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A NOTE ON THE EXPLICIT ANALYTIC SOLUTION TO A NONAUTONOMOUS ABSTRACT EVOLUTION EQUATION

ANDREW P. BASSOM and PHIL HOWLETT✉

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Abstract

We consider a nonautonomous abstract evolution equation on Banach space. If the evolutionary operator and the evolutionary kernel are real analytic at the initial time there is an explicit formula for the coefficients of the operator in terms of the coefficients of the kernel. In this note we show that the converse is also true by finding an explicit formula for the coefficients of the kernel in terms of the coefficients of the operator.

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1. Introduction

Let X be a Banach space and consider the abstract nonautonomous evolution equation

$$dR(t)/dt = A(t)R(t) \tag{1.1}$$

with $R(0) = I$, where the kernel $A \in C^\infty([0,a), \mathcal{B}(X))$ is well defined if $t \in [0,a) \subseteq \mathbb{R}$ for some $a > 0$. Suppose there exists an evolutionary operator $R \in C^\infty([0,r), \mathcal{B}(X))$ satisfying (1.1) for some $r \in [0,a)$ such that $R(t)$ is real analytic for $t \in (-r,r)$ with $R(t) = \sum_{n\in\mathbb{N}-1} R_n t^n$, where $R_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N} - 1$. Bassom *et al.* [5] used basic power series manipulation to prove that $A(t)$ is necessarily real analytic when $t \in (-b,b)$ for some $b > 0$ with $A(t) = \sum_{n\in\mathbb{N}-1} A_n t^n$, where $A_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N} - 1$. They then used the initial condition $R(0) = I$ and the recursive relationship

$$nR_n = \sum_{j=0}^{n} A_j R_{n-j} \tag{1.2}$$

for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ to show that $R(t)$ is necessarily also analytic for $t \in (-b,b)$. Subsequently they proved that

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$$R_n = \sum_{q=0}^{n-1} \Big[\sum_{\boldsymbol{m} \in \mathcal{S}_{n,q}} \pi_{\mathbf{m}} A_{\mathbf{m}} \Big] \in \mathcal{B}(X) \tag{1.3}$$

for each $n \in \mathbb{N}$, where

$$\mathcal{S}_{n,q} \coloneqq \{\mathbf{m} = (m_1, \ldots, m_{n-q}) \in (\mathbb{N}-1)^{n-q} = \{0, 1, 2, \ldots\}^{n-q} \mid \langle \mathbf{m}, \mathbf{1} \rangle = q\} \tag{1.4}$$

is the index set for each $n \in \mathbb{N}$ and $q \in \{0, 1, \ldots, n-1\}$ and where

$$\pi_{\mathbf{m}} \coloneqq \prod_{j=1}^{n-q} \Big[n - q + 1 - j + \Big(\sum_{s=j}^{n-q} m_s \Big) \Big]^{-1} \in \mathbb{R}$$

is the coefficient for each $\mathbf{m} \in \mathcal{S}_{n,q}$. The notation

$$A_{\mathbf{m}} = A_{m_1} A_{m_2} \cdots A_{m_{n-q}}$$

for each $\mathbf{m} \in S_{n,q}$ is used in (1.3) and the notation $\mathbf{1} = \mathbf{1}_{n-q} = (1, \ldots, 1) \in (\mathbb{N}-1)^{n-q}$ is used in (1.4). We note that $\langle \mathbf{m}, \mathbf{1} \rangle = q$ is the total index for the product $A_{\mathbf{m}}$ and $n - q$ is the total exponent. Bassom *et al.* [5] conjectured that it may be possible to find a corresponding explicit nonrecursive formula for the coefficients $\{A_n\}_{n \in \mathbb{N}-1}$ of the kernel in terms of the coefficients $\{R_n\}_{n \in \mathbb{N}-1}$ of the operator.

