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A Combinatorial Approach to the Two-Sided Exit Problem for Left-Continuous Random Walks

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We present a combinatorial lemma that provides a new approach to the two-sided exit problem and related questions for left-continuous random walks (*i.e.*, random walks on the integers whose negative steps have size -1). Some applications to random walks on the circle are also derived.

1. Introduction

Fluctuation theory for real-valued random walks has frequently been developed with the help of combinatorial methods. Some important contributions in this area are the celebrated works of Sparre Andersen [14], Spitzer [16], Wendel [21] and Takács [18]. In the special case of left-continuous random walks (that is, random walks on $\mathbb{Z}$ whose only possible negative steps have size -1), the theory is especially powerful and leads to more explicit results than in the general case (see, *e.g.*, [18, Chapter 2]). This accounts in part for the interest these random walks have raised, another reason being their importance in various domains of applied probability theory, such as queues, dams, insurance risks, and branching processes (see [2, 10], for example).

The aim of this paper is to use a new approach to study the two-sided exit problem for left-continuous random walks, that is, to determine the law of the exit time from a finite interval. This problem naturally arises from the generalization of the classical gambler's ruin problem. Some results in this field can be found in [18].

The key to our study is a general combinatorial result on rational power series involving a particular formulation of the inversion of matrices. Although this method can be considered as standard in combinatorics, its application in probability seems to be new. It allows us to recover in a simple manner several results related to the two-sided exit problem. Moreover, we explain the link between certain quantities involved in this study and the ergodic properties of the random walks, recovering in a discrete setting some

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results that are the analogues of those recently established by Bertoin for completely asymmetric Lévy processes [1].

Effective calculations of exit times are possible for a simple (but not necessarily symmetric) random walk, using the Chebyshev polynomials of the second kind.

We also use the combinatorial method to study random walks on the circle. This gives an alternative approach to a result of Pitman on the invariance by rotation of the occupation measure [9, 5].

The remainder of this paper is organized as follows. In Section 2 we state the basic combinatorial formula. In Section 3 we apply it to the solution of the two-sided exit problem and make explicit the link with ergodic theory. Section 4 is devoted to computations and asymptotic estimates for simple random walks and birth and death processes. Finally, the application of the combinatorial lemma for the study of random walks on the circle is made in Section 5.

2. The combinatorial result

As mentioned in the introduction, the result stated here is a very classical one in combinatorics: see, for instance, [3, Theorems (1.2), (1.3) and the references on page 36]. Nonetheless, in order to make the paper self-contained we shall give a complete presentation.

Consider a finite set V of vertices. To each directed edge (a, a') ($a, a' \in V$ and a' can be equal to a) associate a real number $p(a, a')$, and define the matrix M with entries $M_{i,j} = p(i, j)$. The matrix $Id - tM$, considered as a matrix with coefficients in $\mathbb{R}[[t]]$, is invertible, and its inverse is

$$\sum_{k=0}^{\infty} t^k M^k,$$

the sum above being well defined as a formal series.

Our aim is to establish a combinatorial formula for the computation of the inverse, making use of the classical Cramer formula, which can be stated as follows. Let A be a matrix with coefficients in a commutative ring ($\mathbb{R}[[t]]$ in our case). Then A is invertible if and only if its determinant is invertible, in which case the coefficient (i, j) of A^{-1} is given by

$$A^{-1}_{i,j} = (\det A)^{-1} \mathrm{cof}_{j,i}(A).$$

Here $\mathrm{cof}_{j,i}(A)$ stands for the (j, i) cofactor, that is, the determinant of the matrix A' obtained by replacing all the coefficients in the jth row and ith column in A by 0, except at (j, i) where we put 1.

Notice that in our case $\det(Id - tM)$ is a polynomial with constant coefficient 1 and is therefore invertible as a formal series. We now proceed to the combinatorial part.

We shall say that a *path* on V with length n is any sequence

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n)$$

of adjacent directed edges of the graph. A path is cyclical if $a_0 = a_n$, n being the length of the path. A cyclical path is simple if it contains no subcycle, that is, for every $i, j \in [1, n]$

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Figure 1 A path and a simple cycle

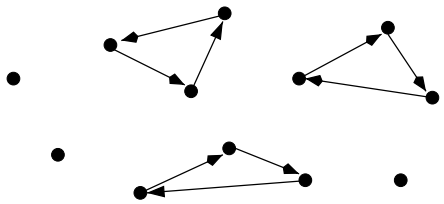


Figure 2 A configuration for $D(t)$

with $i \neq j$, $a_i \neq a_j$. Two cyclical paths are said to be equivalent if they are identical up to a cyclical permutation of the edges. A simple cycle is an equivalence class of simple cyclical paths. See Figure 1.

To each path $\gamma = (a_0, a_1), \ldots, (a_{n-1}, a_n)$ we associate the valuation

$$v(\gamma)(t) = \prod_{i=1}^{n} p(a_{i-1}, a_i)t \in \mathbb{R}[t].$$

As the valuation is invariant under cyclical permutation of the edges, we can similarly define the valuation of a simple cycle.

