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Arbitrage with Fixed Costs and Interest Rate Models

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Abstract

We study securities market models with fixed costs. We first characterize the absence of arbitrage opportunities and provide fair pricing rules. We then apply these results to extend some popular interest rate and option pricing models that present arbitrage opportunities in the absence of fixed costs. In particular, we prove that the quite striking result obtained by Dybvig, Ingersoll, and Ross (1996), which asserts that under the assumption of absence of arbitrage long zero-coupon rates can never fall, is no longer true in models with fixed costs, even arbitrarily small fixed costs. For instance, models in which the long-term rate follows a diffusion process are arbitrage-free in the presence of fixed costs (including arbitrarily small fixed costs). We also rationalize models with partially absorbing or reflecting barriers on the price processes. We propose a version of the Cox, Ingersoll, and Ross (1985) model which, consistent with Longstaff (1992), produces yield curves with realistic humps, but does not assume an absorbing barrier for the short-term rate. This is made possible by the presence of (even arbitrarily small) fixed costs.

I. Introduction

In this paper, we study securities market models with fixed costs. We characterize the absence of arbitrage opportunities and we provide fair pricing rules. We then apply these results to extend some classical interest rate and option pricing models that present arbitrage opportunities in the absence of fixed costs.

Transaction costs are usually said to be fixed if they are constant regardless of the size of the transaction. In this paper, we study more general fixed costs structures in which the cost of transacting has a fixed component, i.e., is bounded away from zero independently of the quantity traded. Some results will more specifically focus on "bounded" fixed costs structures for which transaction costs are bounded (below and above) regardless of the size of the transaction or on "vanishing" fixed costs structures, i.e., transaction costs structures such that the marginal cost tends to zero when the transaction size increases. Note that bounded fixed costs structures are a specific case of vanishing fixed costs structures.

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As underlined by Brennan (1975), such fixed costs structures capture reasonably well the basic features of the commission structure of many stock exchanges, which involves a stated amount plus a declining percentage of the value of the transaction. In fact, investors incur many different fees and charges that depend, among other things, on the traded volume, traded amount, and traded asset. The total fee is in general bounded below by a given amount that corresponds to the ticket charges that have to be paid for each transaction regardless of the size of that transaction and bounded above. For instance, on the GLOBEX clearing system fees are capped (for trades executed for members trading within their division for their own account) at $50 per day per product per operator.[1] We refer to Jones (2002) for a general description of the NYSE's trading costs structure across the 20th century.

Note that such a fixed costs structure is not limited to the modelization of clearing or trading fees imposed by stock exchanges. Indeed, as underlined by Leape (1987):

> Acquiring assets virtually always incurs financial charges such as brokerage fees, which in general have a fixed costs component . . . Even in the cases in which such costs appear proportional to the amount invested, there are typically minimum investment requirements which are equivalent to fixed costs of trading. The costs of acquiring assets also include the opportunity costs and information costs associated with analyzing new assets. Such costs are truly costs of participating in particular markets and are independent of the amount invested (p. 2).

In fact, our fixed costs structure can model brokerage fees, holding costs, fixed investment taxes to gain access to a market (such as a foreign market), and operational and processing costs that typically exhibit strong economies of scale (e.g., through automation). We can also model fixed costs involved in setting up an office and obtaining access to information, the opportunity cost of looking at a market or of doing a specific trade, and brokerage arrangements where marginal fees go to zero beyond a given volume that is reset periodically (a common practice in the industry).

Other frictions studied in the literature include transaction costs that are proportional to the quantity of asset traded. Without minimizing the relevance of this type of friction, the fixed costs specification that we analyze in this paper is of importance for the three major reasons Viard (1995) developed:

> First, as Leape (1987) argued, a number of costs associated with purchasing and holding assets approximately satisfy the fixed-costs specification . . .
>
> Second, such costs are more analytically challenging than proportional costs. The latter can be easily accommodated in the standard CAPM framework with little additional analysis, because such costs simply reduce the net rate of return.[2]

[1] More details on the clearing and execution fees are provided on the Chicago Mercantile Exchange website (http://www.cme.com).

[2] This is challenging in a dynamic setting (see Jouini and Kallal (1995) for a characterization of no-arbitrage in a proportional transaction costs framework).

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> Third, fixed costs offer a simple and plausible explanation for the observation that many investors hold exactly zero of most available assets; although proportional costs would alter an investor's holding of any particular asset, they generally would not drive such holdings to zero (p. 45).

We find that the absence of arbitrage opportunities in models with bounded fixed costs (or more generally with vanishing fixed costs) is equivalent to the existence of a family of nonnegative state-price densities compatible with the asset price processes. The only difference with the frictionless setting lies in the fact that these state-price densities may be nonnegative instead of positive. When the fixed costs are neither bounded nor vanishing, we still have that the existence of nonnegative state-price densities implies the absence of arbitrage opportunity, but the converse implication does not necessarily hold. Note that a zero state-price density for some states of the world does not mean that investors might have consumption for free in those states since they always incur a positive fixed cost. We define fair pricing rules as those that are arbitrage-free and lie below the super replication cost. Indeed, no rational agent would pay more than the super replication cost for a contingent claim since there is a cheaper way to achieve at least the same payoff using a trading strategy. We show that the only fair pricing rules are those that are equal to the sum of a fixed cost function and a linear pricing rule associated with one of the nonnegative state-price densities given by the absence of arbitrage. While we focus on discrete time models, the characterization of the absence of arbitrage opportunities remains essentially the same in a continuous time setting. However, as usual in continuous time and for technical reasons, the concept of no-arbitrage is replaced by that of no free lunch, and the boundedness condition on the fixed costs structure is modified accordingly (see Jouini, Kallal, and Napp (2001)).

The main impact of the introduction of fixed costs, both in discrete time as well as in continuous time, is the enlargement of the set of possible state-price densities from the positive densities to the nonnegative densities. The absence of arbitrage opportunities in a fixed costs framework is therefore weaker than in a frictionless model, and some models that contain arbitrage opportunities in the absence of frictions might become arbitrage-free with the introduction of fixed costs including arbitrarily small fixed costs. In a discrete time setting, this enables us to rationalize some model specifications that are usually rejected for arbitrage considerations, but have other desirable properties (see Section III). The discrete time framework permits us to provide intuitions about how things work and to see exactly how the fixed costs and the zero weights in the state-price densities rationalize the models under consideration.

In particular, we prove that the Dybvig, Ingersoll, and Ross (1996) striking result, which asserts that under the assumption of absence of arbitrage long zero-coupon rates can never fall, is no longer true in models with fixed costs, even arbitrarily small fixed costs. In particular, we consider discrete time versions of the classical Brennan and Schwartz (1979) model. Based on pure expectation the-

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ory, this model assumes a diffusion behavior for long-term rates,[3] and is therefore incompatible with the absence of arbitrage result given by Dybvig, Ingersoll, and Ross (1996). We show that such a model is now compatible with the absence of arbitrage (with fixed costs, even arbitrarily small fixed costs). This enables us to reconcile arbitrage conditions and the empirical observations that do not seem to reject the Brennan and Schwartz (1979) model (see Chan, Karolyi, Longstaff, and Sanders (1992)). In the same spirit, we also provide examples of term structure models adapted from Ho and Lee (1986) in which the long-term zero-coupon rates can both increase and decrease.

Fixed costs also permit us to rationalize models with partially absorbing or reflecting barriers on the asset price processes. In a frictionless framework, such models obviously admit arbitrage opportunities but they are arbitrage-free in the presence of fixed costs, including arbitrarily small fixed costs. In addition, we show that in such models the price of a call option is equal to the price of a barrier option in a model without any barrier. We also propose a discrete time version of a Cox, Ingersoll, and Ross (CIR) (1985) model which, as in Longstaff (1992), produces yield curves with realistic humps. However, our model does not assume an absorbing barrier for the short rate, but instead assumes fixed costs.

Studies on fixed transaction costs include Duffie and Sun (1990), Grossman and Laroque (1990), and Morton and Pliska (1995), among others, who examine the optimal portfolio problem with transaction fees that are proportional to the size of the overall portfolio (as opposed to the size of the specific transaction). In the same spirit, Luttmer (1999) addresses the empirical problem of the required level of transaction costs for observations on consumption choices to be consistent with data on asset returns. Brennan (1975), Goldsmith (1976), Levy (1978), Mayshar (1979), (1981), Leape (1987), and Viard (1995) consider a CAPM model with fixed costs in order to explain the empirical evidence of limited diversification: households hold a very limited set of assets. Jouini, Kallal, and Napp (2001) study the characterization of the assumption of no free lunch as well as viability issues in securities market models with fixed costs in a continuous time framework.

The paper is organized as follows. Section II presents the absence of arbitrage characterization in models with fixed costs, and provides a general description of fair pricing rules. Section III provides applications of these results to interest rate models. All proofs are in the Appendix.

II. Models with Fixed Costs

A. Arbitrage in Models with Fixed Costs

1. Formulation of the Market Model

The model we study here is the so-called event-tree model. The reader who is familiar with such a framework may skip the paragraph below.