1.1. An affirmative answer to the conjecture in Hilbert space. The conjecture in [5] was answered in the affirmative by Neto [16] in the case where X is a Hilbert space. He extended the argument used in [5] to find an explicit solution in Banach space to the nonhomogeneous equation

$$dQ(t)/dt = A(t)Q(t) + B(t), \tag{1.5}$$

where the kernel $A(t) = \sum_{n \in \mathbb{N}-1} A_n t^n$, the evolutionary operator $Q(t) = \sum_{n \in \mathbb{N}-1} Q_n t^n$ and the forcing function $B(t) = \sum_{n \in \mathbb{N}-1} B_n t^n$ are all assumed to be analytic on some nontrivial open interval $(-b, b) \subset \mathbb{R}$. Neto then showed that in Hilbert space

$$dA(t)^*/dt = (-1)Y(t)^* A(t)^* + Z(t)^* \tag{1.6}$$

where $A(0)^* = A_0^*$, $Y(t) = [dR(t)/dt]R(t)^{-1}$, $Z(t) = [d^2R(t)/dt^2]R(t)^{-1}$, $R(t)$ is the evolutionary operator defined by (1.1) and $A(t)^*$, $Y(t)^*$ and $Z(t)^*$ are the respective Hilbert space adjoint operators. Consequently the Maclaurin series coefficients for $A(t)^*$ can be defined explicitly in terms of the Maclaurin series coefficients for $R(t)^*$. See [16, Proposition 6, *et seq.*] for details of the explicit formulas.

1.2. Our contribution: an affirmative answer to the conjecture in Banach space. In this paper we restrict our attention to the homogeneous equation (1.1). When X is a Banach space we use the recursion (1.2) to find an explicit nonrecursive formula for the coefficients $\{A_n\}_{n \in \mathbb{N}-1}$ in terms of the coefficients $\{R_n\}_{n \in \mathbb{N}-1}$. No additional assumptions are required and we find a much simpler expression than the one given in [16].

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2. The main results

Let X be a Banach space and assume that $A(t) = \sum_{n\in\mathbb{N}-1} A_n t^n, R(t) = \sum_{n\in\mathbb{N}-1} R_n t^n$ are real analytic. Let $\mathbf{1} = (1, 1, \ldots) \in (\mathbb{N}-1)^\infty$. Let $\mathbf{p} = (p_1, p_2, \ldots)$, $\mathbf{t} = (t_1, t_2, \ldots) \in (\mathbb{N}-1)^\infty$ denote the respective power and index vectors. Define the operator product

$$R_{\mathbf{t}}^{\mathbf{p}} := R_{t_1,t_2,\ldots}^{p_1,p_2,\ldots} := R_{t_1}^{p_1} R_{t_2}^{p_2} \cdots$$

whenever $\langle \mathbf{p}, \mathbf{1}\rangle < \infty$. If $p_j = 0$ or $t_j = 0$ then $R_{t_j}^{p_j} = I$ and we can use a reduced product with finite length $\langle \mathbf{p}, \mathbf{1}\rangle$. In general, we can write $R_{\mathbf{t}}^{\mathbf{p}} = R_{t_1}^{p_1} R_{t_2}^{p_2} \cdots R_{t_n}^{p_n}$ for some $n \le \langle \mathbf{p}, \mathbf{1}\rangle$ where $\mathbf{p}, \mathbf{t} \in \mathbb{N}^\infty$. The order of terms in the product is important. If $X = \mathbb{R}$ then $a(t) = \sum_{n\in\mathbb{N}-1} a_n t^n$, $r(t) = \sum_{n\in\mathbb{N}-1} r_n t^n$, where $a_n, r_n \in \mathbb{R}$. Let $\mathbf{n} = (1, 2, \ldots) \in (\mathbb{N}-1)^\infty$ and define the real number product

$$r_{\mathbf{n}}^{\mathbf{p}} := r_{1,2,\ldots}^{p_1,p_2,\ldots} := r_1^{p_1} r_2^{p_2} \cdots$$

