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Dynamical approximation and kernels of non-escaping hyperbolic components

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Abstract. Let $\mathscr{F}_n$ be families of entire functions, holomorphically parameterized by a complex manifold M. We consider those parameters in M that correspond to *non-escaping hyperbolic* functions, i.e. those maps $f \in \mathscr{F}_n$ for which the postsingular set $P(f)$ is a compact subset of the Fatou set $\mathcal{F}(f)$ of f. We prove that if $\mathscr{F}_n \to \mathscr{F}_\infty$ in the sense of a certain dynamically sensible metric, then every non-escaping hyperbolic component in the parameter space of $\mathscr{F}_\infty$ is a *kernel* of a sequence of non-escaping hyperbolic components in the parameter spaces of $\mathscr{F}_n$. Parameters belonging to such a kernel do not always correspond to hyperbolic functions in $\mathscr{F}_\infty$. Nevertheless, we show that these functions must be J-stable. Using quasiconformal equivalences, we are able to construct many examples of families to which our results can be applied.

1. *Introduction*

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. From a dynamical point of view, the complex plane naturally splits into two sets: the Fatou set $\mathcal{F}(f)$ of f, consisting of all points that have a neighborhood in which the iterates f^n of f form a normal family, and its complement the Julia set $\mathcal{J}(f) := \mathbb{C}\backslash\mathcal{F}(f)$. The dynamics of such a map f strongly depends on the iterative behavior of its set of *singular values*. This set, which we will denote by $S(f)$, is defined to be the smallest closed set in $\mathbb{C}$ such that $f : \mathbb{C}\backslash f^{-1}(S(f)) \to \mathbb{C}\backslash S(f)$ is a covering map; the closure of the union of its iterated forward images under f is called the *postsingular set* of f and denoted by $P(f)$. (Basic definitions and notation are revised in §2.) In this article, we study non-escaping hyperbolic functions in the space $\mathrm{Hol}^*(\mathbb{C})$ of all entire functions that are not constant or linear; here, a map $f \in \mathrm{Hol}^*(\mathbb{C})$ is said to be *non-escaping hyperbolic* if $P(f)$ is a compact subset of the Fatou set $\mathcal{F}(f)$. For transcendental entire functions, this is the same as the usual notion of hyperbolicity; in the case of polynomials, it corresponds to hyperbolic maps for which no singular value escapes to ∞ (see §2.2). We want to understand how such maps behave under certain small perturbations. It is known that within the space of all polynomials or the space of all

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transcendental entire maps with finitely many singular values, (non-escaping)hyperbolic functions exhibit particularly simple and stable dynamics [**8, 18**].

The space $\mathrm{Hol}^*(\mathbb{C})$ is naturally equipped with the topology of locally uniform convergence. However, this topology is not convenient for our dynamical considerations since maps that are nearby in the corresponding metric often have completely different dynamics (see e.g. Example 2.7). We introduce a new metric χ_{dyn} on $\mathrm{Hol}^*(\mathbb{C})$ which is dynamically more sensible: two entire maps are close with respect to χ_{dyn} if their locally uniform distance *and* the Hausdorff distance between their sets of singular values are small.

There are certainly many examples of entire maps that converge in the metric χ_{dyn} to a map in $\mathrm{Hol}^*(\mathbb{C})$; probably the most prominent is the approximation of the exponential map $E(z) = e^z$ by the polynomials $P_n(z) = (1 + z/n)^n$. In [**6**], Devaney *et al* investigated the relationship between the families $E_\lambda(z) = \lambda e^z$ and $P_{n,\lambda}(z) = \lambda(1 + z/n)^n$. The result was a nice connection between the respective parameter spaces: the authors proved pointwise convergence of non-escaping hyperbolic components (i.e. connected components of the set of those $\lambda \in \mathbb{C}$ for which E_λ or $P_{d,\lambda}$, respectively, is non-escaping hyperbolic) as well as convergence of certain external rays to curves called 'hairs' in the exponential parameter space (see also [**4**]). One can say that this point of view has provided an important conceptional basis for much subsequent work on the exponential family.

In this article, we embed the underlying approximation idea into a formalized and generalized setup. More precisely, let $\widehat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and denote by $\chi_{\widehat{\mathbb{N}}}$ the standard metric on $\widehat{\mathbb{N}}$, let M be a complex manifold with metric χ_M and define $\chi_{\widehat{\mathbb{N}}\times M} := \chi_{\widehat{\mathbb{N}}} + \chi_M$. Denote by $\mathrm{Hol}_b^*(\mathbb{C})$ the set of all maps in $\mathrm{Hol}^*(\mathbb{C})$ with bounded sets of singular values, so $\mathrm{Hol}_b^*(\mathbb{C})$ consists of polynomials of degree at least two and functions in the Eremenko–Lyubich class $\mathscr{B}$. Our main result addresses kernel convergence of non-escaping hyperbolic components of M; roughly speaking, a *kernel* of a sequence of domains O_n is the maximal domain for which every compact subset lies in all but finitely many O_n.

THEOREM 1.1. *For every $n \in \widehat{\mathbb{N}}$, let $\mathscr{F}_n = \{f_{n,\lambda}\} \subset \mathrm{Hol}_b^*(\mathbb{C})$ be a family of entire functions that depend holomorphically on $\lambda \in M$. Assume that for every n, the singular values of all maps in $\mathscr{F}_n$ are holomorphically parameterized by M, and that $F : \widehat{\mathbb{N}} \times M \to \mathrm{Hol}_b^*(\mathbb{C})$, $(n, \lambda) \mapsto f_{n,\lambda}$ is continuous with respect to the metrics $\chi_{\widehat{\mathbb{N}}\times M}$ and χ_{dyn}.*

If $\widetilde{H}$ is a kernel of a sequence of non-escaping hyperbolic components of $\mathscr{F}_n$, then exactly one of the following statements holds.

(i) *The map $f_\lambda \in \mathscr{F}_\infty$ is not non-escaping hyperbolic for any $\lambda \in \widetilde{H}$.*

(ii) *There is a non-escaping hyperbolic component H_∞ of $\mathscr{F}_\infty$ such that $\widetilde{H} = H_\infty$.*

Remark. The theorem also holds when $\chi_{\widehat{\mathbb{N}}\times M}$ is replaced by a metric on $\widehat{\mathbb{N}} \times M$ which can be written as a continuous image of the metrics on $\widehat{\mathbb{N}}$ and M (see Proposition 2.6).

In [**14**], Krauskopf and Kriete studied kernel convergence of non-escaping hyperbolic components for holomorphic families $\mathscr{F}_n = \{f_{n,\lambda} : \lambda \in \mathbb{C}\}$, $n \in \widehat{\mathbb{N}}$, of entire maps for which the sets of singular values are holomorphically parameterized and uniformly bounded in size by some finite integer. They proved the same conclusions as in Theorem 1.1, provided that $\mathscr{F}_n \to \mathscr{F}_\infty$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$. Theorem 1.1 is a

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natural generalization of their result to entire functions whose set of singular values is (not necessarily finite but) bounded. This turns out to be exactly the set of maps for which the notion of (non-escaping)hyperbolicity makes sense (see §2.2). Hence, Theorem 1.1 applies to a much larger set of functions, e.g. to real functions of sine type which we discuss in §4.2. For the proof, it is essential that all singular values of the approximating families behave in a controlled way; Krauskopf and Kriete achieved this by forcing the approximating functions to have at most as many singular values as the functions in the limit family. (Using classical results in complex analysis, one easily derives from local uniform convergence that the number of singular values must then be equal for all considered functions.) In order to study this more general case of functions with possibly infinitely many singular values, we demand that the occurring functions converge with respect to the metric χ_{dyn}.

