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A note on internality of certain differential systems

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Abstract. We prove two results, generalizing certain theorems by Jin and Moosa (2020, *Trans. Amer. Math. Soc.*, 373, 4863–4887), on the internality of the system of differential equations

$$\begin{cases} x' = f(x) \\ y' = g(x)y, \end{cases}$$

where f and g are rational functions in one variable.

Let K be a differential field of characteristic zero. We fix a universal differential field extension[1] U of K and assume that all differential field extensions of K considered in this article are contained in U. For any differential field extension M of K, the subfield of constants of M is denoted by C_M. A *generic solution over* K of a system of first-order differential equations

$$\text{(S)} \qquad \begin{cases} y_1' = f_1(y_1, \ldots, y_n) \\ \quad \vdots \\ y_n' = f_n(y_1, \ldots, y_n), \end{cases}$$

where $f_1, \ldots, f_n$ are rational functions in n variables over K, is a tuple $(y_1, \ldots, y_n) \in U^n$ such that for each $i, 1 \le i \le n$, $y_i' = f_i(y_1, \ldots, y_n)$ and the field transcendence degree $\mathrm{tr.deg}(K(y_1, \ldots, y_n)|K) = n$.

The system (S) is said to be *almost internal to the constants* (respectively, *internal to the constants*) if there are m generic solutions $(y_{i1}, \ldots, y_{in}), 1 \le i \le m$, of (S) such that for every generic solution $(y_1, \ldots, y_n)$ of (S), there exist constants $c_1, \ldots, c_l \in C_U$ such that $y_1, \ldots, y_n$ are algebraic over the field $K(\Omega)$ (respectively, $y_1, \ldots, y_n$ belongs to the field $K(\Omega)$), where $\Omega = \{y_{ij}, c_p \mid 1 \le i \le m, 1 \le j \le n, 1 \le p \le l\}$.

The model-theoretic concept of internality of a system of differential equations is known to be closely related to Kolchin's strongly normal extensions (see [12,

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[1] A differential field extension U of K is said to be a *universal extension* of K if for any finitely generated differential field extension of F of K contained in U and for any finitely generated differential field extension F_1 of F, which is not necessarily contained in U, there exists a differential F-embeddeding from F_1 into U (see [7, p. 133]).

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Section 2.7]), and it has its origin in the works of Rosenlicht [13, Proposition] and Zilber [14]. In [5], Jin and Moosa prove two theorems, which appear as Theorems A and B in their paper, on almost internality of the system of differential equations

$$\text{(L)} \qquad \begin{cases} x' = f(x) \\ y' = xy, \end{cases}$$

where f is a rational function in one variable. In Theorems 4 and 5 of this article, we generalize and extend both these results of Jin and Moosa. In particular, Theorem 4 completely answers the question of when the system (L) is almost internal to the constants, given that $x' = f(x)$ is internal to the constants.

Proposition 1 *Let*

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

be a system almost internal to the constants. Then, there exists a finitely generated differential field extension M of K such that for any generic solution (x, y) over M of the system,[2] *the differential field $M(x, y)$ contains two M-algebraically independent constants. Furthermore, if N is a differential field intermediate to $M(x, y)$ and M with* $\operatorname{tr.deg}(N|M) = 1$, *then* $\operatorname{tr.deg}(C_N|C_M) = 1$.

Proof Since the system is almost internal to the constants, there are generic solutions $(x_1, y_1), \ldots, (x_m, y_m)$ over K of the system such that for any generic solution (x, y) of the system over $M := K(x_1, y_1, \ldots, x_m, y_m)$, there are constants $c_1, \ldots, c_l$ such that x and y are algebraic over $M(c_1, \ldots, c_l)$. We now show that $M(x, y)$ contains two M-algebraically independent constants. Let $c_1, \ldots, c_r$ be an M-transcendence basis of constants for $M(c_1, \ldots, c_l)$. Since $\operatorname{tr.deg}(M(c_1, \ldots, c_l, x, y)|M) = r$, we must have $\operatorname{tr.deg}(M(c_1, \ldots, c_l, x, y)|M(x, y)) = r - 2$. Therefore, there is a nonzero polynomial $P \in M(x, y)[X_1, \ldots, X_r]$ such that $P(c_1, \ldots, c_r) = 0$. Since $c_1, \ldots, c_r$ are constants, it follows that such a polynomial P must belong to $C_{M(x,y)}[X_1, \ldots, X_r]$ [6, Section 14, Theorem 2]. As $\operatorname{tr.deg}(M(c_1, \ldots, c_l)|M) = r$, it also follows that one of the coefficients of P must be a constant nonalgebraic over M. If we call this constant u_1 and replace M by $M(u_1)$, then a similar argument shows the existence of another constant $u_2 \in M(x, y)$, which is nonalgebraic over $M(u_1)$. Observe that either u_1 or u_2 must belong to $M(x, y) \setminus M(x)$.