whenever $\langle \mathbf{p}, \mathbf{1}\rangle < \infty$. In the scalar case the order of terms in the real number product is no longer important. If $p_j = 0$ then $r_j^{p_j} = 1$ and we can use the reduced product

$$r_{\mathbf{n}}^{\mathbf{p}} = r_{j(1),j(2),\ldots,j(n)}^{p_{j(1)},p_{j(2)},\ldots,p_{j(n)}} = r_{j(1)}^{p_{j(1)}} r_{j(2)}^{p_{j(2)}} \cdots r_{j(n)}^{p_{j(n)}}$$

for some $n \le \langle \mathbf{p}, \mathbf{1}\rangle$ where $j(1) < j(2) < \cdots < j(n)$ are the indices for the nonzero powers.

2.1. The general case. In the general case

$$A_q = \sum_{\langle \mathbf{p},\mathbf{t}\rangle = q+1} (-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1 R_{\mathbf{t}}^{\mathbf{p}} \tag{2.1}$$

for each $q \in \mathbb{N}$, where $\langle \mathbf{p}, \mathbf{1}\rangle = p_1 + p_2 + \cdots$ is the power sum (the product length) and $\langle \mathbf{p}, \mathbf{t}\rangle = p_1 t_1 + p_2 t_2 + \cdots$ is the power-weighted index sum.

2.2. The scalar case. In the scalar case the general formula can be replaced by the collective formula

$$a_q = (q+1) \cdot \sum_{k=1}^{q+1} (-1)^{k-1} k^{-1} \Big[\sum_{\mathbf{p}\in\mathcal{P}_{k,q}} \rho_{\mathbf{p}} r_{\mathbf{n}}^{\mathbf{p}} \Big] \tag{2.2}$$

for each $q \in \mathbb{N}$, where

$$\mathcal{P}_{k,q} := \{\mathbf{p} = (p_1, p_2, \ldots) \in (\mathbb{N}-1)^\infty \mid \langle \mathbf{p}, \mathbf{1}\rangle = k \text{ and } \langle \mathbf{p}, \mathbf{n}\rangle = q+1\}$$

for each $k = 1, \ldots, q+1$ and

$$\rho_{\mathbf{p}} := \binom{\langle \mathbf{p}, \mathbf{1}\rangle}{\mathbf{p}} = \binom{p_1 + p_2 + \cdots}{p_1\ p_2\ \cdots} = \binom{k}{p_1\ p_2\ \cdots}$$

for each $\mathbf{p} \in \mathcal{P}_{k,q}$ and $\langle \mathbf{p}, \mathbf{n}\rangle = p_1 + 2p_2 + \cdots$ is the power-weighted ordered index sum.

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3. The intuitive argument

We simply use successive applications of the basic recursion.

3.1. The general case. The initial condition shows that $R_0 = I$ is the identity operator. Therefore, we can rearrange the general recursion to show that $A_0 = R_1$ and

$$A_q = (q+1)R_{q+1} - \sum_{j=0}^{q-1} A_j R_{q-j}$$

for each $q \in \mathbb{N}$. By applying this recursion for $q = 1, 2, 3, 4, \ldots$, we obtain

$$\begin{aligned}
A_1 &= 2R_2^1 - R_1^2,\\
A_2 &= 3R_3^1 - (R_{1,2}^{1,1} + 2R_{2,1}^{1,1}) + R_1^3,\\
A_3 &= 4R_4^1 - ([R_{1,3}^{1,1} + 3R_{3,1}^{1,1}] + 2R_2^2) + [R_{1,2}^{2,1} + R_{1,2,1}^{1,1,1} + 2R_{2,1}^{1,2}] - R_1^4,\\
A_4 &= 5R_5^1 - ([R_{1,4}^{1,1} + 4R_{4,1}^{1,1}] + [2R_{2,3}^{1,1} + 3R_{3,2}^{1,1}])\\
&\quad + ([R_{1,3}^{2,1} + R_{1,3,1}^{1,1,1} + 3R_{3,1}^{1,2}] + [R_{1,2}^{1,2} + 2R_{2,1,2}^{1,1,1} + 2R_{2,1}^{2,1}])\\
&\quad - [R_{1,2}^{3,1} + R_{1,2,1}^{2,1,1} + R_{1,2,1}^{1,1,2} + 2R_{2,1}^{1,3}] + R_1^5,
\end{aligned}$$

