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Summability implies Collet–Eckmann almost surely

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Abstract. We provide a strengthened version of the famous Jakobson's theorem. Consider an interval map f satisfying a summability condition. For a generic one-parameter family f_t of maps with $f_0 = f$, we prove that $t = 0$ is a Lebesgue density point of the set of parameters for which f_t satisfies both the Collet–Eckmann condition and a strong polynomial recurrence condition.

1. *Introduction*

The famous result of Jakobson [**J**] states that maps with stochastic behavior are abundant, in the probabilistic sense, in one-dimensional dynamics, which opened the way to much progress in non-uniformly expanding dynamics. Several alternative proofs and generalizations of this result were obtained in subsequent works, see [**AM05, BC1, BC2, Lu, Ly1, R, T, TTY, V, WT, Yo**] among others. In this paper, we shall provide another generalization of this result.

To state our result, we start with some definitions. Let $f : [0, 1] \to [0, 1]$ be a C^1 map and let $\mathcal{C}(f)$ denote the set of critical points of f. We say that f satisfies the *summability condition* (abbreviated (SC)), if for each $c \in \mathcal{C}(f)$, we have

$$\sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|} < \infty.$$

We say that f satisfies the *Collet–Eckmann condition* (abbreviated (CE)), if for each $c \in \mathcal{C}(f)$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(f(c))| > 0.$$

Furthermore, we say that f satisfies the *polynomial recurrence condition of exponent* β (abbreviated (PR_β)), if there exists $C > 0$ such that for any c, $c' \in \mathcal{C}(f)$ and any $n \geq 1$, we have

$$\operatorname{dist}(f^n(c), c') \geq Cn^{-\beta}.$$

If, for each $\beta > 1$, f satisfies PR_β, then we say that f satisfies the *strong polynomial recurrence condition* (abbreviated (SPR)).

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Let $\mathcal{A}$ be the collection of C^1 interval maps $f : [0, 1] \to [0, 1]$ with the following properties:

- f has no attracting or neutral periodic orbits;
- each critical point of f lies in the interior $(0, 1)$;
- f is C^3 outside $\mathcal{C}(f)$;
- for each critical point c, there exist $\ell > 1$ and a C^3 diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(c) = 0$ and such that $|f(x) - f(c)| = |\varphi(x) - \varphi(c)|^\ell$ holds near c.

Consider a one-parameter C^1 family $f_t : [0, 1] \to [0, 1]$, $t \in [-\delta, \delta]$, with $f_0 \in \mathcal{A}$. We say that this family is *regular* if the following hold:

(1) the map $(t, x) \mapsto f_t(x)$ is C^2 on $\{(t, x) \in [-\delta, \delta] \times [0, 1] : f_t'(x) \neq 0\}$;

(2) there exist C^2 functions $c_i : [-\delta, \delta] \to (0, 1)$, $i = 1, 2, \ldots, d$, such that $0 < c_1(t) < c_2(t) < \cdots < c_d(t) < 1$ and $\mathcal{C}(f_t) = \{c_i(t) : 1 \leq i \leq d\}$;

(3) for each $1 \leq i \leq d$, there exist $\ell_i > 1$, $\varepsilon > 0$ and a C^2 family φ_t of diffeomorphisms of $\mathbb{R}$ such that $\varphi_t(c_i(t)) = 0$, and $|f_t(x) - f_t(c_i(t))| = |\varphi_t(x) - \varphi_t(c_i(t))|^{\ell_i}$ holds when $|x - c_i(t)| < \varepsilon$ and $|t| \leq \delta$.

It is easy to see that if $f_t : [0, 1] \to [0, 1]$, $t \in [-1, 1]$, is a C^3 family such that $f_0 \in \mathcal{A}$ has only non-degenerate critical points, then for $\delta > 0$ small enough, $\{f_t\}_{|t|<\delta}$ is a regular family. In addition, if f_t, $t \in [-1, 1]$, is a real analytic family such that all of the maps f_t have the same number of critical points, and the corresponding critical points have the same order, then f_t is regular.

For a measurable subset X of $\mathbb{R}^n$, let $\mathrm{Leb}_n(x)$ denote the Lebesgue measure of X in $\mathbb{R}^n$. For simplicity, let $|X|$ denote $\mathrm{Leb}_1(X)$ for the measurable set $X \subset \mathbb{R}$.

MAIN THEOREM. *Consider a regular one-parameter family $f_t : [0, 1] \to [0, 1]$, $t \in [-1, 1]$, and denote $F(x, t) = f_t(x)$. Assume that:*

- *f_0 satisfies the summability condition (SC);*
- *the following non-degeneracy condition holds for $t = 0$:*

$$(\mathrm{NV_t}) \qquad \sum_{j=0}^{\infty} \frac{\partial_t F(f_t^j(c), t)}{Df_t^j(f_t(c))} \neq 0 \quad \textit{for any critical point } c \in \mathcal{C}(f_t).$$

Define

$$\mathcal{Z} := \{t \in [-1, 1] : f_t \textit{ satisfies (CE), (SPR) and } (NV_t)\}.$$

Then we have

$$\lim_{\varepsilon \to 0} \frac{|[-\varepsilon, \varepsilon] \cap \mathcal{Z}|}{2\varepsilon} = 1.$$

In particular, $|\mathcal{Z}| > 0$.

Like most of the approaches to the Jakobson's theorem, our proof is purely real analytic. Compared with the previous works, our assumption on f_0 is much weaker and the result on strong polynomial recurrence condition is new. Previously the weakest assumption was given in [**T**], where f_0 satisfies (CE) and the critical points are at most sub-exponentially recurrent. Our analysis on the phase space geometry is based on the recent work [**S**] by the second author, and these estimates are transformed to the parameter space by modifying the argument in [**T**].

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For the family of real quadratic polynomials, our theorem is implicitly contained in [**AM05**], where the complex method developed in [**Ly1**] was applied to relate the phase and parameter spaces. The complex method is powerful for uni-critical maps, but does not work for multimodal maps.

The non-degeneracy condition ($\mathrm{NV_t}$) was introduced in [**T**]. In [**AM03**], a geometric interpretation of this condition was given: for a real analytic family f_t of unimodal maps for which f_0 satisfies (SC), ($\mathrm{NV_t}$) holds at $t = 0$ if and only f_t is transversal to the topological conjugacy class of f_0. In [**A, Le1**], it was proved that for the family of quadratic maps $Q_t(z) = z^2 + t$, if Q_{t_0} satisfies (SC) then the condition ($\mathrm{NV_t}$) automatically holds at $t = t_0$. By [**Ly2**], for almost every $t \in \mathbb{R}$, Q_t is either uniformly hyperbolic or satisfies (SC). Thus, our theorem gives a new proof of Theorem A and a part of Theorem B of [**AM05**].

Recently this transversality result has been generalized to higher-degree polynomials in [**Le2**]. With this result, we can extend our Main Theorem to the high-dimensional version. More precisely, for any positive integer n and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, let $P_{\mathbf{a}} = \sum_{i=1}^{n} a_i x^i + (1 - \sum_{i=1}^{n} a_i)x^{n+1}$. Hence, $P_{\mathbf{a}}(0) = 0$ and $P_{\mathbf{a}}(1) = 1$.

Let $\mathcal{P}$ be the collection of polynomial maps P with the following properties:

- $P([0, 1]) \subset [0, 1]$;
- $P|_{[0,1]} \in \mathcal{A}$ and $P|_{[0,1]}$ satisfies (SC).

COROLLARY. *Fixing a positive integer n, we define parameter sets*

$$\Lambda = \{\boldsymbol{a} \in \mathbb{R}^n : P_{\boldsymbol{a}} \in \mathcal{P}\},$$

and

$$\Lambda_0 = \{\boldsymbol{a} \in \Lambda : P_{\boldsymbol{a}}|_{[0,1]} \textit{ satisfies (CE) and (SPR) conditions}\}.$$

Then we have $\mathrm{Leb}_n(\Lambda \backslash \Lambda_0) = 0$.

Proof. Consider the parameter set $\Lambda_1 = \{\mathbf{a} \in \mathbb{R}^n : P_{\mathbf{a}} \text{ has degenerate critical points}\}$. For any $\mathbf{a} \in \Lambda_1$, the discriminant $\Delta(\mathbf{a})$ of $P'_{\mathbf{a}}$ is equal to zero. Since $\Delta(\mathbf{a})$ is a polynomial in $\mathbf{a}$, the set Λ_1 has codimension one in $\mathbb{R}^n$, hence $\mathrm{Leb}_n(\Lambda_1) = 0$.

Define

$$\Pi = \{\mathbf{a} \in \mathbb{C}^n : \text{ all critical points of } P_{\mathbf{a}} \text{ are non-degenerate}\}.$$

Fix $\mathbf{a}_* \in \Pi$. For $\mathbf{a}$ in a small neighborhood of $\mathbf{a}_*$, the critical points of $P_{\mathbf{a}}$, $c_1(\mathbf{a}), c_2(\mathbf{a}), \ldots, c_n(\mathbf{a})$, depend on $\mathbf{a}$ analytically. Letting $v_j(\mathbf{a}) = P_{\mathbf{a}}(c_j(\mathbf{a}))$ for $j = 1, 2, \ldots, n$, by [**Le3**, Proposition 1], $\{v_1(\mathbf{a}), v_2(\mathbf{a}), \ldots, v_n(\mathbf{a})\}$ is a local analytic coordinate.

Now let $\mathbf{a}_* \in \Lambda \backslash \Lambda_1$. Suppose that $c_1, c_2, \ldots, c_r$ are all the critical points of $P_{\mathbf{a}_*}$ in $(0, 1)$. By [**Le2**, Theorem 1], the rank of the matrix

$$\mathbf{L} = (L(c_j, v_k))_{1 \le j \le r, 1 \le k \le n}$$

is equal to r, where

$$L(c_j, v_k) := \lim_{m \to \infty} \frac{\partial P^m_{\mathbf{a}}(c_j)/\partial v_k|_{\mathbf{a}=\mathbf{a}_*}}{(P^{m-1}_{\mathbf{a}_*})'(P_{\mathbf{a}_*}(c_j))}.$$

Note that $\{a_1, a_2, \ldots, a_k\}$ is a global analytic coordinate, and we define

$$L(c_j, a_k) := \lim_{m \to \infty} \frac{\partial P^m_{\mathbf{a}}(c_j)/\partial a_k|_{\mathbf{a}=\mathbf{a}_*}}{(P^{m-1}_{\mathbf{a}_*})'(P_{\mathbf{a}_*}(c_j))}.$$

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Hence, the rank of the matrix

$$\widehat{\mathbf{L}} = (L(c_j, a_k))_{1\le j\le r, 1\le k\le n}$$

is equal to r and all entries of $\widehat{\mathbf{L}}$ are real numbers.

For any direction $\mathbf{u} \in S^{n-1}$, let $F^{(\mathbf{u})}(x, t) := P_{\mathbf{a}_*+t\mathbf{u}}(x)$, then we have

$$\sum_{m=0}^{\infty} \frac{\partial_t F^{(\mathbf{u})}(P^m_{\mathbf{a}_*}(c_j), 0)}{DP^m_{\mathbf{a}_*}(P_{\mathbf{a}_*}(c_j))} = (L(c_j, a_1), L(c_j, a_2), \ldots, L(c_j, a_k)) \cdot \mathbf{u}.$$

Thus, the (NV_0) condition holds for $F^{(\mathbf{u})}(x, t)$ if and only if all entries of $\widehat{\mathbf{L}} \cdot \mathbf{u}$ are non-zero. Since the rank of the matrix $\widehat{\mathbf{L}}$ is equal to r, all rows of $\widehat{\mathbf{L}}$ are non-zero. If the kth entry of $\widehat{\mathbf{L}} \cdot \mathbf{u}$ is equal to zero, then $\mathbf{u}$ is contained in the intersection of a hyperplane in $\mathbb{R}^n$ and S^{n-1}. Thus, for almost all $\mathbf{u}$ in S^{n-1} (endowed with the Lebesgue measure on S^{n-1}), all entries of $\widehat{\mathbf{L}} \cdot \mathbf{u}$ are non-zero.