A finite set Ω of states of the world is specified and fixed. Also specified is a time horizon T, which is the terminal date for all economic activity under

[3]Long rates are defined as the limit zero-coupon rate when maturity goes to infinity. This term does not correspond to what is usually called the long rate on the markets, namely, the rate associated with the longest-dated maturities (e.g., 30 years).

consideration. The information arrival is given by a family of increasingly finer partitions, $\{F_0, \ldots, F_T\}$ of Ω such that $F_0 = \{\Omega\}$ and $F_T = \{\{\omega\} : \omega \in \Omega\}$. A date t node is an element of F_t. We denote by Σ_t the set of all date t nodes and for $\sigma_t \in \Sigma_t, f(\sigma_t) = \{\sigma_{t+1} \in \Sigma_{t+1} : \sigma_{t+1} \subset \sigma_t\}$ can be interpreted as the set of the immediate successors of the date t node σ_t.

There are $n+1$ assets, and for a date t node σ_t, we denote by $Z^k(\sigma_t)$ the price at time t and in the node σ_t of asset k. We assume that $Z^0 \equiv 1$: this amounts to assuming the existence of a numéraire asset, i.e., with positive prices at every period, and considering discounted price processes.

In this framework, a portfolio is described by a vector $\theta \in \mathbb{R}^{n+1}$, where θ_k is the quantity of asset k held by the investor. A portfolio strategy is then an adapted process $(\theta_t)_{t=0,\cdots,T}$, where $\theta(\sigma_t)$ is the portfolio held at date t in the event σ_t.

Each time investors revise their portfolio of risky assets, they pay a given transaction cost. The amount paid might depend on which assets are traded, the quantity of assets that are traded, the traded amount, as well as on portfolio composition and previous trades. For a given portfolio strategy $(\theta_t)_{t=0,\cdots,T}$, the transaction cost paid by the investors at date t and in the node σ_t is given by $c^\theta(\sigma_t)$. We also denote by $C_t^\theta(\sigma_t)$ the total transaction costs paid by the investor up to time t. We impose the following conditions:

i) For a given strategy θ, the process c^θ is nonnegative and adapted to the information structure $\{F_0, \ldots, F_T\}$.

ii) If there is a trade at date t and in the state of the world σ_t (i.e., $\theta_t(\sigma_t) \neq \theta_{t-1}(\sigma_{t-1})$ for $\sigma_{t-1} \in \Sigma_{t-1}$ such that $\sigma_t \in f(\sigma_{t-1})$), then $C_t^\theta(\sigma_t) > \mathbf{c}$, where $\mathbf{c}$ is a given positive minimum transaction cost.

The first condition means that the "fixed costs" are nonnegative and depend only on present and past information; the second condition means that investors have to pay a given minimum amount at least once in order to trade.

We shall say that the "fixed costs" are vanishing if

iii) for any strategy θ, any date t, and any date t node σ_t, we have $\lim_{\lambda\to\infty} (c^{\lambda\theta}(\sigma_t)/\lambda) \to 0$.

This condition reflects the declining marginal rate of the transaction costs structure, and is satisfied when the transaction costs are bounded.

Many transaction costs structures are taken into account with this modelization including the following.

i) Clearing fees and brokerage arrangements where the amount to be paid is a function of the traded amount and where marginal fees go to zero beyond a given amount. We then have $c_t^\theta = \sum_{k=1}^n f^k((\theta_t^k - \theta_{t-1}^k)Z_t^k)$, where f^k, the fee structure associated with asset k, satisfies $f^k(x) > \mathbf{c}$ for $x \neq 0$ and $\lim_{\lambda\to\infty}(f^k(\lambda x)/\lambda) \to 0$.

ii) Trading fees and brokerage arrangements where the amount to be paid is a function of the quantity of assets that are traded and where marginal fees go to zero beyond a given volume. We then have $c_t^\theta = \sum_{k=1}^n f^k(\theta_t^k - \theta_{t-1}^k)$, where f^k, the fee structure associated with asset k, satisfies the same conditions as above.

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iii) Holding costs where the amount to be paid is a function of the quantity of assets held in the portfolio. We then have $c_t^\theta = \sum_{k=1}^n f^k(\theta_t^k)$, where f^k still satisfies the same conditions as above.

iv) Fixed investment taxes to gain access to a market or fixed costs involved in setting up an office and obtaining access to information where the amount is paid once before the first access to that market. Denote by C^k the amount to be paid in order to obtain later access to the kth market. We then have $C^\theta(\sigma_t) \geq C^k$ at any date where $\theta_t^k(\sigma_t) \neq \theta_{t-1}^k(\sigma_{t-1})$ for $\sigma_{t-1} \in \Sigma_{t-1}$ such that $\sigma_t \in f(\sigma_{t-1})$.

v) Operational and processing costs that exhibit strong economies of scale where the amount paid might be a function of the traded amount or the traded quantity with a possible fixed initial amount. The costs structure is then a combination of the structures presented above.

In all the examples above, we assume that there is no cost associated with transactions on the riskless asset denoted by 0. In the case of transaction costs, this assumption is made without any loss of generality since any trade on the riskless asset implies a trade on at least one other asset. In the case of holding costs, this assumption means that there is no holding fee on the riskless asset. Such an assumption is made in order to make the arguments clearer and can be relaxed without difficulty.

2. Arbitrage Opportunities

We now introduce the notion of arbitrage. As usual, an arbitrage opportunity is a plan that yields, through some combination of buying and selling securities, a positive gain in some circumstances without a countervailing threat of loss in other circumstances. More precisely:

Definition 1. A two-period arbitrage opportunity with fixed costs consists of a portfolio[4] θ such that $\sum_{k=0}^n \theta_k Z^k(\sigma_t) + c^\theta(\sigma_t) \leq 0$ and $\sum_{k=0}^n \theta_k Z^k(\sigma_{t+1}) - c^\theta(\sigma_{t+1}) \geq 0$ for all $\sigma_{t+1} \in f(\sigma_t)$ with at least one strict inequality, for some t and some σ_t in Σ_t.

Before going further, we provide some intuition for how things work and illustrate our results. A strategy that leads, exclusive of transaction costs, to a nonnegative and nonzero net gain that is zero in some states would be an arbitrage opportunity in the frictionless framework, but is not an arbitrage in the fixed costs setting. Indeed, since positive transaction costs have to be paid, the net gain (inclusive of transaction costs) becomes negative in some states. For instance, a corporate bond and a government bond with the same characteristics can have the same price in a model with fixed costs even if the corporate bond has a positive probability of default and the government bond is considered default-free. Indeed, buying the government bond and selling the corporate bond would be an arbitrage in the frictionless setting but not in the setting with fixed costs. Hence, the assumption of absence of arbitrage in the frictionless case is not equivalent to the assumption of absence of arbitrage in the case with fixed costs.

[4]Strictly speaking, the transaction costs process c^θ is only defined for strategies θ and not for portfolios. However, for a given date t node σ_t, a portfolio θ can obviously be identified with the strategy that consists in buying that portfolio at date t in the event of σ_t and selling it at date $t+1$.

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In order to clarify the link between the concepts of arbitrage with fixed costs and of frictionless arbitrage, we note the following:

Proposition 2. If there is a two-period arbitrage opportunity with fixed costs, then there exists a portfolio θ such that $\theta \cdot Z(\sigma_t) < 0$ and $\theta \cdot Z(\sigma_{t+1}) \geq 0$ for all $\sigma_{t+1} \in f(\sigma_t)$ for some t and some σ_t in Σ_t. The converse implication also holds true for vanishing fixed costs.

The existence of an arbitrage opportunity with vanishing fixed costs is equivalent to the existence of a strong form of frictionless arbitrage, i.e., a portfolio with negative initial cost and nonnegative payoffs. To illustrate this result, consider a two-period binomial model, i.e., a model consisting of one risky security with a value of 1 at date 0 and a value of u in the up state and of d in the down state at date 1, and one riskless security with a value of 1 at date 0 and a value of $(1 + r)$ in both states of the world at date 1. It is easy to see that if $d < (1 + r) < u$, then the model is arbitrage-free, in both cases, with and without fixed costs; if $d = (1 + r)$ or $u = (1 + r)$, then it is arbitrage-free in the model with fixed costs but not in the frictionless model; if $d > (1 + r)$ or $u < (1 + r)$, then the model contains arbitrage opportunities in the frictionless model as well as in the model with vanishing fixed costs.

A characterization of the absence of arbitrage with fixed costs is as follows:

Proposition 3. There is no two-period arbitrage with fixed costs (vanishing fixed costs) if (if and only if) for each date t and each date t node σ_t there exists a nonnegative state-price density on $f(\sigma_t)$ compatible with the asset prices, i.e., there exists a family of nonnegative numbers $(\pi_\rho)_{\rho \in f(\sigma_t)}$ such that

$$Z^k(\sigma_t) \quad = \quad \sum_{\sigma_{t+1} \in f(\sigma_t)} \pi_{\sigma_{t+1}} Z^k(\sigma_{t+1}), \qquad k = 0, \ldots, n.$$

For illustration purposes, consider a two-period financial model with one bond and one stock, where the riskless rate is equal to the lowest possible rate of return of the stock. According to the fundamental theorem of asset pricing, since there is an arbitrage opportunity for the frictionless case, there cannot exist a positive compatible state-price density. Nevertheless, the state-price density that puts all the weight on the states of the world such that the riskless rate is equal to the lowest possible rate of return of the stock is compatible with our financial model.

