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Geodesically Complete Hyperbolic Structures

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Abstract

In the first part of this work we explore the geometry of infinite type surfaces and the relationship between its convex core and space of ends. In particular, we give a geometric proof of a Theorem due to Alvarez and Rodriguez that a geodesically complete hyperbolic surface is made up of its convex core with funnels attached along the simple closed geodesic components and half-planes attached along simple open geodesic components. We next consider gluing infinitely many pairs of pants along their cuffs to obtain an infinite hyperbolic surface. We prove that there always exists a choice of twists in the gluings such that the surface is complete regardless of the size of the cuffs. This generalises the examples of Matsuzaki.

In the second part we consider complete hyperbolic flute surfaces with rapidly increasing cuff lengths and prove that the corresponding quasiconformal Teichmüller space is incomplete in the length spectrum metric. Moreover, we describe the twist coordinates and convergence in terms of the twist coordinates on the closure of the quasiconformal Teichmüller space.

1. *Introduction*

A pair of pants is a metrically complete hyperbolic surface of type $(0, 3)$ with boundary components being either closed geodesics, called *cuffs*, or punctures, and at least one boundary component being geodesic. The pair of pants is made geodesically complete by attaching appropriate size funnels to each geodesic boundary component. A natural way of creating more complicated hyperbolic surfaces is to glue pairs of pants along their cuffs via

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an isometry (where the cuffs that are glued necessarily have the same length). The gluings are not unique and they depend on a real parameter called the *twist*.

Recall that a Riemannian manifold is geodesically complete if and only if it is metrically complete. On the other hand, if the manifold is not metrically complete or has boundary it is natural to ask if it has a geodesic completion. Alvarez and Rodriguez (see [**5**]) showed that a hyperbolic surface constructed from gluing pairs of pants that form a pants decomposition X' has a unique metric completion to the convex core of a geodesically complete hyperbolic surface X by attaching funnels and closed half-planes; conversely, any geodesically complete hyperbolic surface is obtained by attaching funnels and half-planes to the convex core of the surface. We state this as Theorem 3·4 and supply our own proof. We sometimes say that X is the *geodesic completion* of X'. The existence of half-planes glued to the convex core appears in [**6**].

To understand the relationship between the convex core boundary and the space of ends we are led to the notion of a *visible end*. An end is *visible* if there is an open sets worth of tangent vectors that when exponentiated exit the end. A geodesic ray that begins in $C(X)$ and exits a visible end e must intersect a boundary component of the convex core $C(X)$; such boundary components are called the *components of the visible end e* (see Section 2·2 for the definition). For a finite type surface, as is well-known, a boundary component of the convex core is a simple closed geodesic which bounds a funnel. The funnel determines the visible end. More generally for an infinite type surface a boundary component of the convex core may be a simple geodesic isometric to the real line which bounds a half-plane (see Corollary 3·6 and Table 2). The half-planes determine visible ends. That there can be several half-planes determining the same visible end is illustrated by our *flute surface* examples in Section 4. A *flute surface* is a sequence of pairs of pants glued in succession along common length boundaries. The flute surface is not necessarily geodesically complete but always has a natural geodesic completion by Theorem 3·4. It is said to be a *tight flute surface* if in addition all the pants holes that have not been glued along are in fact cusps. The flute surface has a unique *infinite type end* - it is the limit of the isolated ends. See Section 2 for the basics on ends and Section 5 for more on flute surfaces (also cf. [**5–8**, **12**] and Figure 4).

Let Ω be a hyperbolic domain in the complex plane. In the paper [**5**] it is shown that a topological half disc in Ω bounding a maximal interval on $\partial\Omega$ can be straightened to be a maximal hyperbolic half-plane. We study such a situation when Ω is the unit disc minus a countable set of points that accumulate on the boundary. In particular we prove *for any* $n \in \mathbb{N} \cup \{\infty\}$ *there exists a tight flute surface whose unique infinite type end is a visible end with n components*. Though this can be proven using the work of [**5**] in Section 4 we give a geometric proof using the convex core and the notion of visible ends. (see Theorem 4·5),

The half-planes in the above discussion arise for infinite type ends that are not metrically (or equivalently not geodesically) complete. Namely, if we glue infinitely many pairs of pants we obtain an infinite surface which might not be complete as a metric space. Indeed, when the lengths of the cuffs of the glued pairs of pants are going to infinity the distance between two cuffs is going to zero. If we choose the twists to be zero and if the distances between cuffs add up to a finite number then the obtained surface has an open finite length geodesic which leaves every compact set and thus the surface is not metrically complete (see [**6**]). Thus a natural question is whether there is a choice of twists such that the surface is complete regardless of how large the cuffs are. On his way to showing the existence of countable Teichmüller modular groups Matsuzaki (see [**15**]) demonstrated that such phenomena

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exist by demonstrating examples. We show that this is the case for all possible topologies on the infinite surfaces which arise by different patterns of gluing pairs of pants. That is, by choosing the twists judiciously, there is no need to attach half-planes to the infinite type ends since they are already geodesically complete (see Theorem 5·1).

THEOREM 1. *Let X' be a (not necessarily complete) hyperbolic surface with a pants decomposition. Then there exists a choice of twists along the cuffs of the pants so that the induced hyperbolic surface X, after possibly adding funnels, is a geodesically complete hyperbolic surface.*

Shiga [**19**], Alessandrini, Liu, Papadopuolos, Su and Sun (cf. [**2–4**]) and others (cf. [**8**, **14**, **18**]) have studied the Teichmüller spaces (quasiconformal and length spectrum) of infinite surfaces either when there is an upper bound on the cuff lengths or when they are given by an explicit construction. These surfaces are complete either because of the upper bound on the cuff lengths or by the construction. For arbitrary surfaces built from the pairs of pants with unbounded cuffs a choice of twists might lead to an incomplete surface. A priori, one might think that being complete could impose conditions on the speed that the cuff lengths go to infinity (which may influence the Teichmüller theory). Theorem 2 says that the completeness of surfaces does not impose constraints on the speed of convergence to infinity of the cuff lengths and this opens the possibility for studying Teichmüller spaces of infinite surfaces of various topological types and geometrical shapes.

We proceed to analyse Teichmüller spaces of flute surfaces which are obtained by gluing pairs of pants in a chain with cuff lengths rapidly increasing. More precisely, we say a strictly increasing sequence is *rapidly increasing* if the sum of the first n elements is going to infinity slower than the $(n+1)$st element. We choose the twists using Theorem 2 such that the obtained hyperbolic surface X_0 is complete. Our main focus are the twists under the limits of the quasiconformal deformations when we fix the lengths of the cuffs.

The *quasiconformal Teichmüller space* $T_{qc}(X_0)$ consists of all quasiconformal deformations of X_0 modulo postcomposition by conformal maps and homotopy. The quasiconformal Teichmüller space $T_{qc}(X_0)$ has a natural metric given by the $1/2$ of the logarithm of the smallest quasiconformal constant in the homotopy class of a quasiconformal map. The *length spectrum Teichmüller space* $T_{ls}(X_0)$ consists of all homeomorphic transformation of X_0 such that the ratio of the lengths of the corresponding simple closed geodesics is bounded away from 0 and ∞. The length spectrum distance is $1/2$ the absolute value of the logarithm of the ratio of the lengths of the corresponding simple closed geodesics. It is a consequence of the quasi-invariance of closed geodesic length (see [**20–22**]) that $T_{qc}(X_0) \subset T_{ls}(X_0)$.

The next Theorem considers the closure (of the slice with fixed cuff lengths and varying twists of a geodesic pants decomposition) of the quasiconformal Teichmüller space $T_{qc}(X_0)$ in the length spectrum metric. We obtain that the twists can be proportional to the lengths of the closed geodesics which tend to infinity, (see Theorem 8·2).

THEOREM 2. *Let X_0 be a geodesically complete tight flute surface built by gluing pairs of pants with rapidly increasing cuff lengths $\{l_n\}$. Then the closure $\overline{T_{qc}(X_0)}$ of the quasiconformal Teichmüller space $T_{qc}(X_0)$ contains all surfaces with Fenchel-Nielsen coordinates $\{(l_n, t_n)\}_n$, where $-Cl_n \leqslant t_n \leqslant Cl_n$, for $C > 0$, and the lengths $\{l_n\}$ correspond to a marked surface in $T_{qc}(X_0)$.*

Remark. In the above Theorem and the Theorems that follow the base point X_0 of $T_{qc}(X_0)$ corresponds to a fixed choice of twist parameters where the twists $\{t_n\}$ satisfy $0 \leqslant t_n < l_n$.

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We are able to describe the convergence in (the slice with fixed cuff lengths of) $T_{qc}(X_0)$ with respect to the length spectrum metric (see Theorem 8·5).

THEOREM 3. *Let X_0 be a geodesically complete tight flute surface with twists $\{t_n\}$ and rapidly increasing cuff lengths $\{l_n\}$. Let X_k be marked hyperbolic surface with cuff lengths equal to $\{l_n\}$ and twists $t_{X_k}(\alpha_n) = t_n + O(l_n)$. If $\lim_{k\to\infty} t_{X_k}(\alpha_n) = t_n$ for each n, then X_k converges to X_0 in the length spectrum metric.*

Using Theorem 3 we prove that the closure of $T_{qc}(X_0)$ is strictly larger than $T_{qc}(X_0)$ (see Theorem 8·6).

THEOREM 4. *If X_0 is a geodesically complete tight flute surface with rapidly increasing cuff lengths, then the length spectrum Teichmüller space $T_{ls}(X_0)$ is strictly larger than the quasiconformal Teichmüller space $T_{qc}(X_0)$. More precisely, $\overline{T_{qc}(X_0)} - T_{qc}(X_0)$ is non-empty.*

Among the open questions that remain are finding the Fenchel–Nielsen coordinates for the quasiconformal, asymptotically conformal and length spectrum Teichmuller space of the surface in question, and describing the various topologies on these spaces in terms of Fenchel–Nielsen coordinates.

