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# Analytic formulation of the 6-3 fully-parallel manipulator's singularity determination

Raffaele Di Gregorio

*Department of Engineering, University of Ferrara, Via Saragat, 1; 44100 FERRARA (Italy)*

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**SUMMARY**

When a parallel manipulator reaches a singular configuration (singularity), the end effect (platform) pose cannot be controlled any longer, and infinite active forces must be applied in the actuated joints to balance the loads exerted on the platform. Therefore, these singularities must be avoided during motion. The first step to avoid them is to locate all the platform poses (singularity locus) making the manipulator singular. Hence, the availability of a singularity locus equation, explicitly relating the manipulator geometric parameters to the singular platform poses, greatly facilitates the design process of the manipulator. The problem of determining the platform poses, that make the 6-3 fully-parallel manipulator (6-3 FPM) singular, will be addressed. A simple singularity condition will be written. This singularity condition consists in equating to zero the mixed product of three vectors, that are easy to be identified on the manipulator, and it is geometrically interpretable. The presented singularity condition will be transformed into an equation (singularity locus equation) explicitly containing the geometric parameters of the manipulator and the platform pose parameters making the 6-3 FPM singular. Eventually, the singularity locus equation will be reduced to a polynomial equation by using the Rodrigues parameters to parameterize the platform orientation. This polynomial equation is cubic in the platform position parameters and one of sixth degree in the Rodrigues parameters.

KEYWORDS: Parallel manipulators; Kinematics; Mobility analysis; Singularity locus

## 1. INTRODUCTION

Reduced workspace, high stiffness and positioning precision are required by many industrial applications of the manipulators. These requirements are matched by parallel architectures much more easily than by the serial ones when a specific task manipulator is designed.

Parallel manipulators (PM) feature the end-effector (platform) connected to the frame (base) by a number of kinematic chains (legs). A wide family of PMs is the one collecting PMs with platform and base connected to one another by six legs either of type SPS (S and P stand for spherical pair and prismatic pair respectively) or equivalent to the SPS ones from a kinematic point of view, for example, legs of type UPS (U stands for universal joint). The manipulators of this family are named fully-parallel manipulators (FPM). FPMs have six degrees of freedom (dof). Their active joints are the prismatic pairs and their active joint coordinates are the distances (leg lengths) between the spherical pair centers of each leg.

FPMs' architectures are classified[1] according to the number of distinct spherical pairs joining the legs to the base and to the platform. For example, a FPM having m distinct spherical pairs in the base and n distinct spherical pairs in the platform is named m-n FPM. The most general FPM architecture is the one of the 6-6 FPM (Fig. 1), featuring six distinct spherical pairs both in the base and in the platform. All the other architectures can be derived from the 6-6 FPM's one by making two or more spherical pairs coincide in the base and/or in the platform. Figure 2 shows the 6-3 FPM (Fig. 2). The 6-3 FPM features six spherical pairs in the base and three double spherical pairs in the platform. The 6-3 FPM architecture has been proposed by Stewart.[2]

The kinematic analysis of a manipulator consists of three steps: position analysis, velocity analysis and acceleration analysis. In the FPMs, the position analysis is the determination of the relationship between the leg lengths and the platform poses (positions and orientations). The velocity analysis is the determination of the relationship between the first-order time derivatives of the leg lengths and the velocities of the platform points. The acceleration analysis is the determination of the relationship between the second-order time derivatives of the leg lengths and the accelerations of the platform points. Each of these analyses consist of two problem: the direct one and the inverse one.

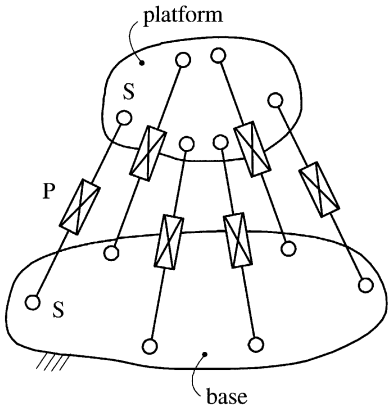



Fig. 1. The 6-6 fully-parallel manipulator.

