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Sampling time-frequency localized functions and constructing localized time-frequency frames

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We study functions whose time-frequency content are concentrated in a compact region in phase space using time-frequency localization operators as a main tool. We obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions of these operators, as well as a local Gabor system covering the region of interest. These would allow the construction of modified time-frequency dictionaries concentrated in the region.

Key words: frames, multipliers, sampling theory, signal theory (filtering, reconstruction), approximation by other special function classes

1 Introduction

When processing audio signals such as music or speech, one sometimes strives for a meticulous separation of signal components in time and frequency. However, it is known that no non-zero function can be compactly supported simultaneously in time and frequency, so that good concentration in one domain usually has to be paid for with increased leakage in the other domain. Due to this trade-off, window design has been an important issue in signal processing. As opposed to traditional approaches, we investigate the optimization of concentration simultaneously in time and frequency. More precisely, we investigate functions that exhibit good concentration in a compact region in the time-frequency plane. Our method is related to the approach introduced by Landau, Slepian, and Pollak, who considered operators composed of consecutive time- and bandlimiting steps, cf. [23, 24, 30]. The operators resulting from consecutive time- and bandlimiting yield the well-known prolate spheroidal functions as eigenfunctions. These functions satisfy some optimality in concentration in a rectangular region in the time-frequency domain.

In [5], Daubechies introduced time-frequency localization operators obtained by restricting the synthesis from time-frequency coefficients to a desired region of interest. Here, we make use of time-frequency localization operators to describe a function's local time-frequency content in regions more general than the rectangles considered in [23, 24, 30]. As in the case of the prolate spheroidal wave functions, the eigenfunctions of the time-frequency localization operators are maximally time-frequency-concentrated

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in the region of interest. We will use these eigenfunctions to characterize a function's time-frequency localization. Based on this characterization, our main contributions are as follows:

- We show to which extent time-frequency localized functions can be approximated using only a finite number of Gabor coefficients, namely, those which are inside some larger cover of the given region. This is influenced by the approximation result formulated by Daubechies in a seminal paper [6]. Similar estimates were also established in [27]. Our results are new since they allow for general localization regions and more general point sets as opposed to the restriction to localization on rectangular regions which can be characterized by separate time- and frequency concentration. (Proposition 4.1, Proposition 4.5)
- Based on the obtained local approximations, we construct global time-frequency frames consisting of atoms which are optimally concentrated in small regions corresponding to a prescribed point set (Theorem 4.10). Each locally concentrated system is constructed by projecting the local Gabor atoms onto the subspace spanned by a finite number of eigenfunctions. Then, by considering a family of such locally concentrated time-frequency dictionaries, we obtain an adaptive frame for all $L^2(\mathbb{R})$. In the construction of these new frames, we are motivated by concrete applications in audio processing, as described in [12,26], for which signal components need to be meticulously separated in time-frequency. The constructed frames are potentially useful for this kind of tasks since localized signal components can be separately processed while keeping distortion from adjacent signal components to a minimum, as also investigated in [11,12]. Note that a similar construction involving prolate spheroidal functions has recently been studied in [21].

This paper is organized as follows. In the next section, we recall the necessary tools from time-frequency analysis, namely, the short-time Fourier transform and Gabor frames. In Section 3, we review some properties of time-frequency localization operators, and we prove characterization and approximation results concerning time-frequency localized functions and eigenfunctions of time-frequency localization operators. Section 4 deals with approximation using a local Gabor system and the construction of the new time-frequency dictionaries. Finally, in Section 5, we present numerical experiments concerning our results and conclude with some perspectives in Section 6.

2 Preliminaries

In this section, we recall some definitions and properties about the short-time Fourier transform and Gabor frames. For a thorough introduction to the field of time-frequency analysis, we refer the reader to [18].

Fix a window function $\varphi \in L^2(\mathbb{R})$ with norm $\|\varphi\|_2 = 1$. The *short-time Fourier transform* (STFT) of $f \in L^2(\mathbb{R})$ with respect to φ is given by

$$\mathcal{V}_\varphi f(z) = \mathcal{V}_\varphi f(x,\omega) = \int_{\mathbb{R}} f(t)\,\overline{\varphi(t-x)}\,e^{-2\pi i\omega\cdot t}dt = \langle f, \pi(z)\varphi\rangle, \tag{2.1}$$

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where $z = (x, \omega) \in \mathbb{R}^2$ and $\pi(z)$ is the time-frequency shift operator given by $\pi(z)f = f(t-x)\, e^{2\pi i \omega \cdot t}$. The STFT is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$, i.e. $\|\mathcal{V}_\varphi f\|_2 = \|\varphi\|_2 \|f\|_2$, and inversion is realized using the formula

$$f = \mathcal{V}_\varphi^* \mathcal{V}_\varphi f = \iint_{\mathbb{R}^2} \mathcal{V}_\varphi f(z) \pi(z) \varphi \, dz, \tag{2.2}$$

where the vector-valued integral above and similar expressions in the sequel are understood in a weak sense, cf. [18, Section 3.2].

The membership of the STFT in $L^p(\mathbb{R}^2)$ provides a definition of a class of function spaces called *modulation spaces*. For a historical account of their development and role in time-frequency analysis, see [13]. For a fixed non-zero window function φ in the Schwartz space $\mathcal{S}(\mathbb{R})$, the modulation space $M^p(\mathbb{R})$ is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $\mathcal{V}_\varphi f \in L^p(\mathbb{R}^2)$. It is a Banach space equipped with the norm $\|f\|_{M^p(\mathbb{R})} := \|\mathcal{V}_\varphi f\|_{L^p(\mathbb{R}^2)}$, where a different window function φ would yield an equivalent norm. Note that for $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$,we have $M^2(\mathbb{R}) = L^2(\mathbb{R})$ since the STFT is isometric. The space $\boldsymbol{S}_0(\mathbb{R}) = M^1(\mathbb{R})$ is also known as Feichtinger's algebra. It is the smallest Banach space isometrically invariant under time-frequency shifts and the Fourier transform, cf. [17]. For the most part, the conditions that we assume on the window functions imply membership in $\boldsymbol{S}_0(\mathbb{R})$.

Discretization of the time-frequency representation via the STFT leads to the theory of Gabor frames. We consider a *Gabor system* $\mathcal{G}(g, \Lambda)$ with window function $g \in L^2(\mathbb{R})$ and a countable set of points Λ in $\mathbb{R}^2$, consisting of time-frequency shifted copies of a function g, i.e. $\mathcal{G}(g, \Lambda) := \{g_\lambda := \pi(\lambda) g\}_{\lambda \in \Lambda}$. We say that $\mathcal{G}(g, \Lambda)$ is a *frame* for $f \in L^2(\mathbb{R})$ if there exist constants A, $B > 0$ such that for all $f \in L^2(\mathbb{R})$

$$A\|f\|_2^2 \leqslant \sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leqslant B\|f\|_2^2. \tag{2.3}$$

If we can choose $A = B$, then we say that $\mathcal{G}(g, \Lambda)$ is a *tight frame*.

Associated with the frame $\mathcal{G}(g, \Lambda)$ is the *frame operator* S given by

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda.$$

The frame conditions (2.3) are equivalent to the invertibility of S, and reconstruction from the coefficients $\{\langle f, g_\lambda \rangle\}_{\lambda \in \Lambda}$ is possible because of the existence of a dual frame $\{\tilde{g_\lambda}\}_{\lambda \in \Lambda}$, the canonical one being $\{S^{-1} g_\lambda\}_{\lambda \in \Lambda}$, having frame bounds B^{-1} and A^{-1}. Every $f \in L^2(\mathbb{R})$ will then have the expansion

$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \tilde{g_\lambda} = \sum_{\lambda \in \Lambda} \langle f, \tilde{g_\lambda} \rangle g_\lambda, \tag{2.4}$$

where both series converge unconditionally in $L^2(\mathbb{R})$. If $\mathcal{G}(g, \Lambda)$ is a tight frame, then $Sf = Af$ so $f = \frac{1}{A} Sf$.

If $\mathcal{G}(g, \Lambda)$ is a frame, then the *analysis operator* U given by $Uf = \{\langle f, g_\lambda \rangle\}_{\lambda \in \Lambda}$ and its adjoint U^*, called the *synthesis operator*, given by $U^* c = \sum_{\lambda \in \Lambda} c_\lambda g_\lambda$, $c = \{c_\lambda\}_{\lambda \in \Lambda}$, are bounded from $L^2(\mathbb{R})$ into $\ell^2(\Lambda)$ and $\ell^2(\Lambda)$ into $L^2(\mathbb{R})$, respectively, with operator norms

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$\|U\|_{\mathrm{Op}} = \|U^*\|_{\mathrm{Op}} \leqslant \sqrt{B}$. For the dual frame $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$, the associated analysis and synthesis operators are also bounded operators with operator norms not exceeding $1/\sqrt{A}$.