This allows us to define the polynomial

$$D(t) = \sum (-1)^r v(\gamma_1)(t) \ldots v(\gamma_r)(t), \tag{2.1}$$

where the summation is extended over all the configurations of r simple cycles $\gamma_1, \ldots, \gamma_r$ whose supports are mutually disjoint, r varying in $\mathbb{N}$ (see Figure 2). We include in the set of configurations the empty configuration with valuation 1.

Likewise, if $i, j \in V$, $i \neq j$, set

$$N_{i,j}(t) = \sum (-1)^r v(\gamma_{i,j})(t) v(\gamma_1)(t) \ldots v(\gamma_r)(t), \tag{2.2}$$

where the summation is extended over all the configurations of a self-avoiding path $\gamma_{i,j}$ from i to j and r simple cycles $\gamma_1, \ldots, \gamma_r$ whose supports are mutually disjoint, and disjoint from the support of $\gamma_{i,j}$, r varying in $\mathbb{N}$ (see Figure 3).

If $i = j$ we define

$$N_{i,i}(t) = \sum (-1)^r v(\gamma_1)(t) \ldots v(\gamma_r)(t), \tag{2.3}$$

where the summation is extended over all the configurations of r simple cycles $\gamma_1, \ldots, \gamma_r$ whose supports are mutually disjoint and do not contain i, r varying in $\mathbb{N}$ (that is, $N_{i,i}(t)$ has the same definition as $D(t)$ if we replace V by $V - \{i\}$).

The key result is as follows.

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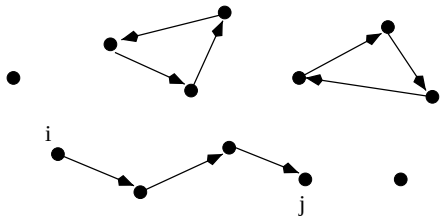


Figure 3 A configuration for $N_{i,j}(t)$

Lemma 2.1. *We have*

(i) $D(t) = \det(Id - tM)$

(ii) $(Id - tM)^{-1}_{i,j} = \dfrac{N_{i,j}(t)}{D(t)}$.

Proof. **(i)** By definition of the determinant we have

$$\det(Id - tM) = \sum_{\sigma} \epsilon(\sigma) w(\gamma_1)(t) \dots w(\gamma_r)(t),$$

where the summation is extended over all the permutations σ of V, $\epsilon(\sigma)$ is the signature of σ, $(\gamma_1 \dots \gamma_r)$ is the decomposition of σ into disjoint cycles, and finally $w(\gamma)(t) = -v(\gamma)(t)$ if γ is a nontrivial cycle, and $w(\gamma)(t) = 1 - p(a,a)t$ if the cycle γ is reduced to the vertex $a \in V$. Developing each term and remarking that $\epsilon(\sigma) = (-1)^{|V|-r}$ gives the result.

(ii) We remark by a similar computation that

$$N_{i,j}(t) = \mathrm{cof}_{i,j}(Id - tM). \tag{2.4}$$

Hence (ii) follows from (i), (2.3) and the Cramer formula. □

Remark. There exists a direct proof of (ii) by means of a combinatorial transformation over the configurations of cycles (Viennot, private communication).

As a consequence, if $(S_n)_{n \geq 0}$ is a homogeneous Markov chain on a countable state space E and V is a finite subset of E, for every directed edge (v, v'), set $p(v, v') = p_{v,v'}$, where $p_{v,v'}$ denotes the transition probability from v to v'. The matrix M defined above is the transition matrix restricted to V. Let i, j be two vertices in V, and let S' denote the Markov chain killed at the first exit time from V. Let $\mathbb{P}_i$ denote the law of S' started at i. It is well known that, for $0 \leq t < 1$,

$$(Id - tM)^{-1}_{i,j} = \sum_{n=0}^{\infty} t^n M^n_{i,j} = \sum_{n=0}^{\infty} \mathbb{P}_i(S'_n = j) t^n.$$

Thus Lemma 2.1 can be restated in the following form.

Corollary 2.2. *The resolvent of the killed Markov chain S' is given by*

$$\sum_{n=0}^{\infty} \mathbb{P}_i(S'_n = j) t^n = \frac{N_{i,j}(t)}{D(t)}.$$ □

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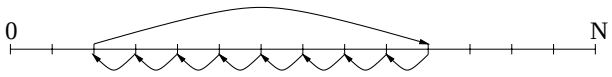


Figure 4 A simple cycle

Remark. The formula above can be viewed as an identity relating formal series. It remains valid as an identity between power series for $|t| < \rho$, ρ being the radius of convergence. In our setting, as the series represent rational functions, ρ is exactly the smallest modulus of a root of D. It is obvious that $\rho \geq 1$, since the coefficients of the power series are real numbers in $[0, 1]$.

3. Application to left-continuous random walks

3.1. Transition probabilities and exit time

The application of the preceding results to left-continuous random walks is particularly simple, since in this situation all the simple cycles with nonzero valuation are equivalent to $(i, j), (j, j-1), \ldots, (i+1, i)$ with $j \geq i$. See Figure 4.