The first case in Theorem 1.1 does indeed occur; an example is given in §3. Nevertheless, parameters that belong to a kernel correspond to maps that are J-stable. We define J-stability analogously to the classical definition for rational maps. Roughly speaking, a map is said to be *J-stable* if it is structurally stable on its Julia set, so a J-stable map is topologically conjugate on its Julia set to any map in a sufficiently small neighborhood (see Definition 3.5). Using the same assumptions and notation as in Theorem 1.1, we prove the following result.

THEOREM 1.2. *Let λ belong to a kernel $\widetilde{H}$. Then $f_\lambda \in \mathscr{F}_\infty$ is a J-stable function.*

Given an entire function f together with a sequence (f_n) of entire maps for which $\chi_{\mathrm{dyn}}(f_n, f) \to 0$ when $n \to \infty$, there is a natural way—using quasiconformal equivalence classes—to construct suitable holomorphic families $\mathscr{F}_n \ni f_n$ that satisfy the assumptions of Theorem 1.1; we explain this method precisely in §4.1. For a lot of explicit functions, including many examples that have been of particular dynamical interest in recent decades, non-trivial approximations in the sense of χ_{dyn} are known. However, for an *arbitrary* (transcendental) entire map there seems to be no general concept of how to establish a non-trivial sequence of maps f_n such that $\chi_{\mathrm{dyn}}(f_n, f) \to 0$. This leaves open an interesting function-theoretic question.

Structure of the article. In §2 we introduce basic definitions, notation and preliminary concepts. Section 3 addresses the proofs of Theorems 1.1 and 1.2. In §4 we discuss examples to which our results can be applied.

2. *Preliminary constructions*

We denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. For a subset V of a metric space M, we denote the ε-neighborhood of V by $U_\varepsilon(V)$. We will write $A \Subset B$ if A is a relatively compact subset of B, i.e. if $\overline{A}$ is compact and contained in B. If not stated differently, the boundary ∂A and the closure $\overline{A}$ of a set $A \subset \mathbb{C}$ are always understood to be taken relative to the complex plane. Throughout this article, we will mainly consider non-constant, nonlinear entire functions $f : \mathbb{C} \to \mathbb{C}$; hence, f will either be a polynomial of degree ≥ 2 or a transcendental entire map. Recall that the space of all such maps is denoted by $\mathrm{Hol}^*(\mathbb{C})$.

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2.1. *Some basics in complex dynamics.* We denote by $C(f) := \{w \in \mathbb{C} : w = f(z) \text{ for some } z \text{ with } f'(z) = 0\}$ the set of all critical values and by $A(f)$ the set of all (finite) asymptotic values of f. Recall that a point $w \in \mathbb{C}$ is an asymptotic value of f if there is a path to ∞ along which f converges to w. Note that a polynomial has no asymptotic values. The set $S(f)$ of *singular values* of f is defined to be the smallest closed set in $\mathbb{C}$ such that $f : \mathbb{C}\backslash f^{-1}(S(f)) \to \mathbb{C}\backslash S(f)$ is a covering map. It is well known that $S(f)$ can be written as $S(f) = \overline{C(f) \cup A(f)}$ (see e.g. [**9**, Lemma 1.1]). Recall that

$$\mathrm{Hol}_b^*(\mathbb{C}) := \{f \in \mathrm{Hol}^*(\mathbb{C}) : S(f) \text{ is bounded}\}$$

denotes the set of all non-constant, nonlinear entire maps with bounded singular sets. The *Eremenko–Lyubich class*

$$\mathscr{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental entire} : S(f) \text{ is bounded}\}$$

is known as the set of all transcendental entire maps in $\mathrm{Hol}_b^*(\mathbb{C})$. Finally, we denote the *postsingular set* of f by $P(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}$.

Let $w \in \mathbb{C}$ be a periodic point of f of period k, that is, $f^k(w) = w$ and $k > 0$ is minimal with this property. The set $O(w) := \{w, f(w), \ldots, f^{k-1}(w)\}$ is called the (forward) *orbit* or *cycle* of w. The *multiplier* $\mu(w)$ of w is defined as $\mu(w) := (f^k)'(w)$. We call w *attracting* if $|\mu(w)| < 1$, *repelling* if $|\mu(w)| > 1$ and *indifferent* if $|\mu(w)| = 1$. An attracting periodic point w with $\mu(w) = 0$ is called *superattracting*. If w is a periodic point, then $\mu(f^i(w)) = \mu(w)$ for all i; hence, we can extend the classification of w to the whole cycle $O(w)$ and speak of the multiplier of the cycle etc. We write $\mathrm{Attr}(f)$ for the set of all attracting periodic points of f.

If O is an attracting periodic cycle of f of period n, then the *basin of attraction* or *attracting basin of* O is the open set $A(O)$ consisting of all points z for which the successive iterates $f^n(z)$, $f^{2n}(z)$, ... converge to some point of O. If $w \in O$, then the component $A^*(w)$ of $A(O)$ that contains w is called the *immediate basin of attraction of* w. (The immediate basin of the cycle O is then the union of the immediate basins of the points in O.) We denote the set of all points that converge to an attracting cycle of f by $\mathcal{A}(f)$.

Recall from the introduction that the Fatou and Julia set of a map $f \in \mathrm{Hol}^*(\mathbb{C})$ is denoted by $\mathcal{F}(f)$ and $\mathcal{J}(f)$, respectively.

Remark. If f is a polynomial, then it extends naturally to a holomorphic map $\hat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\hat{f}(\infty) = \infty$ and, in this case, ∞ can be seen as a critical point and a critical value of $\hat{f}$. However, the dynamically relevant sets such as $\mathcal{F}(f)$, $\mathcal{J}(f)$ or $S(f)$ of a map $f \in \mathrm{Hol}^*(\mathbb{C})$ are deliberately defined as subsets of the plane (rather than the sphere) since the consideration of the point at ∞ does not contribute to the understanding of the dynamics: for every transcendental entire map, ∞ is an essential singularity, while, for a polynomial, ∞ is its only preimage.

The *escaping set* of f is defined to be

$$I(f) := \left\{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\right\}.$$

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A point $z \in I(f)$ is called an *escaping point*. For a polynomial $f \in \mathrm{Hol}^*(\mathbb{C})$, the escaping set belongs to $\mathcal{F}(f)$: it can be considered as the basin of attraction of the superattracting fixed point ∞ of $\hat{f}$. In contrast, if $f \in \mathscr{B}$, then $\mathcal{J}(f) = \overline{I(f)}$ [**7**, p. 344].

Since we use the classification of Fatou components frequently in this work, let us recall that every periodic Fatou component of a map $f \in \mathrm{Hol}^*(\mathbb{C})$ is an immediate attracting basin, an immediate parabolic basin, a Siegel disk or a Baker domain. A Baker domain consists of points that converge to ∞ and, by [**7**, p. 344], it cannot occur for a map in $\mathscr{B}$. In particular, every Fatou component of a map in $\mathrm{Hol}_b^*(\mathbb{C})$ is simply-connected. By Sullivan's theorem [**19**, Theorem 16.4], polynomials have no wandering domains, i.e. Fatou components for which no forward image is periodic. Transcendental entire maps can have wandering domains, but it is an open question whether they can occur for functions in the class $\mathscr{B}$.

There is a close relationship between Fatou components and (post)singular values of a function $f \in \mathrm{Hol}^*(\mathbb{C})$: every periodic cycle of immediate attracting or parabolic basins intersects $S(f)$. (In fact, the intersection with $S(f)$ contains a point with infinite orbit if the basins are not superattracting.) Furthermore, the boundary of a Siegel disk is contained in $P(f)$. Finally, if U is a wandering domain of f, then every limit function of $\{f^n|_U\}$ is constant and contained in the set of accumulation points of $P(f)$ or it is ∞.