Hence, for almost every direction $\mathbf{u}$ in S^{n-1}, the (NV_0) condition holds for the one-parameter family $F^{(\mathbf{u})}(x, t)$. Together with our Main Theorem, it follows that $\mathbf{a}_*$ is a density point of the set Λ_0 along the line $\mathbf{a}_* + t\mathbf{u}$. By [**AM03**, Proposition 5.2], $\mathrm{Leb}_n((\Lambda\backslash\Lambda_1)\backslash\Lambda_0) = 0$. Then the statement follows. □

The paper is organized as follows. In §2, we state a Reduced Main Theorem from which we deduce the Main Theorem. The rest of the paper is devoted to the proof of the Reduced Main Theorem. As described by Adrien Douady, the proof consists of two steps: in §3 we 'plough in the phase space' and in §4 we 'harvest in the parameter space'.

2. *Reduction*

2.1. *Normalization.* A regular family $g_t : [0, 1] \to [0, 1]$, $t \in [-1, 1]$, is said to be *normalized* if the following hold:

(i) the maps g_t, $t \in [-1, 1]$, all have the same critical points (denoted by $c_1, c_2, \ldots, c_d$);

(ii) there exists $\varepsilon_\# > 0$ and for each $i \in \{1, 2, \ldots, d\}$ there exists $\ell(c_i) > 1$ such that $|g_t(x) - g_t(c_i)| = |x - c_i|^{\ell(c_i)}$ holds for all $t \in [-1, 1]$ and $x \in B(c_i, \varepsilon_\#)$;

(iii) $|\partial_t G(x, t)| \le 1$ for all $x \in [0, 1]$ and $|t| \le 1$.

To prove the Main Theorem, we only need to consider a normalized regular family. Indeed, given any regular family $f_t : [0, 1] \to [0, 1]$, $t \in [-1, 1]$, one can find a C^2 family h_t, $t \in [-1, 1]$, of diffeomorphisms from $[0, 1]$ onto itself, such that $\widetilde{f}_t = h_t \circ f_t \circ h_t^{-1}$ all have the same critical points and are normalized as in (ii). Furthermore, take a small constant κ and define $g_t = \widetilde{f}_{t\kappa}$. Then the family $G(x, t) = g_t(x)$, $t \in [-1, 1]$, satisfies all the properties (i)–(iii). Note also that if f_0 satisfies (SC), then g_0 satisfies (SC); and if F satisfies the non-degeneracy condition $(\mathrm{NV_t})$ at $t = 0$, then so does G.

In the remaining of this paper, we assume that F is a normalized regular family. Let $\mathcal{C}$ denote the common set of critical points of f_t, and let

$$\ell_{\max} = \max\{\ell(c) : c \in \mathcal{C}\}, \quad \ell_{\min} = \min\{\ell(c) : c \in \mathcal{C}\}.$$

Moreover, let $f = f_0$ and $\mathrm{CV} = f(\mathcal{C})$.

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2.2. *The (CE) and (PR) conditions.* For each $c \in \mathcal{C}$ and $\delta > 0$, let

$$\widetilde{B}(c;\,\delta) = B(c,\,\delta^{1/\ell(c)}), \quad D_c(\delta) = \frac{\delta}{|\widetilde{B}(c;\,\delta)|} = \frac{1}{2}\delta^{1-\ell(c)^{-1}},$$

and let

$$\widetilde{B}(\delta) = \bigcup_{c\in\mathcal{C}} \widetilde{B}(c;\,\delta).$$

The space $\{f_t\}_{t\in[-1,1]}$ is admissible in the sense of [**S**]. Thus, by [**S**, Theorem 1], we have the following proposition.

PROPOSITION 2.1. *For each $\varepsilon > 0$ small enough, there exist $\Lambda(\varepsilon) > 1$ and $\alpha(\varepsilon) > 0$ such that*

$$\lim_{\varepsilon\to 0} \Lambda(\varepsilon) = \infty, \quad \lim_{\varepsilon\to 0} \alpha(\varepsilon) = 0$$

and the following hold for $|t| < \varepsilon$.

(i) *Let $x \in [0,\,1]$ be such that $\operatorname{dist}(x, \mathrm{CV}) \le 4\varepsilon$, with $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $j = 0,\,1,\,\ldots,\,s-1$ and $f_t^s(x) \in \widetilde{B}(c;\,2\varepsilon)$ for some $c \in \mathcal{C}$. Then,*

$$|Df_t^s(x)| \ge \frac{\Lambda(\varepsilon)}{D_c(\varepsilon)} \exp(\varepsilon^{\alpha(\varepsilon)} s). \tag{2.1}$$

(ii) *Let $x \in [0,\,1]$ be such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $j = 0,\,1,\,\ldots,\,s-1$, then*

$$|Df_t^s(x)| \ge A\varepsilon^{1-\ell_{\max}^{-1}} \exp(\varepsilon^{\alpha(\varepsilon)} s), \tag{2.2}$$

where $A > 0$ is a constant independent of ε.

Remark. This result is our starting point to prove abundance of Collet–Eckmann parameters near a summable one. The proof is based on decomposition of an f_t-orbit into pieces that can be shadowed by f_0-orbits and a delicate choice of the binding periods played a central role.

Let $\mathbb{N} = \{0,\,1,\,\ldots\}$ denote the set of natural numbers. Define

$$q_\varepsilon(x) = \inf\left\{k \in \mathbb{N} : x \notin \bigcup_{c\in\mathcal{C}} \widetilde{B}(c;\,e^{-k\ell(c)}\varepsilon)\right\}. \tag{2.3}$$

Note that for $x \in \widetilde{B}(c;\,\varepsilon)$ with $\varepsilon > 0$ small and $c \in \mathcal{C}$, we have

$$\begin{aligned} |Df_t(x)| &= \ell(c)\operatorname{dist}(x,\,\mathcal{C})^{\ell(c)-1} \ge \ell(c)(e^{-q_\varepsilon(x)\ell(c)}\varepsilon)^{1-\ell(c)^{-1}} \\ &> e^{-q_\varepsilon(x)(\ell(c)-1)} D_c(\varepsilon) > e^{-q_\varepsilon(x)\ell_{\max}} D_c(\varepsilon). \end{aligned} \tag{2.4}$$

Thus, the following is an immediate consequence of Proposition 2.1.

LEMMA 2.1. *Provided that $\varepsilon > 0$ is small enough, the following holds: for any $y \in \widetilde{B}(\varepsilon)$, $t \in [-\varepsilon,\,\varepsilon]$, and $n \ge 1$, putting*

$$m = \#\{1 \le k \le n : f_t^k(y) \in \widetilde{B}(\varepsilon)\},$$

we have

$$|Df_t^n(f_t(y))| \ge A\varepsilon^{1-\ell_{\max}^{-1}} \Lambda(\varepsilon)^m \exp\left(-\ell_{\max} \sum_{k=1}^{n} q_\varepsilon(f_t^k(y))\right) e^{\varepsilon^{\alpha(\varepsilon)} n}. \tag{2.5}$$

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Furthermore, if $f_t^n(y) \in \widetilde{B}(\varepsilon)$, *then*

$$|Df_t^n(f_t(y))| \geq \Lambda(\varepsilon)^m \exp\left(-\ell_{\max} \sum_{k=1}^{n} q_\varepsilon(f_t^k(y))\right). \tag{2.6}$$

Proof. Let $0 = n_0 < n_1 < \cdots < n_m$ be all the integers in $\{0, 1, \ldots, n\}$ such that $f_t^{n_j}(y) \in \widetilde{B}(\varepsilon)$. Note that $\operatorname{dist}(f_t(y), \mathrm{CV}) \leq 2\varepsilon$. Applying Proposition 2.1(i) to obtain lower bounds for $|Df_t^{n_{j+1}-n_j-1}(f_t^{n_j+1}(y))|$, $0 \leq j < m$, applying (ii) to obtain lower bounds for $|Df_t^{n-n_m-1}(f_t^{n_m+1}(y))|$ in the case $n_m < n$, and applying (2.4) give us the desired inequalities. □

For $t \in [-1, 1]$, $\varepsilon > 0$ and $c \in \mathcal{C}$, let $S_1^{(c)}(t;\varepsilon) < S_2^{(c)}(t;\varepsilon) < \cdots < S_n^{(c)}(t;\varepsilon) < \cdots$ be the positive integers such that $f_t^{S_j^{(c)}(t;\varepsilon)+1}(c) \in \widetilde{B}(\varepsilon)$, and let

$$d_j^{(c)}(t;\varepsilon) = q_\varepsilon(f_t^{S_j^{(c)}(t;\varepsilon)+1}(c)). \tag{2.7}$$

2.2.1. *Convention.* If c returns to $\widetilde{B}(\varepsilon)$ at most $n-1$ times, then let $S_n^{(c)}(t;\varepsilon) = \infty$ and $d_n^{(c)}(t;\varepsilon) = 0$.

Given $C > 0$, for each $n = 1, 2, \ldots$, we define

$$X_{n,\varepsilon}(C) = \left\{ t \in [-\varepsilon, \varepsilon] : \sum_{j=1}^{k} d_j^{(c)}(t;\varepsilon) \leq Ck \text{ for any } k < n \text{ and } c \in \mathcal{C} \right\}, \tag{2.8}$$

and

$$X_\varepsilon(C) = \bigcap_{n=1}^{\infty} X_{n,\varepsilon}(C). \tag{2.9}$$

Given $C > 0$ and $\tau > 0$, for each $m = 0, 1, \ldots$, we define

$$Y_\varepsilon^m(C,\tau) = \left\{ t \in X_\varepsilon(C) : \operatorname{dist}(f_t^{k+1}(\mathcal{C}), c) \geq \frac{\varepsilon^{1/\ell(c)}}{(k+1)^\tau} \text{ for } 0 \leq k < m \text{ and } c \in \mathcal{C} \right\} \tag{2.10}$$

and

$$Y_\varepsilon(C,\tau) = \bigcap_{m=0}^{\infty} Y_\varepsilon^m(C,\tau). \tag{2.11}$$

LEMMA 2.2. *We have the following.*

(i) *Given $C > 0$, the following holds provided that $\varepsilon > 0$ is small enough: for any $t \in X_\varepsilon(C)$, f_t satisfies the condition (CE).*

(ii) *For any $C > 0$, $\tau > 1$ and $\alpha > 0$, if $t \in Y_\varepsilon(C,\tau)$ and $\varepsilon > 0$ is small enough, then f_t satisfies the condition* PR_τ.

Proof. For any $t \in X_\varepsilon(C)$, $n \geq 1$ and any $c \in \mathcal{C}$, let

$$m = \#\{1 \leq k \leq n : f_t^k(c) \in \widetilde{B}(\varepsilon)\}.$$

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By Lemma 2.1, we have

$$|Df_t^n(f_t(c))| \geq A\varepsilon^{1-\ell_{\max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)}n} \Lambda(\varepsilon)^m \exp\left(-\ell_{\max} \sum_{j=1}^{m} d_j^{(c)}(t;\varepsilon)\right)$$
$$\geq A\varepsilon^{1-\ell_{\max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)}n} (\Lambda(\varepsilon) e^{-\ell_{\max} C})^m \geq A\varepsilon^{1-\ell_{\max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)}n},$$

provided that $\varepsilon > 0$ is small enough so that $\Lambda(\varepsilon) \geq e^{\ell_{\max} C}$. Hence, f_t satisfies the condition (CE). The second statement is trivial. □

REDUCED MAIN THEOREM. *Let $F = (f_t)$ be a normalized regular family of interval maps. Assume that f_0 satisfies* (SC) *and that the condition* ($\mathrm{NV_t}$) *holds at $t = 0$. Then:*

(i) *given $C > 0$ there exists $K = K(C) > 0$ such that $K(C) \to \infty$ as $C \to \infty$ and such that*

$$|X_{n,\varepsilon}(C) \backslash X_{n+1,\varepsilon}(C)| \leq K^{-n}\varepsilon, \quad n = 1, 2, \ldots,$$

provided that $\varepsilon > 0$ is small enough;

(ii) *given $C > 0$, for any $t \in X_\varepsilon(C)$,* ($\mathrm{NV_t}$) *holds provided that $\varepsilon > 0$ is small enough;*

(iii) *given $C > 0$, $\tau > \tau_0 > 1$, and $\sigma > 0$, we have*

$$|Y_\varepsilon^m(C,\tau) \backslash Y_\varepsilon^{m+1}(C,\tau)| \leq \sigma\varepsilon(m+1)^{-\tau_0}, \quad m = 0, 1, \ldots,$$

provided that $\varepsilon > 0$ is small enough.