We note that the assumption $g \in \boldsymbol{S}_0(\mathbb{R})$ leads to various convenient properties of the associated Gabor frames. For example, the set of windows and lattices which form a Gabor frame is open, hence stable to perturbations, cf. [14]. Moreover, membership in $g \in \boldsymbol{S}_0(\mathbb{R})$ together with the frame property for $L^2(\mathbb{R})$ automatically imply that the dual window is also in $\boldsymbol{S}_0(\mathbb{R})$, hence possesses good localization in time and frequency. Also, these frames are frames in all modulation spaces, [18, Section 13.6]. The assumption $g \in \boldsymbol{S}_0(\mathbb{R})$ is mild enough not to exclude most important windows used in practice and has thus become fairly standard in time-frequency analysis.

3 Time-frequency concentration via the STFT

Time-frequency localization operators as introduced by Daubechies in [5] are built by restricting the integral in the inversion formula (2.2) to a subset of $\mathbb{R}^2$. Its properties, connections with other mathematical topics, and applications have been topics in various works, e.g. [1, 4, 7, 10, 11, 15, 20, 28, 34].

Let Ω be a compact set in $\mathbb{R}^2$, χ_Ω its characteristic (or indicator) function, and φ a window function in $L^2(\mathbb{R})$, with $\|\varphi\|_2 = 1$. The *time-frequency localization operator* H is defined by

$$Hf = \iint_\Omega \mathcal{V}_\varphi f(z)\pi(z)\varphi \, dz = \mathcal{V}_\varphi^*(\chi_\Omega \, \mathcal{V}_\varphi f). \tag{3.1}$$

Note that while we denote a time-frequency localization only by H, we emphasize that it is dependent on the window φ and the region Ω.

The above integral can be interpreted as the portion of the function f that is essentially contained in Ω. Moreover, the following inner product involving H measures the function's energy inside Ω:

$$\langle Hf, f\rangle = \iint_\Omega \mathcal{V}_\varphi f(z)\langle \pi(z)\varphi, f\rangle dz = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz. \tag{3.2}$$

We will say that a function $f \in L^2(\mathbb{R})$ is (ε, φ)-*concentrated* inside Ω if $\langle Hf, f\rangle \geqslant (1-\varepsilon)\|f\|_2^2$ or equivalently $\langle (I-H)f, f\rangle \leqslant \varepsilon\|f\|_2^2$, where I is the identity operator.

The time-frequency localization operator H is a compact and self-adjoint operator so we can consider the spectral decomposition

$$Hf = \sum_{k=1}^{\infty} \alpha_k \langle f, \psi_k\rangle \psi_k, \tag{3.3}$$

where $\{\alpha_k\}_{k=1}^\infty$ are the positive eigenvalues arranged in a non-increasing order and $\{\psi_k\}_{k=1}^\infty$ are the corresponding eigenfunctions. By the min-max theorem for compact, self-adjoint operators, the first eigenfunction has optimal time-frequency concentration inside Ω in the sense of (3.2), i.e.

$$\iint_\Omega |\mathcal{V}_\varphi \psi_1(z)|^2 dz = \max_{\|f\|_2=1} \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz.$$

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In general, the first N eigenfunctions $\psi_1, \psi_2, \ldots, \psi_N$ form an orthonormal set in $L^2(\mathbb{R})$ having optimal cumulative time-frequency concentration inside Ω:

$$\sum_{k=1}^{N} \iint_{\Omega} |\mathcal{V}_{\varphi}\psi_k(z)|^2 dz = \max_{\{\phi_k\}_{k=1}^{N} \text{orthonormal}} \sum_{k=1}^{N} \iint_{\Omega} |\mathcal{V}_{\varphi}\phi_k(z)|^2 dz.$$

If we let V_N be the span of the first N eigenfunctions and if $f \in V_N$, so $f = \sum_{k=1}^{N} \langle f, \psi_k \rangle \psi_k$, then

$$\langle Hf, f \rangle = \sum_{k=1}^{N} \alpha_k |\langle f, \psi_k \rangle|^2 \geqslant \alpha_N \sum_{k=1}^{N} |\langle f, \psi_k \rangle|^2 = \alpha_N \|f\|_2^2 . \tag{3.4}$$

By contrast, functions which are $(1-\alpha_N, \varphi)$-concentrated on Ω need not lie in V_N. The following proposition characterizes a function that is (ε, φ)-concentrated on Ω.

Proposition 3.1 *Let φ, Ω, and ε be given and let N_0 be the integer such that $\alpha_{N_0} \geqslant 1-\varepsilon$ and $\alpha_{N_0+1} < 1-\varepsilon$. Furthermore, let $f_{\ker}$ denote the orthogonal projection of f onto the kernel $\ker(H)$ of H. A function f in $L^2(\mathbb{R})$ is (ε, φ)-concentrated on Ω if and only if*

$$\sum_{k=1}^{N_0} (\alpha_k + \varepsilon - 1) |\langle f, \psi_k \rangle|^2 \geqslant \sum_{k=N_0+1}^{\infty} (1 - \varepsilon - \alpha_k) |\langle f, \psi_k \rangle|^2 + (1-\varepsilon) \|f_{\ker}\|_2^2 .$$

Proof The eigenfunctions $\{\psi_k\}_k$ form an orthonormal subset in $L^2(\mathbb{R})$, possibly incomplete if $\ker(H) \neq \{0\}$; hence, we can write $f = \sum_{j=1}^{\infty} \langle f, \psi_j \rangle \psi_j + f_{\ker}$ and, as in equation (3.4), $\langle Hf, f \rangle = \sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k \rangle|^2$. So the function f is (ε, φ)-concentrated on Ω if and only if

$$\sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k \rangle|^2 \geqslant (1-\varepsilon) \left(\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 + \|f_{\ker}\|_2^2 \right),$$

and the conclusion follows. □

Remark 3.2 *A function f in $L^2(\mathbb{R})$ is $(1-\alpha_N, \varphi)$-concentrated on Ω if and only if $\sum_{k=1}^{N-1} (\alpha_k - \alpha_N) |\langle f, \psi_k \rangle|^2 \geqslant \sum_{k=N+1}^{\infty} (\alpha_N - \alpha_k) |\langle f, \psi_k \rangle|^2 + \alpha_N \|f_{\ker}\|_2^2$.*

In [6, Theorem 3.1], Daubechies bounded the error of a function's local approximation using a finite number of Gabor atoms by means of an estimate based on the function's and its Fourier transform's *projection* onto bounded intervals. In Daubechies' theorem, the local approximations were made with respect to regular lattice points living in a rectangle in the time-frequency plane. In order to achieve a similar bound, but for more general point sets and regions, in Proposition 4.5 in the next section, we consider the projection of a function f onto the best-concentrated eigenfunctions of a localization operator and derive the following estimate. We note that approximations of bandlimited functions via projections onto eigenspaces of approximately time- and bandlimited functions were presented in [22, 32].

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Proposition 3.3 *Let f be (ε, φ)-concentrated on $\Omega \subset \mathbb{R}^2$. For fixed $c > 1$, let $\psi_k, k = 1, \ldots, N$, be all eigenfunctions of H corresponding to eigenvalues $\alpha_k > \frac{c-1}{c}$. Then,*

$$\left\| f - \sum_{k=1}^{N} \langle f, \psi_k \rangle \psi_k \right\|_2^2 \leqslant c\varepsilon \|f\|_2^2 .$$

Proof Without loss of generality, we assume that $\|f\|_2 = 1$. We have, by assumption:

$$\langle Hf, f \rangle = \sum_{k=1}^{\infty} \alpha_k |\langle f, \psi_k \rangle|^2 = \iint_{\Omega} |\mathcal{V}_\varphi f(z)|^2 dz \geqslant (1 - \varepsilon) \|f\|_2^2 .$$

We argue by contradiction; to this end, assume that $\sum_{k=1}^{N} |\langle f, \psi_k \rangle|^2 = K < 1 - c\varepsilon$. Furthermore, we have

$$\|f\|_2^2 = 1 = \sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 + \|f_{\ker}\|_2^2 ,$$

hence

$$\sum_{k=N+1}^{\infty} |\langle f, \psi_k \rangle|^2 = 1 - K - \|f_{\ker}\|_2^2 .$$

We then have

$$\sum_{k=N+1}^{\infty} \alpha_k \, |\langle f, \psi_k \rangle|^2 \leqslant \frac{c-1}{c} \cdot (1 - K - \|f_{\ker}\|_2^2)$$

such that

$$\begin{aligned} \sum_{k=1}^{\infty} \alpha_k \, |\langle f, \psi_k \rangle|^2 &\leqslant K + \frac{c-1}{c} \cdot (1 - K - \|f_{\ker}\|_2^2) \\ &= \frac{c - 1 + K}{c} - \frac{c-1}{c} \|f_{\ker}\|_2^2 \\ &< 1 + \frac{1 - c\varepsilon - 1}{c} - \frac{c-1}{c} \|f_{\ker}\|_2^2 < 1 - \varepsilon, \end{aligned}$$

which is a contradiction. Hence, $\sum_{k=1}^{N} |\langle f, \psi_k \rangle|^2$ must be greater than or equal to $1 - c\varepsilon$. □

We note that while Hf is interpreted as the part of f in Ω, the uncertainty principle prohibits its STFT from having non-zero values only in Ω, cf. [33], and there will always be points $z \in \mathbb{R}^2 \setminus \Omega$ at which $|\mathcal{V}_\varphi Hf(z)| \neq 0$. It can be shown, however, that $|\mathcal{V}_\varphi Hf(z)|$ decays quickly with respect to the distance of z from Ω. Daubechies proved this result in [5] for the case where the window function is the Gaussian $\varphi_0(t) = e^{-\pi t^2}$, showing that the pointwise magnitude of the STFT decays exponentially.