and so on. In order to apply the proposed formula (2.1) to find A_q we must first solve the equation $\langle \mathbf{p}, \mathbf{t} \rangle = q + 1$. When $q = 4$ we must solve the equation $\langle \mathbf{p}, \mathbf{t} \rangle = 5$ with $n \le \langle \mathbf{p}, \mathbf{1} \rangle \le 5$. We can now use (2.1) to write down the relevant terms:

$$\begin{aligned}
&\langle \mathbf{p}, \mathbf{1} \rangle = 1 : 5R_5^1,\\
&\langle \mathbf{p}, \mathbf{1} \rangle = 2 : (-1)R_{1,4}^{1,1},\ (-1)4R_{4,1}^{1,1},\ (-1)2R_{2,3}^{1,1},\ (-1)3R_{3,2}^{1,1},\\
&\langle \mathbf{p}, \mathbf{1} \rangle = 3 : R_{1,3}^{2,1},\ R_{1,3,1}^{1,1,1},\ 3R_{3,1}^{1,2},\ R_{1,2}^{1,2},\ 2R_{2,1,2}^{1,1,1},\ 2R_{2,1}^{2,1},\\
&\langle \mathbf{p}, \mathbf{1} \rangle = 4 : (-1)R_{1,2}^{3,1},\ (-1)R_{1,2,1}^{2,1,1},\ (-1)R_{1,2,1}^{1,1,2},\ (-1)2R_{2,1}^{1,3},\\
&\langle \mathbf{p}, \mathbf{1} \rangle = 5 : R_1^5.
\end{aligned}$$

This confirms the validity of (2.1) when $q = 4$.

3.2. The scalar case. The initial condition shows that $r_0 = 1$. Hence, we can rearrange the general recursion to show that $a_0 = r_1$ and

$$a_q = (q+1)r_{q+1} - \sum_{j=0}^{q-1} a_j r_{q-j}$$

for each $q \in \mathbb{N}$. By applying this recursion for $q = 1, 2, 3, 4, 5, \ldots$, we obtain

$$\begin{aligned}
a_1 &= 2r_2^1 - r_1^2,\\
a_2 &= 3r_3^1 - 3r_{1,2}^{1,1} + r_1^3,\\
a_3 &= 4r_4^1 - (4r_{1,3}^{1,1} + 2r_2^2) + 4r_1^2 r_2 - r_1^4,
\end{aligned}$$

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$$a_4 = 5r_5^1 - (5r_{1,4}^{1,1} + 5r_{2,3}^{1,1}) + (5r_{1,3}^{2,1} + 5r_{1,2}^{1,2}) - 5r_{1,2}^{3,1} + r_1^5,$$
$$a_5 = 6r_6 - (6r_1r_5 + 6r_2r_4 + 3r_3^2) + (6r_1^2r_4 + 12r_1r_2r_3 + 2r_2^3) - (6r_1^3r_3 + 9r_1^2r_2^2) + 6r_1^4r_2 - r_1^6,$$

and so on. In order to apply the proposed formula (2.2) to find a_q we must first find all $\mathbf{p} \in \mathcal{P}_{q,k}$ for each $k = 1, \dots, q+1$. Thus, we must solve the simultaneous equations $\langle \mathbf{p}, \mathbf{n} \rangle = q+1$ and $\langle \mathbf{p}, \mathbf{1} \rangle = k$ for each $k = 1, \dots, q+1$. When $q = 5$ we obtain the following solutions:

$$k = 1 : \mathbf{p} = (0,0,0,0,0,1),$$
$$k = 2 : \mathbf{p} = (1,0,0,0,1),\ (0,1,0,0,1),\ (0,0,2),$$
$$k = 3 : \mathbf{p} = (2,0,0,0,1),\ (1,1,1),\ (0,3,0),$$
$$k = 4 : \mathbf{p} = (3,0,1),\ (2,2),$$
$$k = 5 : \mathbf{p} = (4,1),$$
$$k = 6 : \mathbf{p} = (6).$$