Proof of the Main Theorem. For $\tau > 1$, let $\mathcal{Z}_\tau$ denote the set of parameters $t \in [-1, 1]$ for which f_t satisfies the conditions (CE), ($\mathrm{NV_t}$) and (PR_τ). We shall prove that

$$\lim_{\varepsilon \to 0} \frac{|\mathcal{Z}_\tau \cap [-\varepsilon, \varepsilon]|}{2\varepsilon} = 1. \tag{2.12}$$

Fix $\tau_0 \in (1, \tau)$ and $\eta > 0$. Choose a large constant $C > 0$ and a small constant $\sigma > 0$, such that

$$\frac{2K-3}{K-1} - \sigma \sum_{m=0}^{\infty} (m+1)^{-\tau_0} > 2 - \eta,$$

where $K = K(C)$ is as in the Reduced Main Theorem. Provided that $\varepsilon > 0$ is small enough, we have

$$|X_\varepsilon(C)| = |X_{1,\varepsilon}(C)| - \sum_{n=1}^{\infty} |X_{n,\varepsilon}(C) \backslash X_{n+1,\varepsilon}(C)| \geq \frac{2K-3}{K-1} \cdot \varepsilon,$$

and

$$|Y_\varepsilon(C,\tau)| \geq |X_\varepsilon(C)| - \sum_{m=0}^{\infty} |Y_\varepsilon^m(C,\tau) \backslash Y_\varepsilon^{m+1}(C,\tau)| \geq (2-\eta)\varepsilon.$$

By Lemma 2.2, and the second statement of the Reduced Main Theorem, we have $Y_\varepsilon(C,\tau) \subset \mathcal{Z}_\tau$. Thus,

$$|\mathcal{Z}_\tau \cap [-\varepsilon, \varepsilon]| \geq (2-\eta)\varepsilon.$$

The equality (2.12) follows.

To complete the proof, we shall show that $\mathcal{Z}_2 \backslash \mathcal{Z}$ has zero measure. Since $\mathcal{Z} = \bigcap_{k=1}^{\infty} \mathcal{Z}_{1+k^{-1}}$, we only need to show that for each $\tau > 1$, $\mathcal{Z}_2 \backslash \mathcal{Z}_\tau$ has measure zero. Indeed, for each $t_0 \in \mathcal{Z}_2$ and $\tau > 1$, we can apply the above argument to f_{t_0} instead of f_0, and obtain that t_0 is not a Lebesgue density point of $\mathcal{Z}_2 \backslash \mathcal{Z}_\tau$. By the Lebesgue density theorem, the statement follows. □

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2.2.2. *Notation.* We collect the notation which will be used in the rest of the paper. For each $c \in \mathcal{C}$, let

$$W^{(c)} = \sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < \infty$$

and

$$a^{(c)} = \sum_{n=0}^{\infty} \frac{\partial_t F(f^n(c), 0)}{Df^n(f(c))} \neq 0.$$

For any $x \in [0, 1]$, $t \in [-1, 1]$ and $n \in \mathbb{N}$, we define

$$\mathrm{A}(x, t, n) = \sum_{j=0}^{n-1} \frac{|Df_t^j(x)|}{\mathrm{dist}(f_t^j(x), \mathcal{C})}.$$

So if $f_t^j(x) \in \mathcal{C}$ for some $j < n$, then $\mathrm{A}(x, t, n) = \infty$.

As before let $S_j^{(c)}(t; \varepsilon)$ denote the jth return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, and let $d_j^{(c)}(t; \varepsilon)$ be as defined in (2.7). Define

$$P_j^{(c)}(t; \varepsilon) = \frac{|Df_t^{S_j^{(c)}(t;\varepsilon)}(f_t(c))|}{\mathrm{dist}(f_t^{S_j^{(c)}(t;\varepsilon)+1}(c), \mathcal{C})},$$

$$p_j^{(c)}(t; \varepsilon) = \log \frac{|P_j^{(c)}(t; \varepsilon)|}{\mathrm{A}(f_t(c), t, S_j^{(c)}(t; \varepsilon))},$$

and

$$\widetilde{p}_j^{(c)}(t; \varepsilon) = \min\{p_j^{(c)}(t; \varepsilon), d_j^{(c)}(t; \varepsilon)\}.$$

3. *Ploughing in the phase space*

In this section, we obtain some estimates in the phase space. The main results are Propositions 3.1 and 3.2 below. Lemmas 3.2, 3.4 and 3.5 used in the argument are taken from [**S**]. Note that the non-degeneracy condition ($\mathrm{NV_t}$) plays no role in this section.

3.1. *A uniform summability.*

PROPOSITION 3.1. *Given $\delta > 0$, the following holds provided that $\varepsilon > 0$ is small enough. For any $t \in [-\varepsilon, \varepsilon]$, $c \in \mathcal{C}$, $x \in [0, 1]$ with $\mathrm{dist}(x, f(c)) \leq 4\varepsilon$, if n is a non-negative integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ holds for all $0 \leq j < n$, then*

$$\sum_{j=0}^{n} |Df_t^j(x)|^{-1} \leq W^{(c)} + \delta.$$

Before we prove this proposition, let us state a corollary.

COROLLARY 3.1. *Given $\theta > 0$ and $C > 0$, the following holds provided that $\varepsilon > 0$ is small enough: for each $t \in X_{n,\varepsilon}(C)$ and $c \in \mathcal{C}$, if S_n is the nth return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, then*

$$\sum_{i=0}^{S_n} |Df_t^i(f_t(c))|^{-1} \leq W^{(c)} + \theta.$$

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Proof. Denote $W = \max_{c\in\mathcal{C}} W^{(c)}$ and fix constants $\delta \in (0, \theta)$ and $\Lambda > (W+\theta)/(\theta-\delta)$. Let $S_0 = -1$, and for each $j \geq 1$, let S_j be the jth return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Write $y_j = f_t^{S_j+1}(c)$, $x_j = f_t(y_j)$. Provided that $\varepsilon > 0$ is small enough, by Proposition 3.1, we have

$$H_k := \sum_{i=0}^{S_{k+1}-S_k-1} |Df_t^i(x_k)|^{-1} \leq W + \delta,$$

for each $k = 1, 2, \ldots$. Moreover, by Lemma 2.1, we have $|Df_t^{S_k+1}(f_t(c))| \geq \Lambda^k$. Thus,

$$\sum_{i=0}^{S_n} |Df_t^i(f_t(c))|^{-1} \leq W^{(c)} + \delta + \sum_{k=1}^{n-1} |Df_t^{S_k+1}(f_t(c))|^{-1} H_k < W^{(c)} + \theta. \qquad \square$$

Fix $\varepsilon_0 > 0$ small such that Proposition 2.1 holds for all $\varepsilon \in (0, 4\varepsilon_0]$ with $\Lambda(\varepsilon) \geq 4$. For each $\varepsilon, \varepsilon' \in (0, 4\varepsilon_0]$ and $c \in \mathcal{C}$, let $\mathcal{D}^{(c)}(\varepsilon, \varepsilon')$ be the collection of all triples (x, t, n) with the following properties: $|x - f(c)| \leq 4\varepsilon'$, $|t| \leq \varepsilon$, and n is a non-negative integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for all $0 \leq j < n$, and let

$$\widehat{L}^{(c)}(\varepsilon, \varepsilon') = \sup\left\{\sum_{i=0}^{n} |Df_t^i(x)|^{-1} : (x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon')\right\}.$$

Moreover, let

$$L^{(c)}(\varepsilon) = \widehat{L}^{(c)}(\varepsilon, \varepsilon),$$
$$L_*^{(c)}(\varepsilon) = \sup\{L^{(c)}(\varepsilon') : \varepsilon' \in [\varepsilon, 4\varepsilon_0]\}, \quad L_*(\varepsilon) = \max_{c\in\mathcal{C}} L_*^{(c)}(\varepsilon),$$

and

$$\widehat{L}(\varepsilon, \varepsilon') = \max_{c\in\mathcal{C}} \widehat{L}^{(c)}(\varepsilon, \varepsilon').$$

By Proposition 2.1(ii), $1 \leq L_*(\varepsilon) < \infty$ for each $\varepsilon > 0$.

LEMMA 3.1. *For any $0 < \varepsilon \leq \varepsilon' \leq 2\varepsilon_0$, we have*

$$\widehat{L}(\varepsilon, \varepsilon') \leq 4L_*(\varepsilon)\left(\frac{\varepsilon'}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}. \tag{3.1}$$

Proof. It suffices to prove that for any integer $k \geq 0$ such that $2^k\varepsilon \leq 4\varepsilon_0$, we have

$$\widehat{L}(\varepsilon, 2^k\varepsilon) \leq 2L_*(\varepsilon) \cdot 2^{k(1-\ell_{\max}^{-1})}. \tag{3.2}$$

Indeed, this implies that for any $\varepsilon' \in [2^{k-1}\varepsilon, 2^k\varepsilon]$, we have

$$\widehat{L}(\varepsilon, \varepsilon') \leq \widehat{L}(\varepsilon, 2^k\varepsilon) \leq 2L_*(\varepsilon) \cdot 2^{k(1-\ell_{\max}^{-1})} < 4L_*(\varepsilon)\left(\frac{\varepsilon'}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}. \tag{3.3}$$

Let us prove (3.2) by induction on k. By definition, the case $k = 0$ is clear. Assume that (3.2) holds for all k not greater than some j. Let us consider the case $k = j+1$ with $2^{j+1}\varepsilon \leq 4\varepsilon_0$. For $c \in \mathcal{C}$ and $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, 2^{j+1}\varepsilon)$, we need to prove that

$$\sum_{i=0}^{n} |Df_t^i(x)|^{-1} \leq 2L_*(\varepsilon) \cdot 2^{(j+1)(1-\ell_{\max}^{-1})}. \tag{3.4}$$

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If $f_t^i(x) \notin \widetilde{B}(2^{j+1}\varepsilon)$ holds for all $0 \le i < n$, then $(x, t, n) \in \mathcal{D}^{(c)}(2^{j+1}\varepsilon, 2^{j+1}\varepsilon)$, so (3.4) holds by the definition of L_*. Otherwise, let $m \in \{1, 2, \ldots, n-1\}$ be minimal such that $f_t^m(x) \in \widetilde{B}(2^{j+1}\varepsilon)$. Then

$$\sum_{i=0}^{m} |Df_t^i(x)|^{-1} \le L_*(2^{j+1}\varepsilon) \le L_*(\varepsilon).$$

Let $c_* \in \mathcal{C}$ be the critical point closest to $f_t^m(x)$ and $\varepsilon_* = |f_t^{m+1}(x) - f_t(c_*)|$. By Proposition 2.1(i), we have

$$|Df_t^{m+1}(x)| \ge \Lambda(2^{j+1}\varepsilon)\left(\frac{\varepsilon_*}{2^{j+1}\varepsilon}\right)^{1-\ell(c_*)^{-1}} \ge 4\left(\frac{\varepsilon_*}{2^{j+1}\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

Note that $|f_t^{m+1}(x) - f(c_*)| \le 2\varepsilon_*$. If $\varepsilon_*/2 \le \varepsilon$, we have that

$$\sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \le L_*(\varepsilon) \le 4L_*(\varepsilon)\left(\frac{\varepsilon_*}{2\varepsilon}\right)^{1-\ell_{\max}^{-1}},$$

where for the last inequality we have used $\varepsilon_* \ge \varepsilon$. Otherwise, $\varepsilon < \varepsilon_*/2 \le 2^j\varepsilon$, and by induction and (3.3), we have

$$\sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \le \widehat{L}(\varepsilon, \varepsilon_*/2) \le 4L_*(\varepsilon)\left(\frac{\varepsilon_*}{2\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

Thus,

$$\begin{aligned}\sum_{i=0}^{n} |Df_t^i(x)|^{-1} &= \sum_{i=0}^{m} |Df_t^i(x)|^{-1} + |Df_t^{m+1}(x)|^{-1} \sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \\ &< 2L_*(\varepsilon)2^{(j+1)(1-\ell_{\max}^{-1})}.\end{aligned}$$

□

To complete the proof, we shall need the following result which is a reformulation of [**S**, Proposition 5.2].

