key research question is the interplay of the production of jobs from the IVQs and stability of the standard queues.

Another class of models where work is added whenever a server becomes idle are *queues with vacations*. If a queue is empty the server goes away to serve at some other location, and returns thereafter. If he/she finds customers waiting, he/she starts servicing, otherwise takes 'another vacation', and so on; see [3] for a survey. Wireless sensor networks are another application; for details we refer to [15].

2.2. The traffic equations

In Jackson's theorem for ergodic networks [8] the overall arrival rate at nodes is needed to compute the steady state distribution. These are obtained as solutions of the traffic equations for ergodic networks. Goodman and Massey [4] observed that in the case of nonergodic networks with overloaded nodes a modified set of traffic equations provides information about the individual nodes' asymptotic behavior. In [15] traffic equations for a Jackson network with infinite supply are derived under the condition $r(j, j) = 0$, for all j, which we do not require to hold. The general principle of *flux in = flux out* is in the case that all nodes are neither fully loaded nor overloaded modified with respect to Theorem 1 as follows.

Definition 3. The (standard) traffic equations of a Jackson network with infinite supply are

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j, i) + \sum_{j \in V} \mu_j r(j, i), \qquad i \in \tilde{J}. \tag{1}$$

Example 2. Continuing our Example 1 (see [1] and [15]), we find $\eta_i = \mu_{3-i} r(3 - i, i)$, $i = 1, 2$.

Lemma 1. *The traffic equations (1) have a unique solution* $\eta = (\eta_1, \ldots, \eta_J)$.

Proof. To solve (1), write it in matrix notation partitioned according to V and W: $\eta_W = \lambda_W + \eta_W R_{W,W} + \mu_V R_{V,W}$, $\eta_V = \lambda_V + \eta_W R_{W,V} + \mu_V R_{V,V}$. From the irreducibility of $\boldsymbol{R}$, $(\boldsymbol{I} - R_{W,W})^{-1}$ exists and is positive. Therefore, $\eta_W = (\lambda_W + \mu_V R_{V,W})(\boldsymbol{I} - R_{W,W})^{-1}$. Inserting this into the second equation yields the unique complete solution. □

When there are nodes with an IVQ which, on average, are fully loaded or overloaded by high priority customers, we need more general traffic equations. These combine the traffic equations from [15] and [4] into a unique setting. For this we observe the following in equilibrium.

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(i) For node $j \in V$ with an IVQ the output rate is μ_j, which usually is greater than the arrival rate η_j.

(ii) For stable node $j \in W$ (no IVQ) the overall arrival rate η_j is the departure rate as well.

(iii) For an unstable node $j \in W$ the overall arrival rate η_j in general cannot be met by the node's capacity, because it can maximally process at rate μ_j.

(iv) The arguments in (ii) and (iii) lead to departure rates $\min(\eta_j, \mu_j)$ from nodes $j \in W$.

Definition 4. The *general traffic equations* for Jackson networks with IVQs are

$$\eta_i = \lambda_i + \sum_{j \in W} \min(\eta_j, \mu_j) r(j,i) + \sum_{j \in V} \mu_j r(j,i), \qquad i \in \tilde{J}. \tag{2}$$

Node i is *stable* if η_i determined by (2) fulfills $\eta_i < \mu_i$, otherwise node i is *unstable*.

Lemma 2. *The general traffic equations (2) have a unique solution,* $\eta = (\eta_1, \ldots, \eta_J)$.

In the proof of Lemma 2, the main tool is an algorithm to determine the unique solution of (2) in at most J steps. The structure of the algorithm is similar to that of [4] for networks without IVQs, but the proof of the correctness of the algorithm is different.

Algorithm 1. Solves (2) in at most J iterations and provides overall arrival rates $\eta = (\eta_i : i \in \tilde{J})$, and $S, U \subseteq \tilde{J}$, the set of stable (respectively unstable) nodes.
Initialize: $S_0 := \varnothing$, $U_0 := \tilde{J}$.

[$*$] Solve for $\eta = (\eta_1, \ldots, \eta_J)$: $\eta_i = \lambda_i + \sum_{j \in S_0 \cap W} \eta_j r(j,i) + \sum_{j \in U_0 \cup V} \mu_j r(j,i)$, $i \in \tilde{J}$,

and set $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$.
If $(S \setminus S_0) \cap W = \varnothing$ then $U := \tilde{J} \setminus S$. Output η, S, U. Stop.
Else $S_0 := S$, $U_0 := U$. Go to [$*$].

Proof of Lemma 2. The interaction of S_0 and S (respectively of U_0 and U) will be demonstrated by sequential enumeration using $S(n)$ and $U(n)$, $n \geq 0$.