The fact that state-price densities can be zero when the physical probability is nonzero seems quite striking and counterintuitive. This result does not suggest that investors have free access to units of consumption in these states since access to units of consumption is linked to the payment of a fixed cost in all states. We can wonder whether the situation with zero state-price densities in states of positive physical probability is compatible with equilibrium or at least utility maximization. Here again, the above result does not mean that the agent's marginal utility in these states is zero. Rather, the result suggests that the fixed cost structure and the agent's utility function are such that at some point of consumption it is not possible to increase the agent's utility, since any marginal increase in consumption in these states (with zero state-price densities) would be compensated by a marginal increase in the fixed cost to pay in all other states. An example of such a

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situation is provided in the Appendix. Note that this example is not a singular one and we have the following general result: even if there are zero state prices, a large class of agents will find a positive optimal solution to the consumption investment problem, namely, those agents whose marginal utility function is decreasing faster than the marginal fixed cost. This condition is satisfied if, for instance, we assume that the fixed transaction costs structure is vanishing but not bounded above and that the utility functions are in the HARA class (i.e., $-(u'/u'')(x) = \alpha + \eta x$) and satisfy the empirically reasonable condition $\eta < 1$.

The same results can be obtained in a dynamic setting. Furthermore, in such a setting an arbitrage opportunity can be defined as a strategy on the whole tree instead of a strategy between a node and its successors. The next result provides a straightforward extension of the no-arbitrage characterization to such a dynamic framework.

Definition 4. A dynamic arbitrage opportunity is a strategy described by a nonnull dynamic portfolio θ (i.e., a portfolio $\theta(\sigma_t)$ for all dates $t = 0, \ldots, T-1$ and all date t nodes σ_t) satisfying for all t, for all $\sigma_t \in \Sigma_t$, and for all $\sigma_{t+1} \in f(\sigma_t)$ the following self-financing condition,

$$\theta(\sigma_t) \cdot Z(\sigma_{t+1}) \quad = \quad \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) + c^{\theta}(\sigma_{t+1}), \tag{1}$$

such that

$$\theta(\sigma_0) \cdot Z(\sigma_0) + c_0^{\theta} \quad \leq \quad 0,$$

and, for all $\sigma_T \in \Sigma_T$,

$$\begin{aligned} \theta^k(\sigma_T) \quad &= \quad 0, \quad k = 1, \ldots, n, \\ \theta(\sigma_T) \cdot Z(\sigma_T) - c^{\theta}(\sigma_T) \quad &\geq \quad 0. \end{aligned}$$

The condition $\theta^k(\sigma_T) = 0, k = 1, \ldots, n$ means that the portfolio is liquidated at the final date. We then have:

Proposition 5. The existence of a dynamic arbitrage opportunity is equivalent to the existence of a two-period arbitrage opportunity.

As a corollary to Proposition 3, in the presence of vanishing fixed costs the no-arbitrage condition implies the existence of a nonnegative compatible state-price density π on the whole tree, i.e., $\pi = (\pi(\sigma_t))_{t=0,\cdots,T,\sigma_t \in \Sigma_t}$ such that at each date $t = 0, \cdots, T-1$ and at each date t node σ_t, we have

$$\pi(\sigma_t) Z(\sigma_t) \quad = \quad \sum_{\sigma_{t+1} \in f(\sigma_t)} \pi(\sigma_{t+1}) Z(\sigma_{t+1}).$$

Indeed, it suffices to multiply the different state-price densities obtained at each node between that node and its successors to construct a nonnegative state-price density compatible with the asset prices.

Such a state-price density can be seen as a nonnegative (instead of positive as in the frictionless case) subjective probability on the space Ω of the states of the world and the compatibility condition between the state-price density and the

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price processes then becomes a martingale condition on the traded asset prices under this subjective probability.

Note, however, the existence of a nonnegative compatible state-price density is not a sufficient condition to rule out any arbitrage opportunity. Consider the following model for which $T = 2$, $\Omega = \{\omega_1, \ldots, \omega_4\}$, $F_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, $n = 1$, and the prices are given by

For the bond:	1	1	1
For the risky security:			1.5
		1	
			0.5
	1		
			1
		0.5	
			1.

There exists a nonnegative state-price density given by

		0.5
	1	
		0.5
1		
		0
	0	
		0.

Nevertheless, as we have seen earlier, the model is not arbitrage-free: we only need at date 1 and in the lower date 1 node $\{\omega_3, \omega_4\}$ to buy one unit of the risky security and to sell one unit of the bond and then to clear the position at date 2 to obtain an arbitrage opportunity.

In order to completely characterize the assumption of absence of arbitrage, we need to introduce a whole family of nonnegative state-price densities or, equivalently, martingale measures with nonnegative weights. We start with some definitions.

For a given date $i \in \{0, \cdots, T-1\}$ and a given date i node σ, we define the submodel S^σ by a set $\Omega^\sigma = \sigma$ of states of the world, a family of increasingly finer partitions $(F_t^\sigma)_{t \in \{0,\ldots,T-i\}}$ with $F_t^\sigma = \{\sigma' \in F_{t+i} : \sigma' \subset \sigma\}$, and asset prices defined for $\sigma' \in F_t^\sigma$, for t in $\{0, \ldots, T-i\}$, by $Z^\sigma(\sigma') = Z(\sigma')$. In fact, the submodel S^σ can be described by the subtree starting in σ.

Theorem 6. There is no dynamic arbitrage opportunity with vanishing fixed costs (fixed costs) if and only if (if) for all dates i and all date i nodes σ, there exists a nonnegative state-price density $\pi^\sigma = (\pi^\sigma(\sigma_t))_{t=0,\cdots,T-i,\sigma_t \in F_t^\sigma}$ compatible with the asset prices in the submodel S^σ, i.e., for all $t \in \{0, \cdots, T-i\}$ and all $\sigma_t \in F_t^\sigma$,

$$\pi^\sigma(\sigma_t) Z^\sigma(\sigma_t) = \sum_{\sigma_{t+1} \in f(\sigma_t)} \pi^\sigma(\sigma_{t+1}) Z^\sigma(\sigma_{t+1}).$$

The absence of arbitrage opportunities is then characterized by the existence of martingale measures with nonnegative weights for every subtree.

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B. Pricing in Models with Fixed Costs

In this subsection, we restrict our attention to vanishing fixed costs structures. A contingent claim x is defined by its payoff in each terminal node $\sigma_T \in \Sigma_T$, paying, at date T, the amount $x(\sigma_T)$ at the node σ_T.

A contingent claim x is said to be attainable (in the frictionless model) if there exists some dynamic portfolio θ satisfying, for all t, for all $\sigma_t \in \Sigma_t$, and for all $\sigma_{t+1} \in f(\sigma_t)$, the self-financing frictionless condition $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) = \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1})$ such that $\theta(\sigma_T) \cdot Z(\sigma_T) = x$ for all $\sigma_T \in \Sigma_T$. Let M denote the set of all attainable contingent claims. Note that M is a linear space.

Definition 7. A fair pricing rule on the set of attainable claims M is a function p defined on M such that

1. p induces no arbitrage, i.e., it is not possible to find a dynamic portfolio θ satisfying the self-financing condition $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) = \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) + c^\theta(\sigma_{t+1})$ and contingent claims $m_1, \ldots, m_d$ in M for which $\sum_{i=1}^d p(m_i) + \theta(\sigma_0) \cdot Z(\sigma_0) + c_0^\theta \leq 0$, $\sum_{i=1}^d m_i + \theta \cdot Z - c_T^\theta \geq 0$ on Σ_T and one of the two is nonnull.

2. $p(m) \leq p_s(m)$, where $p_s(m) \equiv \inf\{\theta(\sigma_0) \cdot Z(\sigma_0) + c_0^\theta$, θ is self-financing, and $\theta \cdot Z - c_T^\theta \geq m$ on $\Sigma_T\}$.

Part 1 of Definition 7 is the usual no-arbitrage condition; it encompasses the fact that the model is arbitrage-free and could be replaced in a frictionless model by $\sum_{i=1}^d p(m_i) \leq 0$, $\sum_{i=1}^d m_i \geq 0$, and one of the two is nonnull. Part 2 of the definition says that a fair price for m is smaller than its super replication price. This latter condition corresponds to a monotonic assumption; we impose that if it is possible to obtain a better payoff than m at a cost $p_s(m)$, then no one will pay more than $p_s(m)$. Note that since m is attainable by a frictionless self-financing strategy, and since the total cost incurred for any strategy is bounded, there always exists at least a self-financing (inclusive of transaction costs) strategy dominating m.