We can now use (2.2) to write down the corresponding terms:

$$k = 1 : \binom{1}{1} r_6 = 6r_6^1;$$
$$k = 2 : (-1)3\binom{2}{1\ 1} r_1r_5 = (-1)6r_{1,5}^{1,1}, \quad (-1)3\binom{2}{1\ 1} r_2r_4 = (-1)6r_{2,4}^{1,1}, \quad (-1)3\binom{2}{2} r_3^2 = (-1)3r_3^2;$$
$$k = 3 : 2\binom{3}{2\ 1} r_1^2r_4 = 6r_{1,4}^{2,1}, \quad 2\binom{3}{1\ 1\ 1} r_1r_2r_3 = 12r_{1,2,3}^{1,1,1}, \quad 2\binom{3}{3} r_2^3 = 2r_2^3;$$
$$k = 4 : 6(-1)4^{-1}\binom{4}{3\ 1} r_1^3r_3 = (-1)6r_{1,3}^{3,1}, \quad 6(-1)4^{-1}\binom{4}{2\ 2} r_1^2r_2^2 = (-1)9r_{1,2}^{2,2};$$
$$k = 5 : 6 \cdot 5^{-1}\binom{5}{4\ 1} r_1^4r_2 = 6r_{1,2}^{4,1};$$
$$k = 6 : 6(-1)6^{-1}\binom{6}{6} r_1^6 = (-1)r_1^6.$$

This confirms the validity of (2.2) when $q = 5$.

4. Literature review

We refer readers to our recent paper [5] for a brief historical review of the extensive literature on this topic. We will simply summarise that review here. Suffice it to say that Kato and Tanabe [11, 19] and Sobolevski [18] found conditions on the evolutionary kernel $A(t)$ that guarantee existence of an evolutionary operator $R(t)$ satisfying (1.1) subject to the initial condition $R(0) = I$ (see [21, Section XIV.5, pages 440–445]

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for an extended discussion and also [2, 8, 10, 12–14, 20]). If $A(t) \in \mathcal{B}(X)$ is real analytic at $t = 0$, the Peano–Baker series [3, 4, 7, 17] exactly reproduces the first $n + 1$ terms of the Maclaurin series for $R(t)$ after n iterations but does not reveal an explicit formula. The Magnus expansion [6, 15] seeks an infinite series solution in the form of an operator exponential. Explicit formulas for the exponents are notoriously complicated [6, Section 2.2, page 20] so recursive calculation is usually preferred. See also fundamental work on the so-called CBHD exponential theorem of Campbell, Baker, Hausdorff and Dynkin [1, 9].

The current paper addresses a question posed in a recent paper by Bassom *et al.* [5] and a conditional answer to that question in a subsequent paper by Neto [16].

In [5] the authors assumed that the exponential operator and the exponential kernel are both real analytic at $t = 0$. They justified this assumption by proving that the evolutionary operator is real analytic at $t = 0$ if and only if the evolutionary kernel is real analytic at $t = 0$. To begin they noted two well-known general results.

LEMMA 4.1. *Let $F \in C^\infty([0, a], \mathcal{B}(X))$ for some $a \in \mathbb{R}$ with $a > 0$. The function $F(t)$ is real analytic for $t \in [0, b]$ where $b \in (0, a] \subseteq \mathbb{R}$ if and only if there exist $M, c > 0$ with $\sup_{t\in[0,b]} \|F^{(k)}(t)/k!\| \le M/c^k$ for all $k \in \mathbb{N} - 1$.*

COROLLARY 4.2. *If $F(t)$ is real analytic for $t \in (-b, b)$ for some $b > 0$, the extended function defined by $F(z) := \sum_{k\in\mathbb{N}-1} F^{(k)}(0)z^k/k!$ is analytic for all $z \in \mathbb{C}$ with $|z| < b$.*

They then proved two specific results.