Algorithm 1 solves the traffic equations (2) recursively, building a sequence $\eta(n) = (\eta_1(n), \ldots, \eta_J(n))$, $n \in \mathbb{N}_+$, together with a sequence $S(n) := \{i : \eta_i(n) < \mu_i\}$ of node sets, detected within the first n steps as being stable, for which the following hold:

(i) $S(n-1) \subseteq S(n)$ for all $n \geq 1$, with $S(0) := \varnothing$,

(ii) there exists exactly one $0 < n^* \leq J$ such that $S(n^*-1) \subset S(n^*) = S(n^*+1)$,

(iii) $\eta(n+1)$ solves the $S(n) \cap W$-partition of traffic equations with $U(n) := \tilde{J} \setminus S(n)$, i.e.

$$\begin{aligned} &\eta(n+1)_{S(n)\cap W} \\ &\quad = \lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W,\, S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n)\cap W}, \quad (3) \\ &\eta(n+1)_{U(n)\cup V} \\ &\quad = \lambda_{U(n)\cup V} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W,\, U(n)\cup V} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, U(n)\cup V}. \quad (4) \end{aligned}$$

We will show that

$$(*) \qquad \eta_i(n) \geq \eta_i(n+1) \quad \text{holds for all } i \in \tilde{J}, n \in \mathbb{N}_+.$$

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Then, since, for all $n \in \mathbb{N}$,

$$\eta_i(n) \geq \eta_i(n+1) \quad \text{for all } i \in \tilde{J} \quad \Longrightarrow \quad S(n) \subseteq S(n+1),$$

we obtain (i) and (ii) above.

To prove that $(*)$ holds, we use induction over n.

Basis $(n = 1)$. We have

$$\eta(1) = \lambda + \mu R_{\tilde{J},\tilde{J}} = \lambda + \mu_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V,\tilde{J}}, \tag{5}$$

$$\eta(2) = \lambda + \eta(2)_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V,\tilde{J}}, \tag{6}$$

so $\eta(1) \geq \eta(2)$ if and only if

$$\mu_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} \quad \text{(component-wise)}.$$

Note that $S(1) \neq \varnothing$ is not trivial: with (5) and (6), we have $\eta(1) = \eta(2) + \mu_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}}$ and from $\mu_{S(1)} > \eta(1)_{S(1)}$, we have $\mu_{S(1)\cap W} > \eta(1)_{S(1)\cap W}$ and

$$\mu_{S(1)\cap W} > \eta(2)_{S(1)\cap W} + \mu_{S(1)\cap W} R_{S(1)\cap W,\, S(1)\cap W} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W,\, S(1)\cap W}$$

if and only if

$$\mu_{S(1)\cap W}(\boldsymbol{I} - R_{S(1)\cap W,\, S(1)\cap W}) > \eta(2)_{S(1)}(\boldsymbol{I} - R_{S(1)\cap W,\, S(1)\cap W}).$$

Multiplying by $(\boldsymbol{I} - R_{S(1)\cap W, S(1)\cap W})^{-1}$ yields $\mu_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W,\tilde{J}}$.

Inductive step $(n \curvearrowright n+1)$. Assume that $\eta(n-1) \geq \eta(n)$ holds for some $n \in \mathbb{N}_+$. We show that $\eta(n) \geq \eta(n+1)$ follows. With $S' := S(n) \setminus S(n-1)$, we have

$$\begin{aligned} \eta(n) &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} + \mu_{U(n-1)\cup V} R_{U(n-1)\cup V,\tilde{J}} \qquad (7)\\ &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} + \mu_{S'\cap W} R_{S'\cap W,\tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V,\tilde{J}}, \\ \eta(n+1) &= \lambda + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W,\tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V,\tilde{J}} \qquad (8)\\ &= \lambda + \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} + \eta(n+1)_{S'\cap W} R_{S'\cap W,\tilde{J}} \\ &\quad + \mu_{U(n)\cup V} R_{U(n)\cup V,\tilde{J}}. \end{aligned}$$

So $\eta(n) \geq \eta(n+1)$ if and only if

$$\eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} + \mu_{S'\cap W} R_{S'\cap W,\tilde{J}} \geq \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W,\tilde{J}}. \tag{9}$$

For $S' \neq \varnothing$, from (7), we have

$$\begin{aligned} \eta(n)_{S(n-1)\cap W} = \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\, S(n-1)\cap W} + \mu_{S'\cap W} R_{S'\cap W,\, S(n-1)\cap W} \\ + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n-1)\cap W} + \lambda_{S(n-1)\cap W} \end{aligned}$$

and

$$\begin{aligned} \eta(n)_{S'\cap W} = \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\, S'\cap W} + \mu_{S'\cap W} R_{S'\cap W,\, S'\cap W} \\ + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S'\cap W} \end{aligned}$$

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if and only if

$$\mu_{S'\cap W} = \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\, S'\cap W} + \mu_{S'\cap W} R_{S'\cap W,\, S'\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W}.$$

With $\eta^*(n)_{S(n)\cap W} := (\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W})$ and $\lambda^*_{S(n)\cap W} := (\lambda_{S(n-1)\cap W}, \lambda_{S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W})$, we have

$$\eta^*(n)_{S(n)\cap W} = \lambda^*_{S(n)\cap W} + \eta^*(n)_{S(n)\cap W} R_{S(n)\cap W,\, S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n)\cap W}$$

and

$$\eta^*(n)_{S(n)\cap W} = (\lambda^*_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n)\cap W})(\boldsymbol{I} - R_{S(n)\cap W,\, S(n)\cap W})^{-1}.$$

Similarly, we obtain the solution of $\eta(n+1)_{S(n)\cap W}$ (from (8)), i.e.

$$\lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W,\, S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n)\cap W},$$

as

$$\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V,\, S(n)\cap W})(\boldsymbol{I} - R_{S(n)\cap W,\, S(n)\cap W})^{-1}.$$

By definition, $\mu_{S'} > \eta(n)_{S'}$, so $\mu_{S'\cap W} > \eta(n)_{S'\cap W}$ and $\lambda^*_{S(n)\cap W} \geq \lambda_{S(n)\cap W}$. Thus,

$$\eta^*(n)_{S(n)\cap W} \geq \eta(n+1)_{S(n)\cap W}$$

if and only if

$$(\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W}) \geq (\eta(n+1)_{S(n-1)\cap W}, \eta(n+1)_{S'\cap W})$$

implies that

$$\begin{aligned}\eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} &\geq \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W,\tilde{J}} \wedge \mu_{S'\cap W} R_{S'\cap W,\tilde{J}} \\ &\geq \eta(n+1)_{S'\cap W} R_{S'\cap W,\tilde{J}},\end{aligned}$$

which yields (9).