Let $N > 0$ be an integer and set $I_N = \{0, 1, \ldots, N\}$. The transition probabilities will be denoted by $p_i := p_{0,i}$. We shall use the notation M_N for the transition matrix restricted to I_N and $D_N(t) = \det(Id - tM_N)$. For general references on Toeplitz matrices see [16, Chapter 5].

We first give a recurrence formula for $D_N(t)$ that enables us to characterize this quantity in terms of the law of S. Let G be the generating function of the one-step distribution:

$$G(t) = \mathbb{E}_0(t^{S_1}) = \sum_{n=-1}^{\infty} p_n t^n.$$

Proposition 3.1. *Put $D_{-1}(t) = 1$. Then the following recurrence formula holds:*

$$D_N(t) = D_{N-1}(t) - \sum_{n=0}^{N} p_n t (p_{-1} t)^n D_{N-n-1}(t). \tag{3.1}$$

As a consequence, for all $t, r \in (0, 1)$,

$$\sum_{n=-1}^{\infty} D_n(t) r^n = \frac{1}{r^2 [tG(p_{-1} tr) - 1]}. \tag{3.2}$$

Proof. Formula (3.1) follows immediately from (2.1) by decomposing the sum according to the (possibly empty) cycle containing 0. Rewrite it in the following manner:

$$D_{N-1}(t) = \sum_{n=-1}^{N} D_n(t) p_{N-n-1} t (p_{-1} t)^{N-n-1}.$$

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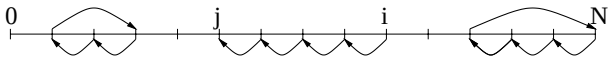


Figure 5 A configuration for $N_{i,j}$, $i \geq j$

Then

$$\begin{aligned} K(t,r) &:= \sum_{n=-1}^{\infty} D_n(t) r^n \\ &= \sum_{n=-1}^{\infty} \left(\sum_{m=-1}^{m} D_m(t) p_{n-m} t (p_{-1}t)^{n-m} \right) r^n \\ &= t \left(K(t,r) G(p_{-1}tr) - \frac{p_{-1} D_{-1}(t)}{p_{-1} t r^2} \right) \\ &= tK(t,r) G(p_{-1}tr) - 1/r^2. \end{aligned}$$

□

Remark. The function $W(n) = D_{n-1}(1)$ is known as the scale function and plays an important role in fluctuation theory. Specializing the results of Proposition 3.1 for $t = 1$, we get $W(0) = 1$,

$$W(n+1) = W(n) - \sum_{k=0}^{n} p_k (p_{-1})^k W(n-k),$$

and, for every $r \in (0,1)$,

$$\sum_{n=0}^{\infty} W(n) r^n = \frac{1}{r^2 [G(p_{-1}r) - 1]}.$$

This corresponds to formulas (3) and (8) on page 19 of [18]. The scale function enables us to determine the probability that the first exit from I_N occurs at the lower boundary (see Proposition 3.3 below). The advantage of introducing the variable t is that we obtain simultaneously the law of the exit time. Our method also allows us to view these quantities as determinants, which does not seem to have been observed before. This will be a key point for the discussion of the ergodic properties in Section 3.2.

Next we restate the $N_{i,j}$s in terms of the D_ns. The result is as follows.

Proposition 3.2. *Let $i, j \in I_N$. Then*

$$N_{i,j}(t) = (p_{-1}t)^{i-j} \left(D_{N-i-1}(t) D_{j-1}(t) - 1_{(i<j)} D_N(t) D_{j-i-2}(t) \right).$$

Proof. If $i \geq j$, then a self-avoiding path from i to j necessarily has the form $(i, i-1), \ldots, (j+1, j)$ and its valuation is $(p_{-1}t)^{i-j}$. For each configuration contributing to $N_{i,j}$, the supports of the simple cycles are either in $[0, j-1]$ or in $[i+1, N]$. See Figure 5.

Hence

$$N_{i,j}(t) = (p_{-1}t)^{i-j} D_{N-i-1}(t) D_{j-1}(t).$$

The study of the case $i < j$ involves a combinatorial transformation of the configurations of cycles. We refer to [8, Chapter 2] for the proof. □

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The expression of $N_{i,j}$ and D_N, together with Corollary 2.2, gives the resolvents of the killed random walk. Analogous results for spectrally positive Lévy processes were derived analytically by Suprun [17]. See also [1] for a probabilistic approach.

The analysis of the two-sided exit problem follows easily from the computation of the resolvents. Let T_N denote the first exit time from I_N, $T_N = \inf\{n \in \mathbb{N},\ S_n \notin I_N\}$.

Proposition 3.3. *Let the random walk S start at $i \in I_N$. Then we have, for every $t \in (0,1)$,*

$$\mathbb{E}_i(t^{T_N},\ S_{T_N} = -1) = (p_{-1}t)^{i+1}\frac{D_{N-i-1}(t)}{D_N(t)}. \tag{3.3}$$

If k is an integer greater than N, then, for every $t \in (0,1)$,

$$\mathbb{E}_i(t^{T_N},\ S_{T_N-1} = j, S_{T_N} = k) = p_{k-j}t\frac{N_{i,j}(t)}{D_N(t)}.$$

Proof. If the random walk starts at i and exits from I_N at -1, then its path up to the T_N can be decomposed into a path on I_N from i to 0 followed by a negative jump. By Proposition 3.2 we have

$$N_{i,0}(t) = (p_{-1}t)^i D_{N-i-1}(t),$$

which yields the result.