LEMMA 3.2. *For $\varepsilon > 0$ sufficiently small and each $c \in \mathcal{C}$, there exists a constant $\Lambda_0(\varepsilon) > 0$ and a positive integer $M = M_c(\varepsilon) \ge 1$ such that $\lim_{\varepsilon\to 0} \Lambda_0(\varepsilon) = \infty$ and such that the following holds: for any $t \in [-\varepsilon, \varepsilon]$ and $y \in [0, 1]$ with $|y - f(c)| \le 4\varepsilon$, we have*

$$y_j := f_t^j(y) \notin \widetilde{B}(2\varepsilon) \quad \text{for all } 0 \le j < M; \tag{3.5}$$

$$e^{-1}|Df^j(f(c))| \le |Df_t^j(y)| \le e|Df^j(f(c))| \quad \text{for all } 0 \le j \le M. \tag{3.6}$$

If $f_t^M(y) \notin \widetilde{B}(\varepsilon_0)$, then

$$|Df_t^{M+1}(y)| \ge \Lambda_0(\varepsilon)\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}. \tag{3.7}$$

If $f_t^M(y) \in \widetilde{B}(\varepsilon_0)$ and $f_t^M(y) \notin \widetilde{B}(\varepsilon)$, then

$$|Df_t^{M+1}(y)| \ge \Lambda_0(\varepsilon)\left(\frac{\operatorname{dist}(f_t^{M+1}(y), \mathrm{CV})}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}. \tag{3.8}$$

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LEMMA 3.3. *Let $\delta > 0$ be given. Then for $\varepsilon > 0$ small enough, and any $c \in \mathcal{C}$,*

$$L^{(c)}(\varepsilon) \le W^{(c)} + \delta L_*(\varepsilon).$$

Proof. In the following, we assume $\varepsilon > 0$ small. We need to prove that for each $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon)$,

$$\sum_{j=0}^{n} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta L_*(\varepsilon). \tag{3.9}$$

Let $M = M_c(\varepsilon)$ be as in Lemma 3.2. We first prove

$$\sum_{j=0}^{\min(n,M)} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta L_*(\varepsilon)/2. \tag{3.10}$$

Take N large enough such that

$$\sum_{j=0}^{N} |Df^j(f(c))|^{-1} \ge W^{(c)} - \delta/(4e).$$

By continuity, we have

$$\sum_{j=0}^{\min(n,N)} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta/4. \tag{3.11}$$

So (3.10) holds when $\min(n, M) \le N$. If $\min(n, M) > N$, then by (3.6),

$$\sum_{j=N+1}^{\min(n,M)} |Df_t^j(x)|^{-1} \le e \sum_{j=N+1}^{M} |Df^j(f(c))|^{-1} \le \delta/4 \le \delta L_*(\varepsilon)/4, \tag{3.12}$$

since $L_*(\varepsilon) \ge 1$. Together with (3.11), this implies (3.10).

In particular, the inequality (3.9) holds if $n \le M$. Let us assume now that $n > M$, so that $f_t^M(x) \notin \widetilde{B}(\varepsilon)$. To complete the proof, we need to prove that

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le \delta L_*(\varepsilon)/2. \tag{3.13}$$

We distinguish two cases.

Case 1. Assume that $f_t^M(x) \in \widetilde{B}(c_*; \varepsilon_0)$ for some $c_* \in \mathcal{C}$. Let $\varepsilon_* := \operatorname{dist}(f_t^{M+1}(x), f(c_*))$. Then $\varepsilon_* \in [\varepsilon, 2\varepsilon_0]$ and $(f_t^{M+1}(x), t, n - M - 1) \in \mathcal{D}^{(c_*)}(\varepsilon, \varepsilon_*)$. By Lemma 3.1, we have

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le \frac{\widehat{L}(\varepsilon, \varepsilon_*)}{|Df_t^{M+1}(x)|} \le \frac{4L_*(\varepsilon)}{|Df_t^{M+1}(x)|} \left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

Together with (3.8), this implies that

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le 4\Lambda_0(\varepsilon)^{-1} L_*(\varepsilon) < \delta L_*(\varepsilon)/2.$$

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Case 2. Assume that $f_t^M(x) \notin \widetilde{B}(\varepsilon_0)$. Let k be the maximal integer with $M < k \le n$ and such that $f_t^j(x) \notin \widetilde{B}(\varepsilon_0)$ for all $M < j < k$. By Proposition 2.1(ii), there exists a constant $C > 0$ such that

$$\sum_{j=0}^{k-M-1} |Df_t^j(f_t^{M+1}(x))|^{-1} \le C, \tag{3.14}$$

$$|Df_t^{k-M-1}(f_t^{M+1}(x))| \ge 1/C. \tag{3.15}$$

Thus, by (3.7),

$$\sum_{j=M+1}^{k} |Df_t^j(x)|^{-1} \le C|Df_t^{M+1}(x)|^{-1} \le \frac{C}{\Lambda_0(\varepsilon)}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-\ell_{\max}^{-1}} < \delta L_*(\varepsilon)/4. \tag{3.16}$$

In particular, (3.13) holds if $k = n$. Assume that $k < n$. Then there exists $c_* \in \mathcal{C}$ such that $f_t^k(x) \in \widetilde{B}(c_*; \varepsilon_0)$. Let $\varepsilon_* := \operatorname{dist}(f_t^{k+1}(x), f_t(c_*)) \in [\varepsilon, \varepsilon_0]$. Then

$$(f_t^{k+1}(x), t, n-k-1) \in \mathcal{D}^{(c_*)}(\varepsilon, \varepsilon_*).$$

So, by Lemma 3.1,

$$\sum_{j=k+1}^{n} |Df_t^j(x)|^{-1} \le |Df_t^{k+1}(x)|^{-1} 4L_*(\varepsilon)\left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

On the other hand,

$$|Df_t(f_t^k(x))| = \ell_{c_*}\varepsilon_*^{1-\ell_{c_*}^{-1}} \ge \varepsilon_*^{1-\ell_{\max}^{-1}},$$

so, by (3.15) and (3.7),

$$\begin{aligned} |Df_t^{k+1}(x)| &= |Df_t^{M+1}(x)||Df_t^{k-M-1}(f_t^{M+1}(x))||Df_t(f_t^k(x))| \\ &\ge \frac{\Lambda_0(\varepsilon)\varepsilon_0^{1-\ell_{\max}^{-1}}}{C}\left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}. \end{aligned}$$

Therefore,

$$\sum_{j=k+1}^{n} |Df_t^j(x)|^{-1} \le \frac{4CL_*(\varepsilon)}{\Lambda_0(\varepsilon)\varepsilon_0^{1-\ell_{\max}^{-1}}} \le \delta L_*(\varepsilon)/4.$$

Together with (3.16), this implies (3.13). This completes the proof of the lemma. □

Proof. (Completion of proof of Proposition 3.1) By Lemma 3.3, it suffices to show that $L_*(\varepsilon)$ is uniformly bounded. Arguing by contradiction, assume that this is not the case. As $L_*^{(c)}(\varepsilon)$ is monotone decreasing in ε for each c, it follows that $L_*(\varepsilon) \to \infty$ as $\varepsilon \to 0$. By the definition of $L^{(c)}(\varepsilon)$, this implies that there exists $\varepsilon_k \to 0$ and $c \in \mathcal{C}$ such that $2L^{(c)}(\varepsilon_k) \ge L_*(\varepsilon_k)$. However, by Lemma 3.3, we have

$$L_*(\varepsilon_k) \le 2L^{(c)}(\varepsilon_k) \le 2W^{(c)} + \tfrac{1}{2}L_*(\varepsilon_k),$$

provided that k is large enough. It follows that $L_*(\varepsilon_k) \le 4W^{(c)}$, a contradiction. □

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3.2. *Essential returns.*

Definition 3.1. We say that $S_n^{(c)}(t;\varepsilon)$ is an *essential return time* of $f_t(c)$ into $\widetilde{B}(\varepsilon)$ if

$$P_n^{(c)}(t;\varepsilon) \geq 3^{n-k} P_k^{(c)}(t;\varepsilon) \quad \text{for all } 1 \leq k < n.$$

Given $C_0 > 0$, we define

$$\mathcal{T}_{\text{ess}}^{(c)}(t;\varepsilon) = \{k \geq 1 : S_k^{(c)}(t;\varepsilon) \text{ is an essential return time of } f_t(c) \text{ into } \widetilde{B}(\varepsilon)\},$$

and

$$\widehat{\mathcal{T}}_{\text{ess}}^{(c)}(C_0, t;\varepsilon) = \{k \in \mathcal{T}_{\text{ess}}^{(c)}(t;\varepsilon) : \widetilde{p}_k^{(c)}(t;\varepsilon) > C_0\}.$$

Refer to the end of §2 for the definition of the notation $P_n^{(c)}$, $\widetilde{p}_k^{(c)}$, etc.

The goal of this section is to prove the following proposition.

PROPOSITION 3.2. *Given $C > 0$, $C_0 > 0$, $\tau > 1$ and $\gamma \in (0, 1)$, the following hold provided that $\varepsilon > 0$ is small enough.*

(i) *For $t \in X_{n,\varepsilon}(C) \backslash X_{n+1,\varepsilon}(C)$, $n = 1, 2, \ldots$, there exists $c \in \mathcal{C}$ such that*

$$\sum_{k \in \widehat{\mathcal{T}}_{\text{ess}}^{(c)}(C_0,t;\varepsilon), k \leq n} \widetilde{p}_k^{(c)}(t;\varepsilon) \geq (\gamma C - C_0)n.$$

(ii) *For $t \in Y_\varepsilon^m(C,\tau) \backslash Y_\varepsilon^{m+1}(C,\tau)$, $m = 0, 1, 2, \ldots$, there exists $c \in \mathcal{C}$ and $n \in \widehat{\mathcal{T}}_{\text{ess}}^{(c)}(C_0, t;\varepsilon)$ such that $m = S_n^{(c)}(t;\varepsilon)$ and*

$$p_n^{(c)}(t;\varepsilon) \geq \gamma\tau \log(m+1).$$

We shall need the following lemma which is [**S**, Proposition 5.6].

LEMMA 3.4. *For any $\varepsilon > 0$ small enough, there exists a constant $\kappa(\varepsilon) > 0$ such that for $|t| \leq \varepsilon$ and $x \in [0, 1]$, if n is an integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $0 \leq j < n$ and $f_t^n(x) \in \widetilde{B}(c;\varepsilon)$ for some $c \in \mathcal{C}$, then*

$$\mathrm{A}(x, t, n) \leq \kappa(\varepsilon) \cdot \frac{|Df_t^n(x)|}{|\widetilde{B}(c;\varepsilon)|} \tag{3.17}$$

and such that

$$\kappa(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.18}$$

We shall also need the following lemma which is [**S**, Lemma 2.1].