Existence of a solution of (2). For the existence of a solution of the general traffic equations we need to show for all $n \in \tilde{J}$ that the $S(n) \cap W$-partition of (2), (3), and (4), has a solution. Transforming (3) into $\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V, S(n)\cap W})(\boldsymbol{I} - R_{S(n)\cap W, S(n)\cap W})^{-1}$ yields the unique solution of (3) and inserting this solution into (4) yields the unique solution of (4).

Uniqueness of a solution of (2). Suppose that η and $\hat{\eta}$ are solutions of (2). Then

$$\begin{aligned}\sum_{i\in W} |\eta_i - \hat{\eta}_i| &= \sum_{i\in W} \left| \sum_{j\in W} (\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j)) r(j,i) \right| \\ &\leq \sum_{j\in W} |(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \\ &\leq \sum_{j\in W} |\eta_j - \hat{\eta}_j|. \qquad (10)\end{aligned}$$

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Since $|\eta_i - \hat{\eta}_i| \geq |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))|$ for all $i \in W$ holds, (10) yields $|(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))|$ for all $i \in W$. So $\{i \in W : \eta_i < \mu_i\} = \{i \in W : \hat{\eta}_i < \mu_i\} =: S \cap W$ and therefore η and $\hat{\eta}$ are the solutions of the same $S \cap W$-partition of the traffic equation (which has a unique solution), which means $\eta = \hat{\eta}$. □

If there is at most one 'bottleneck' in W, Algorithm 1 can be simplified by a shortcut modification when a test on $|W \cap S| \geq |W| - 1$ is positive; see [14, Algorithm 2].

2.3. The ergodic case

In this section we prove complements and slight extensions of [15]. Other than [15] for ergodic networks with an IVQ we allow feedback at nodes, i.e. $r(i, i) > 0$.

Theorem 2. (Local equilibrium analysis.) *Consider a Jackson network with an IVQ from Definition 2. Assume $\eta_i < \mu_i$ for all $i \in \tilde{J}$, where $\eta = (\eta_1, \ldots, \eta_J)$ is the unique solution of the traffic equations (1).*

(i) *For nodes without an IVQ, the joint marginal limiting distribution is of product form, i.e.*

$$\lim_{t\to\infty} \mathbb{P}(X_i(t) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{11}$$

for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and this is a stationary distribution for the subnetwork on W as well.

(ii) *If the system is started with (11) as a marginal joint queue lengths distribution on W, the arrival stream at $i \in V$ from $j \in W$ is a Poisson stream with rate $\eta_j r(j, i)$.*

(iii) *If the system is started with (11) as a marginal joint queue lengths distribution on W, then the limiting distribution for a node $i \in V$ with an IVQ and no immediate feedback, i.e. $r(i, i) = 0$, is*

$$\lim_{t\to\infty} \mathbb{P}(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{12}$$

for all $n_i \in \mathbb{N}$, and this is the stationary distribution of X_i as well.

We emphasize the condition 'If the system is started with (11) as a marginal joint queue lengths distribution on W' which is needed in Theorem 2(ii) and (iii). Without this assumption, streams from W to V are not Poisson. Before proving Theorem 2 we summarize 'arguments' which occur in the companion proof for ergodic networks in [15], and will be reused in the nonergodic case. For details see [14, pp. 11–12].

2.3.1. *Subnetwork argument.* The subnetwork W of nodes without IVQs develops as a Jackson network, see Definition 1, where the complementary subnetwork $\{0\} \cup V$, represents source and sink. The corresponding queueing process $\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$ is Markov in its own right. The traffic equations of this subnetwork W are

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j\in W} \tilde{\eta}_j r(j, i), \quad i \in W,$$

where $\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j, i)$, so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$.

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2.3.2. *The M/M/1 argument.* If the subnetwork W is in equilibrium, a node $i \in V$ with an IVQ and without immediate feedback (i.e. $r(i,i) = 0$) behaves as an M/M/1 system in its own right. The corresponding queue length process $\hat{X}$ is a birth–death process on $\mathbb{N}$ with birth rates $\hat{\lambda}_i = \eta_i$ and death rates μ_i.

Proof of Theorem 2. (i) Evoking the 'subnetwork argument' for node set W we conclude that by Jackson's theorem [8] $\tilde{X}$ has the unique stationary and limiting distribution (11) because $\eta_i < \mu_i$ holds for all $i \in W$. Thus, even if the subnetwork V of nodes with IVQs is not in equilibrium, the equilibrium on the subnetwork W of nodes without IVQs is preserved if the initial distribution has the joint marginal (11).