Proposition 8 gives the fair pricing rules on M through the use of the nonnegative state-price densities on the whole tree π^Ω introduced in Theorem 6. Notice that the linear function l given by $l[(\theta(\sigma_T) \cdot Z(\sigma_T))_{\sigma_T \in \Sigma_T}] = \theta(\sigma_0) \cdot Z(\sigma_0)$ for all frictionless self-financing strategies θ is well defined on M: indeed, if two frictionless self-financing strategies θ and θ' generate the same contingent claim m, i.e., if their terminal values are both equal to m, then their initial values must also be equal; if not, the strategy given by $\theta - \theta'$ or by $\theta' - \theta$ would be a strong form of frictionless arbitrage opportunity, which would lead to the existence of an arbitrage opportunity with vanishing fixed costs as in Proposition 1. We find that any fair pricing rule can be written as the sum of the linear function l and a vanishing fixed cost, or can be expressed in terms of the nonnegative state-price densities in the preceding section.

Proposition 8. If the fixed costs structure is bounded (vanishing), then any fair pricing rule p on the set of attainable claims M is given by

$$p(m) = \sum_{\sigma_T \in \Sigma_T} \pi^\Omega(\sigma_T) m(\sigma_T) + c(m) \qquad \text{for all } m \text{ in } M,$$

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where π^{Ω} is any nonnegative state-price density as determined in Theorem 6 and where c is bounded (vanishing, i.e., $c(\lambda m)/\lambda \to_{\lambda\to\infty} 0$). Moreover, if $p(\lambda x) \leq \lambda[p(x)]$ for λ sufficiently large, then the fixed cost function $c(\cdot)$ is nonnegative.

This implies that $p(\lambda m)/\lambda \to_{\lambda\to\infty} \sum_{\sigma_T\in\Sigma_T} \pi^{\Omega}(\sigma_T)m(\sigma_T)$ for any attainable contingent claim m. Therefore, the unit price of any attainable contingent claim m is given by $\sum_{\sigma_T\in\Sigma_T} \pi^{\Omega}(\sigma_T)m(\sigma_T)$ in the limit of large quantities traded.

As usual, we say that the market is complete (in the frictionless model) if any contingent claim is attainable (again, in the frictionless model). It is then easy to see that if the market is arbitrage-free and complete, there exists a unique nonnegative state-price density and there exists a unique fair price for any contingent claim in the limit of large quantities.

If we further assume that the frictionless model is arbitrage-free, the unique state-price density is then positive and the unique fair price for a contingent claim in the model with vanishing fixed costs is equal to the frictionless price in the limit of large quantities. If the market is incomplete, the set of possible prices[5] in the model with vanishing fixed costs for a contingent claim is equal to the set of possible frictionless prices (and their limits).

III. Applications

In this section, we apply the results of Section II to show that popular term structure of interest rate models that are rejected by the results in Dybvig, Ingersoll, and Ross (1996) are indeed arbitrage-free in the presence of fixed costs (including arbitrarily small fixed costs). We also show that fixed costs allow us to use models where asset prices are bounded and reach their bound. In particular, we obtain yield curve shapes as in Longstaff (1992), in a model based on the CIR (1985) model but without assuming an absorbing barrier for the short rate. As in Section II, all the models under consideration are in a discrete time framework. This permits us to provide simple intuitions about how things work. It would be possible to generalize the results in continuous time through some technical adjustment. For instance, to construct arbitrage-free models where asset prices are bounded and reach their bound, it suffices to consider processes that have a positive probability to be absorbed when they reach the bound. The absolutely continuous state-price density will then put zero weight on all the trajectories that are not absorbed and the whole weight on the absorbed trajectories. The same kind of considerations apply in the Dybvig, Ingersoll, and Ross (1996) property discussion: it suffices to consider interest rate models where the nondecreasing trajectories have positive weight under the initial probability and to put all the weight on these trajectories in the state-price density.

A. Long Rates Can Fall

We start by proving that the quite striking result obtained by Dybvig, Ingersoll, and Ross (1996), which asserts that under the assumption of absence of

[5]The set of possible prices is defined, as usual, as the set of prices given by all possible fair pricing rules.

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arbitrage long-term zero-coupon rates can never fall, is not true in models with fixed costs. We then show that in term structure interest rate models of the type modeled by Ho and Lee (1986), long-term rates can fluctuate if there are fixed costs. Finally, we show that with fixed costs the Brennan and Schwartz (1979) model is indeed arbitrage-free.

1. A Simple Example

Let us begin by recalling the framework of Dybvig, Ingersoll, and Ross (1996). There are traded zero-coupon bonds of all integer maturities. For $t < t'$, we define $\nu(t, t')$ as the discount bond price, i.e., the price at t of receiving a unit payoff at some later date t'. The zero-coupon rate $z(t, t')$ is defined implicitly by

$$\nu(t, t') \equiv \frac{1}{[1 + z(t, t')]^{t'-t}}.$$

The long-term zero-coupon rate at time t is given by $z_L(t) = \lim_{t' \to \infty} z(t, t')$ if the limit exists.

A free lunch is defined as a sequence of net trades such that i) the payoff tends uniformly to a nonnegative random variable that is positive with positive probability and the price tends to zero, or ii) the price tends to a negative number but the payoff tends uniformly to a nonnegative random variable.

Dybvig, Ingersoll, and Ross prove that under the assumption of no free lunch if for $s < t$, the long-term zero-coupon rate exists at time s and exists with probability one at time t, then $z_L(s) \leq z_L(t, \omega)$ with probability one (Dybvig, Ingersoll, and Ross (1996), Theorem 2, p. 9). To do so, they use an arbitrage argument of the form i). However, if we transpose their result in our discrete time and finite states of the world framework, the set of attainable payoffs is closed and a free lunch is in fact an arbitrage.

Their result can then be rephrased in our framework as follows. An arbitrage is a net trade with zero cost and nonnegative, nonzero terminal payoff. Under the assumption of no-arbitrage, if for $s < t$ the long-term zero-coupon rate exists at time s and exists with probability one at time t, then $z_L(s) \leq z_L(t, \omega)$ with probability one.

As a consequence of the Dybvig, Ingersoll, and Ross (1996) result, models where the long-term rate follows a specified nonmonotonic behavior and, in particular, models where the long-term rate follows the discrete time analog of a diffusion are rejected by the no-arbitrage condition. However, this does not mean that models where the long-term yield is a risk factor as in Duffie and Kan (1996) have to be systematically rejected. Indeed, in these models the long-term yield corresponds to a yield associated with a long but finite horizon, while in Dybvig, Ingersoll, and Ross (1996), the long-term rate is the limit rate when the horizon goes to infinity. In fact, it is possible to construct models where: for any t, the function $t' \to z(t, t')$ is nonmonotonic; for any T, the function $t \to z(t, t + T)$ is nonmonotonic; but $t \to z_L(t)$ exhibits monotonic nondecreasing behavior. Such an example is provided in the Appendix.

We show on a simple counterexample that in models with fixed costs, long-term zero-coupon rates can fall. Suppose it is known that the spot rate will be r_1

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until s, and that at time s it will be revealed to either shift to $r_2(< r_1)$ forever or to remain at r_1 forever, which is illustrated by

$$1 \to \cdots \to (1+r_1)^s \to (1+r_1)^{s+1} \quad \to \cdots \to (1+r_1)^t \quad \to \ldots$$
$$\searrow (1+r_1)^s(1+r_2) \to \cdots \to (1+r_1)^s(1+r_2)^{t-s} \to \ldots.$$

We assume that the discount bond prices are given (for $t' \geq t$) by $\nu(t,t') = 1/(1+r_1)^{t'-t}$ for $t \leq s$ and, for $t > s$, by $\nu(t,t') = 1/(1+r_1)^{t'-t}$ (resp. $1/(1+r_2)^{t'-t}$) if r_1 (resp. if r_2) obtains.

In fact, for $t' \geq t > s$ and for $s \geq t' \geq t$, interest rates are deterministic and bond prices are uniquely determined without making any assumption on the state-price density or on the risk premium. For $t' > s \geq t$, our bond prices correspond to a state-price density that puts zero weight on the lower branch of the tree. In a fixed costs framework, by Theorem 6 this model does not contain any arbitrage opportunity in a finite horizon. As far as free lunches are concerned, it is clear that we need not consider strategies starting after date $s+1$. For strategies starting before date s, it is easy to see that any net trade in the state r_1 is equal to 0 if we do not take the fixed costs into account and is smaller than $-\mathbf{c}$ if we do take them into account. Hence, there is no sequence of net trades, taking the fixed costs into account that converges to a nonnegative nonnull random variable.

We then obtain $z_L(t) = r_1$ for $t \leq s$ and for $t > s$, $z_L(t) = r_1$ if r_1 and $z_L(t) = r_2$ if r_2, so that, since $r_2 < r_1$, we do not have $z_L(s) \leq \min_\omega z_L(t,\omega)$ (more precisely, for all $t_1 \leq s$ and all $t_2 > s$, we have $z_L(t_1) \geq z_L(t_2,\omega)$ with probability one and $P[z_L(t_1) > z_L(t_2,\omega)] > 0$).

Therefore, we proved that when there are fixed costs (including arbitrarily small fixed costs), the result in Dybvig, Ingersoll, and Ross (1996) is no longer true: long-term rates can fall. In our example, they fall with a probability that can be chosen arbitrarily, namely, the probability for the short-term rate to shift from r_1 to r_2 at date s.