Let N be a differential field intermediate to $M(x, y)$ and M. Since $\operatorname{tr.deg}(N|M) = 1$, there is a nonzero polynomial $P \in C_N[X_1, X_2]$ such that $P(u_1, u_2) = 0$. Again, by the same argument as above, there is a constant $v \in N$ which is not algebraic over M. ■

Theorem 2 *Let f be a rational function in one variable over K. The equation $x' = f(x)$ is internal to the constants if and only if $f(x) = a_2x^2 + a_1x + a_0$, where $a_0, a_1, a_2 \in K$.*

[2] Generic solutions over M of the system always exist. To see this, consider the rational function field $M(s, t)$ and equip this field with the derivation induced by the system: $s' = f(s, t)$ and $t' = g(s, t)$. Then, by definition, there is a differential M-embedding of $M(s, t)$ into U. The image of s, t under this embedding provides a generic solution over M of the system.

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Proof Let $x' = f(x)$ be internal to the constants. Then, there are generic solutions $x_1, \ldots, x_m$ over K such that for any generic solution x over $M := K(x_1, \ldots, x_m)$, there are constants $c_1, \ldots, c_l$ such that $x \in M(c_1, \ldots, c_l)$. From [7, Chapter 2, Corollary 2], we know that $M(x)$ is generated over M as a field by a set of constants. Therefore, we may assume that $M(x) = M(c_1, \ldots, c_l)$. We first claim that there exists an algebraic extension $\tilde{M}$ of M such that $\tilde{M}(x) = \tilde{M}(c)$ for some constant c.

Note that $C_{M(x)}$ and M are linearly disjoint over C_M [7, Chapter 2, Corollary 1]. Therefore, $C_M(c_1, \ldots, c_l)$ and M are linearly disjoint over C_M. Since $M(c_1, \ldots, c_l) = M(x)$, we have $\mathrm{tr.deg}(M(c_1, \ldots, c_l)|M) = 1$ and from linear disjointness, we obtain $\mathrm{tr.deg}(C_M(c_1, \ldots, c_l)|C_M) = 1$. Now since $M(c_1, \ldots, c_l) = M(x)$ is a rational function field, there is a finite algebraic extension $\tilde{C}_M$ of C_M such that $\tilde{C}_M(c_1, \ldots, c_l) = \tilde{C}_M(c)$ [1, Theorem 5]. Any extension of a zero derivation on a field to its algebraic closure is again a zero derivation. Therefore, c must also be a constant. Let $\tilde{M} := M\tilde{C}_M$ and observe that

$$\tilde{M}(x) = \tilde{M}(c_1, \ldots, c_l) = \tilde{M}(c).$$

This proves the claim.

Since $\tilde{M}(x) = \tilde{M}(c)$, there exists $\alpha, \beta, \gamma, \delta \in \tilde{M}$ with $\alpha\delta - \beta\gamma \neq 0$ such that

$$c = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

A simple calculation shows that $f(x) = x' = a_2x^2 + a_1x + a_0$ for some $a_0, a_1, a_2 \in \tilde{M}$. Now since $f \in K(x)$, we obtain that $a_0, a_1, a_2 \in K$.

