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MINIMAL AXIOMATIC FRAMEWORKS FOR DEFINABLE HYPERREALS WITH TRANSFER

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Abstract. We modify the definable ultrapower construction of Kanovei and Shelah (2004) to develop a **ZF**-definable extension of the continuum with transfer provable using countable choice only, with an additional mild hypothesis on well-ordering implying properness. Under the same assumptions, we also prove the existence of a definable, proper elementary extension of the standard superstructure over the reals.

§1. Introduction. The usual ultrapower construction of a hyperreal field $\mathbb{R}^\omega/U$ is not functorial (in the category of models of set theory) due to its dependence on a choice of a free ultrafilter U, which can be obtained in **ZFC** only as an application of the axiom of choice, but not as an explicitly definable set-theoretic object. Kanovei and Shelah [12] developed a functorial alternative to this, by providing a construction of a definable hyperreal field, which we refer to below as *the KS construction*.

The KS construction was analyzed in [15], Section 1G of the Online 2007 edition, and generalized in many ways in [5, 10, 13], [11, Chapter 4] among others. We give [14] as the source of the problem of a uniquely definable nonstandard real line and [3, 7, 18, 24] as basic references in nonstandard matters.

Nonstandard analysis is viewed by some as inherently nonconstructive. One of the reasons is that nonstandard models are typically presented in terms of an unspecified choice of a free ultrafilter, which makes the resulting ultrapower hopelessly nondefinable. In fact, as Luxemburg [18] observed, if there is a nonstandard model of the reals, then there is a free ultrafilter on the natural numbers ω. The observation that elements of a nonstandard extension ${}^*A$ correspond to ultralters on A was first exploited in detail by Luxemburg [17].

To circumvent the unspecified choice of a free ultrafilter, the KS construction starts with the collection of ultrafilters U on ω parametrized by surjective maps from a suitable ordinal onto such ultrafilters U. Such maps are ordered lexicographically. This generates a definable linear ordering of ultrafilters in which each of them is included in many copies. The tensor product is applied to merge the ultrafilters into a definable ultrafilter in the algebra of finite support product sets.

Thus, the KS construction can be viewed as a functor which, given a model of set theory, produces a definable extension of the reals in the model.

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The KS construction in [12] (as well as its modifications as in [5,13]) was originally designed to work in Zermelo–Fraenkel set theory **ZFC** with choice. However for it *prima facie* to yield the expected result, it is sufficient to assume the well orderability of the reals. Let $\mathbf{WO}(\mathbb{R})$ be the following statement: the continuum $2^\omega = \{X : X \subseteq \omega\}$ is well orderable. Thus it emerges that the theory $\mathbf{ZF} + \mathbf{WO}(\mathbb{R})$ is sufficient for the KS construction to yield a definable proper elementary extension of the reals.

The goal of this note is to weaken this assumption.

§2. The result. Consider the following two consequences of the axiom of choice in **ZF**:

$\mathbf{AC}_\omega(\mathbb{R})$: countable **AC** for sets of reals, that is, any sequence $\{X_n\}_{n<\omega}$ of sets $\varnothing \neq X_n \subseteq \mathbb{R}$ admits a choice function;

WOB: there exists a free ultrafilter over ω with a well-orderable base. (A set $B \subseteq U$ is a *base* of an ultrafilter U over ω, if and only if there is no ultrafilter $U' \neq U$ over ω with $B \subseteq U'$. In such case we write $U = [B]$.)

THEOREM 2.1 (**ZF**). *There exists an extension* ${}^*\mathbb{R}$ *of the reals* $\mathbb{R}$, *such that both* ${}^*\mathbb{R}$ *and a canonical embedding* $x \longmapsto {}^*x$ *from* $\mathbb{R}$ *into* ${}^*\mathbb{R}$ *are presented by explicitly definable set-theoretic constructions and in addition*:

(i) $\mathbf{AC}_\omega(\mathbb{R})$ *implies that* ${}^*\mathbb{R}$ *is an elementary extension, in the sense of the language* $\mathscr{L}(\mathbb{R})$ *with symbols for all finitary relations on* $\mathbb{R}$;

(ii) **WOB** *implies that* ${}^*\mathbb{R}$ *is a proper extension of* $\mathbb{R}$, *containing infinitesimals and infinitely large numbers.*

It follows by (i) that, instead of $\mathbf{WO}(\mathbb{R})$, the axiom $\mathbf{AC}_\omega(\mathbb{R})$ can be used to establish elementarity. It emerges that proving the transfer principle for the definable extension requires no more choice than proving, for instance, the σ-additivity of the Lebesgue measure; see [9]. Similarly, by (ii), **WOB** successfully replaces $\mathbf{WO}(\mathbb{R})$ in the proof of properness.