(ii) Ergodic Jackson networks with Poisson arrival streams from the source to node i with rate $\tilde{\lambda}_i$ have, in equilibrium, Poisson$(\tilde{\eta}_i \tilde{r}(i,0))$ streams from node i to the sink; see [12, Example 7.1]. From (i) we know that the subnetwork on W behaves like an ergodic Jackson network for its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i \tilde{r}(i,0) = \eta_i \left(1 - \sum_{j \in W} r(i,j)\right) = \eta_i \left(r(i,0) + \sum_{j \in V} r(i,j)\right).$$

Hence, if the global network process is started with an initial distribution which has marginal (11) on W, departures to the sink from nodes $i \in W$ are Poisson$(\eta_i r(i,0))$, and departures to any node $j \in V$ are also Poisson streams with rate $\eta_i r(i,j)$, because a portion, given by $r(i,j)/(r(i,0) + \sum_{j \in V} r(i,j))$, of the departure intensity $\tilde{\eta}_i \tilde{r}(i,0)$ from $i \in W$ is directed to $j \in V$, even if the subnetwork V is not in equilibrium.

(iii) From the 'M/M/1 argument' we conclude that $\hat{X}$ has a stationary distribution $\pi_i(n) = (1 - \eta_i/\mu_i)(\eta_i/\mu_i)^n$, $n \in \mathbb{N}$, because $\eta_i < \mu_i$ was assumed. □

Remark 1. In Theorems 2(i) and 2(ii) feedback is allowed. In Theorem 2(iii) we need $r(i,i) = 0$ for the node under consideration. For the necessity of this, see [14, Remark 9]. In general, ergodic Jackson networks with IVQs (even if $r(i,i) = 0$ for all $i \in V$ holds) do not have stationary distributions of product form. So, even in equilibrium, the queue lengths at the nodes with an infinite supply of work ($i \in V$) are at a fixed time instant neither independent of each other nor independent of the queue lengths of the nodes in W, although all flows between the nodes with infinite supply are Poisson. For a detailed proof see [14, pp. 13–14]. The proof also reveals the following fact. The only class of Jackson networks with IVQs where the stationary queue lengths distribution is of product form is characterized by: customers departing from a node $i \in V$ leave the network directly to the sink with probability 1, i.e. $r(i,0) = 1$ for all $i \in V$.

Example 3. Consider Example 1 from [1] and [15] and assume that the conditions put forward in Theorem 2 hold. If $\eta_i = \mu_{3-i} r(3-i,i) < \mu_i$, $i = 1,2$, then the marginal stationary distribution of node i is geometric with success probability $1 - (\mu_{3-i} r(3-i,i)/\mu_i)$, $i = 1,2$, by Theorem 2 (see (12)). In [1] and [15] it was shown that the queue lengths in equilibrium are not independent for fixed t.

Remark 2. If a Jackson network with IVQs features a subset $\varnothing \neq W$ without IVQs with the property: 'There is only one node, say $i_* = i_*(W)$, from which customers can move directly from W to V, and, for all $j \in V$, $r(j,j) = 0$ and $r(j,i_*) > 0$, hold', then for any fixed $n \in \mathbb{N}$ and $t \geq 0$ in equilibrium the two events {node i_* is empty at a time t} and

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{node $j \in V$ has a queue length of n customers at time t} are independent of each other. The proof is rather involved, see [13, Proposition 4.25]. A similar property was found in [10, Theorem 1] in a two-node network related to Example 1. In equilibrium for any $n \in \mathbb{N}$ the two events are independent of each other, $i = 1, 2$: {node i is empty at a time t} and {node $3 - i$ has a queue length of n customers at time t}.

2.4. The nonergodic case

For Jackson networks with unstable nodes no steady-state distribution exists, but in [4] it was proved that for the set of stable nodes a limiting distribution of product form exists. The message of the next theorem is that for the stable part W the marginal limiting distribution on W exists similar to the result of [4] and, moreover, it is a stationary distribution on W. The latter observation is surprising because in the framework of [4] stationarity of the limiting distribution is *not* proposed; see also Theorem 4, below.

Theorem 3. (Local equilibrium analysis.) *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an IVQ as in Definition 2. Denote by $\eta = (\eta_1, \ldots, \eta_J)$ the unique solution of the traffic equations (2). We assume that all nodes without IVQs are stable, i.e. $U \cap W = \varnothing$. Then the traffic equations (2) reduce to*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \qquad i \in \tilde{J}. \tag{13}$$

(i) *For nodes without an IVQ, the joint marginal limiting distribution is of product form, i.e.*

$$\lim_{t \to \infty} \mathbb{P}(X_i(t) = n_i : i \in W) = \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{14}$$

for all $(n_i,\ i \in W) \in \mathbb{N}^{|W|}$, and this is a stationary distribution on W as well.

(ii) *If the system is started with (14) as a marginal joint queue lengths distribution on W, the arrival stream from $j \in W$ to $i \in V$ is* Poisson$(\eta_j r(j,i))$. *All these Poisson streams are independent.*

(iii) *If the system is started with (14) as a marginal joint queue lengths distribution on W, then for a stable node $i \in V \cap S$ with $r(i,i) = 0$,*

$$\lim_{t \to \infty} \mathbb{P}(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{15}$$

holds for all $n_i \in \mathbb{N}$, and this is the stationary distribution of X_i as well.