To prove the converse, let x, x_1, x_2, x_3 be distinct generic solutions of $x' = a_2x^2 + a_1x + a_0$ over K. It is observed in [11, p. 102] that

$$c := \frac{(x - x_2)(x_3 - x_1)}{(x - x_1)(x_3 - x_2)}$$

is a constant.[3] Since

$$x = \frac{x_2(x_3 - x_1) + cx_1(x_2 - x_3)}{(x_3 - x_1) + c(x_2 - x_3)},$$

we now obtain $x \in K(x_1, x_2, x_3, c)$. This proves that the equation $x' = a_2x^2 + a_1x + a_0$ is internal to the constants. ∎

Remark 3 We note here that a particular case of Theorem 2 can be derived from certain recent results [4, 8]. Suppose that K is an algebraically closed field and that the equation $x' = f(x)$ is both internal to the constants and weakly orthogonal to the constants.[4] Then, it is known that the binding group of the equation $x' = f(x)$ is a linear algebraic group. Therefore, one can apply [4, Proposition 6.5] and conclude

[3] It is easily seen that the first-order differential equation $y' = (a_1(x + x_1 + x_2 + x_3) + a_0)y$ has $(x - x_2)(x_3 - x_1)$ and $(x - x_1)(x_3 - x_2)$ as its solutions and therefore c, being the ratio of these two solutions, must be a constant.

[4] That is, the rational function field $K(x)$ with the extended derivation $x' = f(x)$ has the same field of constants as K.

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that $K(x) = K(y)$, where $y' = ay^2 + by + c$ for some $a, b, c \in K$. Writing

$$x = \frac{\alpha y + \beta}{\gamma y + \delta},$$

where $\alpha, \beta, \gamma, \delta \in K$ with $\alpha\delta - \beta\gamma \neq 0$, and taking derivatives, we obtain the conclusion of Theorem 2 that f is a polynomial over K of degree at most 2.

The conditions that $x' = f(x)$ is both internal to the constants and weakly orthogonal to the constants imply that the differential field $K(x)$ can be embedded in a strongly normal extension of K. Now, applying [8, Theorem 1.1(iii)], we obtain that f is a polynomial over K of degree at most 2. Thus, Theorem 2 provides a generalization of these results, as it requires neither that $x' = f(x)$ be weakly orthogonal to the constants nor that K be an algebraically closed field.

Under the assumption that $x' = f(x)$ is internal to the constants with the binding group not of dimension 3, it is shown in [5, Theorem A] that the system (L) is almost internal to the constants if and only if it splits; which implies that the system can be transformed into a system of the form

$$\begin{cases} x' = f(x) \\ y' = ay, \end{cases}$$

where $a \in K$. In the next theorem, using Theorem 2 and Proposition 1 and without any assumption on the binding group of $x' = f(x)$, we shall completely determine when the system (L) is almost internal to the constants.

Theorem 4 *Let K be a differential field and f be a nonzero rational function in one variable over K. The system (L) is almost internal to the constants and $x' = f(x)$ is internal to the constants if and only if $f(x) = a_2x^2 + a_1x + a_0$, where $a_0, a_1 \in K$ and a_2 is a nonzero rational number.*

Proof We know from Theorem 2 that $f(x) = a_2x^2 + a_1x + a_0$, where $a_0, a_1, a_2 \in K$. Suppose that the system (L) is almost internal to the constants. By Proposition 1, there exists a differential field extension M of K such that for any generic solution (x, y) over M, the differential field $M(x, y)$ contains two M-algebraically independent constants. Then, $M(x, y)$ contains a constant which is not in $M(x)$ and therefore the latter must contain an element z such that $z' = nxz$ for some positive integer n [10, Example 1.11]. Let $z = P/Q$, where $P, Q \in M[x]$. We can assume that P and Q are relatively prime and Q is monic. Write $P = \alpha_p x^p + \alpha_{p-1}x^{p-1} + \cdots$, $Q = x^q + b_{q-1}x^{q-1} + \cdots \in M[x]$ with $\alpha_p \neq 0$. Then,

$$P'Q - Q'P = nxPQ$$

and we obtain $(p - q)a_2 = n$. Since $n \neq 0$, $a_2 = n/(p - q)$ is a nonzero rational number.