Lemma 3.4 (Daubechies, [5]) *Let H be a time-frequency localization operator over the region Ω with the Gaussian as its window function, i.e. $\varphi = \varphi_0$. For any δ between* 0 *and* 1,

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and $f \in L^2(\mathbb{R})$*, one has*

$$|\langle Hf, \pi(z)\varphi_0\rangle| \leqslant \tfrac{1}{\sqrt{2}}\delta^{-\frac{1}{2}}\|f\|_2 \exp[-\tfrac{\pi}{2}(1-\delta)\,\mathrm{dist}(z,\Omega)^2].$$

A similar result involving windows with milder decay conditions is the following.

Lemma 3.5 *Let* $\varphi, g \in L^2(\mathbb{R})$ *such that* $\|\varphi\|_2 = 1$ *and* $|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$*, for some* $C > 0$ *and* $s > 1$*, for all* $z \in \mathbb{R}^2$*. For any* δ *between* 0 *and* 1*, one has*

$$|\mathcal{V}_\varphi Hf(z)| = |\langle Hf, \pi(z)g\rangle| \leqslant C_s\delta^{-\frac{1}{2s}}\|f\|_2(1+(1-\delta)\mathrm{dist}(z,\Omega)^s)^{-1},$$

where $C_s = C\pi\sqrt{\frac{2}{s}\csc\left(\frac{\pi}{s}\right)}$.

Remark 3.6 *We note that the inequality* $|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$ *implies that both* φ *and* g *are in* $\boldsymbol{S}_0(\mathbb{R})$*. On the other hand, the inequality* $|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$ *is satisfied for all* $z \in \mathbb{R}^2$ *when* φ *and* g *are in the Schwartz space* $\mathcal{S}(\mathbb{R})\,(\subset \boldsymbol{S}_0(\mathbb{R}))$*. Moreover, in that case, for every* $s > 0$ *there is a* C *for which the inequality holds.*

Proof of Lemma 3.5 If $z, z' \in \mathbb{R}^2$, then $|\langle \pi(z')\varphi, \pi(z)g\rangle| = |\langle \varphi, \pi(z-z')g\rangle| \leqslant C'_s(1+|z-z'|^{2s})^{-1}$.

For $0 < \delta < 1$,

$$\begin{aligned}
|\langle Hf, \pi(z)g\rangle| &\leqslant \iint_\Omega |\langle f, \pi(z')\varphi\rangle|\,|\langle \pi(z')\varphi, \pi(z)g\rangle|\,dz' \\
&\leqslant C\iint_\Omega |\langle f, \pi(z')\varphi\rangle| \frac{1}{1+|z-z'|^{2s}}\,dz' \\
&\leqslant C\iint_\Omega |\langle f, \pi(z')\varphi\rangle| \frac{1}{\sqrt{1+\delta|z-z'|^{2s}}}\frac{1}{\sqrt{1+(1-\delta)|z-z'|^{2s}}}\,dz' \\
&\leqslant C\sqrt{2}\frac{1}{1+(1-\delta)\inf\limits_{z'\in\Omega}|z-z'|^s}\left(\iint_{\mathbb{R}^2}\frac{1}{1+\delta|z-z'|^{2s}}\,dz'\right)^{\frac{1}{2}} \cdot \\
&\qquad \left(\iint_{\mathbb{R}^2}|\langle f, \pi(z')\varphi\rangle|^2\,dz'\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\iint_{\mathbb{R}^2}\frac{1}{1+\delta|z-z'|^{2s}}\,dz' = \frac{\pi^2}{s\,\delta^{1/s}}\csc\left(\frac{\pi}{s}\right)$, cf. [19, equation 3.241(2)], we have

$$|\langle Hf, \pi(z)g\rangle| \leqslant C\pi\sqrt{\frac{2}{s}\csc\left(\frac{\pi}{s}\right)}\,\delta^{-\frac{1}{2s}}(1+(1-\delta)\inf_{z'\in\Omega}|z-z'|^s)^{-1}\|\varphi\|_2\|f\|_2,$$

and the conclusion follows. □

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4 Local Gabor approximation and new TF dictionaries

We shall make use of the results concerning time-frequency localization to obtain an approximation of a function using a finite Gabor expansion. We expect that if the function is well localized on a region Ω in the time-frequency plane, then f can be approximated with good accuracy using only the Gabor coefficients on a larger region covering Ω. From the local Gabor systems, we will obtain frames for the eigenspace, the collection of which forms a frame for $L^2(\mathbb{R})$.

4.1 Time-frequency localization and local Gabor approximation

In this section, we consider a given region Ω in $\mathbb{R}^2$ and let V_N be the N-dimensional subspace spanned by the first N eigenfunctions of the corresponding localization operator H. Here, the eigenvalues are assumed to be arranged in descending order. Furthermore, we let g be a window function in $L^2(\mathbb{R})$ such that $\|g\|_2 = 1$ and

$$|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$$

for some $C > 0$ and $s > 1$. We then consider the Gabor system $\mathcal{G}(g, \Lambda)$, assume that it forms a frame with lower and upper frame bounds A and B, respectively, and let $\{\tilde{g}_\lambda : \lambda \in \Lambda\}$ be its dual frame.

We first have the following approximation of a function in V_N by local Gabor atoms.

Proposition 4.1 *For any $\varepsilon > 0$, there exists an $\Omega^* \supset \Omega$ such that*

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 \leqslant \varepsilon \|f\|_2, \quad \textit{for all } f \in V_N. \tag{4.1}$$

Proof Let $\varepsilon > 0$ and $f \in V_N$. We first observe that for any $\Omega^* \supset \Omega$,

$$\begin{aligned}
\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2^2 &= \left\| \sum_{k=1}^N \langle f, \psi_k \rangle \left(\sum_{\lambda \notin \Lambda \cap \Omega^*} \langle \psi_k, g_\lambda \rangle \tilde{g}_\lambda \right) \right\|_2^2 \\
&\leqslant \left(\sum_{k=1}^N |\langle f, \psi_k \rangle| \left\| \sum_{\lambda \notin \Lambda \cap \Omega^*} \langle \psi_k, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 \right)^2 \\
&\leqslant \sum_{k=1}^N |\langle f, \psi_k \rangle|^2 \sum_{k=1}^N \left\| \sum_{\lambda \notin \Lambda \cap \Omega^*} \langle \psi_k, g_\lambda \rangle \tilde{g}_\lambda \right\|_2^2 .
\end{aligned}$$

By the boundedness of the synthesis operator of the given dual frame $\{\tilde{g}_\lambda\}$, we obtain

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2^2 \leqslant A^{-1} \|f\|_2^2 \sum_{k=1}^N \sum_{\lambda \notin \Lambda \cap \Omega^*} |\langle \psi_k, g_\lambda \rangle|^2 .$$

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We consider $|\langle \psi_k, g_\lambda \rangle|$ and note that $|\langle \psi_k, g_\lambda \rangle| = \frac{1}{\alpha_k} |\langle H\psi_k, g_\lambda \rangle|$. Since g satisfies $|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$, it follows from Lemma 3.5 that $|\langle \psi_k, g_\lambda \rangle| \leqslant \frac{1}{\alpha_k} C_s 2^{\frac{1}{2s}} (1 + \frac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-1}$, where δ is taken to be $\frac{1}{2}$, which gives us