Quite obviously $\mathbf{WO}(\mathbb{R})$ implies $\mathbf{AC}_\omega(\mathbb{R})$ and **WOB** in **ZF**. The failure of the inverse implication is dealt with in 4.1 below.

The proof of Theorem 2.1 appears in Section 3. We also show, in 4.3, how the theorem can be generalized in order to obtain even a nonstandard superstructure over ${}^*\mathbb{R}$.

§3. What it takes: array of ultrafilters. Let an *array of ultrafilters* be any sequence $\{D_a\}_{a\in A}$, where $A = \langle A, <_A\rangle$ is a linearly ordered set and each D_a is an ultrafilter over ω.

PROPOSITION 3.1 (in $\mathbf{ZF} + \mathbf{AC}_\omega(\mathbb{R})$). *Assume that* $\{D_a\}_{a\in A}$ *is a definable array of ultrafilters over* ω, *with at least one free ultrafilter* D_{a_0}. *Then there is a definable* (as in Theorem 2.1) *proper extension* ${}^*\mathbb{R}$ *of* $\mathbb{R}$, *elementary w. r. t. the language* $\mathscr{L}(\mathbb{R})$ *containing all finitary relations on* $\mathbb{R}$.

PROOF (Sketch, based on the proof in [12]). The following is defined:

- the *index set* $I = \omega^A = \{x : x \text{ is a map } A \to \omega\}$;
- the algebra $\mathscr{X} = \mathscr{X}(A)$ of *finite-support subsets* of $I = \omega^A$, so that a set $X \subseteq \omega^A$ is in $\mathscr{X}$ if and only if there is a finite $u \subseteq A$ such that

$$\forall\, x,\, y \in \omega^A \left(x \restriction u = y \restriction u \implies (x \in X \iff y \in X)\right);$$

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– the collection $F = F(A)$ of *finite-support functions* $f : I \to \mathbb{R}$, so that $f : I \to \mathbb{R}$ belongs to F if and only if there is a finite set $u \subseteq A$ such that

$$\forall\, x,\ y \in \omega^A \ \big(x \restriction u = y \restriction u \implies (f(x) = f(y))\big).$$

The *tensor, or Fubini product* $D = \bigotimes_{a\in A} D_a$ consists then of all sets $X \subseteq I$ such that for a finite subset $u = \{a_1 <_A \dots <_A a_n\} \subseteq A$ we have:

$$D_{a_n} k_n \dots D_{a_2} k_2\ D_{a_1} k_1\ (\langle k_1, \dots, k_n\rangle \in\!\uparrow X),$$

where $\langle k_1, \dots, k_n\rangle \in\!\uparrow X$ means that every $x \in I$ satisfying $x(a_1) = k_1, \dots, x(a_n) = k_n$ belongs to X, and $D_a k\, \Phi(k)$ means that the set $\{k : \Phi(k)\}$ belongs to D_a. It turns out that D is an ultrafilter in the algebra $\mathscr{X}$, which allows to define the *ultrapower* ${}^*\mathbb{R} = F/D = \{[f]_D : f \in F\}$, where $[f]_D = \{g \in F : f =^D g\}$ and $f =^D g$ means that $\{x \in I : f(x) = g(x)\} \in D$. All finitary relations in $\mathscr{L}(\mathbb{R})$ extend to ${}^*\mathbb{R}$ naturally.

In addition, we send every real r to the equivalence class ${}^*r = [c_r]_D$ of the constant function $c_r \in F$ with value r. The axiom $\mathbf{AC}_\omega(\mathbb{R})$ is strong enough to support the ordinary proof of the Łoś lemma, and hence $r \longmapsto {}^*r$ is an elementary embedding in the sense of the language $\mathscr{L}(\mathbb{R})$. To prove that the embedding is proper, make use of the assumption that at least one of D_a is a free ultrafilter. Finally, the extension ${}^*\mathbb{R}$ is definable since the given array of ultralters $\{D_a\}_{a\in A}$ is definable by hypothesis. ⊣

Proof (Theorem 2.1). To define a suitable array of ultrafilters, let ϑ be the least ordinal such that for any well-orderable set $Z \subseteq \mathbb{R}$ there is a surjective map $a : \vartheta \xrightarrow{\text{onto}} Z$. Let A consist of all maps $a : \vartheta \to \mathscr{P}(\omega)$ such that the set $B_a = \operatorname{ran} a = \{a(\gamma) : \gamma < \vartheta\}$ is a base of an ultrafilter on ω, and let $D_a = [B_a]$ be this ultrafilter. The set A is ordered lexicographically: $a <_A b$, if and only if there exists an ordinal $\gamma < \vartheta$ such that $a \restriction \gamma = b \restriction \gamma$ and $a(\gamma) < b(\gamma)$, in the sense of the lexicographical linear order $<$ on $\mathscr{P}(\omega)$. Then $\{D_a\}_{a\in A}$ is a definable array of ultrafilters. Assuming **WOB**, it contains at least one free ultrafilter D_a, and we apply Proposition 3.1. ⊣

§4. **Remarks.** Here we add some related remarks, starting with a model of $\mathbf{ZF} + \mathbf{AC}_\omega + \mathbf{WOB}$ in which the continuum is not well orderable. This demonstrates that Theorem 2.1 is an actual strengthening of the key result of [12].