For the case of an exit at the upper boundary, we simply remark that, if $S_{T_N} = k > n$, then the path up to T_N can be decomposed into a path on I_N from S_0 to $j \in I_N$, followed by a jump from j to k. □

Remark. Formula (3.1) could be deduced from (3.3) by decomposing a path of the killed random walk starting at 0, up to the first hitting time of -1, into its first jump of size, say i, followed by $i+1$ paths starting at k up to the first hitting time of $k-1$, $0 \le k \le i$. This kind of path decomposition, applied to the non-killed random walk, is used by Takács to derive an analogue of (3.2) in the case when $t = 1$ ([18, Chapter 2.7]), and also constitutes the basis of the link with the Lagrange expansion formula (see Raney [12] or Wendel [22]).

Notice that if $i = N$ then $N_{N,j}(t) = (p_{-1}t)^{N-j}D_{j-1}(t)$ according to Proposition 3.2, and

$$\mathbb{E}_N(t^{T_N},\ S_{T_N-1} = j,\ S_{T_N} = k) = p_{k-j}t(p_{-1}t)^{N-j}\frac{D_{j-1}(t)}{D_N(t)}.$$

Hence, if T_+ denotes the first hitting time of $[1,\infty)$ of the random walk started at 0, and if I stands for the infimum process $I_n = \inf\{S_k,\ 0 \le k \le n\}$, then, for all integers $N, k > 0$ and $j \in [0, N]$, and for every real $t \in (0,1)$,

$$\begin{aligned}\mathbb{E}_0(t^{T_+},\ S_{T_+-1} = -j,\ S_{T_+} = k,\ I_{T_+} \ge -N)\\ = \quad p_{k+j}t(p_{-1}t)^j\frac{D_{N-j-1}(t)}{D_N(t)}.\end{aligned}$$

This is to be compared with a result of Prabhu [11] describing the excursions away from 0 of a spectrally positive Lévy process.

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3.2. Asymptotic properties

A classical consequence of the representation of the resolvent densities as rational series is the study of ergodic properties of the random walk killed as it exits from I_N (we shall denote it by S' as in Section 2).

Suppose for simplicity that the matrix M_N is irreducible and aperiodic. Then by the Perron–Frobenius theorem there exists a real number $\lambda > 0$ that is a simple root of D_N and has the smallest modulus among all the roots of D_N. As $D_N(t) = \det(Id - tM_N)$, we see that $1/\lambda$ is the largest eigenvalue of M_N and it is clear that $\lambda \geq 1$. We shall let μ denote the second-smallest modulus of a root of D_N.

The results can be summarized as follows.

Proposition 3.4.

(i) *$\lambda^n M_N^n$ converges as n tends to infinity to a matrix $P = \hat{U}^t U$, where U and $\hat{U}$ are right and left eigenvectors of M_N with eigenvalue $1/\lambda$.*

(ii) *More precisely, U, the invariant function of P, is proportional to*

$$(N_{i,0}(\lambda),\ 0 \leq i \leq N) = ((p_{-1}\lambda)^i D_{N-i-1}(\lambda),\ 0 \leq i \leq N),$$

whereas $\hat{U}$, the invariant measure, is proportional to

$$(N_{N,j}(\lambda),\ 0 \leq i \leq N) = ((p_{-1}\lambda)^{N-i-1} D_i(\lambda),\ 0 \leq i \leq N).$$

(iii) *The rate of convergence to P is given by*

$$\frac{\log(||P - \lambda^n M_N^n||)}{n \log \mu} \to 1$$

as n tends to infinity. If μ corresponds to a unique simple root, then

$$||P - \lambda^n M_N^n|| \sim c\mu^n.$$

Proof. Assertion (i) is a classical result of elementary ergodic theory (see, for instance, Seneta [13, Chapter 1]).

To make use of Corollary 2.2, note that, if $\lambda_1, \ldots, \lambda_k$ are the (complex) roots of the polynomial D_N (or, equivalently, the inverse of the eigenvalues of M_N) with respective orders $r_1, \ldots, r_k$, then, by classical results on rational series, the transition probabilities of the killed Markov chain have the form

$$p'_n(i,j) := \mathbb{P}_i(S'_n = j) = \sum_{m=1}^{k} Q_{m,i,j}(n)\lambda_m^{-n}, \tag{3.4}$$

where $Q_{m,i,j}$ is a polynomial of degree r_m. According to the definition of λ, it is straightforward that

$$p'_n(i,j) \sim c_{i,j}\lambda^{-n}.$$

Moreover, decomposing the ratio in (3.4) yields

$$c_{i,j} = \lim_{t \to \lambda} (1 - t/\lambda) N_{i,j}(t)/D_N(t),$$

and (ii) follows by taking $j = 0$ and $i = N$.