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2^2 \leqslant A^{-1} \|f\|_2^2 C_s^2 2^{\frac{1}{s}} \left(\sum_{k=1}^{N} \frac{1}{\alpha_k^2} \right) \sum_{\lambda \notin \Lambda \cap \Omega^*} (1 + \tfrac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-2}.$$

The right-hand side of the above inequality approaches 0 as Ω^* gets larger. In particular, given $\varepsilon > 0$, one can choose Ω^* so that the sum

$$\sum_{\lambda \notin \Lambda \cap \Omega^*} (1 + \tfrac{1}{2}\operatorname{dist}(\lambda, \Omega)^s)^{-2} < \varepsilon^2 / \left(A^{-1} C_s^2 2^{\frac{1}{s}} \sum_{k=1}^{N} \frac{1}{\alpha_k^2} \right)$$

which gives the conclusion of the proposition. □

We show an example for the case where the window function is the Gaussian φ_0 and the region Ω is the disk $B(O, R)$ with centre at the origin O and with radius R. We will make use of the decay of the STFT of H in Lemma 3.4. First, we prove the following lemma that gives an estimate on the decay of the tail of the sum of samples of the two-dimensional Gaussian outside the disk $B(O, R^*)$. Let $Q(j) = [j_1 - \frac{1}{2}, j_1 + \frac{1}{2}) \times [j_2 - \frac{1}{2}, j_2 + \frac{1}{2})$, $j = (j_1, j_2) \in \mathbb{Z}^2$.

Lemma 4.2 *Let Λ be a set of points in $\mathbb{R}^2$ with $\sup_{z \in \mathbb{R}^2} \#(\Lambda \cap Q(z)) =: N_\Lambda < \infty$. Fix $R > 0$. If $R^* > R$, then*

$$\sum_{\lambda \in \Lambda,\, |\lambda| > R^*} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2) \leqslant C_\Lambda \exp\left(-\tfrac{\pi}{4}\left(\tfrac{(R^*)^2}{4} - R^2 \right) \right), \tag{4.2}$$

where $C_\Lambda = 8\exp(\frac{5\pi}{4}) N_\Lambda$.

Remark 4.3 *We note that a set of points $\Lambda \subset \mathbb{R}^2$ satisfying $\sup_{z \in \mathbb{R}^2} \#(\Lambda \cap Q(z)) < \infty$ is said to be* relatively separated.

Proof of Lemma 4.2 Let $R^* > R$ and define the sets

$$\mathcal{J}_{R^*} = \{ j \in \mathbb{Z}^2 \,:\, Q(j) \cap (\mathbb{R}^2 \setminus B(O, R^*)) \neq \emptyset \} \quad \text{and}$$
$$\Lambda_{R^*, j} = \{ \lambda \in \Lambda \,:\, |\lambda| > R^*,\, \lambda \in Q(j) \}.$$

We are then able to rewrite the left-hand side of equation (4.2) as

$$\sum_{\lambda \in \Lambda,\, |\lambda| > R^*} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2) = \sum_{j \in \mathcal{J}_{R^*}} \sum_{\lambda \in \Lambda_{R^*, j}} \exp(-\tfrac{\pi}{2}(|\lambda| - R)^2). \tag{4.3}$$

If $\lambda, z \in Q(j)$, then $|\lambda| \geqslant |z| - \sqrt{2}$. And since $-(|z| - R - \sqrt{2})^2 \leqslant -\left(\frac{(|z| - R)^2}{2} - 2 \right)$, we have

$$e^{-\frac{\pi}{2}(|\lambda| - R)^2} \leqslant e^{-\frac{\pi}{2}(|z| - R - \sqrt{2})^2} \leqslant e^{\pi} \exp\left(-\tfrac{\pi}{4}\left(|z| - R \right)^2 \right).$$

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Using the inequalities $-(|z|-R)^2 \leqslant -\frac{|z|^2}{2}+R^2$ and $-(R^*-\sqrt{2})^2 \leqslant -\frac{(R^*)^2}{2}+2$, we estimate (4.3) as follows:

$$\begin{aligned}\sum_{j\in\mathcal{J}_{R^*}}\sum_{\lambda\in\Lambda_{R^*,j}}\exp(-\tfrac{\pi}{2}(|\lambda|-R)^2) &\leqslant \sum_{j\in\mathcal{J}_{R^*}}\sum_{\lambda\in\Lambda_{R^*,j}}\iint_{Q(j)} e^{\pi}\exp\big(-\tfrac{\pi}{4}\big(|z|-R\big)^2\big)\,dz\\ &\leqslant N_\Lambda\, e^{\pi}\iint_{\mathbb{R}^2\setminus B(O,R^*-\sqrt{2})}\exp\big(-\tfrac{\pi}{4}\big(|z|-R\big)^2\big)\,dz\\ &\leqslant N_\Lambda\, e^{\pi}e^{\frac{\pi R^2}{4}}\iint_{|z|>R^*-\sqrt{2}}\exp\big(-\tfrac{\pi|z|^2}{8}\big)\,dz\\ &= 8N_\Lambda\, e^{\pi}e^{\frac{\pi R^2}{4}}\exp\big(-\tfrac{\pi(R^*-\sqrt{2})^2}{8}\big)\\ &\leqslant 8N_\Lambda\, e^{\frac{5\pi}{4}}\exp\big(-\tfrac{\pi}{4}(\tfrac{(R^*)^2}{4}-R^2)\big).\end{aligned}$$

By taking $C_\Lambda = 8N_\Lambda\, e^{\frac{5\pi}{4}}$, we get the conclusion of the lemma. □

Example 4.4 Suppose that the Gabor system $\{\varphi_{0,\lambda} := \pi(\lambda)\varphi_0 : \lambda\in\Lambda\}$ forms a frame having upper and lower frame bounds A and B, respectively, and let the system $\{\widetilde{\varphi_{0,\lambda}} : \lambda\in\Lambda\}$ be its dual frame. Then, for any ε between 0 and 1, there exists $R_\varepsilon > 0$ such that if Ω is a disk centred at the origin with radius R, the following inequality holds for all $f\in V_N$:

$$\Big\|f-\sum_{|\lambda|\leqslant R+R_\varepsilon}\langle f,\varphi_{0,\lambda}\rangle\widetilde{\varphi_{0,\lambda}}\Big\|_2 \leqslant \varepsilon\|f\|_2\,. \tag{4.4}$$

Here, we can take $R_\varepsilon \geqslant -R+\sqrt{4R^2-\frac{16}{\pi}\ln(\varepsilon^2/A^{-1}C_\Lambda\sum_{k=1}^N\frac{1}{\alpha_k^2})}$, where $C_\Lambda = 8N_\Lambda\, e^{\frac{5\pi}{4}}$.

Proof Following the proof of Proposition 4.1, we have for $\Omega^*\supset\Omega$,

$$\Big\|f-\sum_{\lambda\in\Lambda\cap\Omega^*}\langle f,\varphi_{0,\lambda}\rangle\widetilde{\varphi_{0,\lambda}}\Big\|_2^2 \leqslant A^{-1}\|f\|_2^2\sum_{k=1}^N\sum_{\lambda\notin\Lambda\cap\Omega^*}\frac{1}{\alpha_k^2}|\langle H_{\Omega,\varphi_0}\psi_k,\varphi_{0,\lambda}\rangle|^2\,.$$

We can take Ω^* to be a disk centred at the origin with radius $R^* := R+R_\varepsilon > R$. We use Lemma 3.4 (with $\delta=\frac{1}{2}$) and Lemma 4.2 to estimate the double sum on the right side as follows:

$$\begin{aligned}\sum_{k=1}^N\sum_{\lambda\notin\Lambda\cap\Omega^*}\frac{1}{\alpha_k^2}|\langle H\psi_k,\varphi_{0,\lambda}\rangle|^2 &\leqslant \sum_{k=1}^N\sum_{\lambda\notin\Lambda\cap\Omega^*}\frac{1}{\alpha_k^2}\exp(-\tfrac{\pi}{2}\operatorname{dist}(\lambda,\Omega)^2)\|\psi_k\|_2^2\\ &= \sum_{k=1}^N\sum_{|\lambda|>R^*}\frac{1}{\alpha_k^2}\exp(-\tfrac{\pi}{2}(|\lambda|-R)^2)\\ &\leqslant \Big(\sum_{k=1}^N\frac{1}{\alpha_k^2}\Big)C_\Lambda\exp\big(-\tfrac{\pi}{4}\big(\tfrac{(R^*)^2}{4}-R^2\big)\big).\end{aligned}$$

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Now, $\exp\left(-\frac{\pi}{4}\left(\frac{(R^*)^2}{4}-R^2\right)\right) \leqslant \varepsilon^2/A^{-1}C_\Lambda \sum_{k=1}^N \frac{1}{\alpha_k^2}$ whenever R^* is greater than or equal to $\sqrt{4R^2 - \frac{16}{\pi}\ln\left(\varepsilon^2/A^{-1}C_\Lambda \sum_{k=1}^N \frac{1}{\alpha_k^2}\right)}$, and the conclusion follows. □

The following proposition gives a localization result by means of a given Gabor frame whose atoms are known to have sufficient TF-localization, provided by the condition $|\mathcal{V}_\varphi g(z)| \leqslant C(1+|z|^{2s})^{-1}$.