(iv) *If the system is started with (14) as a marginal joint queue lengths distribution on W, then for unstable nodes with an IVQ, $i \in U \subseteq V$ with $r(i,i) = 0$, the limit of the marginal queue length probability is*

$$\lim_{t \to \infty} \mathbb{P}(X_i(t) = n_i) = 0, \quad \textit{for all } n_i \in \mathbb{N}. \tag{16}$$

The message of parts (i) and (iii) is that instability of nodes with an IVQ does not matter for the (local) limiting and stationary behavior of W and of stable nodes in V.

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Proof of Theorem 3. (i) We start by evoking the 'subnetwork argument', which yields in this case (13). By Jackson's theorem [8], $\tilde{X}$ has the unique stationary and limiting distribution (14) if and only if $\tilde{\eta}_i < \mu_i$ holds for all $i \in W$. This is equivalent to $\eta_i < \mu_i$ for all $i \in W$, which was assumed. So the subnetwork on W is in equilibrium if and only if the network process on $\tilde{J}$ is started with (14).

(ii) We see that Jackson networks in equilibrium with external Poisson($\tilde{\lambda}_i$) arrivals to node i have Poisson($\tilde{\eta}_i \tilde{r}(i,0)$) departure streams from node i to the sink; see [12, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network in its own right with $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i \tilde{r}(i,0) = \eta_i \left(1 - \sum_{j \in W} r(i,j)\right) = \eta_i \left(r(i,0) + \sum_{j \in V} r(i,j)\right).$$

Hence, if subnetwork W is in equilibrium, departures to the sink from node $i \in W$ constitute a Poisson process with rate $\eta_i r(i,0)$ and departures to any node $j \in V$ are also a Poisson with rate $\eta_i r(i,j)$, because a portion $r(i,j)/(r(i,0) + \sum_{j \in V} r(i,j))$ of the departure stream $\tilde{\eta}_i \tilde{r}(i,0)$ from node $i \in W$ is directed to $j \in V$.

(iii) and (iv) We start by evoking the 'M/M/1 argument' showing that the resulting birth and death process $\hat{X}$ of high priority customers has a unique limiting and stationary distribution $\pi_i(n) = (1 - \eta_i/\mu_i)(\eta_i/\mu_i)^n$, $n \in \mathbb{N}$, if and only if $\eta_i < \mu_i$ holds. If $\eta_i \geq \mu_i$, i is unstable and the limiting queue length distribution of $\hat{X}$ degenerates to a one-point distribution in ∞. This completes the proof. □

The next theorem can be termed the 'Goodman–Massey theorem' for a subnetwork of a generalized Jackson network when there are nodes with an IVQ outside of the subnetwork W. If $V = \varnothing$ it is the result of [4]. A statement like (15) for stable nodes with IVQs seems not to be possible by arguments as above.

Theorem 4. (Local limiting analysis.) *With $\eta = (\eta_1, \ldots, \eta_J)$ as a solution of (2), we have for any initial distribution for all $n_i \in \mathbb{N}$:*

$$\lim_{t \to \infty} \mathbb{P}(X_i(t) = n_i : i \in S \cap W) = \prod_{i \in S \cap W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{17}$$

$$\lim_{t \to \infty} \mathbb{P}(X_i(t) = n_i) = 0, \quad \textit{for all } i \in U \cap W. \tag{18}$$

Proof. We start by evoking the 'subnetwork argument'. The traffic equations of the described subnetwork W now are

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \min(\tilde{\eta}_j, \mu_j) r(j,i), \qquad i \in W,$$

where $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$, so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$; see (2). According to [4, Theorem 1], $\tilde{X}$ has limiting marginal joint queue length distribution (17) on $S \cap W$ and limiting marginal queue length probabilities (18) for nodes in $U \cap W$. □

Example 4. If in Example 1 from [1] and [15], under the initial conditions put forward in Theorem 3, it holds that $\eta_1 = \mu_2 r(2,1) < \mu_1$ and $\eta_2 = \mu_1 r(1,2) \geq \mu_2$, then the marginal limiting and stationary distribution of node 1 is geometric with success probability

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$1-(\mu_2 r(2,1)/\mu_1)$, whereas the queue length of node 2 diverges; see (15) and (16). It is not possible that both nodes are unstable: $U \neq \tilde{J}$ holds in any case.

3. Applications

Important performance metrics are the system oriented throughput and the customer oriented passage times. We will give prototype examples for both problems. Eventually, we will discuss bottleneck analysis.

3.1. Evaluation of system performance

3.1.1. *Ergodic systems.* In this case, long-time average returns can be approximated via stationary characteristics if these are explicitly known, which is not the case for ergodic Jackson networks with IVQs. Exploiting results of Section 2.3 it is possible for ergodic networks to approximate accumulated costs over a long time horizon in a slightly weakened form.

