102.48 The function $f(x)/x$ is v-shaped for convex f

A function $f : I \to \mathbb{R}$ defined on an interval I of the real line is called *convex* if for any two distinct points x, y in I and for all $0 < t < 1$ the inequality

$$f\big(tx + (1 - t)y\big) \leqslant tf(x) + (1 - t)f(y) \tag{1}$$

holds. A function f is called *concave* if $-f$ is convex.

Geometrically, convexity means that the graph of a function between any two points on the graph lies below the line segment joining them.

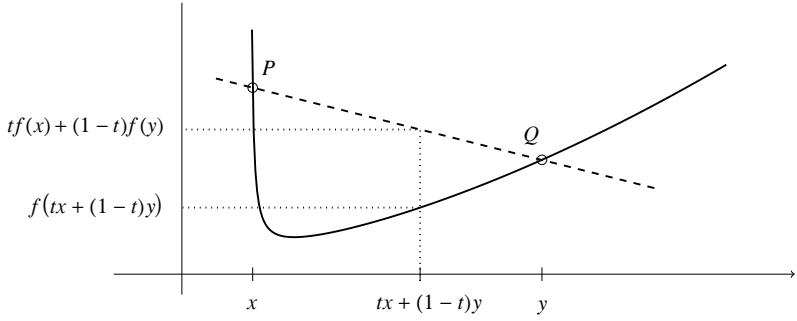


FIGURE 1: Graph of a convex function lies below the segment PQ

A function will be called *v-shaped* if there is a point $m \in I$ such that f decreases for $x < m$ and increases for $x > m$. The figure above shows the function $x^2 / \ln x$ which is both convex and v-shaped on the interval $(1, \infty)$.

Marek Kuczma proved, but never published, the following:

Theorem (Kuczma's Theorem):

If f is convex on an interval $I = (a, b) \subset (0, \infty)$, then the mapping $x \to f(x)/x$ is either monotonic or v-shaped.

The examples below illustrate the above theorem and show that in general the convexity of the function $f(x)/x$ is unpredictable. All the functions below are defined on the interval $(0, \infty)$.

(1) Let $f(x) = x^3$. Then $\dfrac{f(x)}{x} = x^2$ is strictly increasing and convex.

(2) For $f(x) = x \ln x$ we obtain $\dfrac{f(x)}{x} = \ln x$ which is strictly increasing and concave.

(3) The function $f(x) = \dfrac{1}{x}$ produces $\dfrac{f(x)}{x} = \dfrac{1}{x^2}$ which is strictly decreasing and convex.

(4) If $f(x) = x^3 - 1$ then $\dfrac{f(x)}{x} = x^2 - \dfrac{1}{x}$ which is strictly increasing but neither convex nor concave in its domain.

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(5) From $f(x) = x^2 + 1$ one gets $\dfrac{f(x)}{x} = x + \dfrac{1}{x}$ which is v-shaped and convex.

(6) The function $f(x) = 1 + x\sqrt{x}$ gives $\dfrac{f(x)}{x} = \dfrac{1}{x} + \sqrt{x}$ which is v-shaped but neither convex nor concave.

In fact the proof by Kuczma assumed differentiability and it was natural to ask whether this condition can be dropped. During the Problems and Remarks session of the 6th International Conference on General Inequalities in Oberwolfach, János Aczél and Che Tat Ng presented a geometric proof of this fact [1, p. 463] and Heinz König produced an analytic proof [1, p. 471].

The aim of this Note is to give two more proofs of Kuczma's Theorem. Both proofs are based on the following characterisation of convex functions.

Property 1: A function $f : I \to \mathbb{R}$ is convex if, and only if, for every $y \in I$ its divided difference $\dfrac{f(x) - f(y)}{x - y}$ increases in $I \setminus \{y\}$.

Geometrically this property means that if we consider two points on the graph of a convex function, fix one of them and let the other move from left to right, then the slope of the line segment joining them increases.

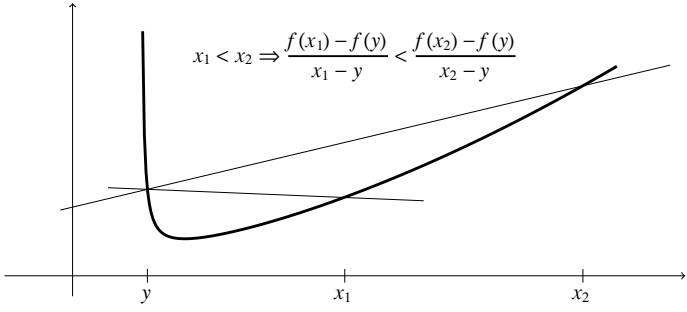


FIGURE 2: Slope of the secant increases

We were not able to find any popular book with the proof of this equivalence. Most of the books just refer to this fact or leave it as an exercise for the reader. Nevertheless this property is quite useful (see [2]) and this Note shows its power.