2. Models Similar to Ho and Lee (1986)

Dybvig, Ingersoll, and Ross (1996) show that in the Ho and Lee (1986) model, long-term rates can only increase. We prove that with the introduction of fixed costs long-term rates can decrease in models similar to Ho and Lee (1986). More precisely, we first give an example of an arbitrage-free Ho and Lee model in which long-term rates fall with probability one. We then give an example of a trinomial model, analogous to Ho and Lee, for which long-term rates can both increase or decrease by introducing fixed costs (even arbitrarily small fixed costs). Suppose that $\Sigma_t = \{u,d\}^t$. In such a context, a date t node σ_t is a sequence $(\rho_1, \cdots, \rho_t)$ of elements of $\{u,d\}$; for $\sigma_t = (\rho_1, \cdots, \rho_t)$, $f(\sigma_t) = \{(\sigma_t,u),(\sigma_t,d)\}$, where $(\sigma_t,u) = (\rho_1, \cdots, \rho_t, u)$ and $(\sigma_t,d) = (\rho_1, \cdots, \rho_t, d)$.

Assume now as in Ho and Lee (1986) that the value of any portfolio is independent from the chosen path and that discount bond prices satisfy

$$\nu(t,T,(\sigma_{t-1},\rho_t)) \quad = \quad \frac{\nu(t-1,T,\sigma_{t-1})}{\nu(t-1,t,\sigma_{t-1})} H(T-t,\rho_t),$$

where $H(\cdot,\rho) = h(\cdot)$, if $\rho = u$, $H(\cdot,\rho) = h^*(\cdot)$, if $\rho = d$, and $h \neq h^*$.

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Proposition 9. If there are fixed costs, the functions h and h^* can be chosen such that h is constant equal to 1 and $h^*(n) = [h^*(1)]^n > 1$, and the model is then arbitrage-free. Furthermore, in such a framework we have $1 + z_L(t, (\sigma_{t-1}, u)) = 1 + z_L(t-1, \sigma_{t-1})$, $1 + z_L(t, (\sigma_{t-1}, d)) < 1 + z_L(t-1, \sigma_{t-1})$. Hence, we have proved that the Ho and Lee (1986) model can lead to nonincreasing long-term rates in the presence of fixed costs including arbitrarily small fixed costs. In fact, in our setting the long-term rates fall with a probability equal to one.

To construct a model where the long-term rates can fluctuate (i.e., increase and decrease depending on the states of the world), we extend the Ho and Lee (1986) model to a trinomial setting. To construct such a trinomial generalization of the Ho and Lee (1986) model, we still assume that the value of any portfolio is independent from the chosen path and

$$\nu(t, T, (\sigma_{t-1}, \rho_t)) = \frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} H(T-t, \rho_t),$$

but with $\Sigma_t = \{u, m, d\}^t$. In Proposition 10, this model is called a generalized trinomial Ho and Lee model. We have the following result:

Proposition 10. In a fixed costs framework, there exists an arbitrage-free specification of the generalized trinomial Ho and Lee (1986) model such that

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, u)) &> 1 + z_L(t-1, \sigma_{t-1}), \\ 1 + z_L(t, (\sigma_{t-1}, m)) &= 1 + z_L(t-1, \sigma_{t-1}), \\ 1 + z_L(t, (\sigma_{t-1}, d)) &< 1 + z_L(t-1, \sigma_{t-1}). \end{aligned}$$

Hence, we construct a Ho and Lee-like model, where long-term rates can fluctuate. More precisely, they increase along the upper branches of the trinomial tree and they decrease along the lower branches. Consequently, models where long-term rates have nonmonotonic behavior are no longer rejected for arbitrage considerations.[6]

3. The Brennan and Schwartz (1979) Model

In the Brennan and Schwartz (1979) model, the long-term rate is assumed to follow a diffusion process. Dybvig, Ingersoll, and Ross (1996) show that such a model is not compatible with the no-arbitrage condition. In this section, we consider a discrete time version of the Brennan and Schwartz model and prove that a model wherein the long-term rates follow a diffusion process (or a discrete time approximation of a diffusion process) is compatible with the no-arbitrage condition in the presence of fixed costs, even arbitrarily small fixed costs. Consequently, even if in such a model the long-term rates fluctuate, decreasing in some states of the world, we cannot reject it based on arbitrage considerations if there are fixed costs (including arbitrarily small fixed costs). This enables us to reconcile arbitrage conditions and empirical observations that do not seem to reject the

[6] As Dybvig, Ingersoll, and Ross (1996) note, in a frictionless framework "It is not permissible to specify a term structure model with a stochastic factor which is the long (asymptotic) end of the zero-coupon yield curve unless this factor can only increase over time" (p. 22).

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Brennan and Schwartz (1979) model (see Chan, Karolyi, Longstaff, and Sanders (1992)).

Recall that in the Brennan and Schwartz (1979) model, the short-term rate and the long-term rate, respectively, denoted by ℓ and r, are governed by the diffusion equation,

$$\begin{cases} d\ell_t &= \mu(\ell_t, r_t, t)dt + \sigma_1 \ell_t dW_t^1 \\ dr_t &= a(\ell_t - r_t)dt + \sigma_2 \sqrt{r_t} dW_t^2, \end{cases}$$

where a, σ_1, and σ_2 are given positive constants and μ is a given bounded function of the current date and both the long- and short-term rates.

Next, we consider the following model for the long- and short-term rates. It is not recombining, but our aim is to show the acceptability of such a model and obtain simple pricing formulas. Let

$$\ell_{n+1} = \ell_n \begin{cases} \exp\left\{\left(\frac{\mu\left(\ell_n, r_n, n\frac{T}{N}\right)}{\ell_n} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} + \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right\} & \text{with probability } p_u \\ 1 & \text{with probability } p_m \\ \exp\left\{\left(\frac{\mu\left(\ell_n, r_n, n\frac{T}{N}\right)}{\ell_n} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} - \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right\} & \text{with probability } p_d \end{cases}$$

$$r_{n+1} = r_n \begin{cases} \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N} + \sqrt{2}\frac{\sigma_2}{\sqrt{r_n}}\sqrt{\frac{T}{N}}\right\} & \text{with probability } q_u \\ \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N}\right\} & \text{with probability } q_m \\ \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N} - \sqrt{2}\frac{\sigma_2}{\sqrt{r_n}}\sqrt{\frac{T}{N}}\right\} & \text{with probability } q_d, \end{cases}$$

where T is a given horizon, $\tilde{\sigma}_1, a, \sigma_2$ are positive constants, and μ is a given bounded function.

This model consists of nine different states of the world at each date, with a probability $p_i q_j$ for each state $(i,j)_{\substack{i=u,m,d \\ j=u,m,d}}$. We have the following result:

Proposition 11. There exists a specification of the probabilities (p_u, p_m, p_d) and (q_u, q_m, q_d) such that the model above is a discrete time approximation of the Brennan and Schwartz (1979) model. Furthermore, in a fixed costs setting, there exists an arbitrage-free bond price structure compatible with this model, i.e., such that the short-term rate is given by r and the long-term rate by ℓ.

Hence, we have proved that in the presence of fixed costs, the Brennan and Schwartz (1979) model (or more precisely a discrete time approximation of this model) is compatible with the no-arbitrage condition. This compatibility is due to the introduction of fixed costs since in such a model the long-term rates fluctuate and following Dybvig, Ingersoll, and Ross (1996) this property leads to arbitrage opportunities in a frictionless setting. There obviously might be many other frictions that might lead to the same result; however, the introduction of fixed costs is the only one[7] for which arbitrage opportunities vanish and such that by Proposition 8 the structure of bond prices remains the same as in the frictionless setting (at least in the limit of large quantities).

[7] Among the classical frictions studied in the literature (proportional transaction costs, short-selling costs, and portfolio constraints, among others).

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B. Price Processes with Upper and Lower Bounds or Absorbing Barriers

We now turn to security pricing models that are bounded and can potentially reach their bounds. We then propose a discrete time version of the CIR (1985) model which, as in Longstaff (1992), produces yield curves with realistic humps, but without assuming an absorbing barrier for the short rate by introducing fixed costs.

1. The Binomial Model of Cox, Ross, and Rubinstein (1979) with Bounds

The introduction of fixed costs enables us to consider where the asset price processes reach some bounds as arbitrage-free models, and to determine in such a framework a fair pricing rule for contingent claims. Indeed, if we assume that the discounted price process of a given asset is bounded by an upper barrier and reaches this barrier without being absorbed, then it is clear that there is no positive state-price density compatible with this asset price process. This price process therefore leads to arbitrage opportunities in a frictionless setting. However, a state-price density that, once the barrier is reached, puts all the weight on the constant discounted price path might be compatible with this asset price process. By Theorem 6, the presence of upper bounds on the discounted asset prices may then be compatible with the no-arbitrage condition in a fixed costs setting.