4.1. Separating WOB + $\mathbf{AC}_\omega$ from WO($\mathbb{R}$). Pincus and Solovay conjectured in [20, p. 89] that iterated Sacks extensions may be useful in the construction of choiceless models with free ultrafilters. Working in this direction, we let $\mathfrak{M}$ be an ω_1-iterated Sacks extension of $\mathbf{L}$, the constructible universe, as in [1]. Let $\mathfrak{N}$ be the class of all sets hereditarily definable from an ω-sequence of ordinals in $\mathfrak{M}$. Then **WOB** is true in $\mathfrak{N}$ since some free ultrafilters in $\mathbf{L}$ (basically, all selective ultrafilters) remain ultrafilter bases in $\mathfrak{M}$ and in $\mathfrak{N}$ by [1, Section 4], and $\mathbf{AC}_\omega$ is true as well (even the principle of dependent choices **DC** holds). Meanwhile, $\mathbf{WO}(\mathbb{R})$ fails in $\mathfrak{N}$ (2^ω is not well orderable) by virtue of arguments, based on the homogeneous structure of the Sacks forcing, and similar to those used in the classical studies of the choiceless Solovay model S' as in [23, Part III, proof of Theorem 1].

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If we let $\mathfrak{M}$ be an ω_2-iterated Sacks extension of $\mathbf{L}$, then the class $\mathfrak{N}$ of all sets that are hereditarily definable from an ω_1-sequence of ordinals in $\mathfrak{M}$, still will be a model of $\mathbf{WOB} + \neg\mathbf{WO}(\mathbb{R})$, in which even $\mathbf{DC}_{\omega_1}$ and $\mathbf{AC}_{\omega_1}$ hold instead of the simple $\mathbf{DC}$ and $\mathbf{AC}_\omega$. Longer iterations make little sense here as each further Sacks real collapses all smaller cardinals down to ω_1.

We know nothing about any model of $\mathbf{WOB} + \mathbf{AC}_\omega(\mathbb{R})$ in which $\mathbf{WO}(\mathbb{R})$ fails, different from the ones just described. (However see Section 4.5 below.) This can be a difficult problem, yet not uncommon in studies of choiceless models.

4.2. Keisler-style representation. Keisler's influential monograph [15] contains (in Section 1G) a somewhat modified exposition of the construction of a definable nonstandard extension of [12], by an explicit *amalgamation* of all ultrapowers of $\mathbb{R}$ via different ultrafilters on ω into one large hyperreal field. A similar Keisler-style modification of the construction readily works in the $\mathbf{ZF} + \mathbf{AC}_\omega(\mathbb{R}) + \mathbf{WOB}$ setting.

4.3. Superstructure over ${}^*\mathbb{R}$. Let $V(\mathbb{R}) = \bigcup_{n\geq 0} V_n(\mathbb{R})$ be the superstructure over the reals, where $V_0(\mathbb{R}) = \mathbb{R}$ and $V_{n+1}(\mathbb{R}) = V_n(\mathbb{R}) \cup \mathscr{P}(V_n(\mathbb{R}))$ for all n, see [2, Section 4.4]. To build a nonstandard superstructure over ${}^*\mathbb{R}$ as in Section 3, we let F_n be the set of all functions $f : \omega^A \to V_n(\mathbb{R})$ of finite support, and then define the ultrapower ${}^*V_n(\mathbb{R}) = F_n/D$ and the elementary embedding $x \longmapsto {}^*x$ from $V_n(\mathbb{R})$ to ${}^*V_n(\mathbb{R})$ as above. (And we need $\mathbf{AC}_\omega$ for subsets of $V_n(\mathbb{R})$ to prove the elementarity.) Then each element of ${}^*V_n(\mathbb{R})$ can be identified with a certain subset of ${}^*V_{n-1}(\mathbb{R})$ or an element of ${}^*V_{n-1}(\mathbb{R})$, so that each ${}^*V_n(\mathbb{R})$ emerges as a subset of $V_n({}^*\mathbb{R})$. This completes the nonstandard superstructure construction under $\mathbf{WOB} + \mathbf{AC}_\omega$.