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Notice that the invariant measure is the invariant function for the dual random walk $-S$ and *vice versa*. These results are to be compared with Theorem 2 in [1].

Finally, the study of the rate of convergence is an easy consequence of (3.4). □

Remark. In the case of a spectrally positive Lévy process X, there exists a function $W : (q, x) \in \mathbb{R} \times \mathbb{R}_+ \mapsto W^q(x)$. which is the analogue of a determinant. More precisely, letting $T(x)$ denote the exit time from the interval $[0, x]$, we have

$$\mathbb{E}_y \left(\exp(-qT(x)),\ X_{T(x)} = 0\right) = \frac{W^q(y)}{W^q(x)},$$

which is the analogue of (3.3), where W plays the role of D, q plays the role of t and x plays the role of N. In particular, the semigroup of the process killed at $T(x)$ decays exponentially with rate ρ, where ρ is the first negative solution of the equation $W^\rho(x) = 0$: see [1]. However, if one works directly in the continuous setting, the interpretation of W as a determinant is not obvious (actually, W is defined by a formula analogous to (6) in [1]). In turn, using our approach , it is obvious that the numerator in the right-hand side of (3.3) is a determinant, since it can be expressed by a sum on configurations of cycles.

4. The simple random walk

4.1. Asymptotic computations

We suppose in this section that S is a simple random walk, that is, $p_1 + p_{-1} = 1$ and $\beta := \sqrt{p_1 p_{-1}} \neq 0$. In that case formula (3.1) leads to

$$D_{N+2}(t) = D_{N+1}(t) - \beta^2 t^2 D_N(t),$$

and $D_1(t) = D_0(t) = 1$.

Recall that the Chebyshev polynomials of the second kind are defined by $U_0(t) = 1$, $U_1(t) = 2t$ and

$$U_{N+2}(t) = 2tU_{N+1}(t) - U_N(t).$$

See, for instance, [4, p. 45]. A characterization of these polynomials is

$$U_N(\cos x) = \frac{\sin(N+1)x}{\sin x},$$

for any real number x and any positive integer N. Comparing the two recursion formulas we get the following result.

Proposition 4.1. *For a simple random walk with $p_1 = p$, $p_{-1} = q = 1 - p$, the polynomials $D_N(t)$ defined in* (3.1) *can be restated as*

$$D_N(t) = (\beta t)^{N+1} U_{N+1}(1/2\beta t),$$

where $\beta = \sqrt{pq}$ and U_N is the Nth Chebyshev polynomial of the second kind. □

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The roots of D_N are the real numbers of the form

$$\frac{1}{2\beta\cos(k\pi/(N+2))},$$

with $k \in \mathbb{N} - \{0\}$ and $-(N+2)/2 < k < (N+2)/2$. Note that the transition matrix of the random walk is periodic, which explains why there exist two different roots with the smallest modulus.

We now specialize to the symmetric case where $2\beta = 1$. According to Donsker's theorem, $(n^{-1}S_{n^2 s})_{s\geq 0}$ converges in distribution to a linear Brownian motion as n tends to infinity, so it is natural to fix two positive integers a and b, to replace i by an, N by $(a+b)n$ and t by e^{λ/n^2} in Proposition 3.3 and to let n tend to infinity. Recall that $\arccos(1-\epsilon) \sim \sqrt{2\epsilon}$ for small $\epsilon > 0$. Moreover, there exists a positive λ_0 such that, for every positive $\lambda < \lambda_0$ and every positive integer n, $e^{-\lambda/n^2} < \cos(\pi/(a+b)n)$. For such λ we get

$$\begin{aligned}
\mathbb{E}_{an}\left(\exp\left(\lambda\frac{T_{(a+b)n}}{n^2}\right),\ S_{T_{(a+b)n}} = -1\right) &= \frac{U_{bn+1}(e^{-\lambda/n^2})}{U_{(a+b)n+1}(e^{-\lambda/n^2})} \\
&= \frac{\sin((bn+2)\arccos(e^{-\lambda/n^2}))}{\sin((an+bn+2)\arccos(e^{-\lambda/n^2}))} \\
&\to \frac{\sin(b\sqrt{2\lambda})}{\sin((a+b)\sqrt{2\lambda})}.
\end{aligned}$$

By analytic continuation, we recover for $\lambda < 0$ a well-known formula for Brownian motion (see, for example, Karatzas and Shreve [7, p. 100]).

4.2. The range process

We go back to the case of a simple, non-perturbed random walk S on $\mathbb{Z}$ starting at 0 with $p_1 = p$ and $p_{-1} = q = 1-p$. Define the range as the difference between the maximum and the minimum:

$$R_n = \max\{X_i, i \leq n\} - \min\{X_i, i \leq n\}.$$

Put $\tau_a = \min\{n, R_n = a\}$ for $a \in \mathbb{N}$. Vallois [19, 20] computed the distribution of τ_N and X_{τ_N} separately. Our combinatorial method enables us to get the joint law. Indeed, if $S_{\tau_N} = k$ let τ_N^k be the first hitting time of the minimum before τ_a, that is, $\tau_N^k = \inf\{n, S_n = k - N\}$. Applying the strong Markov property at τ_N^k and the results of Section 4.1 yield the following result.