We will illustrate this property on a version of the binomial model of Cox, Ross, and Rubinstein (1979) with bounds. This model has been studied in Sondermann (1988) for other purposes.[8]

More precisely, denote by $(W_t)_{t\in\{0,\cdots,T\}}$ the stochastic process given by

$$W_t \quad = \quad Z_0 \cdot \prod_{n=1}^{t}(1+\rho_n),$$

where (ρ_n) is a family of independent random variables such that

$$\rho_n \quad = \quad \begin{cases} \delta + \frac{1}{2}\delta^2 \stackrel{def}{=} \delta_u & \text{with probability } p \\ -\delta + \frac{1}{2}\delta^2 \stackrel{def}{=} \delta_d & \text{with probability } 1-p \end{cases}$$

for fixed $2 > \delta > 0$, $0 < p < 1$. The process $(W_t)_{t\in\{0,\cdots,T\}}$ is a geometric random walk with compensated stochastic drift and corresponds to the security price process in the Cox, Ross, and Rubinstein model (1979). Note that in such a context, a date t node σ_t is a sequence $(\rho_1, \cdots, \rho_t)$. As we have mentioned, we assume that for some reason, the price process cannot go beyond some a priori given bounds. Here we fix an upper bound M and we assume for simplicity of exposition that M belongs to the set of values taken by the process $(W_t)_{t\in\{0,\cdots,T\}}$. The price process of our security is given by $(Z_t)_{t\in\{0,\cdots,T\}}$ with $Z_0 = W_0$ and where, for $\sigma_t = (\rho_1, \cdots, \rho_t)$ and $\sigma_{t+1} = (\rho_1, \cdots, \rho_{t+1}) \in f(\sigma_t)$,

$$\begin{aligned} Z(\sigma_{t+1}) \quad &= \quad Z(\sigma_t)(1+\rho_{t+1}) \quad \text{if } Z(\sigma_t) < M \text{ and, if } Z(\sigma_t) = M, \\ Z(\sigma_{t+1}) \quad &= \quad \begin{cases} M & \text{if } \rho_{t+1} = \delta_u \\ Z(\sigma_t)(1+\rho_{t+1}) & \text{if } \rho_{t+1} = \delta_d \end{cases}. \end{aligned}$$

[8] This paper might be difficult to consult. See also Sondermann (1987).

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In Proposition 12, this model is called the generalized Cox, Ross, and Rubinstein model (1979) with bounds. The aim is to value a European call $x = (Z_T - K)_+$ for some a priori given bound $M \geq Z_0$. We have the following result:

Proposition 12. The generalized Cox, Ross, and Rubinstein model (1979) with bounds is arbitrage-free in a fixed costs setting. Furthermore, if the fixed costs structure is vanishing, the price of a European call option in this model is equal (in the limit of large quantities) to the price of a barrier option with the same strike and maturity in the frictionless Cox, Ross, and Rubinstein model (1979).

Sondermann (1988) shows that the corresponding option pricing formula can be simplified for small grid sizes. Letting the grid size go to zero by the Central Limit Theorem, the analog of the Black and Scholes (1973) formula is obtained for such options and then for European call options with bounds on the underlying asset. These formulas reduce to the Black and Scholes formula if the upper bound goes to infinity. The same kind of results obtain if we introduce a lower bound on the asset prices and the pricing formulas reduce to the Black and Scholes formula if the lower bound approaches zero.

Hence we have proved that the presence of fixed costs (including arbitrarily small fixed costs) allows us to use models with bounds on the discounted asset price processes. Such bounds might be subjectively assumed by some investors in order to model the fact that, from their point of view, the price of a given asset cannot exceed a given level. They can also be seen as objective bounds due to policy constraints as in target zones where some bounds are imposed on exchange rates (as in the previous European Monetary System). They finally can be seen as objective bounds resulting from the choice of a given model.

Such situations can be encountered, for instance, in interest rate models where $r=0$ is an absorbing barrier as in Longstaff (1992). Indeed, in such models, the price of a given bond, when the barrier $r = 0$ is reached, is equal to 1, which corresponds to an upper bound for a bond price in models where the short-term rates are always nonnegative. Longstaff (1992) introduced such models in order to enlarge the set of possible shapes of the interest rate term structure in a CIR (1985) model.

C. The Cox, Ingersoll, and Ross (1985) Model with a Partially Absorbing Barrier

The CIR (1985) model is empirically tested by Brown and Dybvig (1986) and Brown and Schaefer (1994). These authors show that the volatility of the short-term rate conforms relatively well to empirical observations, but they argue that the model is not flexible enough to capture the variety of shapes encountered in practice. More precisely, while this model allows for inverted shapes at the short end of the curve, it fails to capture yield curves with a hump. Longstaff (1992) proposes an extension of this model that allows for an absorbing barrier at $r = 0$ for the short-term rate. He then provides analytical pricing formulas for the bonds and argues that this model leads to more realistic term structures of interest rates. The aim of this section is to show that Longstaff's (1992) approach and formulas (or, more precisely, their discrete time version) are compatible with

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the no-arbitrage condition with fixed costs, even if $r = 0$ is not an absorbing barrier. This permits us to enlarge the set of possible shapes of the interest rate term structure, and consistent with Longstaff (1992), to generate possible humps without imposing the unrealistic absorbing barrier condition. Indeed, it is difficult to accept that when the short-term rate reaches zero it will stay at that level forever.

Consider the following model for the short-term rate,

$$r_{n+1} = r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{with probability } 1/4 \\ a(\ell - r_n)\frac{T}{N} & \text{with probability } 1/2 \\ a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{with probability } 1/4. \end{cases}$$

Following Nelson and Ramaswamy (1990), this model is a discrete time approximation of the CIR (1985) model, where the short-term rate is governed by the following diffusion equation,

$$dr_t = a(\ell - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

For this reason, the discrete time model introduced above is referred to below as the discrete CIR model.

If $2\sigma^2 < a\ell$, as imposed by CIR (1985), and if $r_0 > 0$, then both models lead to a (strictly) positive short-term rate at any date with probability one. The CIR (1985) model does not explore the case where the condition $2\sigma^2 < a\ell$ is not satisfied and where the short-term rate might be equal to zero with positive probability.

However, Longstaff (1992) points out that this condition can be relaxed if we impose a boundary condition on the bond prices when the short-term rate is equal to zero. In particular, he considers an absorbing barrier at $r = 0$ for the short-term rate and imposes bond prices equal to one once the barrier is reached. He then provides analytical pricing formulas for the bonds for this case.

The discrete time version of Longstaff's (1992) absorbing barrier model is given by

$$r_{n+1} = r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & 1/4 \\ a(\ell - r_n)\frac{T}{N} & 1/2 \quad \text{if } r_n \neq 0 \\ \left[a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}}\right]_+ & 1/4 \end{cases}$$

$$r_{n+1} = r_n \text{ if } r_n = 0.$$

Below we refer to the discrete time version of the bond pricing formulas obtained by Longstaff (1992) in the absorbing barrier framework as Longstaff's bond prices.

Let us modify the discrete CIR model as

$$r_{n+1} = r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & 1/4 \\ a(\ell - r_n)\frac{T}{N} & 1/2 \\ [\varphi(r_n)]_+ & 1/4, \end{cases}$$

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where

$$\varphi(r_n) = \begin{cases} a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{if } r_n > 0 \\ 0 & \text{if } r_n = 0. \end{cases}$$

This model corresponds to a generalization of the discrete CIR model with a partially absorbing barrier at $r=0$. Indeed, when r_n is positive, we define r_{n+1} as in the discrete CIR model with only one modification: $a(\ell - r_n)(T/N) - \sqrt{2}\sigma\sqrt{r_n}\sqrt{T/N}$ in the lower branch is replaced by its positive part in order to ensure that the short-term rate remains positive. When r_n is equal to zero, there are three possible states of the world at date $n+1$; for one of them the short-term rate is absorbed and remains equal to zero and for the two others the short-term rate is reflected. Below we refer to this model as the partially absorbing CIR model.

Proposition 13. For λ small enough and N large enough

$$\left(\text{more precisely, for } \ |\lambda| < \frac{a}{\sqrt{2}} \text{ and } \lambda^2\frac{T}{N} \leq \sigma^2 \inf\left\{\frac{1}{r_0}, \left(\frac{2ah}{\sqrt{2}\sigma + \sqrt{2\sigma^2 + 4a^2\ell h^2}}\right)^2\right\}\right),$$

Longstaff's bond prices are compatible with the partially absorbing CIR model with fixed costs (in the sense that these prices can be derived in an arbitrage-free way from the partially absorbing model for the short-term rate).

To obtain the same term structure as in the absorbing barrier model we choose a risk-neutral probability for which zero is actually an absorbing barrier. This is made possible by the fact that in the presence of fixed costs the risk-neutral probability and the actual probability do not have to be equivalent. A given barrier can then be absorbing under only one of these probabilities.

Our term structure is therefore the discrete time analog of the specific example of Longstaff ((1992), Section 3) and allows for humps at longer maturities. However, the short-term rate dynamics is not the same as we do not assume that zero is an absorbing barrier. We only impose that when the short-term rate reaches zero, there is a nonnull probability that it continues to equal zero for at least one period. As underlined by Longstaff (1992), the behavior of Treasury-bill yields appears to be more consistent with such a sticky boundary than with a totally reflecting boundary: extremely low levels of yields tend to persist rather than immediately increasing back toward higher levels.