Take nondecreasing bounded cost functions $g_i\colon \mathbb{N} \to \mathbb{R}$ and $g_W\colon \mathbb{N}^{|W|} \to \mathbb{R}$ associated with the respective queue lengths. The time average of accumulated costs of the system over time horizon $[0, T]$ is

$$d(T) = \sum_{i\in V} \frac{1}{T}\int_0^T g_i(X_i(t))\,\mathrm{d}t + \frac{1}{T}\int_0^T g_W(X_i(t)\colon i \in W)\,\mathrm{d}t. \tag{19}$$

If $\eta_i < \mu_i$ holds for all $i \in \tilde{J}$ and $r(i,i) = 0$ for all $i \in V$, by Theorem 2(i)–(iii) we approximate for large T the averaged integrals in (19) by respective state space averages, which yield

$$\begin{aligned} d(T) \approx{}& \sum_{i\in V}\sum_{n_i\in\mathbb{N}} g_i(n_i)\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i} \\ &+ \sum_{(n_i\colon i\in W)\in\mathbb{N}^{|W|}} g_W(n_i\colon i\in W)\prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \end{aligned}$$

3.1.2. *Nonergodic Jackson networks with an IVQ.* In this case, we utilize the idea which leads to the 'subnetwork argument' (see the proof of Theorem 2). For a nondecreasing cost function $g_W\colon \mathbb{N}^{|W|} \to \mathbb{R}$ almost surely

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g_W(X_i(t)\colon i\in W)\,\mathrm{d}t = \sum_{(n_i\colon i\in W)\in\mathbb{N}^{|W|}} g_W(n_i\colon i\in W)\pi_W(n_i\colon i\in W).$$

So, even in *nonergodic* Jackson networks with infinite supply, the pathwise evaluated time averages on the subnet W for a time horizon $[0, T]$ with large T can be approximated by state space averages over the limiting distributions:

$$\frac{1}{T}\int_0^T g_W(X_i(t)\colon i\in W)\,\mathrm{d}t \approx \sum_{(n_i\colon i\in W)\in\mathbb{N}^{|W|}} g_W(n_i\colon i\in W)\prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

3.2. Throughput

3.2.1. *Throughput computation in ergodic networks with infinite supply.* This case is given by $S = \tilde{J}$, $V \neq \varnothing$. It requires schemes which are slightly different from the case of classical

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networks. Due to ergodicity, the stationary throughput T_i of node i (mean number of departures from i per time unit) is

$$\mathrm{T}_i =: \sum_{(n_1,\ldots,n_J)\in\mathbb{N}^J} \pi(n_1,\ldots,n_J)\mu_i \mathbf{1}_{\mathbb{N}_+}(n_i) = \eta_i \quad \text{for } i \in W, \qquad \mathrm{T}_j = \mu_j \quad \text{for } j \in V.$$

The stationary throughput T, i.e. the mean number of departures from the network to the outside is

$$\mathrm{T} = \sum_{j\in\tilde{J}} \mathrm{T}_j r(j,0) = \sum_{j\in W} \eta_j r(j,0) + \sum_{j\in V} \mu_j r(j,0) = \lambda - \sum_{j\in V} \underbrace{(\eta_j - \mu_j)}_{<0} > \lambda. \tag{20}$$

Here the last equality follows from

$$\lambda = \sum_{i\in\tilde{J}} \lambda_i = \sum_{i\in W} \eta_i r(i,0) + \underbrace{\sum_{i\in V} \mu_i r(i,0)}_{(*)} + \sum_{i\in V} (\eta_i - \mu_i),$$

where in $(*)$ customers from an IVQ which immediately depart after service are counted as proper customers.

3.2.2. *Throughput computation in nonergodic networks with infinite supply.* This case is given by $S \subset \tilde{J}, V \neq \varnothing$. It is possible as a computation of 'locally stationary throughput' by use of Theorem 3 with its local equilibrium. If nodes without an IVQ are stable, the stationary throughput T_i of node $i \in W$ is

$$\mathrm{T}_i := \sum_{(n_j:\ j\in W)\in\mathbb{N}^{|W|}} \pi_W(n_j:\ j \in W)\mu_i \mathbf{1}_{\mathbb{N}_+}(n_i) = \eta_i,$$

and if $r(j,j) = 0$, for all $j \in V \cap S$, the stationary throughput T_j of a stable node $j \in V$ with an IVQ is

$$\mathrm{T}_j := \sum_{n_j\in\mathbb{N}} \pi_j(n_j)\mu_j = \mu_j.$$

If nodes without an IVQ are stable and if $r(j,j) = 0$, for all $j \in V \cap S$, the stationary throughput T_S of the subnetwork of stable nodes (mean number of departures from the subnetwork S to the outside (sink)) is

$$\mathrm{T}_S = \sum_{i\in W} \mathrm{T}_i r(i,0) + \sum_{j\in V\cap S} \mathrm{T}_j r(j,0) = \lambda - \sum_{j\in V} (\eta_j - \mu_j) - \sum_{j\in V\cap U} \mu_j r(j,0),$$

where the second equality follows by the same arguments as the last equality in (20).

3.3. Travel times

Travel times of customers inside a network or passage times for customers through the whole network are the most important performance indices from a customer's point of view. We will discuss these topics and show that our results from Section 2 lay the ground for, for example, finding feasible paths to traverse the network, finding shortest feasible paths between two nodes, and determining expected travel times.

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Definition 5. (*Paths.*) A path in the network is a sequence of connected nodes, where nodes may occur repeatedly, i.e. a sequence of nodes $\langle j_1, j_2, \ldots, j_m\rangle$ with the property $r(j_k, j_{k+1}) > 0$, for all $k = 1, \ldots, m-1$. A path in the network is *feasible* if it does not contain unstable nodes.