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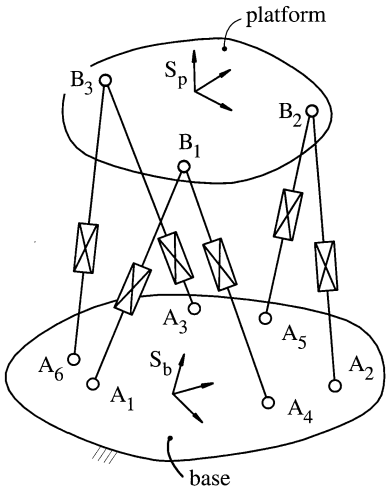



Fig. 2. The 6-3 fully-parallel manipulator.

The direct problem is the computation of the parameters (either position or velocity or acceleration) in the platform's space (Cartesian space) once the parameters (leg lengths and their time derivatives) in the legs' space (joint space) are assigned. The inverse problem is the computation of the joint space parameters once the Cartesian space parameters are given.

The inverse position analysis of the FPMs is straightforward. On the contrary, the direct position analysis of the FPMs is difficult to be solved since it involves the solution of nonlinear equation systems.

The direct position analysis of the 6-3 FPM has been solved by Innocenti and Parenti-Castelli.[3] Innocenti and Parenti-Castelli demonstrated that at most sixteen platform poses correspond to a given set of leg lengths, since they reduced the solution of the 6-3 FPM closure equation system to the solution of a sixteen degree univariate polynomial equation. Then, Parenti-Castelli and Di Gregorio[4] demonstrated that the 6-3 FPM platform pose is uniquely determined when the value of one passive joint coordinate is measured besides the six leg lengths.

On the other side, the velocity and acceleration analyses involve the solution of linear systems. The linear systems, to be solved in the velocity analysis, have the same coefficient matrices of the unknowns as the ones, to be solved in the acceleration analysis. These coefficient matrices (Jacobian matrices) depend on the manipulator configuration. The Jacobian matrices are two[5]: one involved in the solution of the direct problem and the other involved in the solution of the inverse problem. If one or both of the Jacobian matrices are singular the solution of the related linear systems can be either impossible or undetermined.

When the solution is impossible, in the direct problem, the assigned joint parameter values, in the inverse problem, the assigned Cartesian space parameter values cannot be reached by the manipulator.

When the solution is undetermined, in the direct problem, the platform can be moved even if the leg lengths are constant; in the inverse problem, one or more than one leg length can change even if the platform pose is locked.The manipulator configurations making one or both of the Jacobian matrices singular are called singularities. Gosselin and Angeles[5] have classified the parallel mechanisms' singularities.

In the FPMs, the inverse problem singularities occur when at least one leg length is zero.[6] This condition is easy to be identified and is not reachable in practice because of the legs' finite sizes.

On the contrary, the direct problem singularities are not easy to be identified. Moreover, they can be located inside the manipulator's workspace. When the manipulator reaches one of these singularities, the platform pose cannot be controlled any longer, and infinite active forces must be applied in the prismatic pairs to balance the loads exerted on the platform. Therefore, these singularities must be avoided during motion. The first step to avoid them is to locate all the platform poses (singularity locus) making the FPM singular. Hence, the availability of a singularity locus equation, explicitly relating the FPM geometric parameters to the singular platform poses, greatly facilitates the FPM design process. Such equations have been presented for some parallel manipulators.[6–10]

Merlet[11] has characterized the 6-3 FPMs' singularities by using geometric concepts (Grassmann geometry). The use of geometric conditions in the three-dimensional space is difficult to be employed during the 6-3 FPM design.

This paper presents the 6-3 FPM's singularity locus equation explicitly relating the 6-3 FPM geometric parameters to the singular platform poses.

Moreover, a scalar relationship among vectors, that is satisfied by the singularities and can be geometrically interpreted, will be written. The vectors used in this relationship are easy to be located on the manipulator.

Eventually, the use of the Rodrigues parameters to parameterize the platform orientation will allow the singularity locus equation to be transformed into a polynomial equation, which is cubic in the platform position parameters and one of sixth degree in the Rodrigues parameters.

## 2. SINGULARITY CONDITION OF THE 6-3 FPM

Figure 2 shows a 6-3 FPM. With reference to Fig. 2, the $B_i$, $i=1, 2, 3$, points are the platform's spherical pair centers. The $A_j$, $j=1, \ldots, 6$, points are the base's spherical pair centers. The $S_b$ and $S_p$ reference systems are fixed to the base and the platform, respectively.