LEMMA 3.5. *There exists a constant $\theta_0 > 0$ such that for any $(x, t) \in [0, 1] \times [-1, 1]$ and any integer $n \geq 1$ with $\mathrm{A}(x, t, n) < \infty$, putting*

$$J = \left[x - \frac{\theta_0}{\mathrm{A}(x, t, n)}, x + \frac{\theta_0}{\mathrm{A}(x, t, n)}\right] \cap [0, 1],$$

we have that $f_t^n | J$ is a diffeomorphism and

$$e^{-1}|Df_t^j(x)| \leq |Df_t^j(y)| \leq e\,|Df_t^j(x)|$$

holds for all $y \in J$ and $0 \leq j \leq n$.

In the following, fix $C > 1$, $\gamma \in (0, 1)$ and denote $\rho = 1 - \sqrt{\gamma}$, $\rho_1 = \rho/4$, $\rho_2 = \rho_1/(2\ell_{\max})$. Let $\varepsilon > 0$ denote a small constant and we fix a parameter $t \in [-\varepsilon, \varepsilon]$. For simplicity, we shall drop t, ε from the notation. So $S_i^{(c)} = S_i^{(c)}(t;\varepsilon)$, $d_i^{(c)} = d_i^{(c)}(t;\varepsilon)$, etc.

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3.2.1. *Free returns.* Define

$$\widehat{S}_i^{(c)} = \sup\{S > S_i^{(c)} : \mathrm{A}(f_t^{S_i^{(c)}+2}(c), t, S - S_i^{(c)}) \le \theta_0 e^{(d_i^{(c)}-1)\ell(c')}\varepsilon^{-1}\},$$

and

$$\widetilde{S}_i^{(c)} = \inf\{S > \widehat{S}_i^{(c)} : f_t^{S+1}(c) \in \widetilde{B}(\varepsilon)\},$$

where c' denotes the critical point of f which is closest to $f_t^{S_i^{(c)}+1}(c)$.

LEMMA 3.6. *Consider $t \in X_{n,\varepsilon}(C)$, $c \in \mathcal{C}$ and $1 \le i < n$. Then*

$$\sum_{k=S_i^{(c)}+2}^{\widehat{S}_i^{(c)}+1} q_\varepsilon(f_t^k(c)) < \rho_1 \cdot d_i^{(c)}. \tag{3.19}$$

Moreover, if there exists $j \le n$ such that $S_j^{(c)} = \widetilde{S}_i^{(c)}$, then

$$\log \frac{P_j^{(c)}}{P_i^{(c)}} > d_j^{(c)} - \rho_1 d_i^{(c)} + (\log 3) \cdot (j - i). \tag{3.20}$$

Proof. Assume $\varepsilon > 0$ small and let $a = 2\ell_{\max}/(\ell_{\min} - 1)$, $\varepsilon' = e^a\varepsilon$. Let c_k denote the critical point of f which is closest to $f_t^{S_k^{(c)}+1}(c)$. For simplicity of notation, we shall write $S_k = S_k^{(c)}$, $\widehat{S}_k = \widehat{S}_k^{(c)}$ and $d_k = d_k^{(c)}$ for each k. Let $y = f_t^{S_i+1}(c)$, $x = f_t(y)$, $v = f_t(c)$ and $v_i = f_t(c_i)$. Note that $\mathrm{A}(x, t, \widehat{S}_i - S_i) \le \theta_0/|v_i - x|$. So by Lemma 3.5, for $0 \le k < \widehat{S}_i - S_i$, we have

$$e^{-1}|Df_t^{k+1}(x)| \le |Df_t^{k+1}(v_i)| \le e|Df_t^{k+1}(x)|, \tag{3.21}$$

and

$$|Df_t^{k+1}(x)| \ge e^{-1}\frac{\mathrm{dist}(f_t^{k+1}(v_i), f_t^{k+1}(x))}{|v_i - x|}. \tag{3.22}$$

We shall first prove that

$$M := \#\{1 \le k \le \widehat{S}_i - S_i : f_t^k(v_i) \in \widetilde{B}(\varepsilon')\} \le \rho_2 d_i (C + a + 1)^{-1} < n. \tag{3.23}$$

and

$$\sum_{k=0}^{\widehat{S}_i - S_i} q_{\varepsilon'}(f_t^{k+1}(c_i)) \le (C + a + 1)M \le \rho_2 d_i. \tag{3.24}$$

Indeed, $q_{\varepsilon'}(z) \le q_\varepsilon(z) + a + 1$ holds for each $z \in [0, 1]$, and thus

$$t \in X_{n,\varepsilon}(C) \subset X_{n,\varepsilon'}(C + a + 1). \tag{3.25}$$

Therefore, the inequality (3.24) will follow once we prove (3.23). Let $T_1 < T_2 < \cdots$ be all of the positive integers such that $f_t^{T_k+1}(c_i) \in \widetilde{B}(\varepsilon')$ and p_k be the critical point of f which is closest to $f_t^{T_k+1}(c_i)$. Then, for each $1 \le m < n$, by Lemma 2.1, we have

$$|Df_t^{T_m+1}(f_t(c_i))| \ge (\Lambda(\varepsilon')e^{-\ell_{\max}(C+a+1)})^m, \tag{3.26}$$

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hence

$$\begin{aligned}\mathrm{A}(v_i, t, T_m+1) &\geq \frac{|Df_t^{T_m}(f_t(c_i))|}{|f_t^{T_m+1}(c_i)-p_m|} = \frac{|Df_t^{T_m+1}(f_t(c_i))|}{|Df_t(f_t^{T_m+1}(c_i))||f_t^{T_m+1}(c_i)-p_m|}\\ &\geq |Df_t^{T_m+1}(f_t(c_i))|(\varepsilon')^{-1} \geq (\Lambda(\varepsilon')e^{-\ell_{\max}(C+a+1)})^m(\varepsilon')^{-1}.\end{aligned}$$

On the other hand, by Lemma 3.5, we have

$$\mathrm{A}(v_i, t, \widehat{S}_i - S_i) \asymp \mathrm{A}(x, t, \widehat{S}_i - S_i) \leq \theta_0 e^{(d_i-1)\ell(c_i)}\varepsilon^{-1}.$$

The inequality (3.23) follows.

Let us now prove (3.19). Indeed, by (3.21), for each $0 \leq k < \widehat{S}_i - S_i$, we have $|Df_t(f_t^k(v_i))| \leq e^2|Df_t(f_t^k(x))|$, so $q_{\varepsilon'}(f_t^k(v_i)) \geq q_\varepsilon(f_t^k(x))$. Thus,

$$\sum_{i<k<j} d_k \leq \sum_{k=0}^{\widehat{S}_i-S_i} q_{\varepsilon'}(f_t^{k+1}(c_i)) \leq \rho_2 d_i, \tag{3.27}$$

which implies (3.19) since $\rho_2 < \rho_1$.

To obtain (3.20) it suffices to prove the following two inequalities:

$$|Df_t^{S_j-S_i-1}(x)| \geq \Lambda_1(\varepsilon)^{j-i}(D_{c_j}(\varepsilon))^{-1}, \tag{3.28}$$

and

$$|Df_t^{S_j-S_i-1}(x)| \geq \kappa \exp(\ell(c_i)d_i - \rho_2\ell_{\max}d_i)(D_{c_j}(\varepsilon))^{-1}, \tag{3.29}$$

where $\Lambda_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $\kappa > 0$ is a constant.

Indeed, combining these two inequalities, we obtain

$$U := |Df_t^{S_j-S_i-1}(x)|D_{c_j}(\varepsilon) \geq 3^{j-i}\exp(\ell(c_i)d_i - \rho_1 d_i + \ell_{\max}).$$

Since

$$\frac{P_j^{(c)}}{P_i^{(c)}} = U\frac{|Df_t(y)||y-c_i|}{|f_t^{S_j+1}(c)-c_j|D_{c_j}(\varepsilon)} \geq U\exp(-\ell(c_i)d_i + d_j - 1),$$

the inequality (3.20) follows.

Let us prove (3.28). Applying Proposition 2.1(i), we obtain

$$|Df_t^{S_j-S_{j-1}-1}(f_t^{S_{j-1}-S_i}(x))| \geq \Lambda(\varepsilon)/D_{c_j}(\varepsilon).$$

Thus, the inequality (3.28) holds with $\Lambda_1(\varepsilon) = \Lambda(\varepsilon)$ if $j = i+1$. When $j > i+1$, $S_{j-1} - S_i$ is of the form $T_m + 1$ for some $j - i - 1 \leq m \leq M < n$, so combining (3.26) with the last inequality, we obtain that (3.28) holds with a suitable choice of $\Lambda_1(\varepsilon)$.

Finally let us prove (3.29). We may certainly assume $(\ell_{\max}-1)\rho_2 d_i \geq 2$. Let

$$A_k = \frac{|Df_t^{S_k-S_i-1}(x)|}{\mathrm{dist}(f_t^{S_k-S_i-1}(x), \mathcal{C})} \quad \text{and} \quad A'_k = \frac{|Df_t^{S_k-S_i-1}(x)|}{|\widetilde{B}(c_k;\varepsilon)|}$$

for $i < k \leq j$. Clearly, $A_k \geq A'_k$. By Proposition 2.1(i), we have

$$\frac{A'_j}{A_k} = |Df_t^{S_j-S_k}(f_t^{S_k+1}(c))|\frac{\mathrm{dist}(f_t^{S_k+1}(c), \mathcal{C})}{|\widetilde{B}(c_j;\varepsilon)|} \geq \Lambda(\varepsilon)^{j-k}\exp\Big(-\ell_{\max}\sum_{k\leq l<j} d_l\Big),$$

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which, by (3.27), implies

$$\frac{A'_j}{A_k} \geq \Lambda(\varepsilon)^{j-k} e^{-\rho_2 \ell_{\max} d_i}. \tag{3.30}$$

Let $\theta = \theta_0/(2e^{\ell_{\max}})$. We distinguish two cases.

Case 1. Assume that $\mathrm{A}(x, t, S_j - S_i - 1) \geq \theta e^{d_i \cdot \ell(c_i)} \varepsilon^{-1}$. Then, by Lemma 3.4, we have

$$\sum_{k=i+1}^{j-1} A_k + A'_j \geq \frac{1}{1+\kappa(\varepsilon)} \mathrm{A}(x, t, S_j - S_i - 1) \geq \theta e^{d_i \cdot \ell(c_i)} (2\varepsilon)^{-1}.$$

Together with (3.30), this implies $A'_j \geq \theta \exp(\ell(c_i) d_i - \rho_2 \ell_{\max} d_i)(4\varepsilon)^{-1}$, provided that $\varepsilon > 0$ is small enough. Thus (3.29) holds in this case.

Case 2. Assume that $\mathrm{A}(x, t, S_j - S_i - 1) < \theta e^{\ell(c_i) \cdot d_i} \varepsilon^{-1}$. In particular, we have $S_j - 1 \leq \widehat{S}_i$ which implies $\widehat{S}_i = S_j - 1$. By the maximality of $\widehat{S}_i$ we have

$$A_j = A(x, t, S_j - S_i) - A(x, t, \widehat{S}_i - S_i) \geq \theta e^{\ell(c_i) \cdot d_i} \varepsilon^{-1}.$$

So (3.29) holds if $d_j \leq \rho_2 \ell_{\max} d_i$. Assume that $d_j > \rho_2 \ell_{\max} d_i$. By (3.24),

$$q_{\varepsilon'}(f_t^{S_j - S_i - 1}(v_i)) \leq \rho_2 d_i \leq \rho_2 \ell_{\max} d_i - 2.$$

Thus, there exists a constant $\kappa_1 > 0$ such that

$$\operatorname{dist}(f_t^{S_j - S_i - 1}(v_i),\, f_t^{S_j - S_i - 1}(x)) \geq \kappa_1 e^{-\rho_2 \ell_{\max} d_i} |\widetilde{B}(c_j; \varepsilon)|.$$

Thus, by (3.22),

$$|Df_t^{S_j - S_i - 1}(x)| \geq e^{-1} \frac{\operatorname{dist}(f_t^{S_j - S_i - 1}(v_i),\, f_t^{S_j - S_i - 1}(x))}{|v_i - x|} \geq \frac{\kappa_1 \exp(\ell(c_i) d_i - \rho_2 \ell_{\max} d_i)}{D_{c_j}(\varepsilon)}.$$

So the inequality (3.29) holds. □

Given $c \in \mathcal{C}$, we define positive integers $i_1 < i_2 < \cdots$ in the following way: $i_1 = 1$. Once i_k and $S_{i_k}^{(c)}$ are both well-defined, let i_{k+1} be such that $\widetilde{S}_{i_k}^{(c)} = S_{i_{k+1}}^{(c)}$. The procedure stops whenever $S_{i_k}^{(c)}$ or $\widetilde{S}_{i_k}^{(c)}$ is not well-defined. The positive integers S_{i_k}, $k = 1, 2, \ldots$, are called *free return times* of $f_t(c)$ into $\widetilde{B}(\varepsilon)$.