An immediate consequence of Property 1 is

Property 2: A convex function is monotonic or v-shaped.

Indeed, a convex function cannot have a local maximum, since the slope of the secant line at a maximum changes in the wrong direction.

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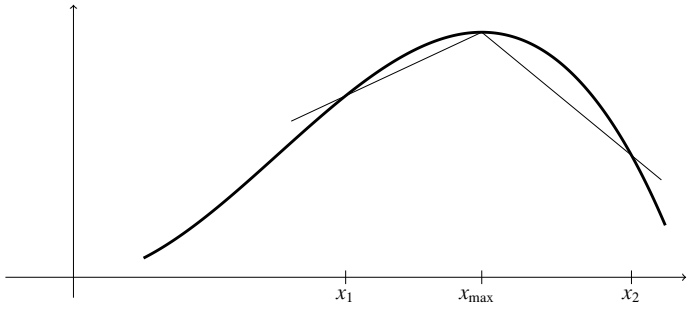


FIGURE 3: Slope at maximum changes in wrong direction

The first proof of Kuczma's Theorem uses a little-known property of convex functions. (The property can be found in a different formulation in [3, Theorem 119, p. 97]).

Lemma: Let $0 \leqslant a < b \leqslant \infty$. A function $f : (a, b) \to \mathbb{R}$ is convex if, and only if, the function $g : \left(b^{-1}, a^{-1}\right) \to \mathbb{R}$ given by the formula $g(x) = xf\left(x^{-1}\right)$ is convex. (We assume here $0^{-1} = \infty$ and $\infty^{-1} = 0$.)

Proof: Fix an arbitrary $y \in \left(b^{-1}, a^{-1}\right)$. For $x \in \left(b^{-1}, a^{-1}\right)$, $x \neq y$, we have

$$\begin{aligned}\frac{f\left(x^{-1}\right) - f\left(y^{-1}\right)}{x^{-1} - y^{-1}} &= xy\frac{f\left(x^{-1}\right) - f\left(y^{-1}\right)}{y - x}\\ &= y\frac{xf\left(x^{-1}\right) - yf\left(y^{-1}\right)}{y - x} + y\frac{yf\left(y^{-1}\right) - xf\left(y^{-1}\right)}{y - x}\\ &= -y\frac{g(x) - g(y)}{x - y} + yf\left(y^{-1}\right).\end{aligned}$$

If x increases then by Property 1 the left-hand side decreases, and so does the right-hand side. Also since $-y$ is negative we see that $\dfrac{g(x) - g(y)}{x - y}$ increases. This proves convexity of g, as required. To prove the equivalence note that the mapping $T(f)(x) = xf\left(x^{-1}\right)$ is an involution, i.e. $T^2(f) = f$.

First proof of Kuczma's Theorem

By the Lemma, the function $xf\left(x^{-1}\right) = \dfrac{f\left(x^{-1}\right)}{x^{-1}}$ is convex. Hence, by Property 2, $xf\left(x^{-1}\right)$ is monotonic or v-shaped. But the change of variable $t \leftrightarrow x^{-1}$ preserves both monotonicity and v-shapedness, and so it follows that the function $f(t)/t$ is either monotonic or v-shaped, as required.

The second proof employs the argument used to justify the Property 2.

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Second proof of Kuczma's Theorem

Let $g(x) = \dfrac{f(x)}{x}$. Fix y and suppose f to be convex. Then for $x \neq y$

$$\begin{aligned}\frac{g(x) - g(y)}{x - y} &= \frac{1}{xy}\,\frac{yf(x) - xf(y)}{x - y}\\ &= \frac{1}{xy}\,\frac{yf(x) - yf(y) + yf(y) - xf(y)}{x - y}\\ &= \frac{1}{x}\left(\frac{f(x) - f(y)}{x - y} - \frac{f(y)}{y}\right).\end{aligned}$$

Since, by Property 1, the expression $\dfrac{f(x) - f(y)}{x - y} - \dfrac{f(y)}{y}$ increases with x, and since x is positive, the right-hand side can change sign from minus to plus only. The same applies then to the left-hand side. This means that g cannot have any local maximum for the same reason as stated immediately after Property 2. Thus g must be monotonic or v-shaped.

Note

If $f : (0, b) \to \mathbb{R}$ is convex and $\lim\limits_{x \to 0^+} f(x) \leqslant 0$, then the function $f(x)/x$ increases. The reason is that we can extend f to a convex function on $[0, b)$ by setting $f(0) = 0$ and apply Property 1 with $y = 0$.

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