When the prismatic pairs are locked, the 6-3 FPM becomes an isostatic structure (6-3 structure) (Fig. 3). If the configuration assumed by the 6-3 FPM is singular, the platform of the 6-3 structure obtained after having locked the prismatic pairs can accomplish either finite motions (singularity in the large) or infinitesimal motions (singularity in the small). In the 6-3 structure (Fig. 3), the $B_i$, $i=1, 2, 3$, platform points are constrained to belong to the $A_iA_{i+3}B_i$, $i=1, 2, 3$, triangles respectively. Moreover, each $A_iA_{i+3}B_i$ triangle can rotate around the axis passing through $A_i$ and $A_{i+3}$. Therefore, if the platform can accomplish motions, the velocity, $\dot{\mathbf{B}}_i$, of the $B_i$ point must be perpendicular to the plane of the $A_iA_{i+3}B_i$ triangle. This condition is analytically expressed by the following relationship

$$\dot{\mathbf{B}}_i = c_i \mathbf{u}_i \times \mathbf{v}_i \qquad i=1, 2, 3 \tag{1}$$

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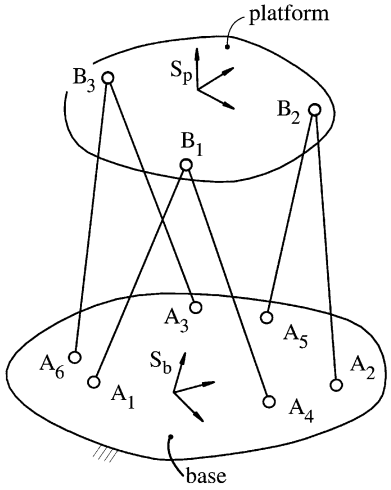



Fig. 3. The 6-3 structure.

where $c_i$, $i=1, 2, 3$, are coefficients different from zero, whereas the $\mathbf{u}_i$ and $\mathbf{v}_i$ vectors are defined as follows

$$\mathbf{u}_i=(\mathbf{B}_i-\mathbf{A}_i) \qquad i=1, 2, 3 \tag{2.1}$$

$$\mathbf{v}_i=(\mathbf{B}_i-\mathbf{A}_{i+3}) \qquad i=1, 2, 3 \tag{2.2}$$

Since the $B_i$, $i=1, 2, 3$, points belong to one rigid body, the platform, the $\dot{\mathbf{B}}_i$, $i=1, 2, 3$, velocities must satisfy the following relationships

$$(\dot{\mathbf{B}}_2-\dot{\mathbf{B}}_1)\cdot(\mathbf{B}_2-\mathbf{B}_1)=0 \tag{3.1}$$

$$(\dot{\mathbf{B}}_3-\dot{\mathbf{B}}_1)\cdot(\mathbf{B}_3-\mathbf{B}_1)=0 \tag{3.2}$$

$$(\dot{\mathbf{B}}_3-\dot{\mathbf{B}}_2)\cdot(\mathbf{B}_3-\mathbf{B}_2)=0 \tag{3.3}$$

Taking into account expressions (1), Eqs. (3) become as follows

$$[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)]\,c_2-[\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_2-\mathbf{B}_1)]\,c_1=0 \tag{4.1}$$

$$[\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)]\,c_3-[\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_3-\mathbf{B}_1)]\,c_1=0 \tag{4.2}$$

$$[\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_2)]\,c_3-[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_2)]\,c_2=0 \tag{4.3}$$

System (4) in matrix form is

$$\mathbf{J}\begin{Bmatrix}c_1\\c_2\\c_3\end{Bmatrix}=0 \tag{5}$$

where

$$\mathbf{J}=\begin{bmatrix}-\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_2-\mathbf{B}_1) & \mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1) & 0\\ -\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_3-\mathbf{B}_1) & 0 & \mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)\\ 0 & -\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_2) & \mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_2)\end{bmatrix} \tag{6}$$

System (5) is a three equation linear and homogeneous system in the three unknowns $c_i$, $i=1, 2, 3$.