These rather classical results (and their proofs) can be found in [**2**, Theorem 7], [**1**, Lemma 2.1] and [**3**]. For more background in holomorphic dynamics, we refer the reader to the expository works [**2, 19**].

2.2. *Non-escaping hyperbolic maps.* Let us start with the definitions of hyperbolic and non-escaping hyperbolic functions.

Definition 2.1. A map $f \in \mathrm{Hol}^*(\mathbb{C})$ is called *hyperbolic* if $S(f)$ is bounded and $P(f) \subset \mathcal{F}(f)$. We say that f is *non-escaping hyperbolic* if f is hyperbolic and $S(f) \cap I(f) = \emptyset$.

By definition, every hyperbolic map is contained in $\mathrm{Hol}_b^*(\mathbb{C})$; a hyperbolic transcendental entire map also lies in the class $\mathscr{B}$. Clearly, every non-escaping hyperbolic map is also hyperbolic. In fact, for transcendental entire maps both terms are equivalent since the escaping set of a map in the class $\mathscr{B}$ belongs to the Julia set. Recall that $S(f)$ is closed by definition; hence, if f is hyperbolic, then $S(f)$ is a compact subset of $\mathcal{F}(f)$.

The next observation will be used frequently; it follows mainly from classical results in holomorphic dynamics but we could not locate a reference; hence, we include a proof for completeness.

PROPOSITION 2.2. *For $f \in \mathrm{Hol}^*(\mathbb{C})$, the following statements are equivalent:*

(i) *f is non-escaping hyperbolic;*
(ii) *$S(f)$ is a bounded subset of $\mathcal{A}(f)$;*
(iii) *$P(f)$ is a bounded subset of $\mathcal{F}(f)$.*

If f is non-escaping hyperbolic, then $\mathrm{Attr}(f)$ *is finite and $\mathcal{F}(f) \setminus I(f) = \mathcal{A}(f)$.*

For $f \in \mathrm{Hol}^*(\mathbb{C})$, the set of singular values is non-empty, so if f is hyperbolic then $\mathcal{F}(f) \neq \emptyset$. Also, recall that $\mathcal{F}(f) = \mathcal{F}(f) \setminus I(f)$ if $f \in \mathscr{B}$.

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Proof. The statement is well known for polynomials [**19**, Theorem 19.1]; hence, we can restrict to transcendental entire maps.

(i) $\Rightarrow$ (ii). Assume that $S(f)$ is bounded and $P(f) \subset \mathcal{F}(f)$, and let U be a component of $\mathcal{F}(f)$. If some image of U, say $\widetilde{U}$, is periodic, then $\widetilde{U}$ is either an immediate attracting or parabolic basin, or a Siegel disk. Since $P(f) \subset \mathcal{F}(f)$, it follows from the results discussed at the end of §2.1 that $\widetilde{U}$ is neither a Siegel disk nor an immediate parabolic basin; hence, U must be some (preimage of an) immediate attracting basin. It also follows that U cannot be a wandering domain, since the limit functions on such a domain would be contained in $\mathcal{J}(f) \cap P(f)$, which is empty. Hence, $\mathcal{F}(f) = \mathcal{A}(f)$.

(ii) $\Rightarrow$ (iii). Assume that $S(f)$ is a bounded subset of $\mathcal{A}(f)$. Then the components of $\mathcal{A}(f)$ form an open cover of $S(f)$ and, since $S(f)$ is compact, there is a finite subcover. Hence, there exist finitely many cycles in $\mathrm{Attr}(f)$ that attract the orbits of all singular values, implying that $P(f)$ is a bounded subset of $\mathcal{F}(f)$, and in fact of $\mathcal{A}(f)$.

The direction (iii) $\Rightarrow$ (i) is obvious.

Now let f be non-escaping hyperbolic. We have shown that only finitely many attracting cycles contain a singular value in their basins and, since every attracting cycle must intersect $S(f)$ [**2**, Theorem 7], it follows that $\mathrm{Attr}(f)$ is finite. □

Finally, we would like to remark that an entire map for which every singular value is absorbed by an attracting cycle does not necessarily have dynamics similar to a hyperbolic function. The requirement in Definition 2.1 that $S(f)$ is a bounded set is indeed crucial for Proposition 2.2 to hold: as [**13**, Example D] shows, there is a transcendental entire map f with wandering domains for which all singular values are mapped by f to an attracting fixed point.

2.3. *Hausdorff and kernel convergence.* Let M be a metric space. Recall that the *Hausdorff distance* between two compact sets $A, B \subset M$ is defined to be

$$d_H(A, B) := \inf\{\varepsilon > 0 : A \subset U_\varepsilon(B),\ B \subset U_\varepsilon(A)\}.$$

For studying convergence of domains (i.e. open connected sets) in M, we will define a concept analogous to the *Carathéodory* or *kernel convergence* for domains in the complex plane. For more details, see [**10**, §5].

Definition 2.3. (Kernel) Let $o \in M$ and let $O_n \subset M$, $n \in \mathbb{N}$, be domains containing o. The *kernel* of the sequence (O_n) (with respect to o) is the largest domain $O \ni o$ such that each compact set $K \subset O$ is contained in all but finitely many O_n.

We call the point o the *marked point* of the sequence (O_n). Clearly, the existence of a kernel is equivalent to the existence of a neighborhood of o which is contained in all but finitely many O_n.

We say that the sequence (O_n) *converges* to O (*as kernels*) and write $O_n \to O$ if O is a kernel of each subsequence of (O_n). The right-hand example in Figure 1 shows a sequence that does not converge to its kernel.

Observe that a sequence (O_n) can have more than one kernel (see the left-hand example in Figure 1), and each of them is specified by the choice of a marked point. Now let M be a locally compact metric space, e.g. an analytic manifold, and let O be a kernel of O_n.

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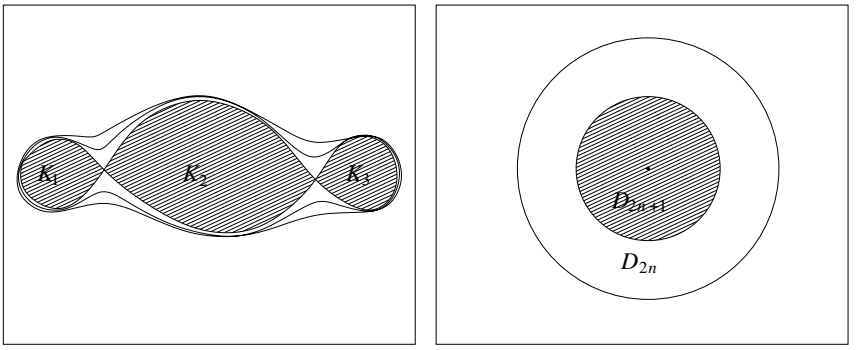


FIGURE 1. *Left*: the interiors of the curves are simply-connected domains with three different kernels K_1, K_2 and K_3. *Right*: the domains $D_n := \{z : |z| < 2 - r_n$, where $n \equiv r_n \bmod 2$, $r_n \in \{0, 1\}\}$ have a unique kernel D_1 but they do not converge to it.

Since, by definition, every kernel is open and since M is locally compact, every point in O has a compact neighborhood which is contained in O. This means that we can choose any point $o \in O$ to be the marked point. Hence, we will talk about the sets O_n and O without mentioning the marked point if it is implicit from the context which point is meant.

We have the following relation between Hausdorff and kernel convergence.

PROPOSITION 2.4. *Let K_n, K be non-empty compact subsets of a locally compact metric space M. Then $d_H(K_n, K) \to 0$ as $n \to \infty$ if and only if the following two conditions hold:*

- *every component O of $K^c := M \setminus K$ is a kernel of a sequence of components O_n of $K_n^c := M \setminus K_n$;*
- *every kernel of an infinite sequence (O_{n_k}) of components of $K_{n_k}^c$ is a component of K^c.*

We will omit the proof since it is elementary and follows mainly from the definitions of kernel and Hausdorff convergence, and since we do not require the statement for any proof in this article; its purpose is more the illustration of how the given concepts of convergence relate to each other. The right-hand example in Figure 1 indicates the necessity of the second requirement in Proposition 2.4.