Proposition 4.2. *The joint law of* (τ_N, S_{τ_N}) *is given by*

$$\mathbb{E}_0(t^{\tau_N}1_{\{S_{\tau_N}=k\}}) = \frac{(qt)^{N-k}(pt)^N D_{k-2}(t)}{D_{N-2}(t)D_{N-1}(t)}$$

for $k > 0$, *the polynomials* D_n *being given by Proposition 4.1.* □

If k is negative it suffices to apply the proposition to the dual random walk $-X$. Summing on k we recover the law of τ_N as in [19], and putting $t = 1$ we get the

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probability that $S_{\tau_N} = k$. In particular, when $p = q = 1/2$ one has the simple formula

$$P(S_{\tau_N} = k) = |k|/N(N+1) \tag{4.1}$$

for any integer $k \in [-N, N]$.

4.3. Birth and death processes

The study of birth and death chains can be carried out using the same ideas as for simple random walks on the integers. Let X be a Markov chain on the positive integers with transition probabilities $p_i = p_{i,i+1}$ and $q_i = p_{i,i-1}$, all the other transition probabilities being zero. Let $X^{(N)}$ denote the Markov chain killed as it exits from I_N. Define $D_k(t)$ by formula (2.1) if we restrict the set of vertices to the subset $[N-k, N]$. The same argument as in Proposition 3.1 entails $D_{-1}(t) = D_0(t) = 1$ and

$$D_{k+1}(t) = D_k(t) - p_{N-k-1}q_{N-k}t^2 D_{k-1}(t).$$

Write $E_k(t) = D_{k+1}(t)/D_k(t)$. Then the previous formula becomes

$$E_k(t) = 1 - \frac{p_{N-k-1}q_{N-k}t^2}{E_{k-1}(t)}.$$

Finally, remarking that $N_{0,0}(t)/D(t) = 1/E_{N-1}(t)$, we get the following expression for the resolvent density of the return times to 0 for $X^{(N)}$:

$$\sum_{n=0}^{\infty} \mathbb{P}_0(X_n^{(N)} = 0)t^n = \cfrac{1}{1 - \cfrac{p_0q_1t^2}{1 - \cfrac{1}{1 - \cfrac{\cdots}{1 - p_{N-1}q_Nt^2}}}}.$$

For fixed $t \in [0, 1)$, both sides of the equality converge as N tends to infinity and we conclude the following.

Proposition 4.3 (Good). *The resolvent density of the return times to 0 is given by the following continued fraction, valid for every $t \in [0, 1)$:*

$$\sum_{n=0}^{\infty} \mathbb{P}_0(X_n = 0)t^n = \cfrac{1}{1 - \cfrac{p_0q_1t^2}{1 - \cfrac{p_1q_2t^2}{\cdots}}}.$$

□

The original result can be found in [6]. See also [4] for some related results.

5. Random walks on the circle

Another application of Lemma 2.1 is the determination of the law of additive functionals. Recall that we are considering a random walk X on a finite set V of vertices. Let

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$f\colon V \times V \to \mathbb{R}_+$ be a positive function on the directed edges, and define the additive functional A^f by

$$A_n^f = \sum_{i=1}^{n} f(X_i, X_{i-1}).$$

Lemma 2.1 enables us to make explicit the law of A^f along the bridges of X. More precisely, fix a real $\lambda > 0$, put on each directed edge (i, j) the valuation $p_{i,j}t\exp(-\lambda f(i, j))$ and define the polynomials $D^{\lambda f}(t)$ and $N_{i,j}^{\lambda f}(t)$ associated to these valuations as in (2.1), (2.2) and (2.3). Then we have the following version of the Feynman–Kac formula as a straightforward consequence of Lemma 2.1.

Corollary 5.1. *The Laplace transform of A^f is given by*

$$\sum_{n=0}^{\infty} \mathbb{E}_i\left(\exp(-\lambda A_n^f)t^n 1_{\{X_n=j\}}\right) = \frac{N_{i,j}^{\lambda f}(t)}{D^{\lambda f}(t)}. \qquad \square$$

Note that, if $g\colon V \to \mathbb{R}_+$ is a positive function on the set of vertices, and if $A_n^g = \sum_{i=1}^{n} g(X_i)$ is its associated additive functional, the Laplace transform of A^g can be computed by putting, for all $i, j \in V$, $f(i, j) = g(j)$. Likewise, if f is symmetric, which will be the case from now on, Corollary 5.1 characterizes the law of the number of passage times on the non-directed edges.

As an application, let $V = \{0, 1, \ldots, N-1\}$ be a set of N points on the circle, and consider the nearest-neighbour random walk S on V, starting at 0, where the probability of going clockwise is p and the probability of going anticlockwise is $q = 1 - p$. Let T denote the first time at which this random walk returns to 0 after a complete loop. Pitman proved that the distribution of the number of visits of the points of V, from time 1 to time T, is invariant under rotation.

