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Random Fourier–Stieltjes transforms on partially ordered groups

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1. *Introduction.* If one changes at random the signs of the Fourier coefficients of an L^2 function the result is still the sequence of Fourier coefficients of an L^2 function. L^2 is the only L^p space with this property; in fact if a sequence remains a Fourier transform for every insertion of $\pm$ signs then it is really the transform of an L^2 function ((6), p. 215, Theorem 8·14). Our main theorem is a version of this principle, for sequences defined on suitable subsets of a group; it is phrased in terms of large sets of choices of $\pm$ signs rather than all sequences of $\pm$ signs.

The setting is the following: G is a compact Abelian group whose dual group Γ is partially ordered with non-negative cone S. E, with elements $\gamma_0 < \gamma_1 < \ldots$ is a countable totally ordered subset of S. w is a sequence defined on E so that for all t in a subset of $[0, 1]$, of positive Lebesgue measure, the function which is 0 on $S \sim E$ and is $r_n(t)\, w(\gamma_n)$ on E, where r_n is the nth Rademacher function, is the restriction to S of a Fourier–Stieltjes transform. We show that every such w is in $l^2(E)$.

Gaudry proves this for totally ordered Γ in ((3), Theorem 1); the case when Γ is partially ordered and the $r_n(t)$ are replaced by all sequences of $\pm$ signs is done by the author in ((2), section 3, Theorem 3). The proof to be given here is a modification of the argument in the latter paper. One feature is that unlike many proofs of theorems of this type, it does not make use of properties of the analytic contraction operator or of pointwise summability. It also provides an estimate of the l^2 norm of w when there is a uniform bound on the norms of the measures whose transforms have the desired restriction to S. The proof of the main result is in section 4; before coming to it we shall present a lemma needed in the proof and establish some facts about Walsh–Paley functions.

Our notation will be fairly standard and consistent with (5); in particular, groups will be written additively. Partial orders on groups Γ are induced by sub-semigroups S with the property that $0 \in S$ and for no other element γ of Γ do both γ and $-\gamma$ belong to S; the rule is that $\gamma \leqslant \gamma'$ if and only if $\gamma' - \gamma \in S$. For instance if $\Gamma = Z \times Z$, where Z is the group of integers, we can take:

$$S = \{(m, n) | m \geqslant 0 \quad \text{and} \quad n \geqslant 0\}.$$

Then $(m, n) \leqslant (m', n')$ means that $m \leqslant m'$ and $n \leqslant n'$. This order arises naturally in the study of double power series ((2), section 1). Letting S be some other quadrant would induce a different order on $Z \times Z$; so would taking S to be an angular region in which the argument of $m + in$ was restricted to an interval of length less than π. In general, the larger S is, the more totally ordered sets there will be, and Γ itself will be totally ordered if and only if S is a half space, that is, for each $\gamma \neq 0$ one but not both of γ and $-\gamma$ are in S.

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In section 5 we discuss the extent to which the order assumptions on Γ and E are needed. They are not necessary if it is known that the Fourier transforms with the desired restrictions to E come from $L^p(G)$ functions with $p > 1$.

2. A useful lemma.

The following appears as Lemma 2 in ((2), section 2) and is included here for completeness. It is the crucial step in our proof of Theorem 2.

LEMMA. *Let H be a Hilbert space and $M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ be closed subspaces of H. Let $A_0, A_1, \ldots, A_n, \ldots$ be unitary linear operators on H with $A_1 M_1 \subset A_2 M_2 \subset \ldots$. Suppose that g and h are elements of H satisfying*:

(i) $A_n h \in A_{n+1} M_{n+1}$ *for all* $n \geqslant 0$,

(ii) g *is orthogonal to the subspaces* $A_{n+1} M_n$ *for all* $n \geqslant 1$.

Then $\left(\sum_0^\infty |\langle g, A_n h\rangle|^2\right)^{\frac{1}{2}} \leqslant 2\|g\| \cdot \|h\|$.

Proof. The homogeneity of the conclusion makes it enough to consider the case $\|g\| = \|h\| = 1$.