For the converse, suppose that $f(x) = a_2x^2 + a_1x + a_0$, where $a_0, a_1 \in K$ and $a_2 = m_1/m_2$, $(m_1, m_2) \in \mathbb{Z}^2$ a nonzero rational number. Let (x, y) be a generic solution of the system (L). We shall now find generic solutions (x_1, y_1) and (x_2, y_2) of the system

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and constants c_1, c_2 so that x, y are algebraic over the field $K(y_1, x_1, y_2, x_2, c_1, c_2)$. This will then complete the proof of the theorem.

Let v be an element algebraic over the field $K(x, y)$ satisfying $v^{m_2} = y^{-m_1}$. Then, $v' = -a_2 x v$, $(a_2 x)' = a_2 x' = (a_2 x)^2 + a_1(a_2 x) + a_2 a_0$ and $v'' = a_1 v' - a_2 a_0 v$. Consider the rational function field $K(v, v')(v_1, v_1', v_2, v_2')$ and extend the derivation on $K(v, v')$ to $K(v, v')(v_1, v_2, v_1', v_2')$ by declaring $v_1'' = a_1 v_1' - a_2 a_0 v_1$ and $v_2'' = a_1 v_2' - a_2 a_0 v_2$. Let $N = K(v_1, v_1', v_2, v_2')$. Note that v_1, v_2 are C_U-linearly independent and that the Wronskian of v, v_1, v_2 is zero. Therefore, $v = c_1 v_1 + c_2 v_2$, for some constants c_1, c_2. This implies that $v \in N(c_1, c_2)$ and since $a_2 x = -v'/v \in N(c_1, c_2)$, we obtain that $x \in N(c_1, c_2)$.

Let y_1 and y_2 be elements algebraic over the field N such that $v_1^{m_2} = y_1^{-m_1}$ and $v_2^{m_2} = y_2^{-m_1}$. Let $x_1 := -v_1'/a_2 v_1$ and $x_2 := -v_2'/a_2 v_2$. Then, it is easy to see that (x_1, y_1) and (x_2, y_2) are generic solutions of the system (L). Now we consider the field $K(y_1, x_1, y_2, x_2, c_1, c_2)$. Clearly, v_1 and v_2 are algebraic over $K(y_1, x_1, y_2, x_2, c_1, c_2)$ and since $v = c_1 v_1 + c_2 v_2$, we obtain that v is also algebraic over $K(y_1, x_1, y_2, x_2, c_1, c_2)$. Since $x = -v'/a_2 v$, we obtain that x is algebraic over $K(y_1, x_1, y_2, x_2, c_1, c_2)$. Finally, $v^{m_2} = y^{-m_1}$ implies that y is algebraic over $K(y_1, x_1, y_2, x_2, c_1, c_2)$. ∎

Theorem 5 *Let C be an algebraically closed differential field with zero derivation, and f, g be two nonzero rational functions in one variable over C. The system*

$$\text{(A)} \qquad \begin{cases} x' = f(x) \\ y' = g(x)y \end{cases}$$

is almost internal to the constants if and only if the following two conditions hold:

(1) $\frac{1}{f(x)} = \frac{\partial u}{\partial x}$ *or* $\frac{1}{f(x)} = \frac{c}{u}\frac{\partial u}{\partial x}$ *for some $c \in C\backslash 0$ and $u \in C(x)$.*

(2) $\frac{b+mg(x)}{f(x)} = \frac{1}{v}\frac{\partial v}{\partial x}$ *for some integer $m \in \mathbb{Z}$, $b \in C$ and $v \in C(x)$.*

Proof Let (x, y) be a generic solution of the system (A) and $F = C(x, y)$. We first make the following observations. Let $z \in F\backslash C$ such that $z' = 0$. Then, $z \notin C(x)$; otherwise, $f(x)\frac{\partial z}{\partial x} = 0$, which is a contradiction. Therefore, $z \in F\backslash C(x)$. As noted in the proof of Theorem 4, there exists a positive integer m and an element $v \in C(x)$ such that $v' = mg(x)v$. Therefore, $f(x)\frac{\partial v}{\partial x} = mg(x)v$ and we obtain

$$\frac{mg(x)}{f(x)} = \frac{1}{v}\frac{\partial v}{\partial x}.$$

Thus, if $z \in F\backslash C$ such that $z' = 0$ then $z \in F\backslash C(x)$, and in that case, the condition (2) holds with $b = 0$.