The first result in that direction was established for Brownian motion [9], the main tool being a decomposition of the Brownian path, together with some Ray–Knight theorems for the different subpaths arising from the decomposition. The result for random walks follows easily: see [5].

Our combinatorial method provides a direct approach. More precisely, applying Corollary 5.1 yields the following.

Proposition 5.2. *Let L^i, $1 \le i \le N$ be the number of times the random walk S goes on the non-directed edge $(i-1, i)$ before time T. Then the law of $(L^i,\ 1 \le i \le N)$ is given by*

$$\mathbb{E}\left(\prod_{i=1}^{N} t_i^{L^i}\right) = \frac{(p^N + q^N)\prod_{i=1}^{N} t_i}{D} \tag{5.1}$$

for all $t_1, t_2, \ldots, t_N \in (0, 1)$, where the quantity D is given by

$$D = 1 + \sum_{n=1}^{N}(-1)^n \sum_{\substack{1 \le a_1, \ldots, a_n \le N \\ a_{i+1} - a_i \ge 2 \\ a_1 - a_n + N \ge 2}} \prod_{i=1}^{n} pqt_{a_i}^2. \tag{5.2}$$

$\square$

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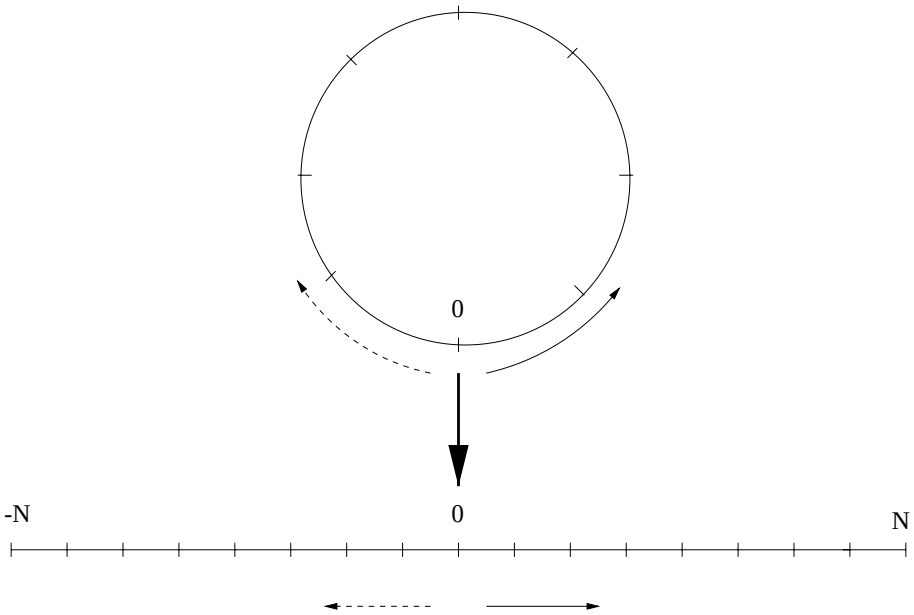


Figure 6

The result of invariance by rotation follows easily.

Corollary 5.3 (Pitman). *The law of the process* $(L_i,\ 1 \le i \le N)$ *is invariant by rotation. More precisely, for every* $k \in \mathbb{Z}$,

$$(L_1, \ldots, L_N) \overset{(d)}{=} (L_{k+1}, \ldots, L_{k+N}),$$

where $j + k$ *means* $j + k \mod N$.

Similarly, if L'_i *denotes the number of visits of the vertex* i *before time* $T - 1$, *then, for every* $k \in \mathbb{Z}$,

$$(L'_0, L'_1, \ldots, L'_{N-1}) \overset{(d)}{=} (L'_k, L'_{k+1}, \ldots, L'_{k+N-1}).$$

Proof. The study of S up to T reduces to the study of a random walk S' on a linear graph H formed by the points $-N, -(N-1), \ldots, N-1, N$ and the directed edges $(-(N-1), -N), (-(N-1), -(N-2)), (-(N-2), -(N-1)), \ldots, (N-2, N-1), (N-1, N-2), (N-1, N)$ with transition probabilities $p_{i,i+1} = p$ and $p_{i,i-1} = q$ for $-(N-1) \le i \le N-1$, all the other transition probabilities being zero. Indeed, a point i on H represents the point $i \mod N$ on V, and the fact that the transition probabilities from N and $-N$ to any other point is zero means that we kill the random walk S' after the first visit of $\{-N, N\}$ (which corresponds to a loop of the original Markov chain S). See Figure 6.

Our problem reduces to the study of the paths of S' from 0 to N and from 0 to $-N$. Fix $\lambda > 0$ and put, for every $i \in \mathbb{Z}$,

$$f(i, i+1) = f(i+1, i) = -\ln(t_i)/\lambda.$$

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Here again, i means i mod N. Applying Corollary 5.1 with $t = 1$ and the function f we get

$$\mathbb{E}\left(\prod_{i=1}^{N} t_i^{L^i}\right) = \frac{(p^N \prod_{i=1}^N t_i)D_1 + (q^N \prod_{i=1}^N t_i)D_2}{B},$$

where B denotes the determinant associated to the graph H and D_1 (resp. D_2) stands for the determinant associated to the subgraph of H containing the points $-(N-1), \ldots, -1$ (resp. $1, \ldots, N-1$) and the corresponding edges.