Notation and contents. For the convenience of the reader in Table 1 we gather some of the notation used in this paper. The Section listed is the first place aside from Section 1 that the notation appears. As a matter of convention we often use the prime notation such as X' to denote a not necessarily complete hyperbolic surface.

In Section 2 we discuss the basics of the topology of surfaces including the classification of surfaces using the space of ends. Then we move to the basics of the geometry of ends (2·2). In Section 3 we discuss pants decompositions and the relationship between the boundary components of the convex core and visible ends. Section 4 has examples of visible ends having more than one component. In Section 5 we address the question of finding a geodesically complete structure with rapidly increasing cuffs. In Section 6 we define a flute surface whose cuff lengths rapidly increase. In Section 7 we define the quasiconformal and length spectrum Teichmüller spaces. In Section 8 we discuss various facts about the Teichmüller space of flute surface with rapidly increasing cuff lengths.

2. *Topology and geometry of ends*

In this Section we discuss some basics on topology and geometry, introduce the concept of a visible end and set-up notation. As references for the basics on hyperbolic geometry and discrete groups we refer to the books of Beardon [**9**] and Buser [**10**].

2·1. *Topology of ends*

A surface is of *finite topological type* if it has a finitely generated fundamental group. Otherwise we say it is of *infinite topological type*. The classification of infinite type surfaces was first proven by Bela Kérékhárto ([**13**]). Also see Richards ([**16**]). We refer the reader to the paper ([**8**]) for a discussion on ends and notation. All surfaces in this paper are triangulable and orientable. Since we are interested in Riemann surfaces all of our surfaces satisfy these two assumptions.

Fix X a topological surface with non-abelian fundamental group, and $\{X_k\}$ a compact exhaustion of X. Let $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_k \supset \cdots$ be a nested sequence of subsets of X so

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Table 1. *Definitions and notation*

Definition	Section	Notation
hyperbolic plane	2	$\mathbb{H}$
unit disc in complex plane	2	Δ
quasiconformal Teichmüller space	7	T_{qc}
Teichmüller distance	7	d_T
length spectrum Teichmüller space	7	T_{ls}
length spectrum distance	7	d_{ls}
convex core	2	$C(X)$
boundary of convex core	2	$\partial C(X)$
equivalent boundary components of convex core	2	$b_1 \sim b_2$
space of ends	2	$\mathcal{E}_X$
space of non-planar ends	2	$\mathcal{N}_X$
visible ends	2	$\mathcal{V}E_X$
limit set	2	$\Lambda(\Gamma)$
set of discontinuity	2	$\Omega(\Gamma)$
X-length of α	7	$\ell_X(\alpha)$
flute surface	5	
tight flute surface	5	
rapidly increasing sequence	6	

that, for each k, $\mathcal{C}_k$ is a connected component of $X - X_k$. Two such sequences $\{\mathcal{C}_k\}$ and $\{\mathcal{C}'_k\}$ are equivalent if for each subset $\mathcal{C}_k, \mathcal{C}'_{k+n} \subset \mathcal{C}_k$ for large n, and vice-versa. These equivalence classes form the *space of ends* denoted $\mathcal{E}_X$. We usually use a representative sequence to denote the equivalence class of an end. We next describe a basis for the topology on the space of ends. Let U be a connected component of $X - X_k$. Define,

$$U^* = \{e \in \mathcal{E}_X : e = \{\mathcal{C}_k\} \text{ and } \mathcal{C}_k \subset U, \text{ for large } k\}. \tag{1}$$

The set of all such U^* form a basis for the topology of $\mathcal{E}_X$. The topology of $\mathcal{E}_X$ does not depend on the choice of compact exhaustion. The subspace $\mathcal{N}_X \subset \mathcal{E}_X$ denotes the subspace of non-planar ends; an end $e = \{\mathcal{C}_k\}$ is *non-planar* if each $\mathcal{C}_k$ has infinite genus. The non-planar ends form a closed subset of $\mathcal{E}_X$.

A homeomorphism $f : X \to Y$ between surfaces induces a homeomorphism of pairs,

$$f_* : (\mathcal{N}_X, \mathcal{E}_X) \longrightarrow (\mathcal{N}_Y, \mathcal{E}_Y), \tag{2}$$

and hence the pair $(\mathcal{N}_X, \mathcal{E}_X)$ is a topological invariant of X (called the *end invariants* of X). If X is of finite topological type then X is a closed surface with $|\mathcal{E}_X| < \infty$ points deleted, and $\mathcal{N}_X = \varnothing$. More particularly, we will say that $e \in \mathcal{E}_X$ is a *finite topological type end* if e is planar and isolated in $\mathcal{E}_X$. Otherwise, it is an *infinite topological type end*. Clearly, a surface is of infinite topological type if and only if there exists an end of infinite topological type.

THEOREM 2·1 (Kérékhárto [**13**], Richards, [**16**]). *The orientable surfaces X and Y are topologically equivalent if and only if genus(X) = genus(Y) and $(\mathcal{N}_X, \mathcal{E}_X)$ is homeomorphic (as pairs) to $(\mathcal{N}_Y, \mathcal{E}_Y)$.*

Let e be an end of the topological surface X. We say that a sequence of compact sets $\{K_i\}$ *exit the end* e, if $\{K_i\}$ converges to e in the space $X \cup \mathcal{E}_X$. By abuse of language, we also say that a path $\gamma : [0, \infty) \to X$ *exits the end* e if $\gamma(t)$ converges to e as t goes to infinity. In the sequel, we will often be interested in surfaces with a hyperbolic structure and hence

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typically the $\{K_i\}$ will either be sequences of pairs of pants or simple closed geodesics and the paths γ will be geodesic rays.

2·2. *Geometry of ends*

We denote the real part, resp. imaginary part, of a complex number z by $\Re(z)$, resp. $\Im(z)$. A *funnel* is a hyperbolic surface with one geodesic boundary component which is isometric to $D/ < z \mapsto e^{\ell} z >$, where $D = \{z \in U : \Re(z) \leqslant 0\}$ has the induced metric as a subspace of the upper half-plane model U of the hyperbolic plane. Funnels are annuli with one geodesic boundary component whose length ℓ determines the funnel. A (standard) *cusp* is a hyperbolic surface with one horocyclic boundary component which is isometric to the quotient $\{z : \Im(z) \geqslant 1\}/ < z \mapsto z + 1 >$, where again $\{z : \Im(z) \geqslant 1\}$ has the induced metric as a subspace of the upper half-plane model. It is well-known that any finite type geodesically complete hyperbolic surface has ends that are either cusps or funnels. We say that $Y \subset X$ is a *geodesic subsurface* of the hyperbolic surface X if it is a subsurface with geodesic boundary.

A Riemannian manifold is *geodesically complete* if every geodesic can be extended infinitely far in both directions. Geodesic completeness is equivalent to the induced Riemannian (metric) distance being complete. A geodesically complete hyperbolic surface X is the quotient of the hyperbolic plane, $\mathbb{H}$, by a torsion-free discrete non-elementary (*Fuchsian*) group Γ of orientation preserving isometries. The action of Γ on the ideal boundary of the hyperbolic plane breaks up into the limit set $\Lambda(\Gamma)$ and the (possibly empty) set of discontinuity, $\Omega(\Gamma)$. The set of discontinuity is made up of a countable union of *intervals of discontinuity*. It is well-known that the stabiliser in Γ of an interval of discontinuity is either generated by a hyperbolic element or is trivial. Only the first possibility occurs if Γ is finitely generated. That such a stabiliser can be trivial if Γ is infinitely generated is investigated in the paper [**6**]. The *convex core* of X, $C(X)$, is the quotient of the convex hull of the limit set, $CH(\Lambda(\Gamma))/\Gamma$. The convex core is the smallest closed convex subsurface (with boundary) which carries all the homotopy. In particular, all closed geodesics are contained in $C(X)$. Let $X = \mathbb{H}/\Gamma$ be a geodesically complete hyperbolic surface. X (or Γ) is said to be of the *first kind* if $\Lambda(\Gamma) = \partial\mathbb{H}$; equivalently, $C(X) = X$. Otherwise it is of the *second kind*. We say that a sequence of oriented geodesics $\{L_i\}$ in $\mathbb{H}$ converge to the oriented geodesic L if the endpoints of the $\{L_i\}$ converge to the endpoints of L. That is, the space of oriented geodesics can be identified with $\mathbb{S}^1 \times \mathbb{S}^1 - \{diagonal\}$. Sometimes we are not interested in orientation of the geodesics and so we say $\{\text{Ł}_i\}$ converges to L if up to changing orientations the convergence occurs. On a hyperbolic surface a sequence of geodesics γ_i is said to converge to the geodesic γ if the geodesics have lifts to the hyperbolic plane so that $\tilde{\gamma}_i$ converges to $\tilde{\gamma}$.

To study the end geometry of a hyperbolic surface we introduce the notion of a visible end. An end e of X is said to be *visible* if there exists an open set V in the unit tangent bundle of X so that for any $v \in V$, the induced geodesic ray g_v exits e. Otherwise, the end is said to be *non-visible* or *complete*. We denote the visible ends by $\mathcal{V}E_X$. The next Lemma allows us to describe a visible end in three different ways.

LEMMA 2·2. *Let X be a geodesically complete hyperbolic surface. The following are equivalent*:

(i) *e is a visible end*;

(ii) *there exists a point $x \in X$ and a cone of vectors based at x so that their corresponding geodesics rays exit e*;

(iii) *there exists a geodesic ray in X that exits e and leaves $C(X)$ in finite time.*

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Proof. The equivalence of items (1) and (2) is clear. For the equivalence of (2) and (3), suppose there is a cone of vectors based at a point of $X = \mathbb{H}/\Gamma$ for which the corresponding geodesic rays exit e. Lifting these geodesic rays to the universal covering $\mathbb{H}$ and noting that they hit $\partial\mathbb{H}$ in an interval it is clear that this interval must be contained in one of the intervals of discontinuity of Γ. But then these geodesic rays leave $C(X)$ in finite time. The converse follows from the fact that if one such leaves $C(X)$ in finite time then there is a cone's worth that does.