IV. Conclusion

We show that with the introduction of fixed costs (including arbitrarily small fixed costs), we can enlarge the set of possible arbitrage-free models to include popular models of the term structure of interest rates. In particular, models with bounded price processes reaching their bounds become arbitrage-free in the presence of fixed costs. In addition, fixed costs allow us to relax the monotonic constraint imposed for arbitrage considerations on the long-term rates by Dybvig, Ingersoll, and Ross (1996).

We prove that when the initial frictionless model is arbitrage-free, contingent claims prices remain the same in the limit of large quantities after the introduction

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of the fixed costs. When the initial frictionless model contains arbitrage opportunities, the absence of arbitrage in the model with frictions is characterized by the existence of a nonnegative state-price density (or equivalently by a martingale measure possibly with zero weights) and contingent claims prices are given (in the limit of large quantities) by this state-price density.

Appendix

Proof of Proposition 2. The direct implication is immediate. Using the fact that $\lim_{\lambda\to\infty} c^{\lambda\theta}(\sigma_t)/\lambda \to 0$, it suffices to multiply the considered portfolio by a sufficiently large constant in order to establish the converse implication.

Proof of Proposition 3. As usual, this result is a direct application of the Farkas-Minkowski Lemma.

An Example of Utility Maximization with Zero State-Price Densities in Some States. Let us consider a financial market where two securities, denoted by A and B, can be traded at two dates 0 and 1 and in two possible states of the world (with equal probability) s_1 and s_2 at date 1 (security A has a value of 1 at date 0 and a value of 1 or 2 at date 1 in state s_1 or s_2, respectively, and security B is always worth one unit of account). We consider an agent who, endowed with initial wealth w and a CARA utility function with an absolute risk aversion $\alpha = 1$, tries to maximize his expected utility from terminal wealth. Furthermore, assume that a transaction fee $c^\theta = (1 + 2|\theta|)/(2 + 2|\theta|)$ is paid at date 0 only if the agent trades a quantity $\theta \neq 0$ of asset A at date 0. Clearly, the transaction costs increase with the quantity traded and are bounded both above and away from zero for $\theta \neq 0$. It is easy to see that an optimal trade should satisfy the condition $\theta \geq 0$ and that it leads then to a terminal wealth equal to $(w - c^\theta, w + \theta - c^\theta)$.[9] If θ is large enough, we have $w - c^\theta \approx w - 1$, $w + \theta - c^\theta \approx w + \theta - 1$, and any marginal increase of the traded quantity from θ to $\theta + \delta\theta$ would lead to a move of the utility level from u to $u + \delta u$, with

$$\begin{aligned}\delta u &= -\frac{1}{4}\exp\left(-w + c^\theta\right)\frac{1}{(1+\theta)^2} + \frac{1}{2}\exp\left(-w - \theta + c^\theta\right)\left(1 - \frac{1}{2(1+\theta)^2}\right)\\ &\approx -\frac{\exp(-w+1)}{4(1+\theta)^2} + \frac{1}{2}\exp(-w-\theta-1) \quad < \quad 0\end{aligned}$$

for θ large enough. Furthermore, it is easy to see that for $\theta > -\ln(2\exp(-c^\theta) - 1)$, the utility level associated to that trade is higher than the utility level associated with no trade. In particular, for $\theta > -\ln(2\exp(-½) - 1) \approx 1.55$, it is better to trade. Hence, in such a framework our agent chooses to trade at a given optimal level θ^* such that $0 < \theta^* < \infty$.

Proof of Proposition 5. If there exists an arbitrage portfolio θ as in Definition 1 between a node σ_t and its successors, it suffices to consider the dynamic strategy that consists of buying portfolio θ at the node σ_t, and investing the "profits" in bonds.

Conversely, suppose that there is a dynamic arbitrage opportunity described by a portfolio θ then there exists a node σ_t such that $\theta(\sigma_t) \cdot Z(\sigma_t) < 0$. If $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) \geq 0$ for all $\sigma_{t+1} \in f(\sigma_t)$, then according to Proposition 1 we have an arbitrage portfolio at σ_t. If $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) < 0$ for some $\sigma_{t+1} \in f(\sigma_t)$, then by the self-financing condition, we have $\theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) < 0$. We start again from the node σ_{t+1} and there must exist a date τ and a node $\sigma \in \Sigma_\tau$ such that $\theta(\sigma) \cdot Z(\sigma) < 0$ and $\theta \cdot Z \geq 0$ on $f(\sigma)$ in order to ultimately obtain $\theta \cdot Z \geq 0$.

Proof of Theorem 6. If there is no arbitrage opportunity, it suffices to multiply the different state-price densities obtained at each node between that node and its successors starting from σ to construct a nonnegative state-price density compatible with the asset prices.

[9] If $\theta = 0$ then there is no trade and no transaction cost. In that case, we define then c^0 by $c^0 = 0$.

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Conversely, for a given date i node σ, it suffices to consider the state-price density $(\pi_\rho)_{\rho\in f(\sigma)}$ defined on the successors of σ by $\pi_\rho = \pi^\sigma(\rho)$. This state-price density characterizes the absence of arbitrage opportunities between σ and its successors for every σ and in turn the absence of dynamic arbitrage opportunities.

Proof of Proposition 8. It is easy to see that for all m in M,

$$\lim_{\lambda\to+\infty}\frac{p_s(\lambda m)}{\lambda} \quad = \quad \lim_{\lambda\to+\infty}\frac{-p_s(-\lambda m)}{\lambda} \quad = \quad l(m).$$

Since there is no arbitrage, we must have $p(m)\geq -p(-m)$ so that $-\pi_s(-m)\leq -p(-m)\leq p(m)\leq \pi_s(m)$, and the price function p can be written as the sum of a linear function and of a fixed cost function, i.e., for all m, $p(m)=l(m)+c(m)$ where $c(\lambda m)/\lambda\to_{\lambda\to\infty}0$. The case of bounded fixed costs structures can be treated similarly.

If we assume that $p(\lambda m)\leq \lambda[p(m)]$, it is easy to see that the fixed cost function is nonnegative. Furthermore, for all frictionless self-financing dynamic portfolios θ,

$$l\left((\theta(\sigma_T)\cdot Z(\sigma_T))_{\sigma_T\in\Sigma_T}\right) \quad = \quad \sum_{\sigma_T\in\Sigma_T}\pi^\Omega(\sigma_T)\theta(\sigma_T)\cdot Z(\sigma_T),$$

where π^Ω is any nonnegative state-price density as in Theorem 4.

Consequently, the fair price $p(m)$ associated with any attainable contingent claim m is given by

$$p(m) \quad = \quad \sum_{\sigma_T\in\Sigma_T}\pi^\Omega(\sigma_T)m(\sigma_T) + \text{ a fixed cost.}$$

An Example of a Nondecreasing Long-Term Rate with Nonmonotonic Finite Horizon Rates. Let us assume that the sequence of short-term rates r_n between date $t=n$ and $n+1$ is ∞-uniform.[10] It is clear that $z(n,n')$ (resp. $z(n,n+N)$) can either increase or decrease when n' (resp. n) increases. However, $\ln(1+z_L(n))$ is the limit of the average of $\ln(1+r_n)$ and is equal to $\frac{1}{2}\ln 2$. Therefore, we have that $z_L(n)$ is constant, independent of n, and equal to $\sqrt{2}-1$.

Proof of Proposition 9. Proceeding as in the frictionless case, we obtain that the absence of arbitrage with fixed costs is implied by the existence of $\pi\in[0,1]$, independent from n, such that

$$\pi h(n)+(1-\pi)h^*(n) \quad = \quad 1 \quad \text{and} \quad \frac{h(n)}{h^*(n)} \quad = \quad \left[\frac{h(1)}{h^*(1)}\right]^n.$$

Indeed, in this case we find that the discounted process $X^T=(\nu(t,T)/\beta_t)_t$ satisfies for all $\sigma_t\in\Sigma_t$,

$$X_t^T(\sigma_t) \quad = \quad \pi X_{t+1}^T(\sigma_t,u)+(1-\pi)X_{t+1}^T(\sigma_t,d),$$

and the prices do not depend on the chosen path.

[10] A sequence (x_n) with values in $[0,1]$ is said to be k-uniform if for all rectangle $D=\prod_{i=1}^k[a_i,b_i]$ with $0\leq a_i\leq b_i\leq 1$, $i=1,\cdots,k$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}1_D((x_{ki},x_{ki+1},\cdots,x_{k(i+1)-1})) \quad = \quad (b_1-a_1)\cdots(b_k-a_k).$$

A sequence is said to be ∞-uniform if it is k-uniform for all $k\geq 1$.