LEMMA 3.7. *An essential return time is a free return time.*

Proof. By definition, for any consecutive free return times $S_i < S_j$, we have $P_k < P_i$ for all $i < k < j$. So S_k is not an essential return time. The lemma follows. □

LEMMA 3.8. *Assume that $t \in X_{n,\varepsilon}(C)$. Let $c \in \mathcal{C}$. If $1 \leq i < j \leq n$ are such that $S_i^{(c)} < S_j^{(c)}$ are consecutive essential return times of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, then*

$$\sum_{i<k<j} d_k^{(c)} \leq \rho d_i^{(c)}, \tag{3.31}$$

and

$$p_j^{(c)} \geq d_j^{(c)} - \rho d_i^{(c)}. \tag{3.32}$$

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Moreover, if n_0 is the largest integer in $\{1, 2, \dots, n\}$ such that $S^{(c)}_{n_0}$ is an essential return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, then

$$\sum_{n_0<k\le n} d^{(c)}_k \le \rho d^{(c)}_{n_0}. \tag{3.33}$$

Proof. By Lemma 3.7, $S^{(c)}_i$ and $S^{(c)}_j$ are both free return times. Let $i = k_0 < k_1 < \cdots < k_m = j$ be all of the positive integers such that $S^{(c)}_{k_l}$ are free return times. Then, by Lemma 3.6, for each $0 \le l < m$, we have

$$\log \frac{P^{(c)}_{k_{l+1}}}{P^{(c)}_{k_l}} \ge d^{(c)}_{k_{l+1}} - \rho_1 d^{(c)}_{k_l} + (\log 3)(k_{l+1} - k_l), \tag{3.34}$$

and

$$\sum_{k_l<k<k_{l+1}} d^{(c)}_k \le \rho_1 d^{(c)}_{k_l}. \tag{3.35}$$

Summing up both sides of (3.34) for $0 \le l < m-1$, we obtain

$$\log \frac{P^{(c)}_{k_{m-1}}}{P^{(c)}_{k_0}} \ge d^{(c)}_{k_{m-1}} + (1-\rho_1)d^{(c)}_{k_{m-2}} + \cdots + (1-\rho_1)d^{(c)}_{k_1} - \rho_1 d^{(c)}_{k_0} + (\log 3)(k_{m-1} - k_0).$$

Since the left-hand side is smaller than $(\log 3)(k_{m-1} - k_0)$, we obtain

$$d^{(c)}_{k_{m-1}} + (1-\rho_1)d^{(c)}_{k_{m-2}} + \cdots + (1-\rho_1)d^{(c)}_{k_1} \le \rho_1 d^{(c)}_{k_0},$$

which, together with (3.35), implies

$$\sum_{i<k<j} d^{(c)}_k \le \rho_1 d^{(c)}_i + (1+\rho_1)\rho_1(1-\rho_1)^{-1} d^{(c)}_i \le \rho d^{(c)}_i.$$

This proves (3.31). Summing up both sides of (3.34) for $0 \le l < m$,

$$\begin{aligned}\log \frac{P^{(c)}_j}{P^{(c)}_i} &\ge d^{(c)}_j + (1-\rho_1)\sum_{l=1}^{m-1} d^{(c)}_{k_l} - \rho_1 d^{(c)}_{k_0} + (\log 3)(j-i)\\ &\ge d^{(c)}_j - \rho_1 d^{(c)}_i + (\log 3)(j-i),\end{aligned}$$

which is equivalent to

$$P^{(c)}_i \le e^{-(d^{(c)}_j - \rho_1 d^{(c)}_i)} 3^{i-j} P^{(c)}_j. \tag{3.36}$$

Let us now prove that

$$P^{(c)}_k \le 3^{k-i} P^{(c)}_i \quad \text{for any } 1 \le k \le j-1. \tag{3.37}$$

Indeed, for $1 \le k < i$, this inequality follows from that fact that $S^{(c)}_i$ is an essential return time, while for $i < k < j$, it follows from the fact that $S^{(c)}_k$ is not an essential return time.

Combining (3.36) and (3.37), we obtain

$$\sum_{k=1}^{j-1} P^{(c)}_k \le e^{-(d^{(c)}_j - \rho_1 d^{(c)}_i)} 2^{-1} P^{(c)}_j.$$

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By Lemma 3.4, we have

$$\mathrm{A}(f_t(c), t, S_j) \le (1+\kappa(\varepsilon)) \sum_{k=1}^{j-1} P_k^{(c)} + \kappa(\varepsilon) \frac{|Df_t^{S_j}(f_t(c))|}{|\widetilde{B}(c_j;\varepsilon)|},$$

where c_j is the critical point of f which is closest to $f_t^{S_j^{(c)}+1}(c)$ and $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. So when $\varepsilon > 0$ is small, we obtain

$$\mathrm{A}(f_t(c), t, S_j) \le e^{-(d_j - \rho_1 d_i)} P_j^{(c)}.$$

The inequality (3.32) follows.

The inequality (3.33) can be proved in a similar way. □

Proof of Proposition 3.2. (i) By definition, there exists c such that $\sum_{k=1}^{n} d_k^{(c)} \ge Cn$. Let $i_1 < i_2 < \cdots < i_m$ be all of the integers in $\{1, 2, \ldots, n\}$ such that S_{i_j} is an essential return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Then $i_1 = 1$. For convenience of notation, we set $i_0 = 0$ and $d_{i_0}^{(c)} = 0$.

By (3.31) and (3.33) in Lemma 3.8, we have

$$\sum_{j=1}^{m} d_{i_j}^{(c)} \ge (1-\rho) \sum_{j=1}^{n} d_j^{(c)} \ge (1-\rho)Cn.$$

Thus,

$$\sum_{j=1}^{m} (d_{i_j}^{(c)} - \rho d_{i_{j-1}}^{(c)}) \ge (1-\rho)^2 Cn.$$

By (3.32) in Lemma 3.8, for each $2 \le j \le m$ we have

$$p_{i_j}^{(c)} \ge d_{i_j}^{(c)} - \rho d_{i_{j-1}}^{(c)}.$$

By Lemma 3.4, this estimate is also true for $j = 1$. Thus,

$$\widetilde{p}_{i_j}^{(c)} \ge d_{i_j}^{(c)} - \rho d_{i_{j-1}}^{(c)}$$

holds for all $j = 1, 2, \ldots, m$, which implies

$$\sum_{j=1}^{m} \widetilde{p}_{i_j}^{(c)} \ge \sum_{j=1}^{m} (d_{i_j}^{(c)} - \rho d_{i_{j-1}}^{(c)}) \ge (1-\rho)^2 Cn.$$

Consequently,

$$\sum_{k \in \widehat{\mathcal{T}}_{\mathrm{ess}}^{(c)}(C_0,t;\varepsilon), k \le n} \widetilde{p}_k^{(c)} \ge \sum_{j=1}^{m} \widetilde{p}_{i_j}^{(c)} - C_0 n \ge (1-\rho)^2 Cn - C_0 n = (\gamma C - C_0)n.$$

(ii) By definition, there exists c, $c' \in \mathcal{C}$ such that $\mathrm{dist}(f_t^{m+1}(c), c') \le \varepsilon^{1/\ell(c')}(m+1)^{-\tau}$. So there exists $n \ge 1$ such that $m = S_n^{(c)}$ and $d_n^{(c)} \ge \tau \log(S_n^{(c)} + 1)$. Since $d_k^{(c)} < d_n^{(c)}$ holds for each $1 \le k < n$, by (3.31) in Lemma 3.8 it follows that n is an essential return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$ and, hence, $p_n^{(c)} \ge (1-\rho) d_n^{(c)} > C_0$. The statement is proved. □

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4. *Harvest in the parameter space*

In this section, we transfer the estimates in phase space to the parameter space and prove the Reduced Main Theorem. The phase and parameter spaces are related through the maps $\xi_n^{(c)}(t) = f_t^{n+1}(c)$. In §4.1, we define parameter boxes. In §4.3, we prove the Reduced Main Theorem by showing that the bad parameters are contained in certain families of parameter boxes with large total depth. Proposition 3.1 will be used to construct the parameter boxes and Proposition 3.2 will be used to estimate the total depth. The parameter boxes which we use are always mapped into $\widetilde{B}(\varepsilon)$ and they form special families of balls. In §4.2, an abstract lemma about sets of points lying deeply in a special family of balls is proved.

4.1. *Parameter boxes.* Recall that

$$\frac{(\xi_n^{(c)}(t))'}{Df_t^n(f_t(c))} = \sum_{j=0}^{n} \frac{\partial_t F(f_t^j(c), t)}{Df_t^j(f_t(c))} =: M_n^{(c)}(t).$$

Definition 4.1. Given $m \geq 0$, $\lambda > 1$ and $c \in \mathcal{C}$, we say that a ball $B(t_0, r)$ in the parameter space is a *λ-bounded c-parameter box of order m* if the following hold:

- $\xi_m^{(c)} : B(t_0, r) \to [0, 1]$ is a diffeomorphism onto its image such that
$$\sup_{t,s \in B(t_0,r)} \frac{(\xi_m^{(c)})'(t)}{(\xi_m^{(c)})'(s)} \leq \lambda;$$
- for any $t \in B(t_0, r)$, we have
$$\lambda^{-1}|a^{(c)}| \leq |M_m^{(c)}(t)| \leq \lambda |a^{(c)}|;$$
- for each $k = 0, 1, \ldots, m$, we have
$$\sup_{t,s \in B(t_0,r)} \frac{|Df_t^k(f_t(c))|}{|Df_s^k(f_s(c))|} \leq \lambda.$$

The goal of this section is to provide an estimate of the size of a parameter box centered at a given parameter t_0.

PROPOSITION 4.1. *Given $\lambda > 1$, there exist $\theta > 0$ and $N \geq 1$ such that the following holds. Let $|t_0| \leq \theta$, $c \in \mathcal{C}$ and $m > N$ be such that*

$$\sum_{i=0}^{m} |Df_{t_0}^i(f_{t_0}(c))|^{-1} \leq W^{(c)} + \theta,$$

then putting

$$r = \theta / A(f_{t_0}(c), t_0, m),$$

$B(t_0, r)$ is a λ-bounded c-parameter box of order m.

Write $D_n^{(c)}(t) = Df_t^n(f_t(c))$.