2.4. *Dynamical approximation.* Let $\mathrm{Hol}(\mathbb{C})$ be the space of all entire functions. We denote the *locally uniform distance* between $f, g \in \mathrm{Hol}(\mathbb{C})$ by $\chi_{\mathrm{luc}}(f, g)$. The metric $\chi_{\mathrm{luc}}(f, g)$ induces the topology of locally uniform convergence on $\mathrm{Hol}(\mathbb{C})$, so we say that f_n converge to f *locally uniformly* if and only if $\chi_{\mathrm{luc}}(f_n, f) \to 0$ as $n \to \infty$. (For more details, see e.g. [**19**, Ch. 3].) It follows from the Weierstraß approximation theorem [**19**, Theorem 1.4] that the space $\mathrm{Hol}(\mathbb{C})$ is closed with respect to this topology.

For entire maps with a non-empty set of singular values, we introduce a new metric which combines locally uniform convergence with controlled behavior on the set of singular values. Hence, this metric will be more convenient for the study of dynamics of entire functions. Recall that $\mathrm{Hol}^*(\mathbb{C})$ denotes the set of all entire functions which are not constant or linear.

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Definition and Proposition 2.5. The map $\chi_{\mathrm{dyn}} : \mathrm{Hol}^*(\mathbb{C}) \to [0, \infty)$ with

$$\chi_{\mathrm{dyn}}(f, g) := \chi_{\mathrm{luc}}(f, g) + d_H(S(f), S(g))$$

is a metric, where $d_H(S(f), S(g))$ is measured with respect to the spherical metric.

We will say that the sequence (f_n) approximates f *dynamically* if $\chi_{\mathrm{dyn}}(f_n, f) \to 0$ as $n \to \infty$.

Clearly, there are many ways to construct a metric which depends not only on local uniform convergence but also on the Hausdorff distance of the sets of singular values. However, the following observation, which follows from standard calculus, justifies our somewhat 'rigid' definition.

PROPOSITION 2.6. *Let (A, χ_A) and (B, χ_B) be metric spaces, and let $f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous map such that $f(\chi_A, \chi_B)$ defines a metric on $A \times B$. Then the identity map*

$$\mathrm{id} : (A \times B, \chi_A + \chi_B) \to (A \times B, f(\chi_A, \chi_B))$$

is continuous.

Maps which are close in the metric χ_{luc} do not necessarily have the property that their sets of singular values are close in the Hausdorff metric. This does not have to be true even in the case of a family of functions which depends holomorphically on some parameter λ, as the following example shows.

Example 2.7. Let $f_\lambda(z) = e^{-\lambda z^2 + z - 2}$ with $\lambda \in \mathbb{C}$. The map $f_0(z) = e^{z-2}$ has 0 as its only singular value. Now let $\lambda \neq 0$ be any complex number. Then, apart from the asymptotic value 0, the map f_λ has the additional critical value $v_\lambda := e^{1/(4\lambda)-2}$. Clearly, $v_\lambda \to +\infty$ when $\lambda \searrow 0$.

It is well known that if (f_n) is a sequence of entire maps such that $\chi_{\mathrm{luc}}(f_n, f) \to 0$, then for every $w \in S(f)$ there is a sequence $\{w_n : w_n \in S(f_n) \text{ for every } n\}$ such that $w_n \to w$ (see e.g. [**12**, Theorem 2]), yielding lower semi-continuity for the sets of singular values in case of locally uniform convergence. Hence, convergence in the metric χ_{dyn} makes in particular sure that there are no sequences of singular values of the approximating maps f_n that accumulate outside $S(f)$. Furthermore, by using the metric χ_{dyn} we exclude the following pathological case.

PROPOSITION 2.8. *Let f_n, f be entire maps such that $\chi_{\mathrm{dyn}}(f_n, f) \to 0$ as $n \to \infty$. If f is a polynomial, then the same holds for all but finitely many f_n.*

Proof. Let f be a polynomial and assume that there is an infinite sequence (f_n) of transcendental entire maps that converge to f in the metric χ_{dyn}. If there exists a constant $R > 0$ such that the Euclidean disk $U_R(0)$ contains all the sets $S(f_n)$, then, for every n, there is a curve Γ_n to ∞ such that $|f_n(z)| = R$ for all $z \in \Gamma_n$. (One can choose Γ_n to be a suitable component of $f_n^{-1}(\partial U_R(0))$.) But this is impossible since f_n converge to f; hence, there must be a sequence $(s_{n_k}) \subset \mathbb{C}$ such that $s_{n_k} \in S(f_{n_k})$ and $s_{n_k} \to \infty$. This yields a contradiction since $S(f)$ is a bounded subset of $\mathbb{C}$. □

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Another reason why the metric of locally uniform convergence is not suitable for our considerations is the fact that (non-escaping) hyperbolicity is not an open property in this topology. For instance, let $f_\lambda(z) := e^{\lambda z} \cdot z^2$. Then $f_0(z) = z^2$ is non-escaping hyperbolic while, for any sufficiently small $\lambda > 0$, the critical value $4/(e^2\lambda^2)$ escapes to ∞. However, the set of non-escaping hyperbolic entire maps is open in the topology induced by the metric χ_{dyn}.

THEOREM 2.9. (Non-escaping hyperbolicity is an open property) *The set*

$$\mathcal{H} := \{f \in \mathrm{Hol}^*(\mathbb{C}) : f \text{ is non-escaping hyperbolic}\}$$

is open in the topology induced by the metric χ_{dyn}.

Note that $\mathcal{H} \subset \mathrm{Hol}_b^*(\mathbb{C})$. Theorem 2.9 will follow from the following lemma.

LEMMA 2.10. *Let* $f \in \mathrm{Hol}^*(\mathbb{C})$ *and let* $K \subset \mathcal{A}(f)$ *be a compact set. Then* $K \subset \mathcal{A}(g)$ *for all* $g \in \mathrm{Hol}^*(\mathbb{C})$ *that are sufficiently close to* f *in the metric* χ_{luc}.

Proof. The components of $\mathcal{A}(f)$ form an open cover of the compact set K; hence, there is a finite subcover. We can assume without loss of generality that K is contained in the basin of attraction $A(z_0)$ of a fixed point $z_0 \in \mathbb{C}$ of f, since otherwise we can repeat the argument for every attracting periodic point of f. There exists a bounded open set $U \ni z_0$ such that $f(U) \Subset U$. By definition of χ_{luc}, $g(U) \Subset U$ holds for all entire maps g for which $\chi_{\mathrm{luc}}(f, g)$ is sufficiently small. By Montel's theorem, $\{g^k\}_{k\in\mathbb{N}}$ is normal in U and, by the contraction mapping theorem, g has a fixed point in U which is necessarily attracting. Since U is bounded, it follows that $U \subset \mathcal{A}(g)$.