Let M_0 denote the subspace of H consisting of 0 alone; with this understanding condition (ii) also holds when $n = 0$.

For each $n \geqslant 0$ let k_n be the orthogonal projection of h onto M_n. Then the sequence $k_1 - k_0, k_2 - k_1, \ldots$ is orthogonal and

$$\sum_1^\infty \|k_n - k_{n-1}\|^2 \leqslant \|h - k_0\|^2 = 1. \tag{1}$$

For each n, because A_n is unitary, $A_n(h - k_n)$ is orthogonal to $A_n M_n$. But if $m < n$,

$$A_m h \in A_{m+1} M_{m+1} \subset A_n M_n$$

and

$$A_m k_m \in A_m M_m \subset A_n M_n.$$

Therefore, if $m \neq n$,

$$A_m(h - k_m) \quad \text{and} \quad A_n(h - k_n) \tag{2}$$

are orthogonal and have norms at most 1.

Now write

$$\langle g, A_0 h\rangle = \langle g, A_0(h - k_0)\rangle + \langle g, A_0 k_0\rangle$$

and for $n \geqslant 1$

$$\langle g, A_n h\rangle = \langle g, A_n(h - k_n)\rangle + \langle g, A_n(k_n - k_{n-1})\rangle + \langle g, A_n k_{n-1}\rangle.$$

Since $k_0 = 0$,

$$\langle g, A_0 k_0\rangle = 0.$$

By (ii)

$$\langle g, A_n k_{n-1}\rangle = 0 \quad \text{for all} \quad n \geqslant 1.$$

By (2) and Bessel's inequality

$$\sum_0^\infty |\langle g, A_n(h - k_n)\rangle|^2 \leqslant \|g\|^2 = 1.$$

Finally for $n \geqslant 1$

$$|\langle g, A_n(k_n - k_{n-1})\rangle| \leqslant \|g\| \cdot \|A_n(k_n - k_{n-1})\| = \|k_n - k_{n-1}\|$$

and by (1)

$$\sum_1^\infty |\langle g, A_n(k_n - k_{n-1})\rangle|^2 \leqslant 1.$$

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Applying the triangle inequality for l^2 we get,

$$\left(\sum_0^\infty |\langle g, A_n h\rangle|^2\right)^{\frac{1}{2}} \leqslant 2.$$

3. Rademacher functions and Walsh–Paley functions.

The nth Rademacher function, r_n, is defined on $[0, 1]$ by $r_n(t) = \operatorname{sgn} \sin(2^{n+1}\pi t)$. In other words, $r_0(t)$ is 1 for $t \in (0, \frac{1}{2})$, -1 for $t \in (\frac{1}{2}, 1)$, and $r_0(0) = r_0(\frac{1}{2}) = r_0(1) = 0$. If r_0 is extended with period 1 to all real numbers then $r_n(t) = r_0(2^n t)$ for all t.

The Rademacher system is orthonormal on $[0, 1]$ but not complete. A natural completion is provided by the Walsh–Paley functions ψ_m which are defined as follows:

$$\psi_0(t) = 1 \quad \text{for all } t.$$

Consider the dyadic representation of each $m > 0$,

$$m = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_i}$$

as a sum of distinct powers of 2. Then let

$$\psi_m(t) = r_{n_1}(t) \cdot r_{n_2}(t) \ldots r_{n_i}(t).$$

For a discussion of this system and a proof of completeness see ((6), p. 34, examples 6 and 7).

Now $r_n(t)$ is constant on any dyadic interval of the form

$$((j-1)/2^{n'}, j/2^{n'}) \quad \text{for} \quad n' \geqslant n+1.$$

If $n' \leqslant n$, then $r_n(t)$ alternates signs throughout the interval and its integral over the interval is 0. The behaviour of ψ_m is this connection is determined by the factor r_n of largest index in the definition of ψ_m; for it will be the only factor to change sign inside a dyadic interval of length 2^{-n}. Thus the integral of ψ_m over such an interval, and hence over any larger dyadic interval, is 0.