LEMMA 4.3. *Let $A(t), R(t) \in \mathcal{B}(X)$ satisfy (1.1). If the solution $R(t)$ is real analytic at $t = 0$ then the kernel $A(t)$ is real analytic at $t = 0$.*

LEMMA 4.4. *Let $\{A_j\}_{j\in\mathbb{N}-1} \in \mathcal{B}(X)$ and suppose that $A(t) := \sum_{n\in\mathbb{N}-1} A_n t^n$ is real analytic on some nontrivial interval $t \in (-b, b)$. Let $R_0 = I \in \mathcal{B}(X)$ and use the recursive relationship $nR_n = \sum_{j=0}^{n-1} A_j R_{n-1-j}$ for each $n \in \mathbb{N}$ to define a countable collection of coefficients $\{R_j\}_{j\in\mathbb{N}-1} \in \mathcal{B}(X)$. The function $R(t) := \sum_{n\in\mathbb{N}-1} R_n t^n$ is real analytic and satisfies the nonautonomous abstract evolution equation (1.1) for all $t \in (-b, b)$. If $A(t)$ is a polynomial this result extends to all $t \in \mathbb{R}$.*

We refer readers to the original paper [5] for the proofs of these two crucial results.

In [16] Neto used the Ω-calculus and an elementary direct argument to find an explicit Maclaurin series solution to the nonhomogeneous evolution equation (1.5). He then used an ingenious role reversal argument to show that in a Hilbert space setting, the adjoint $A(t)^*$ of the original kernel is the evolutionary operator for a nonhomogeneous equation (1.6) with kernel $[R(t)^*]^{-1}dR(t)^*/dt$ and forcing function $[R(t)^*]^{-1}d^2R(t)^*/dt^2$. Thus, he was able to apply the extended results to find an explicit expression for the coefficients A_n in terms of the coefficients R_n. The role reversal argument is not used here. Consequently, the methodology in [16] is not directly relevant to the current paper.

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5. Justification of the main results

We will prove the general formula (2.1) for A_q by induction.

PROOF. Formula (2.1) is true for $q = 0$. Assume (2.1) is true for $r < q \in \mathbb{N}$. The recursive formula (1.2) shows that

$$A_q = (q+1)R_{q+1} - \sum_{j=0}^{q-1} A_j R_{q-j}. \tag{5.1}$$

for each $q \in \mathbb{N}$. We argue that each individual product term in A_q is generated by precisely one term in the recursive relationship. Suppose A_q contains the term $c_{\mathbf{p},\mathbf{t}} R^{p_1,\dots,p_r}_{t_1,\dots,t_r}$ for some $c_{\mathbf{p},\mathbf{t}} \in \mathbb{R}$ where $\mathbf{p} \in \mathbb{N}^r$ for some $r \le q+1$. The assumption that $\mathbf{p} \in \mathbb{N}^r$ simply means that trivial terms with $p_j = 0$ are omitted from the product. If $r = q+1$, the only possible term is $(q+1)R^1_{q+1}$. Thus, $c_{\mathbf{1},q+1} = q+1$. If $r < q+1$, the term $c_{\mathbf{p},\mathbf{t}} R^{p_1,\dots,p_r}_{t_1,\dots,t_r}$ must be generated by an individual product within the collective product $A_{q+1-t_r} R_{t_r}$. The inductive hypothesis shows that the specific product is

$$(-1)\cdot(-1)^{p_1+\cdots+(p_r-1)-1} t_1 R^{p_1,\dots,p_r-1}_{t_1,\dots,t_r} \cdot R_{t_r} = (-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1 R^{p_1,\dots,p_r}_{t_1,\dots,t_r}$$

where $p_1 t_1 + \cdots + (p_r - 1)t_r = q + 1 - t_r$. Therefore, $c_{\mathbf{p},\mathbf{t}} = (-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1$ and $\langle \mathbf{p}, \mathbf{t}\rangle = q+1$. It is also clear that every term of the form

$$(-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1 R^{p_1,\dots,p_r}_{t_1,\dots,t_r}$$

with $\langle \mathbf{p}, \mathbf{t}\rangle = q+1$ is generated in this way. Thus, (2.1) is also true for $r = q$. □