4.4. Another definable choiceless ultrapower. Consider *the basic Cohen model* $\mathbf{L}(A)$, obtained by adding a set $A = \{a_n : n < \omega\}$ of Cohen generic reals a_n to $\mathbf{L}$, [8, 5.3]. (Not to be confused with *the Feferman model* [4], adding all a_n but not A.) The set A belongs to $\mathbf{L}(A)$ but the map $n \longmapsto a_n$ does not. $\mathbf{AC}_\omega$ badly fails in $\mathbf{L}(A)$ as A is an infinite Dedekind finite set. Yet $\mathbf{L}(A)$ contains a free ultrafilter U over ω, see [21] for a short proof.

Let ${}^*\mathbb{R} = \mathbb{R}^\omega/U$ be the associated ultrapower. Then ${}^*\mathbb{R}$ **is not** an elementary extension of $\mathbb{R}$ in the full relational language $\mathscr{L}(\mathbb{R})$ as in Theorem 2.1, since the formula "$\forall n \in \omega\ \exists x\,(x$ codes an n-tuple of elements of $A)$" is true for $\mathbb{R}$ but false for ${}^*\mathbb{R}$. However ${}^*\mathbb{R}$ **is** an elementary extension of $\mathbb{R}$ with respect to the sublanguage $\mathscr{L}'(\mathbb{R})$ of $\mathscr{L}(\mathbb{R})$, containing only real-ordinal definable finitary relations on $\mathbb{R}$. Note that $\mathscr{L}'(\mathbb{R})$ is a sufficiently rich language to enable an adequate development of nonstandard real analysis.

Both U and ${}^*\mathbb{R}$ are definable in $\mathbf{L}(A)$ by a set theoretic formula with the only parameter A. And this is probably all we can do in $\mathbf{L}(A)$ since the model contains no real-ordinal definable elementary extensions of $\mathbb{R}$.

4.5. A possible WOB model. One may want to extend $\mathbf{L}(A)$ as in 4.4 by a $P(U)$-generic real $c = c_0 \in 2^\omega$, where $P(U)$ is the Mathias forcing with infinite conditions in U. If $\mathbf{L}(A)[c_0]$ happens to have an $\{A, a_0\}$-definable ultrafilter U_1 over ω with $U \subseteq U_1$ then let $c_1 \in 2^\omega$ be a $P(U_1)$-generic real over $\mathbf{L}(A)[c_0]$. Extending this forcing iteration as in [16, A10 in Chapter 8], one may hope to get a final

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extension of $\mathbf{L}(A)$ with a well-ordered ultrafilter base $\{c_\xi : \xi < \omega_1\}$ but with A still not well orderable.

4.6. Least cardinality. What is the least possible cardinality of a definable hyperreal field? A rough estimate for the general definable extension in [12] under AC yields $\leq \exp^3(\aleph_0)$. As for the definable extension ${}^*\mathbb{R}$ in Section 3 of this paper, if the ground set universe is the ω_2-iterated Sacks extension of $\mathbf{L}$ as in 4.1.1 then $\mathrm{card}({}^*\mathbb{R}) = 2^{\aleph_0} = \aleph_2$, which is minimal.

§5. Conclusions. Analysis with infinitesimals presupposes the existence of an extended mathematical universe which, in the tradition of Robinson and Zakon [22], is typically understood as an extended superstructure over the reals, although for some basic applications an extension of the set of reals suffices. Even for certain more sophisticated applications, it is enough for this extension of the mathematical universe to satisfy the Transfer Principle, which means that it is an elementary extension in the sense of model theory.

We have shown that one can find definable extensions of both the set of reals and the superstructure over the reals; more precisely, our extensions are definable by purely set-theoretic means without recourse to well ordering and have the following properties: (I) one can prove the *Transfer Principle* for such extensions from Zermelo–Fraenkel set theory with merely Countable Choice; (II) the existence of infinitesimals and infinitely large numbers in those extensions follows from a mild well-ordering assumption.

The property of *countable saturation*, important for some advanced applications, is not asserted but can be achieved by the ω_1-iteration of the given extension construction, as described in [12, Section 4].

Our results may be of interest to practitioners working with fragments of nonstandard analysis. For instance, the Transfer Principle plus the existence of an infinitely large integer is all that is required to develop Edward Nelson's [19, p. 30] *minimal nonstandard analysis* or the related *minimal Internal Set Theory* [6, pp. 3, 4, 104]. Such fragments of nonstandard analysis have the potential for application in diverse fields, ranging from stochastic calculus and mathematical finance to theoretical quantum mechanics [6].

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