LEMMA 4.1. *Given $\lambda > 1$ there exist $\eta = \eta(\lambda) > 0$ and an integer $N = N(\lambda) \geq 1$ such that the following holds. Let $t \in [-\eta, \eta]$, $c \in \mathcal{C}$ and let $m > N$ be a positive integer.*

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Assume that

$$\sum_{i=0}^{m} |D_i^{(c)}(t)|^{-1} \le W^{(c)} + \eta. \tag{4.1}$$

Then

$$\lambda^{-1}|a^{(c)}| < |M_m^{(c)}(t)| < \lambda|a^{(c)}|.$$

Proof. Take $\delta > 0$ small. Let N be large such that

$$\sum_{i=0}^{N} |Df^i(f(c))|^{-1} > W^{(c)} - \delta \quad \text{and} \quad |M_N^{(c)}(0) - a^{(c)}| < \delta.$$

By continuity, there exists $\eta_0 > 0$ such that for any $t \in [-\eta_0, \eta_0]$, we have

$$\sum_{i=0}^{N} |D_i^{(c)}(t)|^{-1} > W^{(c)} - \delta \quad \text{and} \quad |M_N^{(c)}(t) - a^{(c)}| < \delta.$$

Now let $\eta = \min(\delta, \eta_0)$. If (4.1) holds, then we have

$$\sum_{i=N+1}^{m} |D_i^{(c)}(t)|^{-1} < \delta + \eta \le 2\delta.$$

Since $|\partial_t F| \le 1$, it follows that

$$|M_m^{(c)}(t) - a^{(c)}| \le |M_N^{(c)}(t) - a^{(c)}| + \sum_{i=N+1}^{m} |D_i^{(c)}(t)|^{-1} < 3\delta.$$

The desired inequality follows since $\min_{c\in\mathcal{C}} |a^{(c)}| > 0$. □

Proof of Proposition 4.1. Fix $\lambda > 1$. Let $\lambda_0 = \lambda^{1/4}$ and let $\eta = \eta(\lambda_0)$, $N = N(\lambda_0)$ be given by Lemma 4.1. Let $\theta \in (0, \eta/2)$ and $\lambda_1 \in (1, \lambda_0)$ be such that

$$\lambda_1(W^{(c)} + \theta) \le W^{(c)} + \eta,$$

holds for each $c \in \mathcal{C}$.

Now let t_0, c, m be as in the assumption of this proposition. Then, by continuity, there exists a maximal $r_0 \in (0, \theta]$ such that for each $t \in \overline{B(t_0, r_0)}$ and any $1 \le j \le m$, we have

$$\frac{1}{\lambda_1} \le \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)} \le \lambda_1. \tag{4.2}$$

So

$$\sum_{i=0}^{m} |D_i^{(c)}(t)|^{-1} \le \lambda_1 \sum_{i=0}^{m} |D_i^{(c)}(t_0)|^{-1} \le \lambda_1(W^{(c)} + \theta) \le W^{(c)} + \eta,$$

which implies by Lemma 4.1 that

$$\lambda_0^{-1}|a^{(c)}| \le |M_m^{(c)}(t)| \le \lambda_0|a^{(c)}|.$$

It follows that $B(t_0, r_0)$ is a λ-bounded c-parameter box of order m. So it suffices to prove that $\theta_0 := r_0 \cdot \mathrm{A}(f_{t_0}(c), t_0, m)$ is bounded away from zero. To this end, we only need to show that there exists a constant $C > 0$ such that for each $1 \le j \le m$,

$$\left|\log \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)}\right| \le C\theta_0. \tag{4.3}$$

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Indeed, if $r_0 = \theta$, then $\theta_0 \geq r_0 = \theta$, and if $r_0 < \theta$, then by maximality of r_0, there exists $t_1 \in \overline{B(t_0, r_0)}$ and $j \in \{1, 2, \ldots, m\}$ such that

$$\text{either } D_j^{(c)}(t_1) = \lambda_1 D_j^{(c)}(t_0) \quad \text{or } D_j^{(c)}(t_0) = \lambda_1 D_j^{(c)}(t_1). \tag{4.4}$$

Thus, the inequality (4.3) implies that $\theta_0 \geq \min(\log \lambda_1 / C, \theta)$.

Let us prove (4.3). First note that there exists $C_1 > 0$ such that for each $1 \leq j \leq m$ and any $t \in \overline{B(t_0, r_0)}$, we have $|M_j^{(c)}(t)| \leq C_1$, so

$$|(\xi_j^{(c)}(t))'| = |M_j^{(c)}(t) D_j^{(c)}(t)| \leq C_1 |D_j^{(c)}(t)|.$$

Since F is a normalized regular family, there exists a constant $C_2 > 0$ such that

$$\left| \frac{\partial^2 F}{\partial t \partial x}(x, t) \right| \leq C_2 |Df_t(x)|$$

for all (x, t). (Indeed, for x close to $\mathcal{C}$ the left-hand side of this inequality is zero.) By (4.2), for each $0 \leq i < m$,

$$\frac{|Df_t(\xi_i^{(c)}(t))|}{|Df_{t_0}(\xi_i^{(c)}(t_0))|} = \frac{|D_{i+1}^{(c)}(t)|}{|D_{i+1}^{(c)}(t_0)|} \frac{|D_i^{(c)}(t_0)|}{|D_i^{(c)}(t)|} \in [\lambda^{-1}, \lambda].$$

By non-flatness of the critical points, it follows that there exist constants C_3 and C_4 such that

$$\frac{|D^2 f_t(\xi_i^{(c)}(t))|}{|Df_t(\xi_i^{(c)}(t))|} \leq C_3 \frac{|D^2 f_{t_0}(\xi_i^{(c)}(t_0))|}{|Df_{t_0}(\xi_i^{(c)}(t_0))|} \leq \frac{C_4}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})}.$$

Since $\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C}) \leq 1$, and $|D_i^{(c)}(t)| \geq \lambda_1^{-1} |D_i^{(c)}(t_0)|$ is bounded away from zero, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}
\left| \frac{d Df_t(\xi_i^{(c)}(t))}{dt} \right| &= \left| \frac{\partial^2 F}{\partial t \partial x}(\xi_i^{(c)}(t), t) + D^2 f_t(\xi_i^{(c)}(t))(\xi_i^{(c)}(t))' \right| \\
&\leq C_2 |Df_t(\xi_i^{(c)}(t))| + C_4 C_1 |Df_t(\xi_i^{(c)}(t))| \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})} \\
&\leq C_5 |Df_t(\xi_i^{(c)}(t))| \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left| \frac{d \log |D_j^{(c)}(t)|}{dt} \right| &= \left| \sum_{i=0}^{j-1} \frac{d Df_t(\xi_i^{(c)}(t))/dt}{Df_t(\xi_i^{(c)}(t))} \right| \\
&\leq C_5 \sum_{i=0}^{j-1} \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})} \leq C \, \mathrm{A}(f_{t_0}(c), t_0, j),
\end{aligned}$$

where $C = C_5 \lambda_1$. Since

$$\log \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)} = \int_{t_0}^{t} d \log |D_j^{(c)}(t)|,$$

the inequality (4.3) follows. □

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4.2. *Special family of balls.* Given $B = B(a, r)$ and $x \in \mathbb{R}$, we define

$$\operatorname{dep}(x|B) = \begin{cases} \inf\{k \in \mathbb{N} : |x - a| \geq e^{-k} r\} & \text{if } |x - a| < e^{-2} r, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for each $k \in \mathbb{Z}$, let

$$B^{(k)} = B(a, e^{-k} r). \tag{4.5}$$

A finite family $\mathcal{M} = \{B_i = B(a_i, r_i)\}_{i \in \mathcal{I}}$ is said to be *special* if the following holds: for any $i, j \in \mathcal{I}$, if $a_i \in B_j^{(1)}$, then there exists $k = k(i, j) \geq 1$ such that $B_i \subset B_j^{(k-1)} \setminus B_j^{(k+1)}$. In particular, the centers a_i, $i \in \mathcal{I}$, are pairwise distinct.

Given a special family as above, define

$$\mathcal{I}_0 = \{i \in \mathcal{I} : \text{ for any } j \in \mathcal{I},\ j \neq i, \text{ we have } a_i \notin B_j^{(1)}\},$$

and for each $k \geq 1$, let

$$\mathcal{I}_k = \left\{ i \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m : \text{ for any } j \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m,\ j \neq i, \text{ we have } a_i \notin B_j^{(1)} \right\}.$$

The minimal integer $n \geq 0$ for which $\mathcal{I}_n = \emptyset$ is called the *height* of $\mathcal{M}$. The support of $\mathcal{M}$ is defined as the union of all of the elements of $\mathcal{M}$.

We shall use the next lemma to estimate the measure of sets of bad parameters.

LEMMA 4.2. *For each $0 < \kappa < 1$ there exists $K = K(\kappa) > 1$ such that if $\mathcal{M} = \{B_i\}_{i \in \mathcal{I}}$ is a special family of height at most n and*

$$X_{\mathcal{M}}(N) = \left\{ x \in \operatorname{supp}(\mathcal{M}) : \sum_{i \in \mathcal{I}} \operatorname{dep}(x|B_i) \geq N \right\}, \quad N = 0, 1, \ldots,$$

then

$$|X_{\mathcal{M}}(N)| \leq K^n e^{-(1-\kappa)N} |\operatorname{supp}(\mathcal{M})|.$$

Proof. Fix $0 < \kappa < 1$ and let $K = K(\kappa) = e^5/(1 - e^{-\kappa})$. We shall prove the lemma by induction on the height n. We take the trivial case $n = 0$ as the starting point. Now let n_0 be a positive integer and assume that the lemma holds for $n < n_0$. Let us consider the case $n = n_0$. Let $\mathcal{I}_0$ be defined as above, and let $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}_0$. Let

$$q_0(x) = \sum_{i \in \mathcal{I}_0} \operatorname{dep}(x|B_i), \quad q'(x) = \sum_{i' \in \mathcal{I}'} \operatorname{dep}(x|B_{i'}).$$

For each $q_0 \geq 0$, and $q' \geq 0$, let $V(q_0) = \{x \in \operatorname{supp}(\mathcal{M}) : q_0(x) = q_0\}$ and $U(q_0, q') = \{x \in V(q_0) : q'(x) \geq q'\}$. Let us prove that

$$|U(q_0, q')| \leq e^{-q_0+5} K^{n-1} e^{-(1-\kappa)q'} |\operatorname{supp}(\mathcal{M})|. \tag{4.6}$$

To this end, we first note that the balls $B_i^{(2)}$, $i \in \mathcal{I}_0$, are pairwise disjoint. Thus,

$$V(q_0) \subset \bigcup_{i \in \mathcal{I}_0} B_i^{(q_0-1)}, \tag{4.7}$$

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and for each $k = 0, 1, \ldots,$ we have

$$\sum_{i \in \mathcal{I}_0} |B_i^{(k)}| \le e^{-k+2} |\mathrm{supp}(\mathcal{M})|. \tag{4.8}$$

In particular, $|V(q_0)| \le e^{-q_0+3}|\mathrm{supp}(\mathcal{M})|$, so the inequality (4.6) holds when $q' = 0$. Assume now that $q' > 0$ and let

$$\mathcal{M}'_{q_0} = \{B_{i'} : i' \in \mathcal{I}',\ B_{i'}^{(2)} \cap V(q_0) \neq \emptyset\}.$$

Then $\mathcal{M}'_{q_0}$ is a special family of height $< n$ and $U(q_0, q') \subset X_{\mathcal{M}'_{q_0}}(q')$. By the induction hypothesis, we have

$$|U(q_0, q')| \le |X_{\mathcal{M}'_{q_0}}(q')| \le K^{n-1} e^{-(1-\kappa)q'} |\mathrm{supp}(\mathcal{M}'_{q_0})|. \tag{4.9}$$

Next, let us show

$$\mathrm{supp}(\mathcal{M}'_{q_0}) \subset \bigcup_{i \in \mathcal{I}_0} B_i^{(q_0-3)}. \tag{4.10}$$

In fact, since $\mathcal{M}'_{q_0} \subset \mathcal{M}$, the relation (4.10) holds when $q_0 \le 3$. Assume that $q_0 > 3$. By (4.7), for each $B_{i'} \in \mathcal{M}'_{q_0}$ there exists $i \in \mathcal{I}_0$ such that $B_{i'}^{(2)} \cap B_i^{(q_0-1)} \neq \emptyset$. By the definition of a special family, we have $B_{i'} \subset B_i^{(q_0-3)}$. Thus, the relation (4.10) holds. By (4.8), it follows that

$$|\mathrm{supp}(\mathcal{M}'_{q_0})| \le e^{-q_0+5} |\mathrm{supp}(\mathcal{M})|,$$

which, together with (4.9), implies (4.6).

