A Note on Some Integrals Involving Bessel Functions. By Mr C. Fox, Sidney Sussex College.

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The object of this note is to prove the following results, all of which hold when $|a|<1$.

$$(1) \qquad \int_0^\infty J_x(ax)\,J_{-x}(ax)\cos\pi x\,dx = \frac{1}{4(1-a^2)^{\frac{1}{2}}}.$$

(2) *If r is any positive integer other than zero, then*

$$\int_0^\infty \left[J_x(ax)\,J_{-x}(ax) - \sum_{m=0}^{r-1} \frac{(-1)^m(2m)!\,(\frac{1}{2}ax)^{2m}}{(m!)^2\,\Gamma(x+m+1)\,\Gamma(-x+m+1)}\right] \times \frac{\cos\pi x}{x^{2p}}\,dx$$

$$= \tfrac{1}{4}a^{2r}\frac{\Gamma(r+\frac{1}{2})}{\Gamma(r+1)\,\Gamma(\frac{1}{2})}\,{}_2F_1(r+\tfrac{1}{2}.\quad 1;\ r+1.;\ a^2), \quad p=0,$$

$$= 0, \qquad (p=1,\ 2,\ 3,\ \ldots\ 2r).$$

$$(3) \qquad \int_0^\infty \{J_x(ax)\,J_{-x}(ax)\cos\pi x - 1\}\,\frac{dx}{x^2} = -\tfrac{1}{2}\pi^2.$$

Equations (1) and (3) reduce to well-known results when $a=0$ and the same is true for (2) provided that we divide through by a^{2r} first*.

The three results are all proved in the same way and so I shall give the proof of (1) only.

PROOF. It is known that†

$$J_x(ax)\,J_{-x}(ax) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}ax)^{2m}(2m)!}{(m!)^2\,\Gamma(m+1+x)\,\Gamma(m+1-x)}$$

$$= \frac{\sin x\pi}{x\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(\frac{1}{2}a)^{2m}}{(m!)^2}\,\frac{x^{2m}}{(x^2-1^2)(x^2-2^2)\ldots(x^2-m^2)}$$

* Equations (1) and (3) were suggested for positive values of $a\,(0<a<1)$ by some theorems of mine in null series (*Proc. London Math. Soc.* (2), 26 (1927), p. 54, Theorems 2 and 3). These theorems can be applied to transform Kapteyn series into integrals, but the proofs are very elaborate. For the proof below I am very much indebted to the referee. Apart from its simplicity it is true for complex values of a in the more extensive region $|a|<1$.

† Watson, *Theory of Bessel Functions*, p. 147 (5).

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$$(4) \quad = \frac{\sin x\pi}{x\pi} \sum_{m=0}^{\infty} \frac{(2m)!\ (\frac{1}{2}a)^{2m}}{(m!)^2}$$

$$+ \frac{\sin x\pi}{x\pi} \sum_{m=0}^{\infty} \frac{(2m)!\ (\frac{1}{2}a)^{2m}}{(m!)^2} \left\{ \frac{x^{2m}}{(x^2-1^2)(x^2-2^2)\dots(x^2-m^2)} - 1 \right\}$$

The integrals

$$(5) \quad \int_0^{\infty} \frac{\sin x\pi \cos x\pi}{x} \left\{ \frac{x^{2m}}{(x^2-1^2)\dots(x^2-m^2)} - 1 \right\} dx$$

$$= \tfrac{1}{4} \int_{-\infty}^{\infty} \frac{\sin 2x\pi}{x} \left\{ \frac{x^{2m}}{(x^2-1^2)\dots(x^2-m^2)} - 1 \right\} dx$$

are evidently absolutely convergent and so it follows from a well-known theorem* that when $|a|<1$ we may multiply (4) by $\cos x\pi$ and integrate the right-hand side term by term.

Now from the theory of partial fractions we may write

$$\frac{1}{x} \left\{ \frac{x^{2m}}{(x^2-1^2)\dots(x^2-m^2)} - 1 \right\} = \sum_{r=-m}^{m} \frac{A_r}{x+r},$$

where the A_r's are independent of x, and on multiplying through by x and letting x tend to infinity we see that

$$\sum_{r=-m}^{m} A_r = 0.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin 2x\pi}{x} \left\{ \frac{x^{2m}}{(x^2-1^2)\dots(x^2-m^2)} - 1 \right\} dx$$

$$= \sum_{r=-m}^{m}{}' A_r \int_{-\infty}^{\infty} \frac{\sin 2x\pi}{x+r}\, dx$$

$$= \pi \sum_{r=-m}^{m} A_r$$

$$= 0.$$

On applying this to (4) we get

$$\int_0^{\infty} J_x(ax)\, J_{-x}(ax) \cos \pi x\, dx$$

$$= \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{(2m)!\ (\frac{1}{2}a)^{2m}}{(m!)^2} \int_0^{\infty} \frac{\sin 2x\pi}{x}\, dx$$

$$= \frac{1}{4(1-a^2)^{\frac{1}{2}}}.$$

* Bromwich, *Theory of infinite series*, 2nd Ed., p. 500, Theorem B.

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