Along with these observations, we suppose that the system (A) is almost internal to the constants. Then, by Proposition 1, there exists a differential field extension M of C such that $M(x, y)$ contains two M-algebraically independent constants. Furthermore, since $M(x)$ is a differential subfield of $M(x, y)$ with $\mathrm{tr.deg}(M(x)|M) = 1$, it follows that $\mathrm{tr.deg}(C_{M(x)}|C_M) = 1$. By [13, Proposition], $C(x)$ contains an element u such that either $u' = 1$ or $u' = cu$ for some nonzero $c \in C$. Then, $\frac{1}{f(x)}$ is either $\frac{\partial u}{\partial x}$ or $\frac{1}{f(x)} = \frac{c}{u}\frac{\partial u}{\partial x}$. This implies that condition (1) holds.

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To prove that condition (2) also holds, we consider the following two cases. Suppose that $C_F \neq C$. Then, by the observation made in the first paragraph, the condition (2) holds. Now suppose that $C_F = C$. Since there exists a differential field M such that M and F are free over C and MF contains two M-algebraically independent constants, by [9, Proposition 4.2], there is a differential field L intermediate to C and F such that $\mathrm{tr.deg}(L|C) = 2$ and L can be embedded in a strongly normal extension of C. Since L is contained in the purely transcendental extension $C(x, y)$, by [9, Theorem 1.2], L can be embedded in a Picard–Vessiot extension of C.[5] The differential Galois group of any Picard–Vessiot extension of C is a connected commutative linear algebraic group over C. Therefore, by the fundamental theorem of differential Galois theory, L must also be a Picard–Vessiot extension of C with a differential Galois group isomorphic to $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$. This implies that $L = C(\alpha, \beta)$ where α, β are C-algebraically independent, $\beta' = c_2\beta$ for some nonzero $c_2 \in C$ and either $\alpha' = 1$ or $\alpha' = c_1\alpha$ for some nonzero $c_1 \in C$. Consequently, there exists an element $\zeta \in C(x, y)$ such that $\zeta' = a\zeta$ for some nonzero $a \in C$. By the Kolchin–Ostrowski theorem, there exist integers m, n such that $h := y^m\zeta^n \in C(x)$. Then,

$$h' = f(x)\frac{\partial h}{\partial x} = my^m\zeta^n g(x) + nay^m\zeta^n.$$

From the above equation, we obtain

$$\frac{mg(x) + b}{f(x)} = \frac{1}{h(x)}\frac{\partial h}{\partial x}, \quad na =: b \in C.$$

Thus, we have shown that the condition (2) holds in both cases.

To prove the converse, let (x, y) be a generic solution of the system (A). From conditions (1) and (2), we obtain that $C(x)$ contains an element s such that $s' = 1$ or $s' = cs$ for some nonzero $c \in C$ and that there is an element $t \in C(x, y)\backslash C(x)$ such that $t' = bt$ for some $d \in C$. Note that $s = u$ and $t = y^m/v$. Now let (x_1, y_1) be another generic solution of the system (A). Consider the differential C-isomorphism $\phi : C(x, y) \to C(x_1, y_1)$, where $\phi(x) = x_1$ and $\phi(y) = y_1$. Let $c_2 := t/\phi(t)$. If $s' = 1$, then let $c_1 := s - \phi(s)$ and if $s' = cs$, then let $c_1 := s/\phi(s)$. In any event, note that c_1, c_2 are constants and that $s, t \in C(x_1, y_1, c_1, c_2)$. Since x, y are algebraic over $C(s, t)$, they are also algebraic over $C(x_1, y_1, c_1, c_2)$. This shows that the system (A) is almost internal to the constants. ∎

We conclude this article with the following observations. When $g(x) = x$, we recover Theorem B of [5]. The nonorthogonality of the system (A), under the assumption that $x' = f(x)$ is orthogonal to the constants, has been examined in [3, Corollary 5.5]. Interestingly, the condition (2) of Theorem 5 also appears in their result.

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[5] The existence of the field L can also be obtained using results from model theory (see [2, Theorem 3.5]).

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