If X has a funnel then the end corresponding to the funnel is a visible end, and corresponds to exactly one component of the complement of $C(X)$. Recalling that an end e is of finite (topological) type if it is isolated in $\mathcal{E}_X - \mathcal{N}_X$, a finite type end is not visible if it corresponds to a cusp of the surface, and visible if it corresponds to a funnel. Thus, as is well-known, we have a nice description of the end geometry of a hyperbolic surface with a finitely generated fundamental group.

Let X be a geodesically complete hyperbolic surface. We next define an equivalence relation on the boundary components of $C(X)$. Namely, two boundary components b_1 and b_2 of $\partial C(X)$ are *equivalent*, denoted $b_1 \sim b_2$, if there exist two geodesic rays based in $C(X)$ that go out the same end where one of them crosses b_1 and the other crosses b_2. This is clearly an equivalence relation and we denote the set of boundary components that are equivalent to the boundary component b by $\{b\}$.

Now given $e \in \mathcal{V}E_X$, let γ be a geodesic ray based in $C(X)$ and exiting e. Then γ must intersect a boundary component, say b, of $C(X)$. This defines a well-defined mapping, $B : \mathcal{V}E_X \to \partial C(X)/\sim$ given by $e \mapsto \{b\}$ which is easy to see is a bijection.

Thus a visible end of a geodesically complete hyperbolic surface X corresponds to an equivalence class of connected components of the complement of $C(X)$. The boundary components that correspond (by the bijection) to the visible end e we call the *components* of e. Of course in the case of a funnel there is exactly one component in its equivalence class. That there can be more than one component is investigated in Section 4.

3. *Pants decompositions and the classification of ends*

In this Section we discuss a geometric classification of ends. This is a consequence of Theorem 3·4 which is due to Alvarez and Rodriguez (see [**5**]). We give a different proof of this Theorem which suits our point of view and leads to the geometric classification of ends. See [**6**, section 6, lemma 2] for the need to use closed half-planes in order to geodesically complete the convex core. This half-plane is bounded by a simple open geodesic which is the limit of an infinite sequence of simple closed geodesics. Hence the closed half-plane is also maximal.

A *topological pair of pants* is a sphere with three disjoint closed discs removed. We sometimes include the three boundary circles as part of our topological pants. The context should make it clear. A *geodesic pair of pants* is a sphere with three disjoint closed discs removed endowed with a hyperbolic metric where the boundary curves are geodesic. We allow the possibility that the pair of pants has one or two cusps (a so called *tight pair of pants*). A pair of pants has a natural geodesic completion to a complete hyperbolic structure where each geodesic boundary component is completed by a funnel. By abuse of language we sometimes call the geodesically complete surface a pair of pants. More generally, any surface X' made up of a finite number of pairs of pants glued along common cuffs has a unique geodesic completion X by adding funnels. Furthermore, $X' = C(X)$. In fact, any geodesically com-

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plete hyperbolic surface with finitely generated fundamental group arises in this way. A *topological pants decomposition* of a surface is a locally finite decomposition by pairs of pants where the pants curve are homotopically distinct and non-trivial. A topological pants decomposition is a *geodesic pants decomposition* if the pairs of pants are geodesic pairs of pants. For ease of language, we will often drop the adjective *geodesic* before the terms "pair of pants" and "pants decomposition." The context should make it clear.

As we saw in Section 2 the boundary of $C(X)$ in X is the union of simple closed geodesics and simple open geodesics. Denoting the simple open ones by $\{L_i\}$, consider the surface with boundary $C(X) - \bigcup\{L_i\}$.

PROPOSITION 3·1. *Let X be a geodesically complete hyperbolic surface. Every topological pants decomposition of $C(X) - \bigcup\{L_i\}$ can be straightened to a geodesic pants decomposition.*

It is a consequence of the classification result ([**16**]) that any infinite type surface admits a topological exhaustion by finite type surfaces. Hence if X is a geodesically complete hyperbolic surface this fact coupled with Proposition 3·1 supplies us with a short proof of the following corollary.

COROLLARY 3·2. *Let X be a geodesically complete hyperbolic surface. Then $C(X) - \{L_i\}$ has a geodesic pants decomposition.*

Remark 3·3. In the case that there are no open geodesics on $\partial C(X)$ Corollary 3·2 is a result in [**4**].

The proof of proposition 3·1 follows.

Proof. Let $X = \mathbb{H}/\Gamma$. Since any (topological) pants decomposition of a surface with boundary induces an exhaustion by finite type (topological) subsurfaces and vice-versa, it is enough to show that a topological exhaustion by finite type subsurfaces $\{K_n\}$ of $C(X) - \bigcup\{L_i\}$ straightens to an exhaustion by finite area geodesic subsurfaces. To see this let Y_n, for each n, be the subsurface K_n with boundary curves straightened to geodesics; $\{Y_n\}$ is a geodesic subsurface of $C(X) - \bigcup\{L_i\} \subset X$. Y_n is also homeomorphic to K_n, for each n, which already implies that the $\{Y_n\}$ are locally finite and hence the straightened geodesic pairs of pants are locally finite.

We are left to show that the $\{Y_n\}$ cover $C(X) - \bigcup\{L_i\}$. Now, by way of the isomorphism between the fundamental group of X and the group Γ there exists a torsion-free Fuchsian subgroup Γ_n of Γ so that $Y_n = C_n/\Gamma$, where C_n is the convex hull of the limit set of Γ_n. Moreover the $\{\Gamma_n\}$ can be chosen so that $\Gamma_n \leqslant \Gamma_{n+1}$, and hence $C_n \subseteq C_{n+1}$. Note that since the fundamental group of $C(X) - \bigcup\{L_i\}$ is isomorphic to Γ, $\Gamma =< \Gamma_n >$ and Γ keeps $\overline{\bigcup_n C_n}$ invariant. On the one hand, $\overline{\bigcup_n C_n}$ must be contained in the convex hull of Γ, and hence the boundary at infinity of $\overline{\bigcup_n C_n}$ is contained in the limit set of Γ. Since the limit set of Γ is the smallest Γ-invariant non-empty closed subset of $\partial\mathbb{H}$ it must be that the boundary at infinity of $\overline{\bigcup_n C_n}$ is equal to the limit set. Hence $\overline{\bigcup_n C_n}$ is the convex hull of the limit set, and therefore $\overline{\bigcup_n C_n/\Gamma} = C(X)$. Thus the geodesic subsurfaces $\{Y_n\}$ exhaust $C(X) - \bigcup\{L_i\}$.

The ends of a hyperbolic surface constructed from finitely many pairs of pants (that is, a finite type hyperbolic surface) are wellknown to be geometrically either cusps or funnels. For a hyperbolic surface constructed from an infinite number of pairs of pants we have,

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THEOREM 3·4. *Let X' be a (not necessarily complete) hyperbolic surface constructed from gluing pairs of pants that form a pants decomposition of X'. Then X' has a unique metric completion to the convex core of a geodesically complete hyperbolic surface X so that $X' \subset C(X) \subset X$. Moreover, the geodesic completion of X' is attained by adding funnels and closed hyperbolic half-planes. Conversely, any geodesically complete hyperbolic surface is the geodesic completion of a (not necessarily complete) hyperbolic surface X' constructed from gluing pairs of pants that form a pants decomposition of X'.*

Proof. The pants decomposition induces an exhaustion by finite type geodesic subsurfaces $\{Y_n\}$. Observe that each such Y_n being of finite type has a completion by adding funnels to boundary geodesics. Hence there exists a torsion-free Fuchsian group Γ_n so that $\mathbb{H}/\Gamma_n$ is a complete hyperbolic surface with convex core Y_n. Denote the convex hull of the limit set of Γ_n by C_n. Since $\Gamma_n \leqslant \Gamma_{n+1}$, we have $C_n \subseteq C_{n+1}$. Next set $C := \bigcup_n C_n$, $\Gamma := \lim_{n\to\infty} \Gamma_n =< \Gamma_n >$, and note that Γ is a torsion-free Fuchsian group (see [**6**] for an infinite version of the combination Theorem). Since the $\{C_n\}$ are an increasing nested sequence of domains with geodesic boundary, it must be that as $n \to \infty$ either the geodesics on the boundary ∂C_n go to infinity or converge to a geodesic. In the later case we include the possibility that a component of ∂C_n is a component of ∂C_{n+k}, for all $k \geqslant 0$. Thus C is a convex subspace of $\mathbb{H}$ where:

(i) the boundary of C is made up of complete geodesics; that is, $C = \mathbb{H} - \bigcup D_i$, where the D_i are open half-planes;
(ii) the interior of C is the universal cover of $X' - \partial X'$;
(iii) a boundary component of C in $\mathbb{H}$ has stabiliser in Γ that is either generated by a hyperbolic element or is trivial. In the later case, the boundary component (geodesic) is the limit of axes of simple hyperbolic elements in Γ;
(iv) Γ keeps invariant $\overline{C}$, the closure of C in $\mathbb{H}$;
(v) Γ does not keep invariant any set smaller than $\overline{C}$;
(vi) $CH(\Lambda(\Gamma)) = \overline{C}$, and hence $\overline{C}/\Gamma$ is the convex core of $\mathbb{H}/\Gamma$, and the metric completion of X'.

We can conclude that $\mathbb{H}/\Gamma$ is the geodesic completion of X' obtained by attaching closed half-planes and funnels. The closed half-planes are coming from the half-planes $\{D_i\}$ in item (1).

For the converse, suppose X is a geodesically complete hyperbolic surface and set $X = \mathbb{H}/\Gamma$, where Γ is a torsion-free discrete group. As a consequence of the fact that the boundary of $CH(\Gamma)$ in $\mathbb{H}$ is comprised of geodesics that are either axes of hyperbolic elements or have trivial stabiliser, we can conclude that the boundary of $C(X)$ in X is made-up of closed geodesics and (infinite) open geodesics. we denote the infinite open geodesics by $\{L_i\}$. Then consider $X' = C(X) - \bigcup\{L_i\}$ and note that we are retaining the closed geodesics, if any, on the boundary of $C(X)$. This surface with possible boundary, where the boundary components are simple closed geodesics, admits a topological pants decomposition, and by Proposition 3·1 we straighten this pants decomposition to a geodesic pants decomposition.