REMARK 5.1. It is interesting to follow the generational history of the archetypal term

$$(-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1 R^{\mathbf{p}}_{\mathbf{t}} = (-1)^{\langle \mathbf{p},\mathbf{1}\rangle - 1} t_1 R^{p_1,\dots,p_n}_{t_1,\dots,t_n}$$

where $\langle \mathbf{p}, \mathbf{t}\rangle = q+1$. If each repeated factor R^p_t with $p > 1$ is written as the product of p identical terms $R_t \cdots R_t$, then we may assume without loss of generality that the archetypal term can be rewritten in the form

$$(-1)^{\langle \mathbf{1}_m,\mathbf{1}\rangle - 1} R^{\mathbf{1}}_{\mathbf{u}} = (-1)^{m-1} R^{1,\dots,1}_{u_1,\dots,u_m},$$

where $\langle \mathbf{1}, \mathbf{u}\rangle = q+1$ and $\mathbf{1}_m := \mathbf{e}_1 + \cdots + \mathbf{e}_m$ where $\{\mathbf{e}_j\}_{j\in\mathbb{N}}$ are the standard unit vectors in $\mathbb{R}^\infty$. The ancestral term arises at stage u_1 in the form $u_1 R^1_{u_1}$ as the first term in the recursion

$$A_{u_1-1} = u_1 R_{u_1} - \sum_{j=0}^{u_1-1} A_j R_{u_1-1-j}.$$

At stage $u_1 + u_2$ the term $(-1)R^{1,1}_{u_1,u_2}$ is generated from the parent term $u_1 R^1_{u_1}$ as part of the product $(-1)A_{u_1-1}R_{u_2}$ in the recursion

$$A_{u_1+u_2-1} = (u_1+u_2)R_{u_1+u_2} - \sum_{j=0}^{u_1+u_2-1} A_j R_{u_1+u_2-1-j}.$$

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In general, at stage $u_1 + \cdots + u_r$, the term $(-1)^{r-1}u_1 R^{1,\ldots,1}_{u_1,\ldots,u_r}$ is generated from the parent term $(-1)^{r-2}u_1 R^{1,\ldots,1}_{u_1,\ldots,u_{r-1}}$ as part of the product $A_{u_1+\cdots+u_{r-1}-1}R_{u_r}$ in the recursion

$$A_{u_1+\cdots+u_r-1} = (u_1 + \cdots + u_r)R_{u_1+\cdots+u_r} - \sum_{j=0}^{u_1+\cdots+u_{r-1}-1} A_j R_{u_1+\cdots+u_r-1-j}.$$

The process terminates with the term $(-1)^m u_1 R^{1,\ldots,1}_{u_1,\ldots,u_m}$ when $u_1 + \cdots + u_m = q + 1$.

We will also prove the scalar formula (2.2) for a_q by induction.