Now, let us complete the induction step. Fix $N \ge 0$ and for each $q_0 \ge 0$, let $q_0' = \max(N - q_0, 0)$. So $q_0 + q_0' \ge N$. Since

$$X_{\mathcal{M}}(N) \subset \bigcup_{q_0=0}^{\infty} U(q_0, q_0'),$$

by (4.6), we obtain

$$\begin{aligned} |X_{\mathcal{M}}(N)| \le \sum_{q_0=0}^{\infty} |U(q_0, q_0')| &\le K^{n-1} e^5 |\mathrm{supp}(\mathcal{M})| \sum_{q_0=0}^{\infty} e^{-\kappa q_0} e^{-(1-\kappa)N} \\ &= K^n e^{-(1-\kappa)N} |\mathrm{supp}(\mathcal{M})|. \end{aligned}$$

This completes the proof. □

4.3. *Proof of the Reduced Main Theorem.* For $c \in \mathcal{C}$ and $m \ge 0$, let $\mathcal{C}_m^{(c)}$ denote the set of parameters $t \in [0, 1]$ for which the following hold:

- $f_t^{m+1}(c) \in \mathcal{C}$;
- $f_t^j(c) \cap \mathcal{C} = \emptyset$ for all $1 \le j \le m$.

A c-parameter box $B(t, r)$ of order m is said to be *pre-critical* if $t \in \mathcal{C}_m^{(c)}$.

For $m \ge 0$, $c \in \mathcal{C}$, $t_* \in \mathcal{C}_m^{(c)}$ and $\lambda > 1$, let $r_\lambda(t_*, \varepsilon)$ be the maximal number r which satisfies the following properties:

(i) $r \in (0, \varepsilon]$ and $\xi_m^{(c)}(B(t_*, r)) \subset \widetilde{B}(\varepsilon)$;

(ii) $B(t_*, r)$ is a λ-bounded pre-critical c-parameter box of order m.

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Given a positive integer n, let

$$\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon) = \left\{ B(t_*, r_\lambda(t_*, \varepsilon)) \;\middle|\; \begin{array}{c} t_* \in \mathcal{C}_m^{(c)} \text{ for some } m \geq 0 \text{ and} \\ \text{there exists } t \in B(t_*, r_\lambda(t_*, \varepsilon)) \text{ such that} \\ \#\{0 \leq j \leq m : f_t^{j+1}(c) \in \widetilde{B}(\varepsilon)\} \leq n \end{array} \right\}.$$

LEMMA 4.3. *There exists $\lambda > 1$ such that for each $c \in \mathcal{C}$, each $n \geq 1$ and each $\varepsilon > 0$ small, $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ is a special family of height at most n.*

Proof. Assume that $\lambda > 1$ is very close to one. To prove that $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ is special, let $B_i = B(t_i, r_i)$, $i = 1, 2$, be distinct parameter boxes in $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$, of order m_i, such that $t_1 \in B_2^{(1)}$. We need to prove that $|B_1|/|t_1 - t_2|$ is small. Let $c_1, c_2 \in \mathcal{C}$ be such that $\xi_{m_i}^{(c)}(B_i) \subset \widetilde{B}(c_i; \varepsilon)$. Since $f_{t_1}^{j+1}(c) \notin \mathcal{C}$ for each $0 \leq j \leq m_2$, we have $m_1 > m_2$. By the bounded distortion property of $\xi_{m_2}^{(c)}|B_2$, it suffices to show that

$$\sup_{t \in B_1} \frac{|\xi_{m_2}^{(c)}(t) - \xi_{m_2}^{(c)}(t_1)|}{|\xi_{m_2}^{(c)}(t_1) - c_2|}$$

is sufficiently small. This is clear: for each $t \in B_1$ and for each $0 \leq k \leq m_2 + 1 \leq m_1$,

$$\lambda^{-1}|Df_{t_1}^k(f_{t_1}(c))| \leq |Df_t^k(f_t(c))| \leq \lambda|Df_{t_1}^k(f_{t_1}(c))|,$$

hence $\lambda^{-1}|Df_{t_1}(\xi_{m_2}^{(c)}(t_1))| \leq |Df_t(\xi_{m_2}^{(c)}(t))| \leq \lambda|Df_{t_1}(\xi_{m_2}^{(c)}(t_1))|$, so the statement follows from the local behavior of f near c_2.

Let us prove that the height of $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ does not exceed n. Otherwise, there would exist $B_j \in \mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$, $0 \leq j \leq n$, such that $B_n \subsetneq B_{n-1} \subsetneq \cdots \subsetneq B_0$. Let m_j be the order of B_j. Then, as above, we would have $m_0 < m_1 < \cdots < m_n$. Then, for $t \in B_n$, $\{0 \leq j \leq m_n : f_t^{j+1}(c) \in \widetilde{B}(\varepsilon)\} \supset \{m_0, m_1, \ldots, m_n\}$ would contain at least $n + 1$ elements, a contradiction. □

Now we fix a constant $\lambda > 1$ so that the conclusion of Lemma 4.3 holds. As before, we use $S_j^{(c)}(t; \varepsilon)$ to denote the jth return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Then, by Proposition 4.1, we have the following lemma.

LEMMA 4.4. *There exists a constant $C_0 > 0$ such that for any $C > 0$ the following holds provided that $\varepsilon > 0$ is small enough. For $t \in X_{n,\varepsilon}(C)$, $c \in \mathcal{C}$ and $1 \leq j \leq n$, if $\widetilde{p}_j^{(c)}(t; \varepsilon) > C_0$, then there is a pre-critical c-parameter box $B(t_*, r)$ of order $S_j^{(c)}(t; \varepsilon)$ in $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ such that*

$$\operatorname{dep}(t|B(t_*, r)) \geq \widetilde{p}_j^{(c)}(t; \varepsilon) - C_0.$$

Proof. Fix C. By Corollary 3.1 and Proposition 4.1, provided that $\varepsilon > 0$ is small enough, there is a λ-bounded c-parameter box $B(t, r_0)$ of order $S_j := S_j^{(c)}(t, \varepsilon)$, with $r_0\, \mathrm{A}(f_t(c), t, S_j) = \theta$, where $\theta > 0$ is a constant independent of C. Let c' be the critical point such that $f_t^{S_j+1}(c) \in \widetilde{B}(c'; \varepsilon)$, $p_j = p_j^{(c)}(t, \varepsilon)$ and $d_j = d_j^{(c)}(t, \varepsilon)$. Assume that p_j and d_j are large. Since $|(\xi_{S_j}^{(c)}(t))'| \cdot r_0 \asymp |Df_t^{S_j}(f_t(c))| \cdot r_0 \asymp e^{p_j} \cdot |f_t^{S_j+1}(c) - c'|$, there exists a λ-parameter box $B(t_*, r_*) \subset B(t, r_0)$ of order S_j such that $r_* \asymp r_0$, $t_* \in \mathcal{C}_{S_j}^{(c)}$

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and $\mathrm{dep}(t|B(t_*, r_*)) - p_j$ is bounded away from $-\infty$. Let $r = r_\lambda(t_*, \varepsilon)$. Clearly, $B(t_*, r) \in \mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)$. If $r \geq r_*$, then $\mathrm{dep}(t|B(t_*, r)) \geq \mathrm{dep}(t|B(t_*, r_*))$ and we are done. So, assume that $r < r_*$.

We claim that $\partial(\xi^{(c)}_{S_j}(B(t_*, r))) \cap \partial\widetilde{B}(c', \varepsilon) \neq \emptyset$. Otherwise we would have $r = \varepsilon$. Since $t \in X_{n,\varepsilon}(C)$, by Lemma 2.1, we would have that $|Df^{S_j}_{t_1}(f_{t_1}(c))|$ were much larger than $(D_{c'}(\varepsilon))^{-1}$. It would then follow that $\xi^{(c)}_{S_j}(B(t_*, r)) \supsetneq \widetilde{B}(c', \varepsilon)$, contradicting the definition of r.

By the bounded distortion property of $\xi^{(c)}_{S_j}|B(t_*, r)$, it follows that $\xi^{(c)}_{S_j}(B(t_*, r)) \supset \widetilde{B}(c', \varepsilon')$ holds for some $\varepsilon' \asymp \varepsilon$. Since $|f_t^{S_j+1}(c) - c'| \geq e^{-d_j}|\widetilde{B}(c'; \varepsilon)|$, we conclude that $\mathrm{dep}(t|B(t_*, r)) - d_j$ is bounded away from $-\infty$. The lemma is proved. □

Proof of the Reduced Main Theorem. (i) Let $\lambda > 1$ and C_0 be as above. We may assume that $C > 8C_0$. Consider $t \in X_{n,\varepsilon}(C)\backslash X_{n+1,\varepsilon}(C)$ with $\varepsilon > 0$ small. By Proposition 3.2(i) (taking $\gamma = 1/2$) and Lemma 4.4, there exists $c \in \mathcal{C}$ such that

$$\sum_{B \in \mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)} \mathrm{dep}(t|B) \geq \sum_{k \in \widehat{\mathcal{T}}^{(c)}_{\mathrm{ess}}(C_0, t; \varepsilon), k \leq n} (\widetilde{p}^{(c)}_k(t; \varepsilon) - C_0) \geq Cn/4. \tag{4.11}$$

By Lemmas 4.3 and 4.2 (taking $\kappa = 1/2$), it follows that

$$|X_{n,\varepsilon}(C)\backslash X_{n+1,\varepsilon}(C)| \leq \sum_{c \in \mathcal{C}} K^n e^{-Cn/8} |\mathrm{supp}(\mathcal{M}^{(c)}_{n,\lambda})| \leq (4\varepsilon \# \mathcal{C}) \cdot K^n e^{-Cn/8},$$

where K is a constant.

(ii) This follows from Corollary 3.1 and Lemma 4.1.

(iii) Fix $C > 0$, $\tau > \tau_0 > 1$ and $\kappa > 0$. Let $\gamma \in (0, 1)$ be a constant such that $\gamma^3\tau > \tau_0$. By Proposition 3.2(ii), for each $t \in Y^m_\varepsilon(C, \tau)\backslash Y^{m+1}_\varepsilon(C, \tau)$, there exist $c \in \mathcal{C}$ and $n \in \widehat{\mathcal{T}}^{(c)}_{\mathrm{ess}}(C_0, t; \varepsilon)$ such that $m = S^{(c)}_n(t; \varepsilon)$ and

$$p^{(c)}_n(t; \varepsilon) \geq \gamma d^{(c)}_n(t; \varepsilon) \geq \gamma\tau \log(S^{(c)}_n(t; \varepsilon) + 1).$$

We may certainly assume that $Y^m_\varepsilon(C, \tau)\backslash Y^{m+1}_\varepsilon(C, \tau) \neq \emptyset$. Then,

$$m \geq S(\varepsilon) := \inf\{S^{(c)}_1(t; \varepsilon) : |t| \leq \varepsilon, c \in \mathcal{C}\}$$

is large provided that $\varepsilon > 0$ is small. By Lemma 4.4, there exists $t_* \in \mathcal{C}^{(c)}_m$ such that

$$\mathrm{dep}(t|B(t_*, r_\lambda(t_*, \varepsilon))) \geq \gamma\tau \log(m + 1) - C_0 \geq \gamma^2\tau \log(m + 1).$$

Since these parameter boxes $B(t_*, r_\lambda(t_*, \varepsilon))$, $t_* \in \mathcal{C}^{(c)}_m$, are pairwise disjoint, it follows that

$$|Y^m_\varepsilon(C, \tau)\backslash Y^{m+1}_\varepsilon(C, \tau)| \leq C_1\varepsilon \# \mathcal{C}(m + 1)^{-\gamma^3\tau}\varepsilon \leq \sigma\varepsilon(m + 1)^{-\tau_0},$$

since m is large. □

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