To elaborate on paths we consider the directed transition graph (transition digraph) of the network's routing chain $\boldsymbol{R} = (r(i, j)\colon i, j \in \{0, 1, \ldots, J\})$, defined as $G_{\boldsymbol{R}} = (V_{\boldsymbol{R}}, E_{\boldsymbol{R}})$ with the set $V_{\boldsymbol{R}} = \{0, 1, \ldots, J\}$ of vertices and with edge set $E_{\boldsymbol{R}} \subseteq V_{\boldsymbol{R}}^2$ defined by $(i, j) \in E_{\boldsymbol{R}} \leftrightarrow r(i, j) > 0$. The technical devices in our evaluation will be Algorithm 1, which stops after at most J iterations and provides as output $\eta = (\eta_i\colon i \in \tilde{J})$, the overall arrival rates and $S, U \subseteq \tilde{J}$, the set of stable (respectively unstable) nodes.

Problem 1. Determine a feasible path to cross the network using prescribed entrance and exit sets. We consider a Jackson network with IVQs from Definition 2 with a set $\varnothing \neq \tilde{I} \subseteq \tilde{J}$ of entrance nodes and a set $\varnothing \neq \tilde{O} \subseteq \tilde{J}$ of exit nodes, i.e. $j \in \tilde{I}$ implies that $r(0, j) > 0$ and $j \in \tilde{O}$ implies that $r(j, 0) > 0$. The problem is to determine a feasible path from some $i \in \tilde{I}$ to some $j \in \tilde{O}$.

Solution. Run Algorithm 1. Take the transition digraph $G_{\boldsymbol{R}} = (V_{\boldsymbol{R}}, E_{\boldsymbol{R}})$ of the network's routing chain.

- Delete all nodes in U and all ingoing and outgoing arcs from nodes in U to obtain a subdigraph $G_{\boldsymbol{R}}(-U)$.
- There exists a feasible path to cross the network if and only if in $G_{\boldsymbol{R}}(-U)$ there exists a path from some node in $\tilde{I} \setminus U$ to some node in $\tilde{O} \setminus U$.

From the output of Algorithm 1 we can compute utilizations, the fraction of time the stable nodes are serving high priority jobs.

Definition 6. For a stable node $i \in \tilde{J}$ the utilization by high priority jobs is $\rho_i := \eta_i/\mu_i$. If node $i \in \tilde{J}$ is unstable, its utilization by high priority jobs is defined to be $\rho_i := 1$.

This definition is a little bit subtle, because we usually do not have ergodic systems. As shown in the previous sections, we may have in parts of the network W stationary and limiting distributions (Theorem 3) or in parts of the subnetwork V with IVQs we may find limiting distributions (Theorem 4). Definition 6 will apply in both situations and refer to stationary and to limiting distributions. To investigate travel time distributions (respectively expected travel times), we use 'test customers'. These are, by definition, customers who find at their arrival some path $\langle j_1, j_2, \ldots, j_m\rangle$, i.e. when entering the tail of the queue at node j_1 the other customers in the network are distributed according to the stationary distribution. In network simulation this is known as 'marked customer technique'; see [7, Chapter 4].

Corollary 1. *For a stable node $i \in S$ the asymptotic or stationary sojourn time of a test customer at node i is exponentially distributed with parameter $\mu_i - \eta_i$. For a test customer the travel time* $\mathrm{tr}\langle j_1, j_2, \ldots, j_m\rangle$ *through path $\langle j_1, j_2, \ldots, j_m\rangle$ of stable nodes has mean*

$$\mathbb{E}[\mathrm{tr}\langle j_1, j_2, \ldots, j_m\rangle] = \sum_{k=1}^{m} \frac{1}{\mu_{j_k} - \eta_{j_k}}.$$

Proof. Because a test customer finds the other customers at his/her entrance epoch distributed according to a geometrical distribution with success probability $1 - \eta_i/\mu_i$, the first statement

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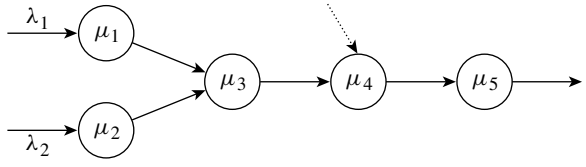


FIGURE 2: Identification of weak bottelnecks.

follows from the node's structure as a single server under FCFS. The second statement follows by additivity of expectations. □

Example 5. Consider the network in Figure 2, where the additional dotted input arrow at node 4 indicates that node 4 has an IVQ. Let $\lambda_1 = \lambda_2 = 1$, $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 3$, $\mu_4 = \frac{5}{2}$, and $\mu_5 = 3$. Then, $\eta_1 = \eta_2 = 1$, $\eta_3 = \eta_4 = 2$, and $\eta_5 = \frac{5}{2}$. Hence, all nodes are stable and the mean waiting times at node i, denoted by W_i, is $W_1 = 1$, $W_2 = \frac{1}{3}$, $W_3 = 1$, $W_4 = 2$, and $W_5 = 2$. By Little's law, the corresponding mean queue lengths, denoted by L_i, are given by $L_1 = 1$, $L_2 = \frac{1}{3}$, $L_3 = 2$, $L_4 = 4$, and $L_5 = 5$.