PROOF. Formula (2.2) is true for $q = 0$. Assume (2.2) is true for $r < q \in \mathbb{N}$. The recursive formula (1.2) shows that

$$a_q = (q + 1)r_{q+1} - \sum_{j=0}^{q-1} a_j r_{q-j}$$

for each $q \in \mathbb{N}$. Within the expansion (2.2), suppose there is a particular vector $\mathbf{p}$ with $\langle \mathbf{p}, \mathbf{1}\rangle = k$ and $\langle \mathbf{p}, \mathbf{n}\rangle = q + 1$. Within $\mathbf{p}$ consider a particular index j with $p_j \geq 1$. Now consider a reduced vector $\widehat{\mathbf{p}}_j$ which is identical in structure to $\mathbf{p}$ except that p_j is replaced by $p_j - 1$. Therefore,

$$\langle \widehat{\mathbf{p}}_j, \mathbf{1}\rangle = k - 1 \quad \text{and} \quad \langle \widehat{\mathbf{p}}_j, \mathbf{n}\rangle = p_1 + 2p_2 + \cdots + j(p_j - 1) + \cdots = q + 1 - j.$$

We propose that $\widehat{\mathbf{p}}_j$ appears in the expansion of exactly one a_s with $s < q$. The inductive hypothesis tells us that the parent term we wish to identify takes the form

$$(s + 1)(-1)^{\ell-1}\ell^{-1}\binom{\ell}{p_1\ p_2\ \cdots\ p_j - 1\ \cdots} r^{\widehat{\mathbf{p}}_j}_{\mathbf{n}}$$

where $\langle \widehat{\mathbf{p}}_j, \mathbf{1}\rangle = \ell$ and $\langle \widehat{\mathbf{p}}_j, \mathbf{n}\rangle = s + 1$. It follows that $\ell = k - 1$ and $s + 1 = q + 1 - j$. Thus, the parent term is

$$(q + 1 - j)(-1)^{k-2}(k - 1)^{-1}\binom{k-1}{p_1\ \cdots p_j - 1\ \cdots} r^{\widehat{\mathbf{p}}_j}_{\mathbf{n}}.$$

The scalar recursion (5.1) shows that the coefficient of $\mathbf{p}$ arising from $\widehat{\mathbf{p}}_j$ is

$$\begin{aligned}(-1)(q + 1 - j)(-1)^{k-2}(k - 1)^{-1}&\binom{k-1}{p_1\ \cdots\ p_j - 1\ \cdots}\\ &= (q + 1 - j)(-1)^{k-1}(k - 1)^{-1}k^{-1}p_j\binom{k}{p_1\ p_2\ \cdots}.\end{aligned}$$

This is the contribution to the coefficient from the parent term when we reduce the power p_j to $p_j - 1$. The total contribution over all possible parent terms is obtained by

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summing over all possible values of j. Hence, the coefficient for the product $r_{\mathbf{n}}^{\mathbf{p}}$ in the expansion for a_q is given by

$$\begin{aligned}
&\sum_{\mathbf{p}\in\mathcal{P}_{k,q},\, p_j\neq 0} (q+1-j)(-1)^{k-1}(k-1)^{-1}k^{-1}p_j\binom{k}{p_1\ p_2\ \cdots}\\
&\quad = (-1)^{k-1}(k-1)^{-1}k^{-1}\binom{k}{p_1\ p_2\ \cdots}\sum_{\mathbf{p}\in\mathcal{P}_{k,q}}[(q+1)p_j - jp_j]\\
&\quad = (-1)^{k-1}(k-1)^{-1}k^{-1}\binom{k}{p_1\ p_2\ \cdots}[(q+1)k-(q+1)]\\
&\quad = (q+1)(-1)^{k-1}k^{-1}\binom{k}{p_1\ p_2\ \cdots}
\end{aligned}$$

as required. This completes the proof. □

REMARK 5.2. Note that in the scalar case with $X = \mathbb{R}$ the exponential operator is defined explicitly in terms of the exponential kernel by the formula $r(t) = \exp[\int_0^t a(\tau)\,d\tau]$.

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ANDREW P. BASSOM, School of Natural Sciences,
University of Tasmania, Hobart, Tasmania 7001, Australia
e-mail: andrew.bassom@utas.edu.au

PHIL HOWLETT, Scheduling and Control Group,
Centre for Industrial and Applied Mathematics, UniSA STEM,
University of South Australia, Mawson Lakes 5095, Australia
e-mail: phil.howlett@adelaide.edu.au

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