For Γ a non-elementary (that is, not virtually abelian) Fuchsian group, the stabiliser of an interval of discontinuity for Γ is non-trivial if and only if the interval is bounded by the axis of a hyperbolic element in Γ. For trivial stabiliser we have the following characterisation,

COROLLARY 3·5. *Let Γ be a non-elementary torsion-free Fuchsian group and I an interval of discontinuity for Γ. Then I has trivial stabiliser in Γ if and only if there ex-*

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ists a sequence of simple hyperbolic elements in Γ *whose axes converge to the geodesic bounding* I.

Proof. Denote by L, the geodesic bounding I. If I has trivial stabiliser then Theorem 3·4 implies that there must exist a sequence of simple hyperbolic elements whose axes converge to L. Conversely, suppose simple hyperbolic elements have axes $\{L_n\}$ converging to L. If L were the axis of a hyperbolic element $\gamma \in \Gamma$ then there would be γ-translates of L_n that transversely intersect $\{L_n\}$ for large n. This violates the assumption that the $\{L_n\}$ are axes of simple hyperbolic elements. Thus the stabiliser of L, and hence I, is trivial.

Let X' be a (not necessarily complete) hyperbolic surface with a pants decomposition and note that $X' \subset C(X) \subset X$. We have,

COROLLARY 3·6 (Visible ends and boundary components). *Let* X *be a geodesically complete hyperbolic surface. Then a boundary component of* $C(X)$ *is either*:

(i) *a simple closed geodesic that bounds a funnel in* X *and corresponds to an isolated end that is visible*; *or*
(ii) *a simple geodesic isometric to the real line that bounds a half-plane in* X *and corresponds to a component of a visible end of infinite type.*

With the aid of this corollary, the following proposition characterizes the geometry of infinite type ends.

PROPOSITION 3·7. *Let* X' *be a (not necessarily complete) hyperbolic surface with a pants decomposition and* X *its geodesic completion.*

(i) *The closure of* X' *in* X *is* $C(X)$, *and hence* X *is of the first kind if and only if* $X = X'$.
(ii) *Suppose* e *is an infinite type end of* X. *Then* e *is not visible if and only if for any geodesic ray* γ *that exits* e *the sequence of pants in the decomposition of* X' *that* γ *intersects also exit* e.
(iii) X *is of the first kind if and only if each end of* X *is not visible.*

Proof. Item (1) follows from Theorem 3·4.

To prove item (2), suppose e is an infinite type end for X that is not visible. Hence by Lemma 2·2 any geodesic ray γ that exits e and starts in $C(X)$ must stay in $C(X)$. Since the closure in X of X' is $C(X)$ the geodesic ray γ must pass through infinitely many pair of pants in the decomposition of X' that also exit e. On the other hand, if e is a visible end then, again by Lemma 2·2, there exists a geodesic ray that exits e and leaves the convex core $C(X)$ in finite time. Since this end is of infinite type it must be that this geodesic ray intersects $\partial C(X)$ in a simple open geodesic. Then by Theorem 3·4 this ray enters into a half plane embedded in X and hence the infinitely many pairs of pants from X' that the ray intersects do not exit the end e. This proves item (2).

To prove item (3), suppose $X = \mathbb{H}/\Gamma$ is of the second kind and let I be an interval of discontinuity for Γ. Let $\beta \subset \mathbb{H}$ be a geodesic ray that limits to a point in I. Project to X the half-plane determined by this interval and the geodesic ray γ, and note that the projected ray determines an end e for which it exits. If the stabiliser of I is infinite cyclic then the end e corresponds to a funnel and hence is visual. If the stabiliser of I is trivial then Theorem 3·4 implies again that the end is visible. To prove the other direction, suppose X has a visible end. Then there exists a geodesic ray that leaves $C(X)$ in finite time. Hence by Corollary 3·6 this can only mean that the geodesic ray intersects the boundary of $C(X)$ and enters into a

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Table 2. *Ends: geometry and topology*

Top. vs. Geom. end.	Not visible (complete)	Visible (boundary components)
finite type	cusp	simple closed geodesic
infinite type	rays that exit end stay in $C(X)$	equiv. class of simple open geodesics

half-plane which it can not escape. Thus $X \neq C(X)$ and we conclude that X is of the second kind.

We have shown that a visible infinite type end (of a geodesically complete hyperbolic surface) has an equivalance class of components of the convex core boundary being simple open geodesics with attached half-planes. Table 2 summarizes the relationship between the topology and geometry of an end.

Even though a surface can have an uncountable number of ends, the hyperbolic metric places restrictions on the geometry of the ends. Namely, the fact that a Fuchsian group has only a countable number of intervals of discontinuity implies that a complete hyperbolic surface has at most a countable set of ends that are visible. In fact, in Section 4 we supply examples to show that a visible end can correspond to n equivalent boundary components of $C(X)$ for any $n = 1, 2, 3, \ldots$ including n being the cardinality of the integers.

4. *Examples: visible ends with equivalent components.*

Recall that the *components* of a visible end are the boundary components of the convex core that correspond to the end. That is, these components bound half-planes that correspond to the end. In this Section we construct examples to show that a visible end can have countable or any finite number of components. These examples first appeared in the paper [**5**]. Our examples are elementary; namely for each $n \in \mathbb{N} \cup \{\infty\}$ we construct a flute surface with the unique infinite type end being a visible end with n components.

Let A be a countable set of points in the unit disc Δ that accumulate to the set K on the boundary $\partial\Delta$; the set K is closed in $\partial\Delta$. We are interested in the plane domain $X = \Delta - A$ with its unique complete hyperbolic structure compatible with the complex structure. X is a tight flute surface and we denote its unique infinite type end by e. We assume that $\partial\Delta - K \neq \varnothing$ and hence $\partial\Delta - K$ is the union of at most countably many open intervals $\{J_i\}$ on the boundary of the unit disc. Set $\widehat{X} = X \cup \{J_i\}$ and suppose Γ is the torsion-free Fuchsian group so that $\mathbb{H}/\Gamma$ is conformally equivalent to X. We denote the covering map by $f : \mathbb{H} \to \mathbb{H}/\Gamma$.

Remark 4·1. Since the plane domain X does not contain a simple homotopy class which bounds an annulus (in fact, only punctures and discs), we can conclude that the stabiliser of any interval of discontinuity is trivial. Or equivalently, Γ moves any half-plane bounding an interval of discontinuity disjointly away from itself.

LEMMA 4·2. *If $\{z_n\}$ is a sequence in $\mathbb{H}$ that accumulates to an interval of discontinuity, then $\{f(z_n)\}$ must accumulate to $\partial\Delta$.*

Proof. We first note that since the $\{z_n\}$ eventually enter into the complement of the convex hull of Γ and f restricted to any component of the complement is injective (Remark 4·1), it must be that $\{f(z_n)\}$ must leave every compact subset of X. Then the only way $\{f(z_n)\}$ does

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not accumulate to $\partial\Delta$ is if there is a subsequence of $\{f(z_n)\}$ which converge to one of the points of A. But since the points of A are punctures, there would have to be a subsequence of the $\{z_n\}$ that limit to a parabolic fixed point. Since a parabolic fixed point cannot be contained in an interval of discontinuity we have a contradiction.

PROPOSITION 4·3. *The covering map $f : \mathbb{H} \to X$ analytically extends to the set of discontinuity of Γ so that the extension F maps an interval of discontinuity to one of the $\{J_i\}$. Moreover, $F : \mathbb{H} \cup \Omega(\Gamma) \to \widehat{X}$ satisfies*:

(i) *F restricted to any interval of discontinuity is injective*;

(ii) *F establishes a one to one correspondence between Γ-equivalence classes of intervals of discontinuity and the intervals $\{J_i\}$.*

Proof. Using Lemma 4·2, we can apply the reflection principle (see [**1**]) to conclude that $f : \mathbb{H} \to X$ has an analytic extension which maps an interval of discontinuity to one of the intervals $\{J_i\}$. We denote the extension by $F : \mathbb{H} \cup \Omega(\Gamma) \to \widehat{X}$. Furthermore, since f restricted to any half-plane Y that bounds an interval of discontinuity I is injective (Remark 4·1) and since F is an analytic extension, it must be that F is injective on I. This verifies item (1).

So far we have shown that F injectively maps each interval of discontinuity into one of the $\{J_i\}$. To verify item (2), we need to show that this map is onto one of the $\{J_i\}$, and that each $\{J_i\}$ occurs as the image of an interval of discontinuity. To this end, suppose J is one of the open intervals in $\{J_i\}$ and let β be a simple differentiable curve in X with endpoints being the endpoints of J. Denoting J with its endpoints as $\bar{J}$ we have that $\bar{J} \cup \beta$ is the boundary of a simply connected region $X_J \subset X$ (cf. Figure 1). The set $f^{-1}(X_J)$ has countably many simply connected components and the restriction of f to each component is a one-to-one conformal map by the simple connectedness. Fix one such component Y and define $\phi : X_J \to Y$ by $\phi = (f|_{X_J})^{-1}$. Now let $\{z_n\} \in X_J$ such that $\{z_n\}$ accumulate to J as $n \to \infty$. Then, since $f \circ \phi = \mathrm{id}|_{X_J}$, it must be that $\{\phi(z_n)\} \in Y$ accumulate on $\partial\mathbb{H}$. The reflection principle applies to ϕ and hence there exists an analytic extension $\Phi : X_J \cup J \to \mathbb{H} \cup \partial\mathbb{H}$ of ϕ. Since Φ is an analytic extension, J is mapped onto the interior of an arc I' of $\partial\mathbb{H}$. Furthermore since ϕ is injective it must be that Φ is injective on J, and hence I' is contained in an interval of discontinuity which we call I. Since by the identity Theorem, $\Phi^{-1} = F$, for all $z \in Y \cup I'$, we may conclude that $I' = I$ and thus F maps I onto J. Since the argument above is natural with respect to the action of Γ we have shown that each $\{J_i\}$ arises as the image of a Γ-equivalence class of intervals of discontinuity.