There exists $N \in \mathbb{N}$ such that $f^N(K) \subset U$, so again, by locally uniform convergence, $g^N(K) \subset U \subset \mathcal{A}(g)$ if $\chi_{\mathrm{luc}}(f, g)$ is sufficiently small. The claim now follows from the complete invariance of $\mathcal{A}(g)$ under the map g. □

Proof of Theorem 2.9. Let f be non-escaping hyperbolic; hence, $S(f)$ is bounded and contained in $\mathcal{A}(f)$. Choose $\delta > 0$ sufficiently small such that $K := \overline{U_\delta(S(f))} \subset \mathcal{A}(f)$. Since K is compact, it follows from Lemma 2.10 that there is a constant $\varepsilon > 0$ such that $K \subset \mathcal{A}(g)$ for all g with $\chi_{\mathrm{luc}}(f, g) < 2\varepsilon$. Now choose $\eta = \min\{\varepsilon, \delta\}$. Then, for all maps g with $\chi_{\mathrm{dyn}}(f, g) < \eta$, we obtain

$$S(g) \subset U_\delta(S(f)) \Subset \mathcal{A}(g);$$

hence, g is non-escaping hyperbolic. □

3. *Stability of non-escaping hyperbolic parameters*

Recall that our goal is to prove that under certain conditions, a kernel of a sequence of non-escaping hyperbolic components equals a non-escaping hyperbolic component of the limit family. As we will see, it is not hard to show that every such component of $\mathscr{F}_\infty$ is contained in a kernel of a sequence of non-escaping hyperbolic components of $\mathscr{F}_n$. This statement requires even fewer restrictions than those stated in Theorem 1.1. For the other inclusion to hold, we have to construct a more sensible setup.

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We denote by $\widehat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ the one-point compactification of the set of natural numbers. The metric $\chi_{\widehat{\mathbb{N}}}$ on $\widehat{\mathbb{N}}$ defined by $\chi_{\widehat{\mathbb{N}}}(m, n) := 2^{-\min(m,n)-1}$ when $m \neq n$ (and equal to zero otherwise) makes $\widehat{\mathbb{N}}$ a complete metric space. From now on, we assume that M is a complex manifold with a metric χ_M. The relation

$$\chi_{\widehat{\mathbb{N}}\times M}((m, \lambda), (n, \nu)) := \chi_{\widehat{\mathbb{N}}}(m, n) + \chi_M(\lambda, \nu)$$

then defines a metric $\chi_{\widehat{\mathbb{N}}\times M}$ on $\widehat{\mathbb{N}} \times M$.

For every $n \in \widehat{\mathbb{N}}$, let $\mathscr{F}_n = \{f_{n,\lambda} : \mathbb{C} \to \mathbb{C} : \lambda \in M\} \subset \mathrm{Hol}_b^*(\mathbb{C})$ be a family of functions parameterized by M. To simplify the notation, we will skip the first index entry for maps in $\mathscr{F}_\infty$, i.e. we will write f_λ for $f_{\infty,\lambda} \in \mathscr{F}_\infty$.

We want all families $\mathscr{F}_n$, $n \in \widehat{\mathbb{N}}$, to satisfy the following.

Dynamical standing assumption ('dsa'). The map

$$F : \widehat{\mathbb{N}} \times M \to \mathrm{Hol}_b^*(\mathbb{C}), \quad (n, \lambda) \mapsto f_{n,\lambda}$$

is continuous with respect to the metrics $\chi_{\widehat{\mathbb{N}}\times M}$ and χ_{dyn}.

The key feature of 'dsa' is the local uniformity in λ *and* n: let $f_{\lambda_0} \in \mathscr{F}_\infty$ and let $K \subset \mathbb{C}$ be a compact set. Then for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $|f_{n,\lambda}(z) - f_{\lambda_0}(z)| < \varepsilon$ for all $z \in K$, $\lambda \in U_\delta(\lambda_0)$ and $n \geq n_0$.

Remark. By Proposition 2.6, one can replace $\chi_{\widehat{\mathbb{N}}\times M}$ by any metric which can be written as a continuous image of the metrics $\chi_{\widehat{\mathbb{N}}}$ and χ_M without violating 'dsa'.

Notation. For every $n \in \widehat{\mathbb{N}}$, we denote by

$$\mathcal{H}(\mathscr{F}_n) := \{\lambda \in M : f_{n,\lambda} \in \mathscr{F}_n \text{ is non-escaping hyperbolic}\}$$

the parameters corresponding to non-escaping hyperbolic maps in the respective family. We will usually denote a component of $\mathcal{H}(\mathscr{F}_n)$ by H_n, a component of $\mathcal{H}(\mathscr{F}_\infty)$ by H_∞ and a kernel of a sequence (H_n) by $\widetilde{H}$.

PROPOSITION 3.1. *Let H_∞ be a component of $\mathcal{H}(\mathscr{F}_\infty)$. Then there exists a kernel $\widetilde{H}$ of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$ such that $H_\infty \subset \widetilde{H}$.*

Proof. Let $\lambda_0 \in \mathcal{H}(\mathscr{F}_\infty)$. It follows from Theorem 2.9 and the dynamical standing assumption that there exists a neighborhood $U(\lambda_0) \subset M$ such that $U(\lambda_0) \subset \mathcal{H}(\mathscr{F}_n)$ for all sufficiently large $n \in \mathbb{N}$. □

To prove the opposite inclusion, we have to make additional restrictions. Our requirements, that we will assume from now on, are formalized in the following way.

Holomorphic standing assumption ('hsa').

(i) For every $n \in \widehat{\mathbb{N}}$, the maps $f_{n,\lambda}$ depend holomorphically on $\lambda \in M$.

(ii) Let $n \in \mathbb{N}$ and $\lambda_0 \in M$. Then the singular values of f_{n,λ_0} are holomorphically parameterized by M, i.e. for each singular value s of f_{n,λ_0} there exists a holomorphic map $w_n : M \to \mathbb{C}$, $\lambda \mapsto w_n(\lambda)$ such that $w_n(\lambda_0) = s$ and $w_n(\lambda) \in S(f_{n,\lambda})$.

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Note that the second condition of 'hsa' does not imply that a parameterization of $S(f_{n,\lambda_0})$ for some λ_0 needs to be an exhaustion of the set of singular values for another parameter $\lambda \neq \lambda_0$. *A priori*, it is possible that $S(f_{n,\lambda_0}) = \{w_n^i(\lambda_0)\}_{i\in I}$ for some index set I but $\{w_n^i(\lambda)\}_{i\in I} \subsetneq S(f_{n,\lambda})$ for some $\lambda \neq \lambda_0$.

Also note that we do not assume condition (ii) to be satisfied by the maps in $\mathscr{F}_\infty$. The reason is that we only need a *local* holomorphic parameterization of the sets of singular values for maps in $\mathscr{F}_\infty$ and, as the next statement shows, this follows from the assumptions we already made.

THEOREM 3.2. *Let $\widetilde{H}$ be a kernel of a sequence (H_n) of non-escaping hyperbolic components, let $U \ni \lambda_0$ be a simply-connected neighborhood of λ_0 with $U \Subset \widetilde{H}$ and let s be a singular value of f_{λ_0}.*

Then there exists a holomorphic map $w : U \to \mathbb{C}$ such that $w(\lambda) \in S(f_\lambda)$, $w(\lambda_0) = s$ and the family $\{f_\lambda^n(w(\lambda))\}_{n\in\mathbb{N}}$ is normal in U.

Proof. Let $f_{\lambda_0} \in \mathscr{F}_\infty$. By [**2**, Theorem 5] and [**19**, Theorem 13.1], the map f_{λ_0} has infinitely many repelling periodic points (in its Julia set). Let us pick two such points and denote them by $p(\lambda_0)$ and $q(\lambda_0)$; let n_1 and n_2 be their periods. If D is a disk at $p(\lambda_0)$ such that $p(\lambda_0)$ is the only periodic point of f_{λ_0} of period $\leq n_1$ in $\overline{D}$, then it follows from 'dsa' and Rouché's theorem that there are a neighborhood $U(\lambda_0)$ of λ_0 and an integer $n_0 \geq 0$ such that for every $\lambda \in U(\lambda_0)$ and every $n \geq n_0$, the map $f_{n,\lambda}$ has exactly one periodic point $p_n(\lambda)$ of period n_1 in D and no other periodic point of period $\leq n_1$ in $\overline{D}$. By the Cauchy integral formula (after decreasing the initial disk D, if necessary), every such periodic point $p_n(\lambda)$ must be repelling. By the implicit function theorem, every one of these points can be analytically continued as a repelling periodic point of period n_1 in a sufficiently small neighborhood. The previous observation then implies that for every sufficiently large $n \in \widehat{\mathbb{N}}$, there exists an analytic function $p_n : U(\lambda_0) \to \mathbb{C}$ such that $p_n(\lambda) \in D$ is a repelling periodic point of $f_{n,\lambda}$ of period n_1. By construction, $p_n(\lambda) \to p(\lambda)$ when $n \to \infty$. We can repeat the same procedure for $q(\lambda_0)$ and obtain holomorphic maps $q_n : U'(\lambda_0) \to \mathbb{C}$.