Proposition 4.5 *For all $N > 0$ and all $\varepsilon > 0$, there exists a set $\Omega^* \supset \Omega$ in $\mathbb{R}^2$, such that for all $f \in L^2(\mathbb{R})$ with corresponding orthogonal projection f_N onto the TF-localization subspace V_N, the following estimate holds:*

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 \leqslant \left(1 + \sqrt{\tfrac{B}{A}}\right) \|f - f_N\|_2 + \varepsilon \|f\|_2 . \tag{4.5}$$

Proof Since

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 \leqslant \|f - f_N\|_2 + \left\| f_N - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f_N, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 + \left\| \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f_N - f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2$$

the result follows from Proposition 4.1 and the boundedness of the associated analysis and synthesis operators. □

As a corollary, we obtain the following result for local approximation of functions with known time-frequency concentration in a given set Ω by Gabor frame elements.

Corollary 4.6 *For fixed $c > 1$, let ψ_k, $k = 1, \ldots, N$, be the eigenfunctions of H corresponding to eigenvalues $\alpha_k > \frac{c-1}{c}$. For N and $\tilde{\varepsilon} > 0$, choose a set $\Omega^* \supset \Omega$ as in Proposition 4.5. Then, the following approximation holds for all functions f which are (ε, φ)-concentrated on Ω:*

$$\left\| f - \sum_{\lambda \in \Lambda \cap \Omega^*} \langle f, g_\lambda \rangle \tilde{g}_\lambda \right\|_2 \leqslant \left(1 + \sqrt{\frac{B}{A}}\right) \cdot (\sqrt{c\varepsilon} + \tilde{\varepsilon}) \|f\|_2 . \tag{4.6}$$

Proof The result follows immediately from Propositions 3.3 and 4.5. □

Remark 4.7 *We emphasize that by Corollary 4.6, we are able to obtain an approximation estimate that holds* uniformly *for all functions that are (ε, φ)-concentrated in Ω. In contrast, for a* fixed *f, such an inequality can be obtained more simply from the unconditional convergence of the Gabor expansions, cf. [18, Section 5.3].*

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4.2 Local and global frames with TF-localization

The next results deal with the construction of frames for the subspace V_N of eigenfunctions of H, and the whole of $L^2(\mathbb{R})$, respectively. Denote by $\mathcal{P}_N$ the orthogonal projection operator onto the subspace V_N.

Proposition 4.8 *If $\varepsilon < 1$ and inequality* (4.1) *is satisfied, then for all $f \in V_N$,*

$$A(1-\varepsilon)^2\|f\|_2^2 \leqslant \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, g_\lambda\rangle|^2 \leqslant B\|f\|_2^2, \tag{4.7}$$

where A and B are lower and upper frame bounds, respectively, for $\mathcal{G}(g,\Lambda)$. This implies that the system $\{\mathcal{P}_N g_\lambda\}_{\lambda\in\Lambda\cap\Omega^*}$ forms a frame for V_N. More generally, the system $\{\pi(v)\mathcal{P}_N\pi(\lambda)g\}_{\lambda\in\Lambda\cap\Omega^*}$, where $v \in \mathbb{R}^2$, forms a frame for the subspace $\pi(v)V_N := \{\pi(v)f : f \in V_N\}$.

Proof From Proposition 4.1, we get

$$\|f\|_2 - \left\|\sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g}_\lambda\right\|_2 \leqslant \left\|f - \sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g}_\lambda\right\|_2 \leqslant \varepsilon\|f\|_2 .$$

And we obtain

$$\begin{aligned}(1-\varepsilon)^2\|f\|_2^2 &\leqslant \left\|\sum_{\lambda\in\Lambda\cap\Omega^*} \langle f, g_\lambda\rangle \tilde{g}_\lambda\right\|_2^2 \\ &\leqslant \frac{1}{A}\sum_{\lambda\in\Lambda\cap\Omega^*} |\langle f, g_\lambda\rangle|^2 \\ &\leqslant \frac{B}{A}\|f\|_2^2 .\end{aligned}$$

For the subspace $\pi(v)V_N$, we first note that

$$\|f\|_2 = \|\pi(v)f\|_2 \quad \text{and} \quad \langle f, \mathcal{P}_N g_\lambda\rangle = \langle \pi(v)f, \pi(v)\mathcal{P}_N g_\lambda\rangle.$$

The inequality in equation (4.7) can then be reformulated as

$$A(1-\varepsilon)^2\|\pi(v)f\|_2^2 \leqslant \sum_{\lambda\in\Lambda\cap\Omega^*} |\langle \pi(v)f, \pi(v)\mathcal{P}_N g_\lambda\rangle|^2 \leqslant B\|\pi(v)f\|_2^2,$$

for all $f \in V_N$, or $\pi(v)f \in \pi(v)V_N$. □

It follows from Proposition 4.8 that any function $f \in V_N$ can be completely reconstructed from the samples $\{\langle f, g_\lambda\rangle\}_{\lambda\in\Lambda\cap\Omega^*}$. In [17, Theorem 3.6.16], a reconstruction procedure was presented where a function on a closed subspace can be reconstructed from restricted Gabor coefficients. Following a similar approach, we apply the following iterative reconstruction for functions in V_N from the local STFT samples, where $U_{\text{loc}} f = \{\langle f, g_\lambda\rangle\}_{\lambda\in\Lambda\cap\Omega^*}$

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and $\tilde{U}^*_{\mathrm{loc}} c = \sum_{\lambda \in \Lambda \cap \Omega^*} c_\lambda \tilde{g}_\lambda$, $c = \{c_\lambda\}_{\lambda \in \Lambda \cap \Omega^*}$: let $f_0 = 0$ and define recursively

$$f_n = f_{n-1} + \mathcal{P}_N \tilde{U}^*_{\mathrm{loc}}(\langle f, g_\lambda \rangle - U_{\mathrm{loc}} f_{n-1}). \tag{4.8}$$

We implement the above reconstruction procedure in Section 5, observing the dependence of the performance of the algorithm on the choice of Ω^*.

Remark 4.9

(a) *Since* $\{\mathcal{P}_N g_\lambda : \lambda \in \Lambda \cap \Omega^*\}$ *is a frame for* V_N, *in the language of [2], the system* $\{g_\lambda : \lambda \in \Lambda \cap \Omega^*\}$ *is also called an* outer frame *for* V_N. *Related terminologies, e.g.* atomic system, *resp.* pseudoframe *for the subspace* V_N, *appear in [16, 25]. In particular, by Proposition 4.8 and [25, Theorems* 2 *and* 3*], the sequence* $\{\widetilde{g_{\lambda,V_N}}\}$, *where* $\widetilde{g_{\lambda,V_N}} = \mathcal{P}_N (U\mathcal{P}_N)^\dagger g_\lambda$ *is called a* dual pseudoframe sequence *for* V_N *with respect to* $\{g_\lambda\}$. *This sequence coincides with a dual frame to the frame* $\{\mathcal{P}_N g_\lambda\}$ *for* V_N.

(b) *It may sometimes be more natural to use, instead of the projection* $\mathcal{P}_N$, *the* approximate projection *operator* H_N *defined as:* $H_N f := \sum_{k=1}^N \alpha_k \langle f, \psi_k \rangle \psi_k$. *Obviously, since we use a finite sequence of positive weights* $\sqrt{\alpha_k}$, *we obtain an equivalent frame for the subspace* V_N, *if* $\mathcal{P}_N$ *is replaced by* H_N *in Proposition 4.8.*

We now consider a family of time-frequency localization operators H^μ, $\mu \in \tilde{\Lambda}$ over the region Ω_μ with a common window function φ. In [11, Theorem 5.10], Dörfler and Romero showed that under certain conditions on H^μ, one can choose N_μ such that

$$\|f\|_2^2 \asymp \sum_{\mu \in \tilde{\Lambda}} \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu \rangle|^2, \ f \in L^2(\mathbb{R}), \tag{4.9}$$

where the functions $\{\psi_k^\mu\}$ are eigenfunctions of H^μ, and $h_1 \asymp h_2$ means there exist positive constants β and γ such that $\beta h_1 \leqslant h_2 \leqslant \gamma h_1$. We use this result to obtain a frame for $L^2(\mathbb{R})$ consisting of local frame elements on the time-frequency localized subspaces.