Suppose that $k(t)$ is constant on each dyadic interval of length 2^{-n} and consider its Walsh–Paley coefficients

$$a_m = \int_0^1 k(t)\, \psi_m(t)\, dt.$$

The above remarks show that this integral will be 0 if ψ_m has any factor r_j of index $j \geqslant n$. Since ψ_m has such a factor if and only if $m \geqslant 2^n$. it follows that for such a function k

$$a_m = 0 \quad \text{for all} \quad m \geqslant 2^n.$$

Now the product of two Walsh–Paley functions is simply the product of the various Rademacher functions appearing in their representations, except that when a factor r_n appears in both terms it can be omitted in the product as $r_n^2(t) = 1$ almost everywhere. We shall only need to know about products of the form $r_n \psi_m$. If $m < 2^n$, r_n is not a factor of ψ_m and will appear in the representation of the product so that $r_n \psi_m = \psi_{m+2^n}$ in this case. If $m \geqslant 2^n$, r_n is a factor of ψ_m and will be omitted in the representation of the product, so that $r_n \psi_m = \psi_{m-2^n}$.

4. *The main theorems.* Recall our convention that G is a compact Abelian group whose dual group Γ is partially ordered with non-negative cone S.

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DEFINITION. *A subset I of Γ will be called* convex *if $\gamma, \gamma' \in I$ and $\gamma \leqslant \gamma'' \leqslant \gamma'$ together imply that $\gamma'' \in I$.*

Since S itself is convex the result announced in the introduction is a special case of the following.

THEOREM 1. *Let I be a convex set and E, with elements $\gamma_0 < \gamma_1 < \gamma_2 < \ldots$, be a totally ordered subset of I. Suppose that w is a function on E with the property that, for every t in a set D contained in $[0, 1]$ and having positive Lebesgue measure, the function which is 0 on $I \sim E$ and which is equal to $r_n(t)\, w(\gamma_n)$ on E is the restriction to I of a Fourier–Stieltjes transform. Then $w \in l^2(E)$.*

Proof. The assumption is that for each t in D there is a complex valued regular Borel measure μ_t, whose transform $\hat{\mu}_t$ has the desired restriction to I. The first step is to show that the μ_t can also be chosen so that the total variation norm $\|\mu_t\|$ stays bounded as t ranges over a set D_1, contained in D, but still of positive Lebesgue measure. We modify the approach used by Gaudry in ((3), Theorem 1).

For simplicity, let us replace D by a slightly smaller set which is compact, omits the countably many discontinuities of the various r_n, and still has positive Lebesgue measure.

Let $C(G)$ be the space of all continuous functions on G, with the supremum norm, and let C_I be the subspace of continuous functions whose Fourier transforms vanish off I. For each t in D consider the bounded linear functional T_t on C_I given by

$$T_t f = \int_G f(-x)\, d\mu_t(x) \quad \text{for all} \quad f \quad \text{in} \quad C_I.$$

Clearly $\|T_t\| \leqslant \|\mu_t\|$. By the Hahn–Banach theorem, T_t can be extended in a norm preserving fashion to all of $C(G)$. It then has the representation

$$T_t f = \int_G f(-x)\, d\nu_t(x) \quad \text{for all} \quad f \quad \text{in} \quad C(G),$$

where ν_t is a regular Borel measure with $\|\nu_t\| = \|T_t\|$. Moreover $\hat{\nu}_t$ has the same restriction to I as $\hat{\mu}_t$, for if $\gamma \in I$ then the function $\gamma(x)$ is in C_I and

$$\hat{\nu}_t(\gamma) = \int_G \gamma(-x)\, d\nu_t(x) = T_t \gamma = \int_G \gamma(-x)\, d\mu_t(x) = \hat{\mu}_t(\gamma).$$

Replacing μ_t by ν_t, if necessary, we can assume that $\|\mu_t\| = \|T_t\|$.