We may conclude that there is a one to one correspondence between Γ-equivalence classes of intervals of discontinuity and the intervals $\{J_i\}$.

Remark 4·4. The proof of Proposition 4·3 shows that the curves β in X which bound the intervals $\{J_i\}$ can be chosen to be the boundary components of the convex core. See Figure 1 for an illustration.

Note that the plane domain X has exactly one infinite type end and in fact is the geodesic completion of a tight flute surface (see Section 5 for the definition). By Proposition 4·3 the infinite type end is visible and has exactly n components since the components correspond exactly to the intervals $\{J_i\}$. We have established,

THEOREM 4·5. *For any $n \in \mathbb{N} \cup \{\infty\}$ there exists a tight flute surface whose unique infinite type end is a visible end with n components.*

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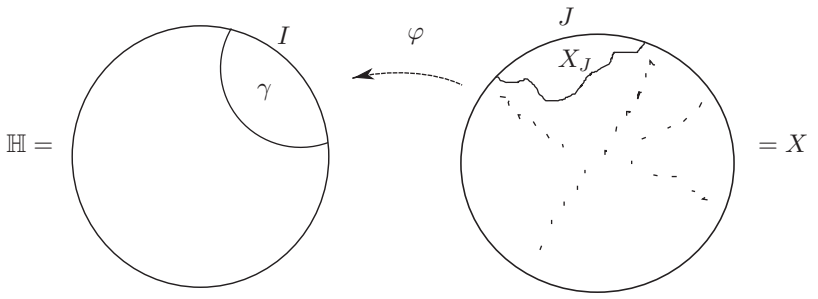


Fig. 1. Plane domain.

5. *Geodesically complete hyperbolic structures*

The purpose of this Section is to prove the following Theorem and investigate some of its consequences.

THEOREM 5·1. *Let X' be a (not necessarily complete) hyperbolic surface with a pants decomposition. Then there exists a choice of twists along the cuffs of the pants so that the induced hyperbolic surface X, after possibly adding funnels, is a geodesically complete hyperbolic surface.*

Thus the induced hyperbolic surface X has the property that all of its infinite ends are not visible.

We will need two Lemmas for the proof of Theorem 5·1. Let Y' be a finite area hyperbolic surface with non-empty geodesic boundary, and fix a boundary component α. Denote the geodesic completion of Y' by Y. We put an orientation on α so that the interior of Y' lies to the left of α. We are interested in unit vectors based in α and directed to the interior of Y'. Such a vector v makes an (oriented) angle θ with α, where $0 < \theta < \pi$, and determines a geodesic ray, $g_v : \mathbb{R}_{\geqslant 0} \to Y$, we call a *$\theta$-ray*. A $\pi/2$-ray is also called an *orthoray*. Denote the vector field of such vectors (based in α) by V_θ. Since the vectors in V_θ may be identified with points in α (take base points of vectors), V_θ inherits a natural measure (called the boundary measure) and topology. This topology is the same as the topology of V_θ as a subspace of the unit tangent bundle of Y. The main Lemma for which the construction below hinges is the following.

LEMMA 5·2. *Fix θ, $0 < \theta < \pi$. Then $V_\theta = A_\theta \dot{\bigcup} O_\theta$ where*:

(i) $A_\theta := \{v \in V_\theta : g_v(t) \subset Y', \text{for all } t \geqslant 0\}$ *is a Cantor set of boundary measure zero*;

(ii) $O_\theta := \{v \in V_\theta : g_v(t) \bigcap \partial Y' \neq \varnothing, \text{ for some } t > 0\}$ *is a countable union of disjoint open intervals in α.*

Proof. Y' is the convex core of the complete hyperbolic surface, $Y = \mathbb{H}/\Gamma$, where Γ is a torsion-free finitely generated Fuchsian group of the second kind. Consider a connected oriented lift of α to the upper half-plane and unit tangent vectors (emanating to the left) that form an angle θ with this lift. The geodesic ray determined by such a vector hits $\partial\mathbb{H}$ in exactly one point. Conversely any point on the left side of the lift of α in $\partial\mathbb{H}$ is the endpoint of such a geodesic ray. Hence there exists a projection map P_θ from $\partial\mathbb{H}$ to the lift. When $\theta = \pi/2$ this map is the usual orthogonal projection. The action of Γ on $\partial\mathbb{H}$ breaks up into the (non-empty) set of discontinuity and the limit set. The set of discontinuity is made of intervals that are bounded by geodesics which are lifts of components of ∂Y. We next note that the geodesic ray determined by a θ-vector $v \in V_\theta$ lifts to a geodesic ray that hits $\partial\mathbb{H}$ at

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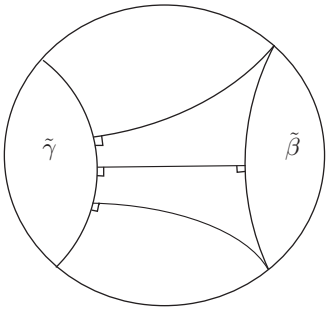


Fig. 2. Interval of orthorays.

a point of discontinuity if and only if the vector is in O_θ. Otherwise the geodesic ray hits the limit set of Γ and hence is in A_θ. Since the limit set of a finitely generated Fuchsian group of the second kind is known to be a measure zero Cantor set we have verified items (1) and (2).

LEMMA 5·3. *Let X' be a (not necessarily complete) hyperbolic surface with a pants decomposition $\mathcal{P}$, and let X be its geodesic completion. Fix γ a simple closed geodesic on the hyperbolic surface X' and let e be an infinite type end of X. The following are equivalent*:

(i) *e is a visible end*;
(ii) *there exists an orthoray based in γ that exits the end e and eventually leaves $C(X)$*;
(iii) *there exists an interval I in γ for which each orthoray based in I exits the end e and eventually leaves $C(X)$*.

Proof. We prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. If e is a visible end for X then by definition there exists a geodesic ray that goes out e and eventually leaves the convex core $C(X)$. Hence the geodesic ray must intersect a component of the boundary of $C(X)$, denoted β, and then enter into a half-plane. By Corollary 3·6, β is necessarily a simple open geodesic. We choose a lift of this simple geodesic $\tilde{\beta}$ and a lift $\tilde{\gamma}$ of γ and observe that the orthogeodesic from $\tilde{\gamma}$ to $\tilde{\beta}$ has finite length. This orthogeodesic extends to an orthoray that enters the half-plane bounded by $\tilde{\beta}$. Pushing this orthoray to the quotient X and noting that this orthoray exits the end e proves item (2). That there is an intervals worth of such orthorays that hit $\tilde{\beta}$ follows from the general fact that the basepoint of an orthogeodesic between any two geodesics $\tilde{\gamma}$ to $\tilde{\beta}$ in $\mathbb{H}$ has an interval of orthorays that all intersect $\tilde{\beta}$ (See Figure 2). Thus $2 \Rightarrow 3$. The final implication $3 \Rightarrow 1$ is clear.

We are now ready to prove Theorem 5·1.

Proof. For each n, let $\{Y_n\}_{n=1}^{\infty}$ be an exhaustion of X' by finite area geodesic subsurfaces which, to make the argument less cumbersome, we assume have the additional property that if a boundary geodesic of Y_n is not a boundary geodesic of X' then it is not a boundary geodesic of Y_{n+1}. Now set $Z_{n+1} = Y_{n+1} - Y_n$ and note that in general Z_n is a finite union of (possibly disconnected) geodesic subsurfaces. Fix γ a simple closed oriented geodesic on Y_1 and choose an ordered countable dense subset $A = \{a_n\}_{n=1}^{\infty}$ of γ. If γ is a boundary geodesic of Y_1, by possibly changing the orientation of γ, we may assume that Y_1 lies to the left of γ. Consider the orthoray g_1 that lies to the left of γ and is based at a_1. If the ray stays inside Y_1 for all time or if it hits ∂Y_1 in a component that is a boundary geodesic of X' then we do nothing and glue Z_2 any way we like. Otherwise, g_1 hits a component, say α, of ∂Y_1, at a point we denote by p_1. Denote the angle g_1 makes with α by θ_1. From Lemma 5·2 the vector

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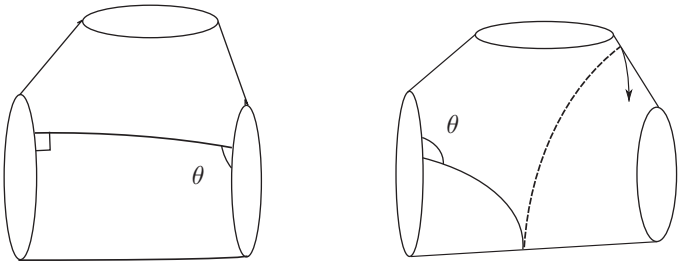


Fig. 3. Angles match.

field $V_{\pi-\theta_1}$ based in α contains a Cantor set of vectors, whose associated geodesic ray stays in Z_2. Pick one and call it v_1. Now glue Z_2 to Y_1 along α so that the orthoray g_1 extends smoothly through p_1 and into Z_2 (see Figure 3). By construction the ray g_1 stays inside $Y_2 = Y_1 \bigcup Z_2$ for infinite time. Next consider the orthoray g_2 with basepoint $a_2 \in A \subset Y_2$. As before if g_2 stays inside Y_2 or hits the ∂Y_2 in a boundary component of X' then we do nothing and just glue Z_3 anyway we like. Otherwise, as before we use Lemma 5·2 to glue Z_3 to Y_2 so that the orthoray g_2 extends smoothly into Z_3 and stays inside $Y_3 = Y_2 \bigcup Z_3$ for infinite time. We can continue this process ad infinitum so that we have constructed X with specified twist parameters so that the orthoray which begins at a_n stays in Y_{n+1}. Now using Lemma 5·3 we may conclude that all the infinite type ends of X are not visible. Finally we add funnels to all the closed boundary geodesics of X' and conclude that the resulting surface is geodesically complete.