Theorema 4.10 *Let* $\{\Omega_\mu\}_{\mu \in \tilde{\Lambda}}$ *be a family of compact regions in* $\mathbb{R}^2$ *such that* $2 < \inf_{\mu \in \tilde{\Lambda}} |\Omega_\mu| \leqslant \sup_{\mu \in \tilde{\Lambda}} |\Omega_\mu| < \infty$ *and* $\sum_{\mu \in \tilde{\Lambda}} \chi_{\Omega_\mu} \asymp 1$, *and let* $\varphi \in \boldsymbol{S}_0(\mathbb{R})$ *such that* $\|\varphi\|_2 = 1$.

Corresponding to each $\mu \in \tilde{\Lambda}$, *choose point sets* Λ_μ *and windows* $g^\mu \in L^2(\mathbb{R})$ *with* $\|g^\mu\|_2 = 1$, $|\mathcal{V}_\varphi g^\mu(z)| \leqslant C_\mu (1 + |z|^{2s_\mu})^{-1}$ *for some* $C_\mu > 0$ *and* $s_\mu > 1$ *such that for all* μ *the system* $\mathcal{G}(g^\mu, \Lambda_\mu)$ *is a frame for* $L^2(\mathbb{R})$ *with frame bounds* A_μ *and* B_μ.

Denote by V_{N_μ} *the span of the first* N_μ *eigenfunctions* $\{\psi_k^\mu\}_{k=1}^{N_\mu}$ *of* H^μ *corresponding to the* N_μ *largest eigenvalues, where each* N_μ *is chosen so that equation* (4.9) *holds.*

If $0 < \varepsilon_\mu < 1$ *such that* $0 < \inf_{\mu \in \tilde{\Lambda}} A_\mu (1 - \varepsilon_\mu)^2 \leqslant \sup_{\mu \in \tilde{\Lambda}} B_\mu < \infty$, *then there exist* $\Omega_\mu^* \supset \Omega_\mu$ *such that the global system* $\bigcup_{\mu \in \tilde{\Lambda}} \{\mathcal{P}_{N_\mu} \pi(\lambda) g^\mu\}_{\lambda \in \Lambda_\mu \cap \Omega_\mu^*}$ *is a frame for* $L^2(\mathbb{R})$.

Proof Let $f \in L^2(\mathbb{R})$ and $\mu \in \tilde{\Lambda}$. We first note that the conditions on the regions Ω_μ and φ ensure that (4.9) holds, cf. [11]. Since $\langle f, \mathcal{P}_{N_\mu} \pi(\lambda) g^\mu \rangle = \langle \mathcal{P}_{N_\mu} f, \pi(\lambda) g^\mu \rangle$, it follows from

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Proposition 4.8 that for every $\varepsilon_\mu \in (0,1)$, there exists $\Omega_\mu^* \supset \Omega_\mu$ such that

$$A_\mu(1-\varepsilon_\mu)^2 \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2 \leqslant \sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{N_\mu}\pi(\lambda)g^\mu\rangle|^2 \leqslant B_\mu \sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2 .$$

By the assumption that $0 < \tilde{A} := \inf_{\mu\in\tilde{\Lambda}} A_\mu(1-\varepsilon_\mu)^2 \leqslant \tilde{B} := \sup_{\mu\in\tilde{\Lambda}} B_\mu < \infty$ and the equivalence in equation (4.9), we get

$$\tilde{A}\sum_{\mu\in\tilde{\Lambda}}\sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2 \leqslant \sum_{\mu\in\tilde{\Lambda}}\sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{N_\mu}\pi(\lambda)g^\mu\rangle|^2 \leqslant \tilde{B}\sum_{\mu\in\tilde{\Lambda}}\sum_{k=1}^{N_\mu} |\langle f, \psi_k^\mu\rangle|^2,$$

and finally $\sum_{\mu\in\tilde{\Lambda}}\sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} |\langle f, \mathcal{P}_{N_\mu}\pi(\lambda)g^\mu\rangle|^2 \asymp \|f\|_2^2$. □

Remark 4.11

(a) *For the special case, where each Ω_μ is just the translated region $\mu+\Omega$, if $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthonormal system of eigenfunctions of H, then one can choose $N \in \mathbb{N}$ and $\Omega^* \supset \Omega$ such that $\bigcup_{\lambda\in\Lambda\cap\Omega^*} \mathcal{G}(\mathcal{P}_N g_\lambda, \tilde{\Lambda})$ is a frame, or a* multi-window *Gabor frame, for $L^2(\mathbb{R})$.*

(b) *This global system forming a frame obtained from local systems is comparable to* quilted Gabor frames *introduced by Dörfler in [8], the difference being the projection of the time-frequency dictionary elements onto the time-frequency localized subspaces. In [29], Romero proved results concerning frames for general spline-type spaces from portions of given frames which provide existence conditions for quilted Gabor frames.*

(c) *If $\sum_{\mu\in\tilde{\Lambda}} \chi_{\Omega_\mu} \equiv 1$, each Λ_μ is a separable lattice, i.e. $\Lambda_\mu = a_\mu\mathbb{Z}\times b_\mu\mathbb{Z}$, $a_\mu, b_\mu > 0$, and the samples $\{\langle f, \pi(\lambda)g^\mu\rangle\}_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*}$ are given, then f can be recovered completely from the samples if the set of sampling functions $\mathcal{G} = \cup_{\mu\in\tilde{\Lambda}}\mathcal{G}(g^\mu, \Lambda_\mu\cap\Omega_\mu)$ form a quilted Gabor frame. In [12], the authors presented an approximate reconstruction of f from the given samples using approximate projection $H_{N_\mu}^\mu$ in Remark 4.9(b). In particular, the following error*

$$\left\| f - \sum_{\mu\in\tilde{\Lambda}}\sum_{\lambda\in\Lambda_\mu\cap\Omega_\mu^*} \langle f, \pi(\lambda)g^\mu\rangle H_{N_\mu}^\mu \pi(\lambda)g^\mu \right\|_2 \tag{4.10}$$

was estimated, and numerical experiments were performed in comparison with the method presented in [26] that used truncated Gabor expansions with weighted coefficients. Furthermore, it was shown, how the local *errors can be decreased by fine-tuning the overlap and number of eigenfunctions, cf. Proposition 1 in [12]. Through the numerical experiments, it was illustrated how equation* (4.10) *can be decreased by having a larger region Ω_μ^* or a larger number N_μ of eigenfunctions. Note that if equation* (4.10) *is less than $\varepsilon\|f\|_2$, $\varepsilon \in (0,1)$, for all $f \in L^2(\mathbb{R})$, then $\{\pi(\lambda)g^\mu : \mu\in\tilde{\Lambda}, \lambda\in\Lambda_\mu\cap\Omega_\mu^*\}$ and $\{H_{N_\mu}^\mu\pi(\lambda)g^\mu : \mu\in\tilde{\Lambda}, \lambda\in\Lambda_\mu\cap\Omega_\mu^*\}$ are said to be* approximately dual frames, *cf. [3], and then each of the sequences is a frame.*

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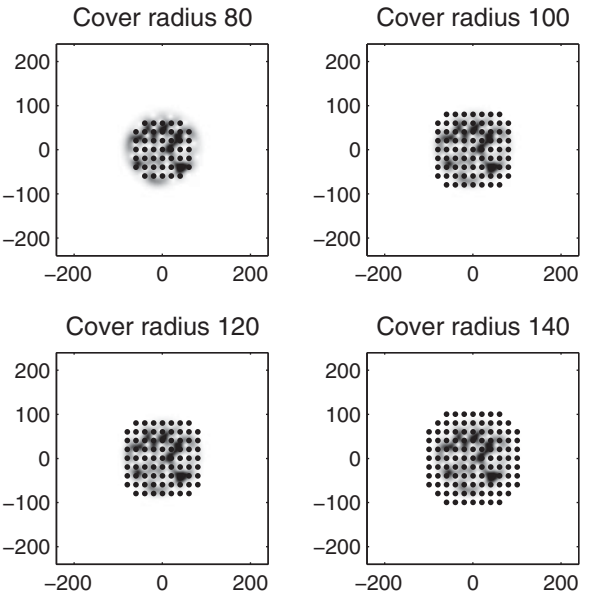


FIGURE 1. Sampling points over various enlargements of the covering region.

5 Numerical examples

In this section, we consider examples in the finite discrete case ($\mathbb{C}^L$, $L = 480$) that illustrate the results in the previous sections. The experiments were done in MATLAB and are available in the following website: http://homepage.univie.ac.at/monika.doerfler/TFlocFrames.html.