Let us now assume that $\lambda_0 \in \widetilde{H}$. Since $\widetilde{H}$ is open, there is a disk $B \subset \widetilde{H}$ centered at λ_0 such that the maps p_n, q_n are defined and holomorphic in B and their images are repelling periodic points of the corresponding maps (and periods). Let $U \supset B$ be a simply-connected bounded domain with closure in $\widetilde{H}$. Since $\widetilde{H}$ is a kernel of a sequence H_n of non-escaping hyperbolic components, the compact set $\overline{U}$ is contained in all H_n for $n \in \mathbb{N}$ chosen sufficiently large. Hence, the maps p_n and q_n ($n \in \mathbb{N}$) can be holomorphically continued to all of U, since otherwise the implicit function theorem would imply that for some $\tilde{\lambda} \in U$, the map $f_{n,\tilde{\lambda}}$ has an indifferent periodic point.

Let us now consider the maps $\Phi_\lambda(z) = (z - p(\lambda))/(q(\lambda) - p(\lambda))$ and $\Phi_{n,\lambda}(z) = (z - p_n(\lambda))/(q_n(\lambda) - p_n(\lambda))$. Conjugating f_λ with Φ_λ and $f_{n,\lambda}$ with $\Phi_{n,\lambda}$, we obtain conformal conjugates such that the points 0 and 1 correspond to our previously considered repelling periodic points. Hence, we can assume that $p(\lambda) = 0$, $q(\lambda) = 1$ for all $\lambda \in B$ and $p_n(\lambda) = 0$, $q_n(\lambda) = 1$ for all $\lambda \in U$. In particular,

$$\{0, 1\} \subset \mathcal{J}(f_\lambda), \mathcal{J}(f_{n,\lambda}) \quad \text{and} \quad S(f_{n,\lambda}) \subset \mathbb{C}\backslash\{0, 1\}$$

for all sufficiently large integers $n \in \widehat{\mathbb{N}}$ and the corresponding values of λ.

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Let s be a singular value of f_{λ_0}. By 'dsa', there is a sequence of singular values s_n of the maps f_{n,λ_0} that converges to s. Due to 'hsa', there are holomorphic maps w_n such that $w_n(\lambda_0) = s_n$ and $w_n(\lambda) \in S(f_{n,\lambda})$ for all $\lambda \in M$. By the previous argument, we have that $w_n(U) \subset \mathbb{C}\backslash\{0, 1\}$; hence, by Montel's theorem, $\{w_n\}_{n\in\mathbb{N}}$ is a normal family on U. Let (w_{n_k}) be a convergent subsequence of (w_n), and let w be a limit function which is necessarily holomorphic. By construction, we have that $w(\lambda_0) = s$, and 'dsa' implies that $w(\lambda) \in S(f_\lambda)$ holds for all $\lambda \in U$. Hence, w is a holomorphic parameterization of the singular value s on U.

Consider now for a fixed ν the family $\{f^\nu_{n,\lambda}(w_n(\lambda))\}_{n\in\mathbb{N}}$ with $\lambda \in U$. Since the Fatou set of an entire map is completely invariant, $w_n(\lambda) \in \mathcal{F}(f_{n,\lambda})$ implies that $f^\nu_{n,\lambda}(w_n(\lambda)) \subset \mathcal{F}(f_{n,\lambda}) \subset \mathbb{C}\backslash\{0, 1\}$. Applying Montel's theorem, it follows that for each ν the above family is normal in U. For simplicity, denote its limit by $S_\nu : U \to \widehat{\mathbb{C}}$, $\lambda \mapsto S_\nu(\lambda) := \lim_{n\to\infty} f^\nu_{n,\lambda}(w_n(\lambda))$. It follows from the local uniform convergence of the maps f_n and w_n that

$$S_\nu(\lambda) = \lim_{n\to\infty} \lim_{m\to\infty} f^\nu_{n,\lambda}(w_m(\lambda)) = \lim_{n\to\infty} f^\nu_{n,\lambda}(w(\lambda)) = f^\nu_\lambda(w(\lambda)).$$

By Hurwitz's theorem, either $S_\nu \subset \mathbb{C}\backslash\{0, 1\}$ or $S_\nu \equiv 0$ or 1. Recall that 0 and 1 are periodic points of f_λ; hence, if $S_\nu \equiv 0$ (respectively 1), then there exists some $m \in \mathbb{N}$ such that $S_{\nu+km} \equiv 0$ (respectively 1) for all $k \in \mathbb{N}$. Applying Montel's theorem once more, we obtain that $\{S_\nu\}_{\nu\in\mathbb{N}}$ is a normal family on U. □

We can now prove the remaining statement, which will then imply Theorem 1.1.

THEOREM 3.3. *Let $\widetilde{H}$ be a kernel of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$. If there is a component H_∞ of $\mathcal{H}(\mathscr{F}_\infty)$ such that $\widetilde{H} \cap H_\infty \neq \emptyset$, then $\widetilde{H} \subset H_\infty$.*

Proof. We will prove the statement by contradiction, so assume that there exists some $\lambda_0 \in \widetilde{H} \cap \partial H_\infty$. With the same notation as in the previous proof, it follows that $S_\nu(B) \subset \mathbb{C}\backslash\{0, 1\}$ for all $\nu \in \mathbb{N}$, since $B' := B \cap H_\infty \neq \emptyset$ (and hence $S_\nu|_B$ cannot be constant 0 or 1).

For a limit function S of S_ν, we have that either $S \equiv c \in \widehat{\mathbb{C}}$, in which case $S \equiv \infty$, or S is a non-constant function with $S(B) \subset \mathbb{C}\backslash\{0, 1\}$.

The case $S \equiv \infty$ clearly cannot occur since this would mean that some singular value converges to ∞ for all parameters in B but the non-empty subset B' of B consists of non-escaping hyperbolic parameters.

Let $S_\nu(B) \subset \mathbb{C}\backslash\{0, 1\}$. For all $\lambda \in B'$, there is a holomorphic map $a : B' \to \mathbb{C}$ such that $a(\lambda)$ is an attracting point of f_λ which attracts $w(\lambda)$. Since $B \supset B'$ is simply-connected, the point $a(\lambda)$ can then be analytically continued to an attracting point on the entire domain B. The image $a(B)$ is bounded; hence, for every $\lambda \in B$ the singular value $w(\lambda)$ is attracted by a finite attracting periodic cycle of f_λ. Since $s = w(\lambda_0)$ was assumed to be an arbitrary singular value of f_{λ_0}, we can repeat this procedure for any of the singular values of f_{λ_0}. Recall that, by assumption, $\mathscr{F}_\infty \subset \mathrm{Hol}^*_b(\mathbb{C})$; hence, the set of singular values is bounded. This implies that $B \ni \lambda_0$ is contained in some non-escaping hyperbolic component H_∞ of $\mathscr{F}_\infty$, contradicting the assumption that $\lambda_0 \in \widetilde{H} \cap \partial H_\infty$. □

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Without the assumption of dynamical approximation (which is part of 'dsa'), we cannot expect that a kernel $\widetilde{H}$ is always a non-escaping hyperbolic component of the family $\mathscr{F}_\infty$. It is easy to find suitable examples. One such example was given in [**15**, Example 5]: the authors approximated a holomorphic family of quadratic polynomials by families of polynomials of degree four, such that a kernel of a sequence of non-escaping hyperbolic components was a proper subset of some non-escaping hyperbolic component of the limit family.