Since the trigonometric polynomials in C_I are dense in C_I,

$$\|T_t\| = \sup |T_t g|,$$

where g runs through all trigonometric polynomials in C_I with $\|g\|_\infty = 1$. For each such g,

$$\begin{aligned} |T_t g| &= \left| \int_G \sum_I \hat{g}(\gamma)\, \gamma(-x)\, d\mu_t(x) \right| \\ &= \left| \sum_I \hat{g}(\gamma)\, \hat{\mu}_t(\gamma) \right| \\ &= \left| \sum_{n=0}^{\infty} \hat{g}(\gamma_n)\, w(\gamma_n)\, r_n(t) \right|, \end{aligned}$$

where all sums are finite as $\hat{g}$ has finite support. Therefore $|T_t g|$ is a continuous function of t on D, and $\|\mu_t\| = \sup |T_t g|$ is lower semi-continuous. Since $\|\mu_t\|$ is a finite measur-

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able function on D there must be a set D_1, contained in D, but still of positive measure, so that $\|\mu_t\|$ stays bounded on D_1. Replacing D by D_1, if necessary, we have reduced Theorem 1 to:

THEOREM 2. *With G, Γ, I, E, w, D, and μ_t all as above, suppose that there is a constant C so that $\|\mu_t\| \leqslant C$ for all t in D. Then there is a constant B, depending only on D, so that*

$$(\sum_E |w(\gamma)|^2)^{\frac{1}{2}} \leqslant B.C.$$

Proof. Let χ_D, the characteristic function of D, have the Walsh–Paley expansion

$$\chi_D(t) = \sum_{m=0}^{\infty} d_m \psi_m(t).$$

Here $d_0 = \int_D dt$ is the Lebesgue measure of D and is positive. Since $\chi_D \in L^2[0, 1]$, there is an integer N with

$$\left(\sum_{2^N}^{\infty} d_m^2 \right)^{\frac{1}{2}} \leqslant \tfrac{1}{4} d_0.$$

Our constant B is just $4 + 5\sqrt{N}$.

Earlier we arranged for $\|\mu_t\|$ to be a measurable function of t; now we want μ_t itself to depend in a regular way on t. To make this possible let us assume for the moment that there is a finite set $L \subset \Gamma$ so that $\hat{\mu}_t$ is supported by L for all t in D. It follows that w is also supported by L. So, there is an integer K with

$$w(\gamma_n) = 0 \quad \text{for all} \quad n \geqslant K.$$

Then the restriction of $\hat{\mu}_t$ to I is 0 except perhaps at $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{K-1}$. This means that, as a function of t, this restriction depends only on the values of

$$r_0(t), r_1(t), \ldots, r_{K-1}(t)$$

which are in turn completely determined by the dyadic interval of the form

$$((j-1)/2^K, j/2^K)$$

to which t belongs. Therefore we can redefine μ_t for all t in $[0, 1]$ in such a way that $\hat{\mu}_t$ has the desired behaviour on I and is still supported by L for all t, and so that the same μ_t is used for all t in the same open dyadic interval of length 2^{-K}. For each such interval J which intersects D, pick a representative t_0 in $J \cap D$ and set $\mu_t = \mu_{t_0}$ for all t in J; on intervals which are disjoint from D and at points $t = j/2^K$ where μ_t is not already defined, make the definition

$$d\mu_t(x) = \left[\sum_{n=0}^{K-1} w(\gamma_n) r_n(t) \gamma_n(x) \right] dx.$$

For each γ in L, $\hat{\mu}_t(\gamma)$ is a constant function of t on open dyadic intervals of length 2^{-K}. Its Walsh–Paley expansion will have the simple form

$$\hat{\mu}_t(\gamma) \sim \sum_{m=0}^{2^K-1} c_m(\gamma) \psi_m(t)$$

with the two sides equal for almost all t.