If system (5) admits solutions different from zero, the platform can perform motions without changing the leg lengths (see Eqs. (1)), i.e., the configuration assumed by the mechanism is singular. System (5) can admit solutions different from zero if and only if the determinant of the **J** matrix vanishes. Therefore, the 6-3 FPM's singular configurations must satisfy the following singularity condition

$$\det(\mathbf{J})=0 \tag{7}$$

Definition (6) of the **J** matrix gives the following explicit expression of det(**J**)

$$\begin{aligned}\det(\mathbf{J})=&-[\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_2-\mathbf{B}_1)][\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)]\\&\cdot[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_2)]+[\mathbf{u}_1\times\mathbf{v}_1\cdot(\mathbf{B}_3-\mathbf{B}_1)]\\&\cdot[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)][\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_2)]\end{aligned} \tag{8}$$

Expression (8) can be transformed as follows (see Appendix A)

$$\det(\mathbf{J})=\mathbf{e}\cdot\mathbf{f}\times\mathbf{g} \tag{9}$$

where

$$\mathbf{e}=(\mathbf{B}_3-\mathbf{B}_1)\times(\mathbf{B}_2-\mathbf{B}_1) \tag{10.1}$$

$$\mathbf{f}=\mathbf{u}_1\times\mathbf{v}_1 \tag{10.2}$$

$$\mathbf{g}=-[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)]\mathbf{u}_3\times\mathbf{v}_3+[\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)]\mathbf{u}_2\times\mathbf{v}_2 \tag{10.3}$$

Taking into account expression (9), singularity condition (7) becomes

$$\mathbf{e}\cdot\mathbf{f}\times\mathbf{g}=0 \tag{11}$$

Equation (11) will be satisfied either if all the **e**, **f** and **g** vectors are parallel to a single plane, or if at least one of the **e**, **f** and **g** vectors is a null vector.

The **e** vector is perpendicular to the plane of the $B_1B_2B_3$ triangle (Fig. 3) which is embedded in the platform. The **f** vector is perpendicular to the plane of the $A_1A_4B_1$ triangle (Fig. 3). The **g** vector lies on a plane parallel to the $\mathbf{u}_k\times\mathbf{v}_k$, $k=2, 3$, vectors that are perpendicular to the planes of the $A_kA_{k+3}B_k$, $k=2, 3$, triangles respectively.

The **e** vector is a null vector when the $B_1B_2B_3$ triangle degenerates into a segment (Fig. 4). Since this condition depends only on the platform shape, it will be verified in any mechanism configuration (architecture singularity[5]) if the platform shape satisfies it.

The **f** vector is a null vector when the $A_1A_4B_1$ triangle degenerates into a segment whilst the **g** vector is a null vector when one of the $A_kA_{k+3}B_k$, $k=2, 3$, triangles degenerates into a segment. Therefore, the mechanism is singular when at least one of the $A_iA_{i+3}B_i$, $i=1, 2, 3$, triangles degenerates into a segment.

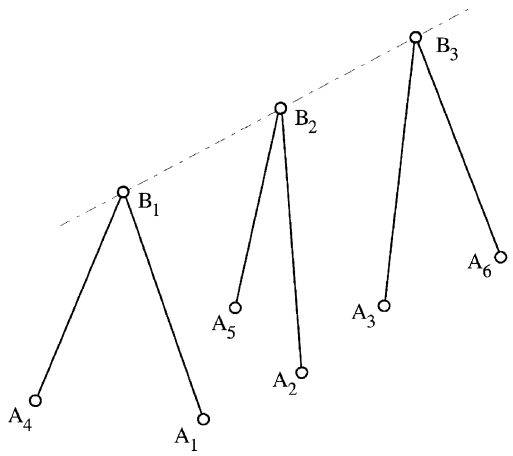



Fig. 4. Architecture singularity.

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A particular case, which all the **e**, **f** and ***g*** vectors are parallel to a single plane in, occurs when two out of the three vectors are parallel. This condition is verified when one out of the $A_iA_{i+3}B_i$, i = 1, 2, 3, triangles lies on the plane of the $B_1B_2B_3$ triangle. This singularity condition was found in different way by Hunt[12], too.

Another particular geometric condition, that makes all the **e**, **f** and **g** vectors parallel to a single plane, occurs when the **e** and **f** vectors are perpendicular to the $(\mathbf{u}_3\times\mathbf{v}_3)\times(\mathbf{u}_2\times\mathbf{v}_2)$ vector, since the **g** vector is always perpendicular to the $(\mathbf{u}_3\times\mathbf{v}_3)\times(\mathbf{u}_2\times\mathbf{v}_2)$ vector.