A *flute surface* is a sequence of pairs of pants glued in succession along common length boundaries. The flute surface as it stands is not necessarily geodesically complete but always has a natural geodesic completion by Theorem 3·4. A flute surface has genus zero, no non-planar ends, and space of ends homeomorphic to $\{1/n\}_{n=1}^{\infty} \bigcup 0$. Denote the successive cuffs of the flute surface by $\{\alpha_n\}$ as in Figure 4. We say that a flute surface is *tight* if each of the pants holes that have not been glued along are in fact cusps.

Remark 5·4. A tight flute surface is geodesically complete if and only if its unique infinite type end is not visible.

THEOREM 5·5. *Fix any positive numbers $\{\ell_n\}$, where $\ell_n \to \infty$. There exists a tight flute surface of the first kind (that is geodesically complete) with $\ell(\alpha_n) = \ell_n$, for all n.*

Proof. Recall that a flute surface has countably many isolated ends which converge to an infinite type end. We construct this surface X' out of tight pairs of pants, $\{P_n\}_{n=0}^{\infty}$, glued in succession. P_0 has cuff lengths, $(0, 0, \ell_1)$ and then generally P_n has cuff lengths $(\ell_n, 0, \ell_{n+1})$ for $n = 1, 2, 3, \ldots$. The gluing parameters are chosen as in Theorem 5·1. With these choices the infinite type end of X' is of the first kind, and hence all ends of X are of the first kind. By Proposition 3·7, $X' = X$ is a tight flute of the first kind.

COROLLARY 5·6. *Let X be the infinite genus surface with one end. There exists a geodesically complete hyperbolic structure on X for which*:

(i) *the hyperbolic structure is of the first kind. In particular, the infinite type end is not visible (complete)*;

(ii) *its length spectrum is discrete.That is, there are finitely many closed geodesics with length less than any given number. In particular, any pants decomposition is not upper bounded but is lower bounded.*

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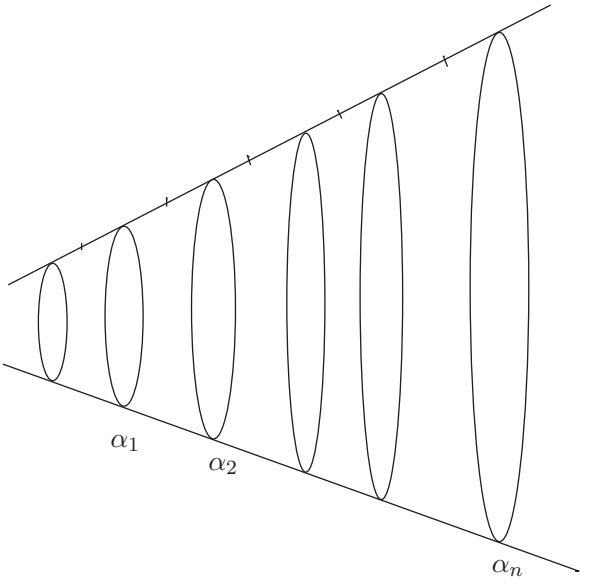


Fig. 4. Flute surface.

Proof. In the paper [**8**] it is shown that for any topological surface there exist hyperbolic structures with a discrete length spectrum. Moreover, the constructions are independent of the twist parameters. In particular, we can construct an infinite genus surface with one end using pairs of pants so that it admits a hyperbolic structure with a discrete length spectrum for any choice of twist parameters. Choosing the twist parameters as in Theorem 5·1 we are guaranteed that the infinite type end is not visible, and hence (since there are no other ends) the hyperbolic structure is of the first kind.

6. *Flute surfaces with rapid increase in cuff lengths*

We consider a tight flute surface (see Figure 4). That is, we have a sequence $\{P_n\}$ of (tight) geodesic pairs of pants whose one cuff is a cusp and two other cuffs are geodesics α_n and α_{n+1}. Except for the first pants which has two cusps. Denote by l_n the length of α_n. We choose one cuff of P_n to be a cusp for the simplicity of the topology of the surface.

We glue P_n to P_{n+1} by identifying α_{n+1} on P_n with α_{n+1} on P_{n+1}. The identification is given by a twist parameter t_n with $0 \leqslant t_n < l_n$. We fix a choice of t_n such that the surface obtained after all the identifications is complete (that is, no visible ends) which is possible by Theorem 2. Thus the surface obtained is a geodesically complete tight flute surface. There are many choices in the gluings such that the obtained surface is geodesically complete. We fix one such choice and denote the geodesically complete tight flute surface by X_0.

We next choose the cuff lengths $\{l_n\}$ of the pants $\{P_n\}$ to be *rapidly increasing*. That is, $\{l_n\}$ (strictly) monotonically goes to ∞ and

$$\sum_{i=1}^{n} l_i = o(l_{n+1}). \tag{3}$$

7. *Teichmüller spaces*

The quasiconformal Teichmüller space $T_{qc}(X_0)$ of a geodesically complete hyperbolic surface X_0 (without visible ends) consists of all quasiconformal mappings $f : X_0 \to X$

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modulo post-compositions by conformal maps and homotopy. The Teichmüller distance between two points $[f : X_0 \to X_1]$ and $[g : X_0 \to X_2]$ is given by

$$d_T([f],[g]) = \frac{1}{2} \inf_{h \sim f \circ g^{-1}} \log K(h),$$

where the infimum is over all quasiconformal maps h homotopic to $f \circ g^{-1}$ and $K(h)$ is the quasiconformal constant of h.

The length spectrum Teichmüller space $T_{ls}(X_0)$ of a geodesically complete hyperbolic surface X_0 consists of all homeomorphisms $f : X_0 \to X$ up to isometry and homotopy, where X is a hyperbolic (not necessarily complete) surface with $\sup_\alpha |\log(l_X(\alpha)/l_{X_0}(\alpha))| < \infty$ and the supremum is over all homotopy classes of simple closed curves α. The length spectrum distance on $T_{ls}(X_0)$ is defined by

$$d_{ls}([f],[g]) = \frac{1}{2} \sup_\alpha |\log \frac{l_{f(X_0)}(\alpha)}{l_{g(X_0)}(\alpha)}|,$$

where the supremum is over all simple closed geodesics α on X_0.

The length spectrum Teichmüller space $T_{ls}(X_0)$ is complete in the length spectrum metric (cf. [**4**]) and $T_{qc}(X_0) \subset T_{ls}(X_0)$. When X_0 has a geodesic pants decomposition whose cuff lengths are bounded from above and from below then the length spectrum metric induces the same topology as the Teichmüller metric on $T_{qc}(X_0)$ (cf. [**19**]). When X_0 has a geodesic pants decomposition with upper bounded cuff lengths and a sequence of cuff lengths goes to zero, then the length spectrum metric is incomplete on $T_{qc}(X_0)$ (cf. [**2**]) and thus it does not induce the same topology as the Teichmüller metric. In the case of upper bounded cuff lengths, the Teichmüller spaces $T_{qc}(X_0)$ and $T_{ls}(X_0)$ are parametrised by the Fenchel–Nielsen coordinates (cf. [**2, 3**]), the closure of $T_{qc}(X_0)$ inside $T_{ls}(X_0)$ is described in terms of the Fenchel–Nielsen coordinates (cf. [**18**]) and local bi-Lipschitz structures of $T_{qc}(X_0)$ and $T_{ls}(X_0)$ is described using the Fenchel–Nielsen coordinates (cf. [**3**] and [**18**]). When X_0 has no geodesic pants decomposition with an upper bounded cuff lengths, the parametrisation of $T_{qc}(X_0)$ using the Fenchel–Nielsen coordinates is rather challenging. In what follows we describe some aspects of the Fenchel–Nielsen coordinates when X_0 is a flute surface with rapidly increasing cuff lengths.

8. *The closure of* $T_{qc}(X_0)$

In this Section we assume that X_0 is the geodesically complete hyperbolic flute surface defined in Section 6. Namely, X_0 is obtained by gluing tight pairs of pants with cuff lengths $\{l_n\}$ satisfying

$$\sum_{i=1}^{n} l_i = o(l_{n+1}).$$

In what follows we need the following Lemma.

LEMMA 8·1. *Let Σ be a geodesic pentagon with right angles at A, B, C and D, and an ideal vertex at E. Let a, b and c be the lengths of the sides AB, BC and CD, respectively. Let d be the length of the geodesic segment orthogonal to both AB and DE. Then, for $a > 1$ large enough and $c > a$, there exist constants C_1 and C_2 such that*

$$C_1 + c - a \leqslant d \leqslant C_2 + c - a.$$

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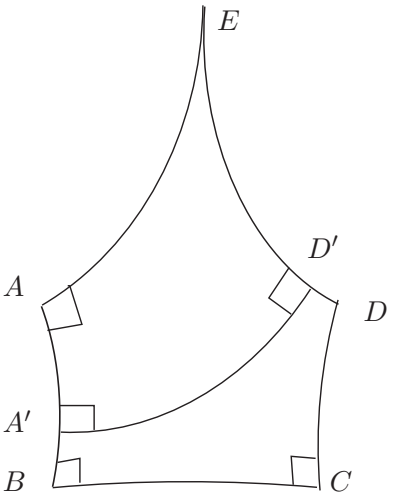


Fig. 5. Lifts.