Each point set Λ that we will use for a Gabor frame is a separable lattice $a\mathbb{Z}_L \times b\mathbb{Z}_L$, where a and b are divisors of L and also called *lattice parameters* of Λ. The *redundancy* of Λ is given by $\frac{L}{ab}$. For more details on Gabor analysis in the finite discrete setting, the reader is referred to [31].

5.1 Experiment 1

We first examine the approximation of time-frequency localized signals, in particular, functions lying in the N-dimensional subspace V_N of eigenfunctions of H, by a local Gabor system as shown in Proposition 4.1. In this example, we take Ω to be a disk centred at the origin with radius 80 and φ to be a normalized Gaussian.

Figure 1 shows the STFT of a random signal in V_N and the sample points taken over circular regions with varying radii, each containing Ω. In each case, the sampling points are obtained by restricting a lattice with parameters $a = b = 20$ (redundancy 1.2) over the circular region. The error of the approximation $\|\mathcal{P}_N - S^{\mathrm{loc}}\mathcal{P}_N\|_{\mathrm{Op}}$, where S^{loc} is a truncated tight frame operator, is shown in Table 1.

We saw in Proposition 4.8 that if $\varepsilon < 1$, corresponding to the operator norm $\|\mathcal{P}_N - S^{\mathrm{loc}}\mathcal{P}_N\|_{\mathrm{Op}}$ being less than 1, then the local Gabor system projected into V_N forms a frame for V_N so perfect reconstruction is possible by the reconstruction algorithm (4.8).

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Table 1. *Error* $\|\mathcal{P}_N - S^{\mathrm{loc}}\mathcal{P}_N\|_{Op}$ *over varying radii for the disk* Ω

Cover radius	No. of samp. pts.	Op. norm error
80	45	0.9650
100	77	0.1105
120	109	0.0194
140	145	0.0031

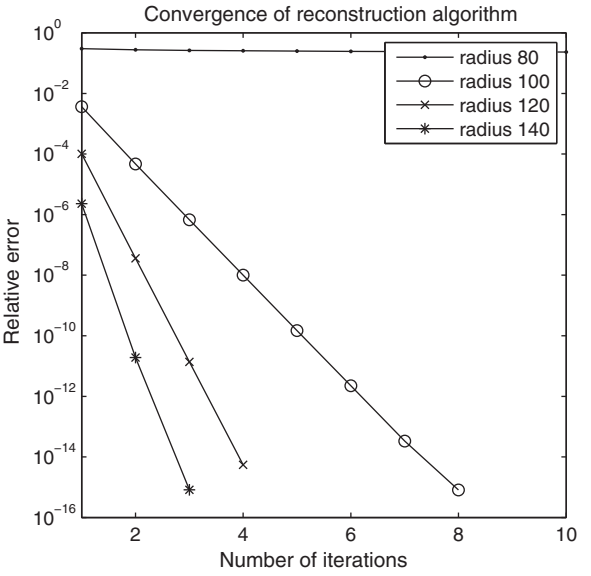


FIGURE 2. Convergence of the reconstruction algorithm from the local samples with the same lattice parameters but with varying radii of the covering regions.

The performance of the reconstruction algorithm is shown in Figure 2. As expected, the larger the covering region, the faster the convergence. This, however, comes at a price of requiring more sampling points.

5.2 Experiment 2

In this next experiment, we look at an example of how the collection of local Gabor systems can form a frame given that the sum of the characteristic functions over the regions is bounded above and below by a positive number. Figure 3 shows ten regions in the TF-plane and Figure 4 shows their sum.

For each region, sampling points from lattices corresponding to various tight global Gabor frames are taken, both with and without overlap between the regions. The lattice parameters assigned to each set are summarized in Table 2, and the sample points are depicted in Figure 5. The left image shows sample points obtained by restricting each lattice over the regions themselves, while the samples in the right image are obtained from the restriction over larger sets containing each region, thus producing more overlap.

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Table 2. *Lattice parameters over the different regions*

Region	(a, b)	Region	(a, b)
1	(20, 20)	6	(15, 15)
2	(16, 20)	7	(12, 15)
3	(20, 16)	8	(12, 12)
4	(16, 16)	9	(10, 12)
5	(15, 16)	10	(10, 10)

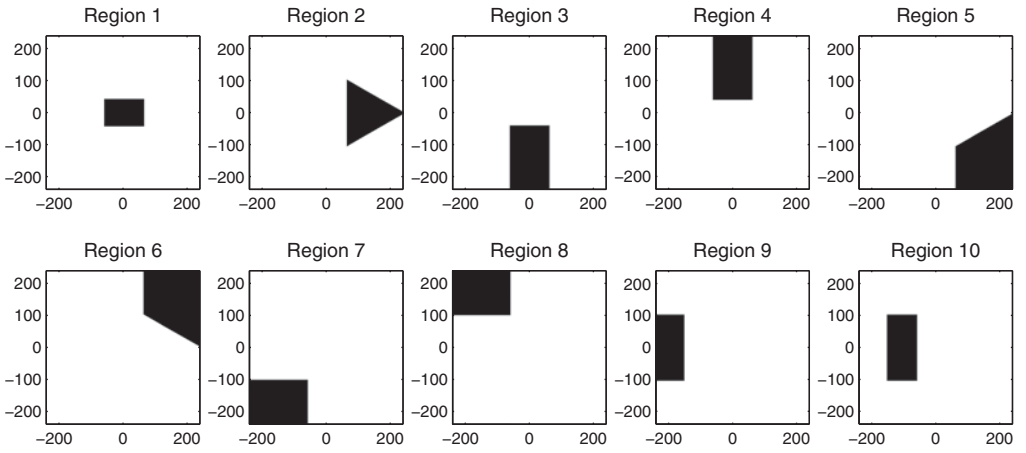


FIGURE 3. Ten regions that partition the time-frequency plane.

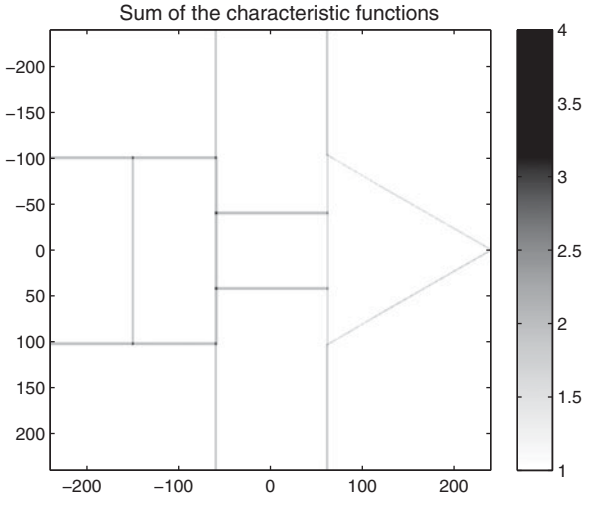


FIGURE 4. Sum of the characteristic functions over the ten regions.

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Table 3. *Average of the error in applying the frame operator to a random signal (average of 1,000 attempts)*

	Without projection	With projection
Less overlap	0.2610	0.1687
More overlap	0.5840	0.1709

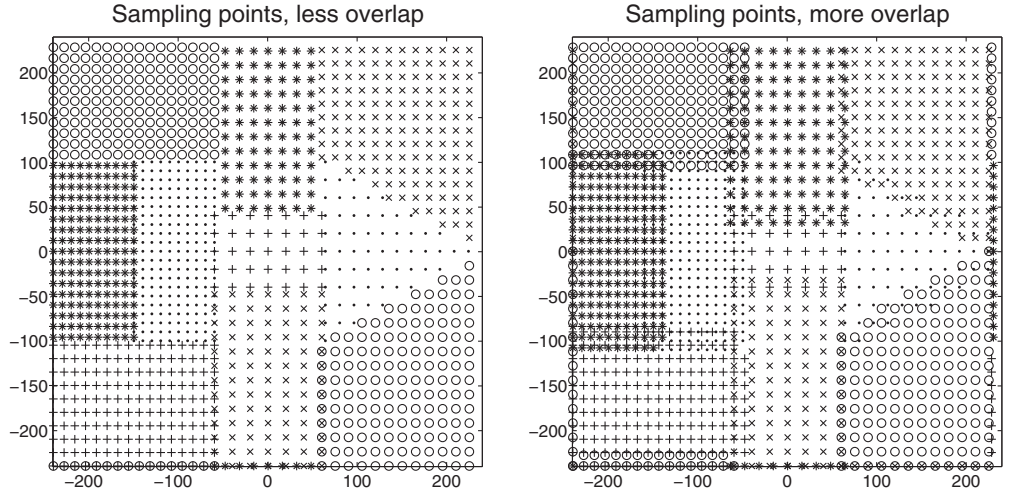


FIGURE 5. Sampling points on the different local patches.