Eventually, it is worth noting that other singular geometric conditions making all the **e**, **f** and **g** vectors parallel to a single plane can be identified by considering the definitions (10) of the **e**, **f** and **g** vectors.

## 3. SINGULARITY LOCUS EQUATION

An equation explicitly containing the platform pose parameters and the mechanism's geometric parameters can be derived from the singularity condition (11), if the **e**, **f** and **g** vectors are written in the $S_b$ reference system, fixed to the base (Fig. 3).

Hereafter, it will be assumed that the platform position parameters are the $B_1$ point coordinates measured in $S_b$. Moreover, $\mathbf{R}_{bp}$ will be the rotation matrix transforming vector components measured in $S_p$ into vector components measured in $S_b$. Bold capital letters will indicate position vectors measured in $S_b$ and the left-hand superscript, p, applied to a vector will indicate that the vector is measured in $S_p$.

The **e** vector measured in $S_b$ has the following explicit expression

$$\mathbf{e}=\mathbf{R}_{bp}[{}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1)] \tag{12}$$

where ${}^p(\mathbf{B}_3-\mathbf{B}_1)$ and ${}^p(\mathbf{B}_2-\mathbf{B}_1)$ are constant vectors depending on the platform geometry. Expression (12) is linear in the $\mathbf{R}_{bp}$ entries.

Taking into account definitions (2) and (10.2), the explicit expression of the **f** vector measured in $S_b$ comes out as follows

$$\mathbf{f}=(\mathbf{B}_1-\mathbf{A}_1)\times(\mathbf{B}_1-\mathbf{A}_4) \tag{13}$$

Expanding the cross product in expression (13), expression (13) becomes

$$\mathbf{f}=\mathbf{B}_1\times(\mathbf{A}_1-\mathbf{A}_4)+\mathbf{A}_1\times\mathbf{A}_4 \tag{14}$$

Expression (14) is linear in the coordinates of $B_1$ measured in $S_b$.

The $\mathbf{B}_k$, k = 2, 3, position vectors have the following explicit expression

$$\mathbf{B}_k=\mathbf{B}_1+\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)\qquad k=2,\,3 \tag{15}$$

Besides, the $\mathbf{u}_k\times\mathbf{v}_k$, k = 2, 3, vectors can be written by using definition (2) as follows

$$\mathbf{u}_k\times\mathbf{v}_k=(\mathbf{B}_k-\mathbf{A}_k)\times(\mathbf{B}_k-\mathbf{A}_{k+3})\qquad k=2,\,3 \tag{16}$$

Expanding the cross product in expression (16), expression (16) becomes

$$\mathbf{u}_k\times\mathbf{v}_k=\mathbf{B}_k\times(\mathbf{A}_k-\mathbf{A}_{k+3})+\mathbf{A}_k\times\mathbf{A}_{k+3}\qquad k=2,\,3 \tag{17}$$

Introducing relationship (15) into expression (17), the following explicit expressions of the $\mathbf{u}_k\times\mathbf{v}_k$, k = 2, 3, vectors result

$$\mathbf{u}_k\times\mathbf{v}_k=\mathbf{B}_1\times(\mathbf{A}_k-\mathbf{A}_{k+3})+(\mathbf{A}_{k+3}-\mathbf{A}_k)\times[\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)]+\mathbf{A}_k\times\mathbf{A}_{k+3}\qquad k=2,\,3 \tag{19}$$

The introduction of expressions (18) and (19) into definition (10.3) yields the following explicit expression of the **g** vector