Proof. Note that the segment orthogonal to AB and DE is necessarily inside the pentagon $ABCDE$. Denote by A' the endpoint in AB and by D' the endpoint in DE of the segment (cf. Figure 5). From the pentagon Σ, we get (cf. Beardon [**9**, page 159])

$$\cosh a \cosh c + 1 = \sinh a \cosh b \sinh c$$

which gives

$$\cosh b = \frac{\cosh a \cosh c + 1}{\sinh a \sinh c}.$$

Further

$$\frac{\cosh a \cosh c + 1}{\sinh a \sinh c} = \frac{(1 + e^{-2a})(1 + e^{-2c}) + 4e^{-(a+c)}}{(1 - e^{-2a})(1 - e^{-2c})}$$

and Taylor's expansion gives, for some constant $C > 0$,

$$1 + Ce^{-2a} \geqslant \cosh b \geqslant 1 + e^{-2a}.$$

Consequently, for a constant $C' > 0$ which depends on C,

$$C'e^{-a} \geqslant \sinh b = \sqrt{\cosh^2 b - 1} \geqslant \sqrt{2}e^{-a}.$$

From the right angled pentagon $A'BCDD'$, we get (cf. Beardon [**9**, page 159])

$$C''e^{-a}e^{c} \geqslant \cosh d = \sinh b \sinh c \geqslant \frac{\sqrt{2}}{4}e^{-a}e^{c}$$

when c is large enough which implies

$$C'''e^{c-a} \geqslant e^{d} \geqslant \cosh d \geqslant \frac{\sqrt{2}}{4}e^{c-a}.$$

Taking logarithms in the above inequality, we get

$$C_2 + c - a \geqslant d \geqslant \log \frac{\sqrt{2}}{4} + c - a.$$

We study the closure $\overline{T_{qc}(X_0)}$ of the quasiconformal Teichmüller space $T_{qc}(X_0)$ for the length spectrum metric in the length spectrum Teichmüller space $T_{ls}(X_0)$. We establish

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THEOREM 8·2. *Let X_0 be a geodesically complete tight flute surface built by gluing pairs of pants with rapidly increasing cuff lengths $\{l_n\}$. Then the closure $\overline{T_{qc}(X_0)}$ of the quasiconformal Teichmüller space $T_{qc}(X_0)$ in the length spectrum metric contains all surfaces with Fenchel-Nielsen coordinates $\{(l_n, t_n)\}_n$, where $-Cl_n \leqslant t_n \leqslant Cl_n$, for $C > 0$, and the lengths $\{l_n\}$ correspond to a marked surface in $T_{qc}(X_0)$.*

Proof. Denote by $\{P_n\}$ the family of tight geodesic pairs of pants that are used to obtain the flute surface X_0. Let α_n and α_{n+1} be the cuffs of P_n that are not cusps. Then α_n has length l_n.

Let $\{(l_n, t'_n)\}_{n\in\mathbb{N}}$ be the Fenchel–Nielsen coordinates of a marked surface X in $T_{qc}(X_0)$. Define $t''_n = t_n - t'_n$ for some t_n with $-Cl_n \leqslant t_n \leqslant Cl_n$ and $C > 0$. Let $f_k : X \to X_k$ be a quasiconformal marking map from X to the surface X_k obtained by twists t''_i around α_i on the surface X for $i = 1, 2, \ldots, k$. It is clear that f_k can be chosen to be a quasiconformal map since we twist around only finitely many geodesics (cf. [**17**]). We prove that the marked surfaces X_k converge in the length spectrum metric to the surface X^* whose Fenchel–Nielsen coordinates are $\{(l_n, t_n)\}_{n\in\mathbb{N}}$.

We divide each pair of pants P_n into two pentagons with one ideal vertex by three simple geodesic arcs: the first arc, denoted by β_n, is orthogonal to α_n and α_{n+1} at its endpoints, the second arc, denoted by β_n^1, is orthogonal to α_n at its endpoint and it finishes in the cusp, and the third arc, denoted by β_n^2, is orthogonal to α_{n+1} at its endpoint and it also finishes in the cusp. Note that the two pentagons are isometric and that they have four straight angles and one zero angle, i.e. one ideal vertex. Let b_n be the length of β_n.

Let γ be an arbitrary simple closed geodesic on X. If γ does not intersect any $\{\alpha_i\}$ for $i \geqslant k$ then $l_{X_k}(\gamma) = l_{X^*}(\gamma)$ and there is nothing to be proved in this case.

Assume that γ intersects α_i for $i = i_0, i_0 + 1, \ldots, j_0$ with $k < j_0$. In this case we need to estimate the size of $l_{X^*}(\gamma)$ compared $l_{X_k}(\gamma)$. We first estimate the size of $l_{X_k}(\gamma)$. Note that γ intersects the pants P_{j_0} whose boundary geodesics are α_{j_0} and α_{j_0+1}. By assumption, γ does not intersect α_{j_0+1} which implies that γ enters and exists the pants P_{j_0} through α_{j_0}. This implies that γ necessarily intersects the geodesic arc $\beta^2_{j_0}$ orthogonal to α_{j_0+1} that ends in the puncture because otherwise γ could be homotoped such that it does not intersect α_{j_0} which is impossible.

We estimate the length of $\gamma \bigcap P_{j_0}$. Consider the lift of the situation to the universal covering $\mathbb{H}$. Fix a single component $\tilde{\alpha}_{j_0}$ of the lift of α_{j_0} to $\mathbb{H}$. Denote by Σ_1 and Σ_2 the two pentagons that the pants P_{j_0} is divided into. Consider all lifts of Σ_1 and Σ_2 that have one side on $\tilde{\alpha}_{j_0}$. A lift $\tilde{\gamma}$ of γ connects $\tilde{\alpha}_{j_0}$ with a lift $\tilde{\beta}^2_{j_0}$ of $\beta^2_{j_0}$ that belongs to a lift of Σ_1 or Σ_2 with one boundary side on $\tilde{\alpha}_{j_0}$. The length of the segment of $\tilde{\gamma}$ that has one endpoint on $\tilde{\alpha}_{j_0}$ and the other endpoint on $\tilde{\beta}^2_{j_0}$ is greater than the length of the common perpendicular geodesic arc to $\tilde{\alpha}_{j_0}$ and $\tilde{\beta}^2_{j_0}$. It is immediate that the common perpendicular geodesic arc p lies in a single lift of one of the two pentagons (cf. Figure 6). Then Lemma 8·1 implies that the length of the common perpendicular arc p is at least $C_1 + (l_{j_0+1} - l_{j_0}/2)$. It follows that

$$l_{X^*}(\gamma), l_{X_k}(\gamma) \geqslant C_1 + l_{j_0+1} - l_{j_0}. \tag{4}$$

Since X^* is obtained by twisting around α_n by the amount t''_n and $|t''_n| \leqslant (C+1)l_n$, we obtain

$$l_{X^*}(\gamma) \leqslant l_{X_k}(\gamma) + \sum_{i=i_0}^{j_0} |t''_i| \leqslant l_{X_k}(\gamma) + o(l_{j_0+1}). \tag{5}$$

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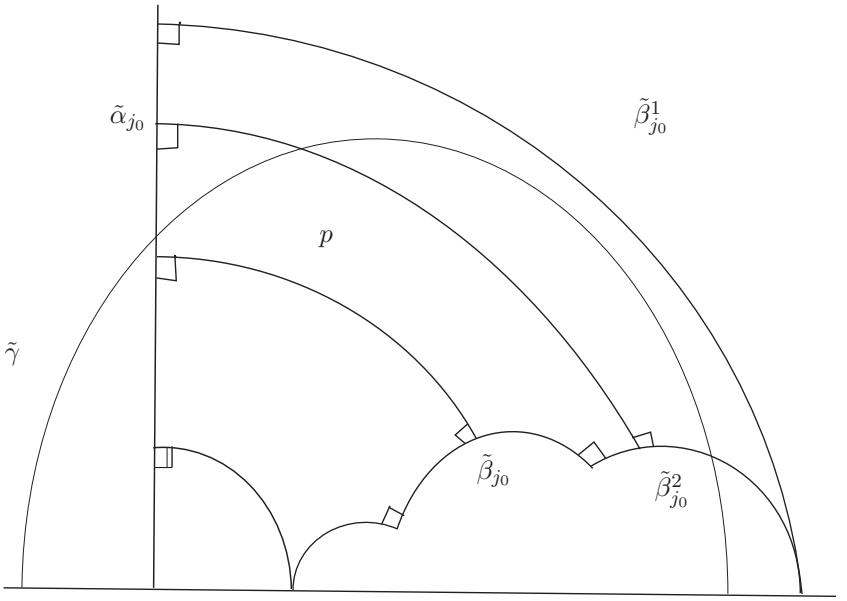


Fig. 6. Lifts.

We get

$$\frac{l_{X^*}(\gamma)}{l_{X_k}(\gamma)} \leqslant 1 + \frac{1}{l_{X_k}(\gamma)} o(l_{j_0+1}) \leqslant 1 + \frac{o(l_{j_0+1})}{C_1 + l_{j_0+1} - l_{j_0}} \leqslant 1 + \frac{o(l_{k+1})}{C_1 + l_{k+1} - l_k} \longrightarrow 1$$

as $k \to \infty$ uniformly in γ.

Notice that X_k is obtained by twisting $-t_i''$ along α_i for $i > k$. If γ is a simple closed geodesic that intersects α_i for $i = i_0, i_0 + 1, \ldots, j_0$. If $j_0 \leqslant k$ then $l_{X_k}(\gamma)/l_{X^*}(\gamma) = 1$. If $j_0 > k$ then

$$l_{X_k}(\gamma) \leqslant l_{X^*}(\gamma) + \sum_{i=i_0}^{j_0} |t_i''| \leqslant l_{X^*}(\gamma) + o(l_{j_0+1})$$

which implies as before that

$$\frac{l_{X_k}(\gamma)}{l_{X^*}(\gamma)} \leqslant 1 + \frac{1}{l_{X^*}(\gamma)} o(l_{j_0+1}) \leqslant 1 + \frac{o(l_{j_0+1})}{C_1 + l_{j_0+1} - l_{j_0}} \leqslant 1 + \frac{o(l_{k+1})}{C_1 + l_{k+1} - l_k} \longrightarrow 1$$

as $k \to \infty$ uniformly in γ.