We form a quilted Gabor frame from the collection of local Gabor systems. Then, by projecting each local Gabor system onto the local subspace corresponding to each region, we likewise obtain a global frame as in Theorem 4.10. Denote by S_1 and S_2 the frame operators corresponding to the quilted Gabor frame (i.e. without projection) and the global frame (i.e. with projection), respectively. We apply S_i to a random signal f and compute the relative error $\frac{\|f-S_if\|_2}{\|f\|_2}$. The average of the relative errors over 1,000 random signals are shown in Table 3.

In both cases of less and more overlap, projecting onto the TF-localized subspaces decreases the relative error between the signal and the approximation by the frame operator. Note that in both quilted Gabor frame and the global frame with projection, having more overlap increases the relative error since we are just comparing f with S_if. Since we are dealing with frames, perfect reconstruction (up to numerical error) is possible via the frame algorithm cf. [18, Algorithm 5.1.1].

We first compare the respective condition numbers of the frame operators for the cases of less and more overlap. The values are shown in Table 4. Once again, in both quilted Gabor frame and the global frame with projection, having more overlap improves the condition number. Note that the large condition number for the frame operator corresponding to the global frame with less overlap can be attributed to the lower frame

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Table 4. *Condition numbers of the resulting frame operators*

	Without projection	With projection
Less overlap	5.1429	16.0406
More overlap	3.5472	1.9845

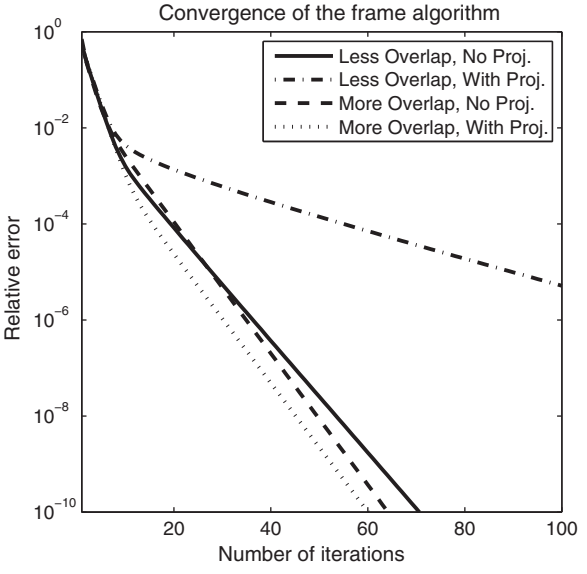


FIGURE 6. Convergence of the frame algorithm.

bound in Theorem 4.10, which is related to the set Ω_μ^* that covers the region Ω_μ – a smaller region Ω_μ^* implies a smaller lower frame bound.

Figure 6 compares the convergence of the frame algorithm for the four cases considered.

5.3 Experiment 3

In this final experiment, we illustrate the approximate reconstruction of a signal from analysis coefficients obtained from a union of tight Gabor systems, each restricted to an enlarged region covering a given region of interest. The union of the local systems forms a quilted Gabor frame, as before. However, in this experiment, the approximate reconstruction is obtained by applying the *approximate projections* $H_{N_\mu}^\mu$ to each local system, as described in Remark 4.11(c).

Similar to the experiment in [12], we consider four rectangular regions and associate tight Gabor frames to each one:

(1) $\mathcal{G}(g_1, 20, 8)$ on the region corresponding to lower frequency and time $t \leqslant L/2$;

(2) $\mathcal{G}(g_2, 24, 10)$ on the region corresponding to lower frequency and time $t > L/2$;

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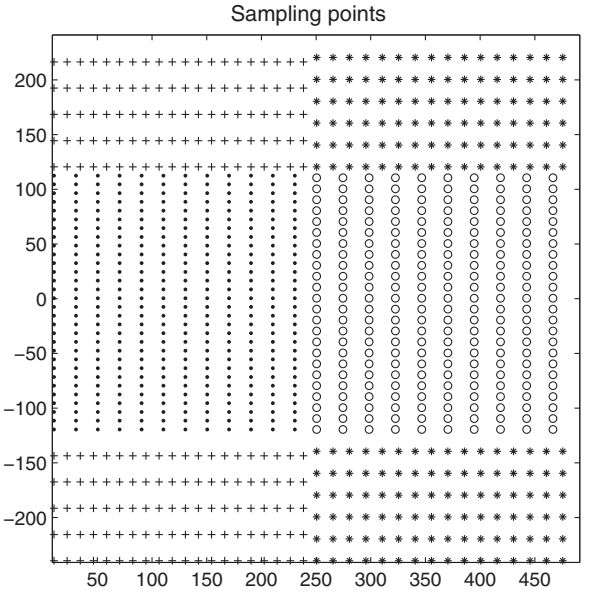


FIGURE 7. Sampling points over the four regions.

(3) $\mathcal{G}(g_3, 12, 24)$ on the region corresponding to higher frequency and time $t \leqslant L/2$;

(4) $\mathcal{G}(g_4, 15, 20)$ on the region corresponding to higher frequency and time $t > L/2$.

The sample points on the four regions are depicted in Figure 7.

We apply an approximate projection onto the subspaces of eigenfunctions of the time-frequency localization operators corresponding to each of the four regions and compute the relative error resulting from the approximate reconstruction. We compare the relative errors over varying overlap d (the amount of increase in the length of a side of the rectangular region) in Figure 8 for three different eigenspace dimensions (nEV). We also include the relative error obtained from resynthesizing with the same quilted Gabor frame elements.

For the case of quilted Gabor frames, an overlap would increase the relative error in the approximate reconstruction, as in the case of Experiment 2 (Table 3). For the cases with approximate projection, we see the decrease in the relative error as the overlap amount increases. Moreover, the relative error improves as the eigenspace dimension is increased.

We finally remark that the main motivation for the construction proposed in this example is the desire to obtain well-separated local components. We refer to detailed explanations in [12] as to how the proposed method leads to significantly improved time-frequency concentration compared to a previously introduced method which relies on weighting the coefficients directly in the time-frequency domain. The significance of the separation of components has been discussed, e.g., in [26].

6 Conclusions and perspectives

In this paper, we investigated the representation of time-frequency localized functions by means of sampling in the time-frequency domain. Motivated by the problem of providing

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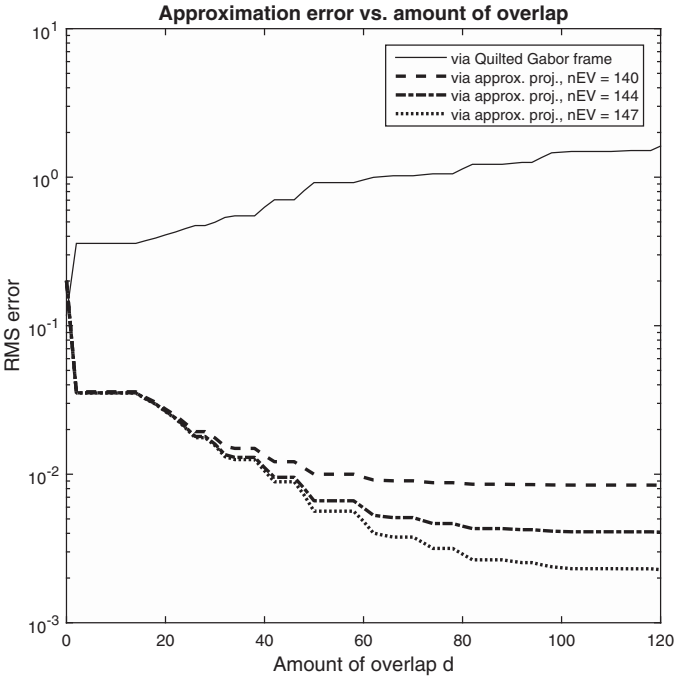


FIGURE 8. Approximation error versus amount of overlap.

dictionaries with good concentration within prescribed regions in time-frequency, we constructed frames of localized time-frequency atoms. The improved localization is obtained by means of projections onto eigenspaces corresponding to time-frequency localization operators. A natural extension of the work would be to investigate such approximation errors in the modulation space $M^p(\mathbb{R})$ norms. It is also worth investigating whether the resulting global frame forms a frame for all modulation spaces $M^p(\mathbb{R})$, as in the classical case, cf. [18] and, for instance, in the setting of non-stationary Gabor frames in [9].

Numerical experiments illustrated the promising potential of the proposed method: providing good reconstruction/approximation quality while preserving the good localization property. However, for applications to real signals, the proposed method would entail having to compute eigenvalues and eigenfunctions of large matrices, which may be numerically cumbersome. The study of efficient numerical methods for the evaluation/approximation of eigenvalues and eigenfunctions of time-frequency localization operators is a topic of future work.

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