$$\begin{aligned}\mathbf{g}=&-\{\mathbf{B}_1\times(\mathbf{A}_2-\mathbf{A}_5)\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_2-\mathbf{B}_1)]+\mathbf{A}_2\times\mathbf{A}_5\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_2-\mathbf{B}_1)]\}\\&\cdot\{\mathbf{B}_1\times(\mathbf{A}_3-\mathbf{A}_6)+(\mathbf{A}_6-\mathbf{A}_3)\times[\mathbf{R}_{bp}{}^p(\mathbf{B}_3-\mathbf{B}_1)]+\mathbf{A}_3\times\mathbf{A}_6\}\\&+\{\mathbf{B}_1\times(\mathbf{A}_3-\mathbf{A}_6)\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_3-\mathbf{B}_1)]+\mathbf{A}_3\times\mathbf{A}_6\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_3-\mathbf{B}_1)]\}\\&\cdot\{\mathbf{B}_1\times(\mathbf{A}_2-\mathbf{A}_5)+(\mathbf{A}_5-\mathbf{A}_2)\times[\mathbf{R}_{bp}{}^p(\mathbf{B}_2-\mathbf{B}_1)]+\mathbf{A}_2\times\mathbf{A}_5\}\end{aligned} \tag{20}$$

Expression (20) is quadratic both in the $B_1$ coordinates measured in $S_b$ and in the $\mathbf{R}_{bp}$ entries.

Introducing expressions (12), (14) and (20) into Eq. (11), the following sigularity locus equation of the 6-3 FPM results

$$-a_2b_3+a_3b_2=0 \tag{21}$$

where

$$a_k=\mathbf{B}_1\times(\mathbf{A}_k-\mathbf{A}_{k+3})\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)]+\mathbf{A}_k\times\mathbf{A}_{k+3}\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)]\qquad k=2,\,3 \tag{22.1}$$

$$\begin{aligned}b_k=&[(\mathbf{A}_1-\mathbf{A}_4)\times(\mathbf{A}_k-\mathbf{A}_{k+3})\cdot\mathbf{B}_1-\mathbf{A}_k\times\mathbf{A}_{k+3}\cdot(\mathbf{A}_1-\mathbf{A}_4)\\&+\mathbf{A}_1\times\mathbf{A}_4\cdot(\mathbf{A}_k-\mathbf{A}_{k+3})]\cdot\{\mathbf{B}_1\cdot[\mathbf{R}_{bp}({}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1))]\}\\&+\{\mathbf{B}_1\times[\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)]\cdot(\mathbf{A}_1-\mathbf{A}_4)-\mathbf{A}_1\times\mathbf{A}_4\cdot\mathbf{B}_1-\mathbf{A}_1\times\mathbf{A}_4\\&\cdot[\mathbf{R}_{bp}{}^p(\mathbf{B}_k-\mathbf{B}_1)]\}\{(\mathbf{A}_k-\mathbf{A}_{k+3})\cdot[\mathbf{R}_{bp}({}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1))]\}\\&+(\mathbf{A}_k\times\mathbf{A}_{k+3}\cdot\mathbf{B}_1)\{(\mathbf{A}_1-\mathbf{A}_4)\cdot[\mathbf{R}_{bp}({}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1))]\}\\&+(\mathbf{A}_1\times\mathbf{A}_4\cdot\mathbf{A}_{k+3})\{\mathbf{A}_k\cdot[\mathbf{R}_{bp}({}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1))]\}\\&-(\mathbf{A}_1\times\mathbf{A}_4\cdot\mathbf{A}_k)\{\mathbf{A}_{k+3}\cdot[\mathbf{R}_{bp}({}^p(\mathbf{B}_3-\mathbf{B}_1)\times{}^p(\mathbf{B}_2-\mathbf{B}_1))]\}\end{aligned}\qquad k=2,\,3 \tag{22.2}$$

Equation (21) is cubic both in the $B_1$ coordinates measured in $S_b$ and in the $\mathbf{R}_{bp}$ entries.

If the platform orientation is parameterized by means of the Rodrigues parameters,[13] the $\mathbf{R}_{bp}$ matrix will be as follows

$$\mathbf{R}_{bp}=\frac{1}{1+\mathbf{x}^2}\,[2(\mathbf{x}\mathbf{x}^T+\mathbf{x}^{sk})+(1-\mathbf{x}^2)\mathbf{I}_{3\times3}] \tag{23}$$

where **x** is the vector $(x_1, x_2, x_3)^T$ of the three Rodrigues parameters $x_i$, i = 1, 2, 3; $\mathbf{I}_{3\times3}$ is the $3\times3$ identity matrix and $\mathbf{x}^{sk}$ denotes the skew matrix associated to the vector **x**. When the Rodrigues parameters become infinite, expression (23) of $\mathbf{R}_{bp}$ fails, even if $\mathbf{R}_{bp}$ is defined (representation singularity). Therefore the only valid values of the Rodrigues parameters are the finite ones.