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The choice of $\pi = 1$ (which, by Theorem 6, is permitted in a fixed costs framework) leads to $h \equiv 1$ and $h^*(n) = [h^*(1)]^n$. As far as long-term rates are concerned, we have

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, u)) &= \lim_{T \nearrow \infty} [\nu(t, T, (\sigma_{t-1}, u))]^{-\frac{1}{T-t}} \\ &= \lim_{T \nearrow \infty} \left[\frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} h(T-t) \right]^{-\frac{1}{T-t}} \\ &= 1 + z_L(t-1, \sigma_{t-1}) \end{aligned}$$

and

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, d)) &= \lim_{T \nearrow \infty} [\nu(t, T, (\sigma_{t-1}, d))]^{-\frac{1}{T-t}} \\ &= \lim_{T \nearrow \infty} \left[\frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} h^*(1)^{T-t} \right]^{-\frac{1}{T-t}} \\ &= \frac{1}{h^*(1)} [1 + z_L(t-1, \sigma_{t-1})] \\ &< 1 + z_L(t-1, \sigma_{t-1}). \end{aligned}$$

Proof of Proposition 10. It suffices to take $H(n, \rho)$ equal to a^n (to 1, to $1/a^n$) if $\rho = u$ ($\rho = m$, $\rho = d$), where $a < 1$ is a given constant. There exists $\pi = (0, 1, 0) \in [0, 1]^3$, such that $\pi_1 a^n + \pi_2 + \pi_3 (1/a^n) = 1$, and with the notation of the previous proof, the process $X^T = (\nu(t, T)/\beta_t)_t$ satisfies

$$X_t^T(\sigma_t) = \pi_1 X_{t+1}^T(\sigma_t, u) + \pi_2 X_{t+1}^T(\sigma_t, m) + \pi_3 X_{t+1}^T(\sigma_t, d).$$

By Theorem 6 this model is arbitrage-free in a fixed costs framework. As far as long-term rates are concerned and using the same approach as in the previous proof, we obtain

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, u)) &= \frac{1}{a} [1 + z_L(t-1, \sigma_{t-1})] \\ &> 1 + z_L(t-1, \sigma_{t-1}) \\ 1 + z_L(t, (\sigma_{t-1}, u)) &= 1 + z_L(t-1, \sigma_{t-1}) \\ 1 + z_L(t, (\sigma_{t-1}, d)) &= a[1 + z_L(t-1, \sigma_{t-1})] \\ &< 1 + z_L(t-1, \sigma_{t-1}). \end{aligned}$$

Proof of Proposition 11. Let us take

$$p_u = \frac{\sigma_1^2 - \frac{\mu}{\ell}\left(\left(\frac{\mu}{\ell} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} - \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right)}{2\tilde{\sigma}_1\left[\tilde{\sigma}_1 + \left(\frac{\mu}{\ell} - \frac{1}{2}\tilde{\sigma}_1^2\right)\sqrt{\frac{T}{N}}\right]}, \quad p_m = \frac{\tilde{\sigma}_1^2 - \sigma_1^2 + \left(\frac{\mu^2}{\ell^2} - \frac{1}{4}\tilde{\sigma}_1^4\right)\frac{T}{N}}{\tilde{\sigma}_1^2 - (\frac{\mu}{\ell} - \frac{1}{2}\tilde{\sigma}_1^2)^2\frac{T}{N}},$$

$$p_d = \frac{\sigma_1^2 - \frac{\mu}{\ell}\left(\left(\frac{\mu}{\ell} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} + \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right)}{2\tilde{\sigma}_1\left[\tilde{\sigma}_1 - \left(\frac{\mu}{\ell} - \frac{1}{2}\tilde{\sigma}_1^2\right)\sqrt{\frac{T}{N}}\right]}, \quad q_u = 1/4, \quad q_m = 1/2, \quad q_d = 1/4,$$

for some positive constant $\sigma_1 < \tilde{\sigma}_1$ and for N sufficiently large. In that case, it is easy to check that $P = (p_u, p_m, p_d)$ defines a probability and following Nelson and Ramaswamy (1990) our model is a discrete approximation of the Brennan and Schwartz (1979) model.

Let the risk-neutral probability

$$[\pi(i, j)]_{\substack{i=u,m,d \\ j=u,m,d}}$$

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be such that $\pi(m,m)=1$ and let π denote the induced probability measure on the terminal states of the world. We then construct bond prices as the discounted expected value under the probability π of the bond's terminal payoff and obtain

$$\nu\left(n,n';(\ell,r)\right) \quad = \quad \left[\left(1+u_n^{\ell,r}\right)\ldots\left(1+u_{n'-1}^{\ell,r}\right)\right]^{-1},$$

where the sequence $u_s^{\ell,r}$ is defined inductively by

$$\left\{\begin{array}{lcl} u_{s+1}^{\ell,r} & = & u_s^{\ell,r}\exp\left\{\frac{a\left(\ell-u_s^{\ell,r}\right)}{u_s^{\ell,r}}\frac{T}{N}\right\} \\ u_n^{\ell,r} & = & r \end{array}\right. .$$

By construction and by Theorem 6, the financial market consisting of these bonds is arbitrage-free in a fixed costs setting and admits r as the short-term rate. Furthermore, we can verify that the long-term rate induced by these bond prices is given by ℓ. Indeed, we have

$$\ln(1+z(n,n';(\ell,r))) \quad = \quad \frac{1}{n'-n}\sum_{k=0}^{n'-n-1}\ln\left(1+u_{n+k}^{\ell,r}\right)$$

and by a classical Cesaro argument we get

$$\ln(1+z_L(n;(\ell,r))) \quad = \quad \lim_{k\to\infty}\ln\left(1+u_k^{\ell,r}\right)$$

when this last limit exists. It is then easy to check that the limit of $u_k^{\ell,r}$ is equal to ℓ and we then obtain $z_L(n;(\ell,r))=\ell$.

Proof of Proposition 12. For $M=\infty$, $(Z_t)_{t\in\{0,\cdots,T\}}$ is equal to $(W_t)_{t\in\{0,\cdots,T\}}$ and there is a unique probability measure, P^*, that makes $(Z_t)_{t\in\{0,\cdots,T\}}$ a martingale, and it is given by the positive transition probability $p^*=½-¼\delta$. By Proposition 8, any fair pricing rule is given by the sum of a fixed cost function and of the expected value with respect to P^* (which corresponds to the option price in a frictionless setting).

It is easy to check that this model is also arbitrage-free in a fixed costs setting for $M<\infty$.

We now turn directly to pricing issues. There is a unique probability measure P_M^* with nonnegative weights that makes (Z_t) a martingale. This probability is equal to the above-mentioned probability measure (with positive weights) P^* on the paths for which the bound M is not reached and puts zero weight on the paths for which the bound is reached and left. Under P_M^*, the process (Z_t) has the same distribution as the primitive price process absorbed at the upper bound M (and denoted by $\tilde{W}_T$) under P^*. Thus, any admissible price for a European call $x=(Z_T-K)_+$ can be written as the sum of a fixed cost and $E^{P_M^*}[(Z_T-K)_+]$, which is equal to $E^{P^*}[(\tilde{W}_T-K)_+]$ and corresponds to the price of a barrier option, where the barrier is set at M, in the classical binomial model where the risk-neutral probability is given by P^*.

Proof of Proposition 13. As in CIR (1985) or Longstaff (1992), we look for a state-price density π compatible with the no-arbitrage condition in a fixed costs setting such that the risk premium is equal to λ for $r\neq 0$.

More precisely, we look for $\pi=(\pi_1,\pi_2,\pi_3)\in\,]0,1[^3$ such that under π, the drift is given by the initial drift minus λr (as in CIR), and the variance remains unchanged (given by $\sigma^2 r(T/N)$). If $|\lambda|<a/\sqrt{2}$ and $\sigma^2\geq\lambda^2\overline{r}(T/N)$,

$$\text{where} \qquad \overline{r} \quad = \quad \sup\left\{r_0,\left(\frac{\sqrt{2}\sigma+\sqrt{2\sigma^2+4a^2lh^2}}{2ah}\right)^2\right\},$$

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the solution is given by

$$\pi_1 = \frac{1}{2}\left[\frac{\sigma^2 + \frac{\lambda^2 T r}{N}}{2\sigma^2} - \frac{\lambda\sqrt{r}}{\sigma\sqrt{2}}\sqrt{\frac{T}{N}}\right], \quad \pi_3 = \frac{1}{2}\left[\frac{\sigma^2 + \frac{\lambda^2 T r}{N}}{2\sigma^2} + \frac{\lambda\sqrt{r}}{\sigma\sqrt{2}}\sqrt{\frac{T}{N}}\right],$$

and $\pi_2 = 1 - \pi_1 - \pi_3$. Furthermore, we want our model to have an absorbing barrier under the risk-neutral probability in order to generate the same prices as in Longstaff (1992). It suffices then to take, for $r = 0$, $(\pi_1, \pi_2, \pi_3) = (0, 0, 1)$.

Letting $\nu(r\,; n, n')$ be given by

$$\nu(r\,; n, n') = E^{\pi}\left[\frac{1}{\prod_{s=n}^{n'-1}(1 + r_s)}\middle| r\right],$$

we obtain an arbitrage-free (with fixed costs) term structure that satisfies $\nu(0\,; n, n') = 1$ (as in Longstaff (1992)) and the market price of interest rate risk is λ (as in CIR (1985) and Longstaff (1992)).

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