It is easy to check that $D_1 = D_2$, since the corresponding valuated graphs are identical up to translation. Moreover, determining whether or not a configuration for B contains the cycle $(0,-1),(-1,0)$ (with valuation pqt_1^2), we have

$$B = D_1 C - pqt_1^2 D_2 C',$$

where C (resp. C') stands for the determinant associated to the subgraph of H containing the points $0, \ldots, N-1$ (resp. $-(N-1), \ldots, -2$) and the corresponding edges.

Finally, the quantity $C - pqt_1^2 C'$ can be restated as in (5.2) (simply write $D = D' + D''t_1^2$, where D and D' do not depend on the variable t_1, and verify that $D' = C$ and $D'' = -pqC'$). This completes the proof of Proposition 5.2.

Notice that formulas (5.1) and (5.2) are the discrete equivalents of (35) and (34) in [9].

It is easy to check that D remains invariant if one replaces t_i by t_{i+k}, $k \in \mathbb{Z}$. Indeed, D is the sum of the valuations of the configurations of cycles on V, where we exclude the two configurations with one cycle over all the vertices (that is, we only consider the configurations with cycles of the form (i,i+1),(I+1,i)). This proves the first part of Corollary 5.3.

On the other hand, it is obvious that L^i necessarily has the form $2n+1$ for some integer n. Furthermore, the number of visits L_i' of a point i between times 0 and $T-1$ is equal to $(L^i + L^{i+1})/2$ and the second part of the corollary follows. □

Other interesting problems for Brownian motion on the circle are related to the cover time C, which is defined as the first time the Brownian motion has visited all the circle. In the discrete setting, one could define it as the first time S has visited all the vertices, or the first time it has gone on all the edges. In both cases, the study of the law of (C, S_C) reduces to Proposition 4.3. In particular, we deduce the following from (5.2).

Corollary 5.4. *Suppose that S is symmetric, that is, $p = q = 1/2$. Define the cover time C to be the first time at which S has visited all the vertices. Then*

$$P(S_C = i) = 1/(N-1)$$

for every integer $i \in [1, N-1]$.

If C' denotes the first time at which S has gone on all the edges, then

$$P(S_{C'} = 0) = 2/(N+1)$$

and

$$P(S_{C'} = i) = 1/(N+1)$$

for $i \neq 0$. □

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Recall that, in the Brownian case, the cover point is uniformly distributed on the circle [9]. For further discussions on the properties of the cover time, we refer to [5] and [9].

Finally, we remark that specializing the variables $t_1, \ldots, t_N$ in Proposition 5.2 enables us to recover the marginals of the joint law of $(L_1, \ldots, L_N)$. Furthermore, one can get asymptotic results related to the convergence in distribution to Brownian motion. To this end, the use of the properties of Chebyshev polynomials, as in Section 4, turns out to be very useful.

To begin with a simple example, let us determine by our method the law of the number of passages on one edge. This is done by putting, say, $t_1 = t$ and $t_i = 1$ for $i \neq 1$ in (5.1). Recall that D can be defined as sum of valuations over certain configurations on V, as in the proof of Corollary 5.4. Considering these configurations one has to determine whether they contain the cycle $(0,1),(1,0)$, and this yields

$$\mathbb{E}(t^{L^1}) = \frac{(p^N + q^N)t}{\beta^N [U_N(a) - t^2 U_{N-2}(a)]}$$

with $a = 1/2\beta$. Hence $L^1 \overset{(d)}{=} 1 + 2Y$ where Y has a geometric law.

In the symmetric case, that is, when $p = q = 1/2$, setting $t = e^{-\lambda/N^2}$ and letting N tend to infinity, we see that the quantity above converges to $1/1 + \lambda$.

Of course, the preceding result could be derived by elementary martingale theory. However, our method applies to more general problems that could not be handled using martingales. Turning our attention to 2 non-adjacent edges, let us take $3 \leq m \leq N - 1$ and put $n = N - m$. Then

$$\mathbb{E}(s^{L^1} t^{L^m}) = \frac{(p^N + q^N)st}{\beta^N A_m}$$

with

$$A_m = U_m(a)U_n(a) - U_{m-1}(a)U_{n-1}(a)(s^2 + t^2) + U_{m-2}(a)U_{n-2}(a)s^2t^2.$$

In the symmetric case, fix a real $k \in (0,1)$ and set $m = [kN]$ ($[kN]$ standing for the integer part of kN), $s = e^{-\lambda/N}$, $t = e^{-\rho/N}$ and let N tend to infinity. Then the quantity above converges to

$$\frac{1}{1 + \lambda + \rho + 2k(1-k)\lambda\rho}.$$

This is the same as (4) and (5) in [9], for a Brownian motion with zero drift.

Other calculations are possible using the properties of Chebyshev polynomials.

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