Here we give an example which respects our standing assumptions, showing that the case $\widetilde{H} \cap \mathcal{H}(\mathscr{F}_\infty) = \emptyset$ in Theorem 1.1 does indeed occur.

Example 3.4. Let

$$P_{\lambda,n}(z) = z^3 - \left(\frac{\lambda - 1}{\sqrt{\mu_n + \lambda - 2}} + \sqrt{\mu_n + \lambda - 2} \right) z^2 + \lambda z, \tag{1}$$

where $\lambda \in \mathbb{C} \setminus [1, 5]$, $|\mu_n| < 1$ does not depend on λ and $\mu_n \to 1$ as $n \to \infty$ (for instance, we can choose $\mu_n = n/(n+1)$). Note that $P_{\lambda,n}(z)$ and $P_\lambda(z) = z^3 - 2\sqrt{\lambda - 1}z^2 + \lambda z$ satisfy 'dsa' and 'hsa'. Every $P_{\lambda,n}$ has 0 as a fixed point of multiplier λ. Furthermore, there is a second fixed point $a_n = \sqrt{\mu_n + \lambda - 2}$ with multiplier μ_n. Thus, if we choose $\lambda \in \mathbb{D}$, then every polynomial $P_{\lambda,n}$ is non-escaping hyperbolic. Hence, there is a kernel $\widetilde{H}$ of components of $\mathcal{H}(\mathscr{F}_n)$ which contains the unit disk $\mathbb{D}$. On the other hand, every P_λ has a parabolic fixed point at $a = \sqrt{\lambda - 1}$; hence, $\mathcal{H}(\mathscr{F}_\infty) = \emptyset$.

Nevertheless, the behavior of f_λ is still stable in the sense of *J-stability* for parameters belonging to a kernel. Here, J-stability is defined analogously to the case of rational maps or transcendental entire maps with finitely many singular values (see e.g. [**8**]).

Definition 3.5. Let $\mathscr{F} = \{f_\lambda : \mathbb{C} \to \mathbb{C} : \lambda \in M\}$ be a holomorphic family of entire functions. A map $f_{\lambda_0} \in G$ is said to be *J-stable* if $\mathcal{J}(f_{\lambda_0})$ moves holomorphically in a neighborhood $\Lambda \subset M$ of λ_0, i.e. if there is a holomorphic motion

$$\Phi : \Lambda \times \mathcal{J}(f_{\lambda_0}) \to \mathbb{C}$$

such that

$$\Phi(\lambda, \mathcal{J}(f_{\lambda_0})) = \mathcal{J}(f_\lambda) \quad \text{and} \quad \Phi(\lambda, f_{\lambda_0}(z)) = f_\lambda(\Phi(\lambda, z))$$

for all $z \in \mathcal{J}(f_{\lambda_0})$ and all $\lambda \in \Lambda$.

Φ being a holomorphic motion means that Φ is injective in z when λ is fixed, holomorphic in λ for fixed z and that $\Phi_{\lambda_0} \equiv \mathrm{id}$. For more details, see [**8**, §8].

The map Φ_λ is a conjugacy between f_{λ_0} and f_λ on their Julia sets; hence, it maps periodic points of f_{λ_0} to periodic points of f_λ. Since periodic points form a dense subset of the Julia set, such a conjugacy is unique if it exists.

THEOREM 3.6. *Let $\widetilde{H}$ be a kernel of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$ and let $\lambda \in \widetilde{H}$. Then $f_\lambda \in \mathscr{F}_\infty$ is J-stable.*

Proof. We will prove the contraposition, so let λ_0 be a parameter for which f_{λ_0} is not J-stable and let Λ be a simply-connected bounded neighborhood of λ_0. Then there is some repelling periodic point, say of period n, of f_{λ_0} which has no analytic continuation as a solution of the equation $f_\lambda^n(z) - z = 0$. Otherwise, it would follow from the λ-lemma [**8**]

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that the closure of all repelling points of f_{λ_0}, which equals the Julia set of f_{λ_0}, would move holomorphically, contradicting that f_{λ_0} is not J-stable.

Let $p(\lambda_0)$ be such a repelling periodic point and let $\gamma : [0, 1] \to \Lambda$, $\gamma(0) = \lambda_0$, be a path along which $p(\lambda_0)$ cannot be continued analytically. By the implicit mapping theorem, the point $p(\lambda)$ for $\lambda = \gamma(1)$ must be indifferent. By the minimum modulus principle, we can then find a nearby path $\tilde{\gamma} \subset \Lambda$ connecting λ_0 to some parameter $\tilde{\lambda}$ along which the considered point becomes attracting. Hence, there is a singular value $s(\tilde{\lambda})$ of $f_{\tilde{\lambda}}$ converging to the attracting periodic point $p(\tilde{\lambda})$.

Now let us assume that our assumption is wrong, meaning that λ_0 belongs to some kernel $\widetilde{H}$. We can assume without loss of generality that the neighborhood Λ was chosen sufficiently small such that $\Lambda \Subset \widetilde{H}$. Then Theorem 3.2 implies that there is a holomorphic parameterization w of the singular value $s(\tilde{\lambda})$ such that $\{f_\lambda^n(w(\lambda))\}$ is a normal family on Λ. But then each $w(\lambda)$ converges to an attracting point $p(\lambda)$ of f_λ, in which case $p(\lambda)$ can be continued analytically to an attracting point of f_λ on the whole of Λ, contradicting the fact that $p(\lambda_0)$ is repelling. □

4. *Construction of examples*

As we have seen in the previous section, a sequence of families to which our results apply has to satisfy two primary conditions, which we have formulated as the standing assumptions 'dsa' and 'hsa'. Hence, starting with an entire map $f \in \mathrm{Hol}_b^*(\mathbb{C})$, we can split the problem into the following two:

(i) find a sequence (f_n) of entire functions which approximates f *dynamically*;

(ii) construct *holomorphic* families $\{f_{n,\lambda}\}$ and $\{f_\lambda\}$ using the functions f_n, f.

We will start with the second problem. It turns out that there is a way to find natural holomorphic families for *any* entire function.

4.1. *Holomorphic families inside quasiconformal equivalence classes.*

Recall that M is a complex manifold. Let μ be a k-Beltrami coefficient of $\mathbb{C}$, i.e. $\mu : \mathbb{C} \to \mathbb{C}$ is a measurable function such that $\|\mu\|_\infty \le k < 1$ almost everywhere (a.e.) in $\mathbb{C}$. By the integrability theorem [**16**, Theorem 4.4], there exists a k-quasiconformal homeomorphism $\Psi : \mathbb{C} \to \mathbb{C}$ whose complex dilatation equals μ a.e. in $\mathbb{C}$.

Let $f \in \mathrm{Hol}_b^*(\mathbb{C})$ be an entire map. Then the *pull-back* $f^*\mu$ of μ by f is given by

$$f^*\mu(z) := (\mu \circ f)\frac{\overline{f'(z)}}{f'(z)}.$$

In particular, $\|\mu\|_\infty \le k$ implies that $\|f^*\mu\|_\infty \le k$.

By the integrability theorem, there exists a quasiconformal homeomorphism Φ whose complex dilatation equals $f^*\mu$ a.e. in $\mathbb{C}$. A formal computation yields that the complex dilatation of the map $g = \Psi \circ f \circ \Phi^{-1}$ is 0 a.e., which means that g is a holomorphic map [**16**, Theorem 1.1]. We say that g is *quasiconformally equivalent* to f.