We now prepare the network to apply standard algorithms to compute fastest routes between prescribed nodes. We need to construct a weighted digraph $G_{\boldsymbol{R}W} = (V_{\boldsymbol{R}}, E_{\boldsymbol{R}}, W)$. We start with the transition digraph $G_{\boldsymbol{R}} = (V_{\boldsymbol{R}}, E_{\boldsymbol{R}})$ of the network's routing chain and associate with each edge $(i, j) \in E_{\boldsymbol{R}}$ a weight

$$w(i, j) = \begin{cases} (\mu_i - \eta_i)^{-1} & \text{if } i \in S, \\ \infty & \text{if } i \in U. \end{cases}$$

Problem 2. Find the shortest paths between two stable nodes. We consider a Jackson network with IVQs from Definition 2 and two stable nodes $i, j \in \tilde{J}$. The problem is to determine a path from i to j with shortest expected travel time for a test customer through all nodes of the path including i and j.

Solution. Run Algorithm 1. Then construct the weighted transition digraph $G_{\boldsymbol{R}W} = (V_{\boldsymbol{R}}, E_{\boldsymbol{R}}, W)$.

- Apply Dijkstra's algorithm [2, pp. 94–97] to determine a shortest path $\mathrm{sp}(i, j) := \langle i = j_1, j_2, \ldots, j_m = j \rangle$ from i to j and the sum of the weights on this path,

$$w(\mathrm{sp}(i, j)) = \sum_{k=1}^{m-1} w(j_k, j_{k+1}). \tag{21}$$

- If (21) yields $w(\mathrm{sp}(i, j)) = \infty$, there is no feasible path from i to j.
- Otherwise: add $(\mu_j - \eta_j)^{-1}$ to obtain the minimal expected travel time from i to j, including the time to pass the initial node and the destination node, i.e.

$$\begin{aligned} \mathbb{E}[\mathrm{tr}(\mathrm{sp}(i, j))] &= \sum_{k=1}^{m-1} w(j_k, j_{k+1}) + \frac{1}{(\mu_j - \eta_j)} \\ &= \mathbb{E}[\mathrm{tr}\langle i, j_2, \ldots, j_{m-1}, j \rangle] \\ &= \sum_{k=1}^{m} \frac{1}{\mu_{j_k} - \eta_{j_k}}. \end{aligned}$$

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TABLE 2.

$\theta \in$	$(0, \frac{1}{4}]$	$(\frac{1}{4}, \frac{1}{2}]$	$(\frac{1}{2}, \frac{2}{3}]$	$(\frac{2}{3}, \frac{4}{5}]$	$(\frac{4}{5}, 1]$
$U(\theta)$	$\{1, 2, 3, 4\}$	$\{1, 3, 4\}$	$\{3, 4\}$	$\{4\}$	$\varnothing$

3.4. Bottleneck analysis

Clearly, the unstable nodes in U are natural bottlenecks of the system. Identifying these 'strong' bottlenecks is straightforward with Algorithm 1, which outputs the set $S = \tilde{J} \setminus U$. Practical experience shows that even in stable networks ($U = \varnothing$ because of $\eta_j < \mu_j$ for all $j \in \tilde{J}$) 'weak' bottlenecks emerge, which have a strong influence on the performance of the network. Heuristically, weak bottlenecks are those nodes where the local queue lengths become large compared to the other nodes' queue lengths.

Algorithm 1 allows for bottleneck analysis in this weak sense as well. To see this, consider an overall stable network and scale the service rates at all stations by a common factor $\theta \in (0, 1]$. Solving the traffic equations and identifying the possible unstable nodes in the reduced service rate setting, identifies possible bottlenecks for the actual (stable) network. This allows us, for example, to identify the stability behavior of a network in the planning phase when the service rates are not exactly known.

Consider a classical Jackson network, i.e. $V = \varnothing = U$. Then the mean queue length at node i is $L_i = \eta_i/(\mu_i - \eta_i)$ and the effect of a marginal increase in μ_i on the mean queue length is given by

$$\frac{\mathrm{d}}{\mathrm{d}\mu_i} L_i = -\frac{\eta_i}{(\mu_i - \eta_i)^2} = -\frac{L_i^2}{\eta_i}.$$

As it should be, a bottleneck may occur if the service rate at node i is decreased. As we will illustrate in the following example, this is different for the nonclassical Jackson networks considered in this paper.

Example 6. We revisit the queuing network presented in Example 5; see Figure 2. Inspecting the mean queue lengths as computed in Example 5, node 5 seems to be the candidate for creating a bottleneck. However, the set of unstable nodes as a mapping of θ, denoted by $U(\theta)$, is given in Table 2.

The analysis shows that node 5 is stable for all values of θ, and that node 4 is the weak bottleneck of the system. Moreover, node 3 is the node that is next critical to node 4, and following this reasoning, the nodes can be ranked. This analysis suggests that node 5 is not critical for stability of the network at all!

This illustrates that mean queue lengths and waiting times depend in a more involved way on service rates than in classical Jackson networks because traffic rates η_i are piecewise linear mappings of the μ_is.

4. Conclusion

We have developed a unified framework for two types of nonstandard Jackson network: networks where some nodes have an additional IVQ, and networks with unstable nodes. Both of these topics have been studied before alone, but in practice both phenomena may occur in parallel, which makes the present work useful.

We obtained closed-form analytical solutions of the steady-state queue length distribution at stable nodes and described the interplay of an IVQ and instability within the network. For

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certain subnetworks we proved stationary distributions (if these exist) (respectively, asymptotic marginal distributions of product form).

A main contribution is an algorithm which elaborates on the modified traffic equations to obtain customer flows in the network, which detects stable and unstable nodes and allows to identify bottlenecks.

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