The left-hand side of Eq. (21) will become a rational function in the six platform pose parameters if expression (23) is substituted for $\mathbf{R}_{bp}$ in Eq. (21). The numerator of this

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rational function is a polynomial, which is cubic in the $B_1$ coordinates and one of sixth degree in the Rodrigues parameters whereas its denominator is $(1+\mathbf{x}^2)^3$. The denominator does not make the rational function vanish for finite values of the Rodrigues parameters. Hence the only platform pose parameters, making Eq. (21) satisfied, are the numerator roots.

As a consequence, the solution of Eq. (21) reduces to the solution of a polynomial equation which is cubic in the $B_1$ coordinates and one of sixth degree in the Rodrigues parameters.

## 4. CONCLUSIONS

The problem of determining the 6-3 FPM's platform poses that make the manipulator singular has been addressed.

A simple singularity condition has been written. This singularity condition consists in equating to zero the mixed product of three vectors, that are easy to be identified on the mechanism, and it is geometrically interpretable.

The presented singularity condition has been transformed into an equation (singularity locus equation) explicitly containing the geometric parameters of the manipulator and the platform pose parameter making the 6-3 FPM singular.

Eventually, the singularity locus equation has been reduced to a polynomial equation by using the Rodrigues parameters to parameterize the platform orientation. This polynomial equation is cubic in the platform position parameters and one of sixth degree in the Rodrigues parameters.

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## APPENDIX A

Collecting the $\mathbf{u}_1\times\mathbf{v}_1$ vector in the right-hand side of relationship (8), relationship (8) becomes as follows

$$\det(\mathbf{J})=\mathbf{u}_1\times\mathbf{v}_1\cdot\{-(\mathbf{B}_2-\mathbf{B}_1)[\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1)]\cdot[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_2)]+(\mathbf{B}_3-\mathbf{B}_1)[\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1)]\cdot[\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_2)]\} \tag{A.1}$$

Besides, the following identity holds

$$(\mathbf{B}_3-\mathbf{B}_2)=(\mathbf{B}_3-\mathbf{B}_1)-(\mathbf{B}_2-\mathbf{B}_1) \tag{A.2}$$

Introducing identity (A.2) into relationship (A.1) and rearranging, relationship (A.1) becomes

$$\det(\mathbf{J})=\mathbf{u}_1\times\mathbf{v}_1\cdot\{(\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1))[-(\mathbf{B}_3-\mathbf{B}_1)\cdot(\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_2-\mathbf{B}_1))+(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1))]+(\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1))[(\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1))-(\mathbf{B}_2-\mathbf{B}_1)\,(\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_1))]\} \tag{A.3}$$

Using the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{A.4}$$

where **a**, **b** and **c** are any vectors, the two expressions in the square brackets that are in the right-hand side of (A.3) can be transformed as follows

$$[-(\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_2-\mathbf{B}_1))+(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_3\times\mathbf{v}_3\cdot(\mathbf{B}_3-\mathbf{B}_1))]=-(\mathbf{u}_3\times\mathbf{v}_3)\times((\mathbf{B}_3-\mathbf{B}_1)\times(\mathbf{B}_2-\mathbf{B}_1)) \tag{A.5}$$

$$[(\mathbf{B}_3-\mathbf{B}_1)(\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_2-\mathbf{B}_1))-(\mathbf{B}_2-\mathbf{B}_1)(\mathbf{u}_2\times\mathbf{v}_2\cdot(\mathbf{B}_3-\mathbf{B}_1))]=(\mathbf{u}_2\times\mathbf{v}_2)\times((\mathbf{B}_3-\mathbf{B}_1)\times(\mathbf{B}_2-\mathbf{B}_1)) \tag{A.6}$$

Taking into account relationships (A.5) and (A.6) in expression (A.3) and rearranging, expression (A.3) becomes as follows

$$\det(\mathbf{J})=\mathbf{f}\cdot\mathbf{g}\times\mathbf{e} \tag{A.7}$$

where the **e**, **f** and **g** vectors are defined by relationship (10).

Using the mixed product properties, expression (A.7) becomes expression (9). Q.E.D.

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