Let $\Lambda \subset M$ be an open connected set and let $\{\Psi_\lambda\}$ be a family of quasiconformal homeomorphisms with uniformly bounded complex dilatations that depend holomorphically on $\lambda \in \Lambda$. The parameterized version of the integrability theorem [**11**, Ch. 4.7] gives the following way to construct such a family, starting from a Beltrami coefficient μ_0.

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Let $(\mu_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of Beltrami coefficients with $\|\mu_\lambda\|_\infty \le k < 1$ that contains μ_0, i.e. $\mu_0 \equiv \mu_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. (One way to embed μ_0 in a holomorphic family of Beltrami coefficients is as follows: let $h : \Lambda \to D_r(0)$ be a holomorphic map, where r is chosen such that $(1+r)\cdot\|\mu_0\|_\infty < 1$ and $h(\lambda_0) = 0$. Then the functions $\mu_\lambda := (1+h(\lambda))\cdot\mu_0$ form a holomorphic family of Beltrami coefficients.) For every λ, let Ψ_λ be a quasiconformal homeomorphism with Beltrami coefficient μ_λ, chosen such that all Ψ_λ have the same parameterization (e.g. all Ψ_λ fix 0, 1 and ∞). Then the integrability theorem implies that $\lambda \mapsto \Psi_\lambda$ is also holomorphic.

As in the previous construction, we obtain a family $\mathscr{F}_\infty = \{f_\lambda = \Psi_\lambda \circ f \circ \Phi_\lambda^{-1}\}_{\lambda\in\Lambda}$ of entire maps. By computing the derivative of the maps in the equation $\Psi_\lambda \circ f = f_\lambda \circ \Phi_\lambda$ with respect to $\overline{\lambda}$, we get that the functions f_λ also depend holomorphically on λ (see e.g. [**5**, Proof of Proposition 13]). Since an entire function f is a covering map when restricted to $\mathbb{C}\backslash f^{-1}(S(f))$, we obtain that $S(f_\lambda) = \Psi_\lambda(S(f))$; hence, the set of singular values moves holomorphically in the family $\mathscr{F}_\infty$. Furthermore, $S(f_\lambda)$ is bounded uniformly in λ since the maximal dilatations of the quasiconformal maps Ψ_λ are uniformly bounded.

Let (f_n) be a sequence of entire functions which converges to f dynamically. In the same manner as above, we define for each n the holomorphic family $\mathscr{F}_n = \{f_{n,\lambda} = \Psi_\lambda \circ f_n \circ \Phi_{n,\lambda}^{-1}\}_{\lambda\in\Lambda}$ by starting with the same family $\{\Psi_\lambda\}$ of quasiconformal homeomorphisms. As before, the set of singular values $S(f_{n,\lambda}) = \Psi_\lambda(S(f_n))$ moves holomorphically. Hence, it remains to show that the map

$$F : \widehat{\mathbb{N}} \times M \to \mathrm{Hol}_b^*(\mathbb{C}), \quad (n, \lambda) \mapsto f_{n,\lambda}$$

is continuous. In other words, let $\lambda_0 \in M$ and let $(n, \lambda) \to (\infty, \lambda_0)$. We have to show that, in this case,

$$\chi_{\mathrm{luc}}(f_{n,\lambda}, f_{\lambda_0}) + d_H(S(f_{n,\lambda}), S(f_{\lambda_0})) \to 0.$$

Clearly, $d_H(S(f_{n,\lambda}), S(f_{\lambda_0})) = d_H(\Psi_\lambda(S(f_n)), \Psi_{\lambda_0}(S(f))) \to 0$ as $(n, \lambda) \to (\infty, \lambda_0)$, since Ψ_λ depends holomorphically (so in particular continuously) on λ and the maps f_n approximate f dynamically.

To see that $\chi_{\mathrm{luc}}(f_{n,\lambda}, f_{\lambda_0}) \to 0$, we have to look at the sequence of pull-backs

$$f_n^*\mu_\lambda = (\mu_\lambda \circ f_n)\frac{\overline{f_n'(z)}}{f_n'(z)},$$

where μ_λ denotes the complex dilatation of Ψ_λ. By assumption, there exists a constant $k < 1$ such that $\|f_n^*\mu_\lambda\|_\infty \le k < 1$ for all n. Since also $f_n^*\mu_\lambda \to f^*\mu_{\lambda_0}$ a.e., it follows that the (uniquely normalized solutions) $\Phi_{n,\lambda}$ converge locally uniformly to Φ_{λ_0} [**16**, Theorem 4.6], yielding the desired statement.

4.2. *Functions of sine type.* The above construction yields many examples of families to which our main theorems can be applied, provided that we have a sequence of functions approximating f dynamically. There are clearly various ways of approximating an entire function locally uniformly, but it is a very strong requirement to keep control over the sets of singular values, since this set can be arbitrarily complicated (for instance, it can have non-empty interior).

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For certain (families of) transcendental entire functions that were and are of particular interest in holomorphic dynamics, appropriate approximations are known and were extensively studied. For instance, one can approximate the exponential map by the polynomials $P_n(z) = (1 + z/n)^n$ or the function $f(z) = \sin(z)/z$, which has infinitely many singular values, by the maps $T_n(z)/z$, where $T_n(z)$ denotes the Chebyshev polynomial of degree n.

We want to introduce another set of transcendental entire functions for which a dynamical approximation exists, namely *real sine-type maps with real zeros*.

Definition 4.1. A function f is said to be of *sine type* σ, $0 < \sigma < \infty$, if there are positive constants c, C, τ such that

$$ce^{\sigma|\operatorname{Im} z|} \leq |f(z)| \leq Ce^{\sigma|\operatorname{Im} z|},$$

where the upper estimate holds everywhere in $\mathbb{C}$ and the lower estimate holds at least outside the horizontal strip $|\operatorname{Im} z| \leq \tau$.

Let f be a sine-type function and denote by z_n the zeros of f. By [**17**, Lecture 17] or [**21**, Theorem 3], the limit

$$\lim_{R\to\infty} \prod_{|z_n|\leq R} \left(1 - \frac{z}{z_n}\right)$$

exists uniformly on compact subsets of $\mathbb{C}$, and it defines an entire function called the *generating function of the sequence* (z_n), which equals f up to $K \cdot z^m$, where K is a constant and $m \geq 0$ is an integer [**21**, Theorem 2]. By definition, the zeros of f are contained in a horizontal strip around the real axis and f has exactly two tracts over ∞, each of which contains some upper and lower half plane, respectively. (A tract over ∞ is a preimage component of a neighborhood of ∞ that contains no singular values.) It follows from the Ahlfors–Denjoy theorem [**20**, XI, §4, p. 313] that f has at most two asymptotic values. It is now easy to show the following.

PROPOSITION 4.2. *Let f be a real sine-type function for which all zeros are real. Then there exists a sequence p_n of polynomials such that $\chi_{\mathrm{dyn}}(p_n, f) \to 0$ when $n \to \infty$.*

Proof. By a theorem of Laguerre [**22**, Ch. 8.51], all zeros of $f'(z)$ are real and are separated from each other by the zeros of f. Since f is of sine type, the set $C(f)$ and hence $S(f)$ is bounded. By construction, the generating polynomials cannot have 'free' critical values and, by [**12**, Theorem 2], each singular value of f is approximated by a sequence of singular values of the generating polynomials. Hence, the sets of singular values converge in the Hausdorff metric. □

Remark. Standard approximation methods in function theory do not relate to the sets of singular values in a way that would enable us to construct a sequence of entire maps that dynamically approximate a given function f in a non-trivial way. This leaves open an interesting function-theoretic question.

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