We obtained the convergence of $X_k \in T_{qc}(X_0)$ to X^* in the length spectrum metric.

We need the following two Lemmas.

LEMMA 8·3. *Let Σ be a pentagon with four right angles and one ideal vertex, i.e. zero angle. Let a, b and c be the lengths of three finite sides of Σ in the counterclockwise order. Assume that $c > a > 1$, $a \to \infty$ and $c/a \to \infty$. Then*

$$b = 2e^{-a} + o(e^{-a}),$$

where $o(e^{-a})/e^{-a} \to 0$ as $a \to \infty$.

Proof. We have (cf. [**9**])

$$\cosh a \cosh c + 1 = \sinh a \cosh b \sinh c$$

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which gives

$$\cosh b = \frac{\cosh a \cosh c + 1}{\sinh a \sinh c} = \frac{(1+e^{-2a})(1+e^{-2c}) + 4e^{-(a+c)}}{(1-e^{-2a})(1-e^{-2c})} = 1 + 2e^{-2a} + o(e^{-2a}).$$

and the result follows.

LEMMA 8·4. *Let $\mathcal{Q}$ be a hyperbolic quadrilateral with three right angles and a fourth angle $0 < \phi < \pi/2$. Let a_1 and a_2 be the lengths of the sides of $\mathcal{Q}$ with two right angles, and let b_1 and b_2 be the lengths of the opposite sides, respectively. Then*

$$\sinh b_1 = \sinh a_1 \cosh b_2$$

and

$$\tanh a_1 \sinh b_2 \tan \phi = 1.$$

Proof. From [**9**, page 157, theorem 7·17·1], we have

$$\begin{aligned} \sinh a_1 \sinh a_2 &= \cos \phi \\ \cosh a_1 &= \cosh b_1 \sin \phi \\ \cosh a_2 &= \cosh b_2 \sin \phi. \end{aligned} \tag{6}$$

Using (6) and $\sin^2 \phi + \cos^2 \phi = 1$, we get

$$\sinh^2 a_1 \sinh^2 a_2 + \frac{\cosh^2 a_1}{\cosh^2 b_1} = 1$$

which implies

$$\sinh^2 a_1 \sinh^2 a_2 \cosh^2 b_1 + \cosh^2 a_1 = \cosh^2 b_1.$$

Substituting $\sinh^2 a_2 = \cosh^2 a_2 - 1$ above we get

$$\sinh^2 a_1 \cosh^2 a_2 \cosh^2 b_1 - \sinh^2 a_1 \cosh^2 b_1 + \cosh^2 a_1 = \cosh^2 b_1.$$

By diving the third equation with the second equation in (6), we get $\cosh a_2 = (\cosh b_2 / \cosh b_1) \cosh a_1$ and substituting above gives

$$\sinh^2 a_1 (\frac{\cosh b_2}{\cosh b_1} \cosh a_1)^2 \cosh^2 b_1 - \sinh^2 a_1 \cosh^2 b_1 + \cosh^2 a_1 = \cosh^2 b_1$$

which in turn gives

$$\sinh^2 a_1 \cosh^2 b_2 \cosh^2 a_1 - \sinh^2 a_1 \cosh^2 b_1 - \cosh^2 b_1 + \cosh^2 a_1 = 0.$$

Since $\sinh^2 a_1 + 1 = \cosh^2 a_1$, the above gives

$$\sinh^2 a_1 \cosh^2 b_2 \cosh^2 a_1 - \cosh^2 a_1 \cosh^2 b_1 + \cosh^2 a_1 = 0.$$

Finally, dividing with $\cosh^2 a_1$ gives

$$\sinh^2 a_1 \cosh^2 b_2 - \cosh^2 b_1 + 1 = 0$$

and the first formula follows easily.

To prove the second formula, note that (6) implies

$$\sinh^2 a_2 = \frac{\cos^2 \phi}{\sinh^2 a_1}$$

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and then

$$\cosh^2 a_2 = \sinh^2 a_2 + 1 = \frac{\cos^2 \phi}{\sinh^2 a_1} + 1.$$

By using (6) above, we get

$$\frac{\cos^2 \phi}{\sinh^2 a_1} + 1 = \cosh^2 b_2 \sin^2 \phi$$

which gives

$$\cos^2 \phi + \sinh^2 a_1 = \sinh^2 a_1 \cosh^2 b_2 \sin^2 \phi.$$

Further

$$\cos^2 \phi + \sinh^2 a_1 \cos^2 \phi = \sinh^2 a_1 \cosh^2 b_2 \sin^2 \phi - \sinh^2 a_1 \sin^2 \phi$$

and then

$$\cosh^2 a_1 \cos^2 \phi = \sinh^2 a_1 \sinh^2 b_2 \sin^2 \phi$$

and the second formula follows.

The following Theorem proves that pointwise convergence of twists implies the convergence in the length spectrum metric when the lengths of the cuffs in the pants decomposition are fixed.

THEOREM 8·5. *Let X_0 be a geodesically complete tight flute surface with twists $\{t_n\}$ and rapidly increasing cuff lengths $\{l_n\}$. Let X_k be a marked hyperbolic surface with cuff lengths equal to $\{l_n\}$ and twists $t_{X_k}(\alpha_n) = t_n + O(l_n)$. If $\lim_{k\to\infty} t_{X_k}(\alpha_n) = t_n$ for each n, then X_k converges to X_0 in the length spectrum metric.*

Proof. Let $\epsilon > 0$ be given. We need to prove that $|l_{X_k}(\beta)/l_{X_0}(\beta) - 1| < \epsilon$ for all simple closed geodesics β on X_0 and for all $k \geqslant k_0(\epsilon)$, where $k_0(\epsilon) > 0$ depends on ϵ.

Indeed, let β be a simple closed geodesic on X_0. If β is a cuff of the pants decomposition, then $l_{X_0}(\beta) = l_{X_k}(\beta)$ and $|l_{X_k}(\beta)/l_{X_0}(\beta) - 1| = 0$ for all k.

If β transversely intersects cuffs, let α_{n_0} be the cuff with the largest index that β intersects. Then we have

$$l_{X_0}(\beta) - \sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j| \leqslant l_{X_k}(\beta) \leqslant l_{X_0}(\beta) + \sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j|$$

and dividing it with $l_{X_0}(\beta)$ we get

$$1 - \frac{\sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j|}{l_{X_0}(\beta)} \leqslant \frac{l_{X_k}(\beta)}{l_{X_0}(\beta)} \leqslant 1 + \frac{\sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j|}{l_{X_0}(\beta)}.$$

As in the proof of Theorem 8·2, we have

$$l_{X_0}(\beta) \geqslant l_{n_0+1} - l_{n_0} + C.$$

Note that

$$\sum_{j=1}^{n_1} |t_{X_k}(\alpha_j) - t_j| = o(l_{n_1+1})$$

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and we can choose $n_1 = n_1(\epsilon)$ such that

$$1 - \epsilon < \frac{\sum_{j=1}^{n} |t_{X_k}(\alpha_j) - t_j|}{l_{n+1} - l_n + C} < 1 + \epsilon$$

for all $n \geqslant n_1$.

Since $t_{X_k}(\alpha_n) \to t_n$ as $k \to \infty$ for each n, it follows that for any $n_0 \leqslant n_1$ the sum $\sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j| < \epsilon$ for all $k \geqslant k_0 = k_0(\epsilon)$. Therefore, $l_{X_k}(\beta)/l_{X_0}(\beta)$ is ϵ-close to 1 for $n_0 \leqslant n_1$ with $k \geqslant k_0$ large enough.

If $n_0 > n_1$ then

$$\frac{\sum_{j=1}^{n_0} |t_{X_k}(\alpha_j) - t_j|}{l_{X_0}(\beta)} \leqslant \frac{o(l_{n_0+1})}{l_{n_0+1} - l_{n_0} + C} \leqslant \frac{o(l_{n_1+1})}{l_{n_1+1} - l_{n_1} + C} < \epsilon$$

by the choice of n_1 for all k. It follows that $|l_{X_k}(\beta)/l_{X_0}(\beta) - 1| < \epsilon$ when $k > k_0$. The convergence in the length spectrum metric is proved.

Using Theorem 8·2 we obtain

THEOREM 8·6. *If X_0 is a geodesically complete tight flute surface with rapidly increasing cuff lengths, then the length spectrum Teichmüller space $T_{ls}(X_0)$ is strictly larger than the quasiconformal Teichmüller space $T_{qc}(X_0)$. More precisely, $\overline{T_{qc}(X_0)} - T_{qc}(X_0)$ is non-empty.*

Proof. Let $\{t_n\}$ and $\{l_n\}$ be the twists and the length parameters of X_0 for the pants decomposition $\mathcal{P} = \{\alpha_n\}$ as above. Define $t_k(\alpha_n) = t_n + |l_n|$ if $k \leqslant n$, and define $t_k(\alpha_n) = t_n$ if $k > n$. Define $l_k(\alpha_n) = l_n$ for all k, n. The marked surface X_k with the Fenchel–Nielsen coordinates $\{(t_k(\alpha_n), l_k(\alpha_n))\}$ is a quasiconformal deformation of X_0. Indeed, the twists are positive and there are only finitely many of them which implies that the left earthquake given by the positive twists has Thurston bounded earthquake measure. Then the left earthquake induces a quasiconformal deformation (cf. [**17**]).

Let X_0' be the surface with lengths $\{l_n\}$ and twists $\{t_n + |l_n|\}$. Then X_k converges to X_0' as $k \to \infty$ in the length spectrum metric because $t_{X_k}(\alpha_n) = t_{X_0'}(\alpha_n) + O(l_n)$ for all k, n and $t_{X_k}(\alpha_n) = t_{X_0'}(\alpha_n)$ for $n \leqslant k$ (cf. Theorem 8·5). However, the limit X_0' is a not a quasiconformal deformation of X_0 since it is obtained by left earthquake with unbounded Thurston's norm (cf. [**17**]).

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