Let

$$P(x, t) = \sum_{\gamma \in L} \sum_{m < 2^K} c_m(\gamma) \psi_m(t) \gamma(x) \sim \sum_{\gamma \in L} \hat{\mu}_t(\gamma) \gamma(x)$$

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so that $d\mu_t(x) \sim P(x,t)\,dx$. P is a finite sum of measurable functions on $G \times [0,1]$ and is itself measurable. We have no useful hold on its L^1 norm, however, and we consider instead

$$F(x,t) = \chi_D(t)\,P(x,t).$$

Then

$$\begin{aligned}\int_0^1 \int_G |F(x,t)|\,dx\,dt &= \int_D \int_G |P(x,t)|\,dx\,dt \\ &= \int_D \int_G d|\mu_t|(x)\,dt \\ &\leqslant \int_D C\,dt = C d_0.\end{aligned}$$

So, F is an L^1 function on $G \times [0,1]$ with norm at most Cd_0. For each fixed t, $F(x,t)$ as a function of x is in $L^1(G)$ and has Fourier coefficients

$$\begin{aligned}F(.,t)^\wedge(\gamma) &= \chi_D(t)\,P(\cdot,t)^\wedge(\gamma) \\ &= \chi_D(t)\,\hat{\mu}_t(\gamma) \quad \text{for almost all } t \\ &= 0 \quad \text{if} \quad \gamma \in I \quad \text{but} \quad \gamma \notin E.\end{aligned}$$

If $\gamma = \gamma_n \in E$, we have

$$\begin{aligned}F(\cdot,t)^\wedge(\gamma_n) &= \chi_D(t)\,\hat{\mu}_t(\gamma_n) \quad \text{for almost all } t \\ &= \chi_D(t)\,r_n(t)\,w(\gamma_n) \sim w(\gamma_n)\,r_n(t) \sum_{m=0}^{\infty} d_m \psi_m(t).\end{aligned}$$

Now define

$$Q(x,t) = \sum_{n=0}^{\infty} \sum_{m=2^N}^{\infty} d_m\,w(\gamma_n)\,r_n(t)\,\psi_m(t)\,\gamma_n(x),$$

where the series converges in $L^2(G \times [0,1])$ for, as m and n vary, the functions

$$r_n(t)\,\psi_m(t)\,\gamma_n(x)$$

are orthonormal in $L^2(G \times [0,1])$, and

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} \sum_{m=2^N}^{\infty} |d_m\,w(\gamma_n)|^2\right)^{\frac{1}{2}} \\ &\quad = \left(\sum_{n=0}^{\infty} |w(\gamma_n)|^2\right)^{\frac{1}{2}} \left(\sum_{m=2^N}^{\infty} d_m^2\right)^{\frac{1}{2}} \\ &\quad \leqslant \left(\sum_{n=0}^{K-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}} \cdot \tfrac{1}{4} d_0.\end{aligned}$$

From the definition of Q it follows that

$$Q(\cdot,t)^\wedge(\gamma) = 0 \quad \text{if} \quad \gamma \notin E$$

and

$$Q(\cdot,t)^\wedge(\gamma_n) \sim w(\gamma_n)\,r_n(t) \sum_{m=2^N}^{\infty} d_m \psi_m(t)$$

for all n.

Finally let

$$f(x,t) = F(x,t) - Q(x,t).$$

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Using $\|\cdot\|_p$ to denote L^p norms on $G\times[0,1]$ we have

$$\|f\|_1 \leqslant \|F\|_1 + \|Q\|_1 \leqslant Cd_0 + \|Q\|_2$$
$$\leqslant Cd_0 + \left(\sum_0^{K-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}} \cdot \tfrac{1}{4}d_0.$$

Also,
$$f(\cdot,t)^\wedge(\gamma) = F(\cdot,t)^\wedge(\gamma) - Q(\cdot,t)^\wedge(\gamma)$$
$$= 0 \quad \text{if} \quad \gamma \in I \sim E \tag{3}$$

and
$$f(\cdot,t)^\wedge(\gamma_n) \sim w(\gamma_n) r_n(t) \sum_{m=0}^{2^N-1} d_m \psi_m(t) \quad \text{for all} \quad n.$$

In order to simplify the latter expansion we assume temporarily that

$$w(\gamma_n) = 0 \quad \text{for all} \quad n < N.$$

Therefore, when $n < N$,
$$f(\cdot,t)^\wedge(\gamma_n) = 0. \tag{4}$$

When $n \geqslant N$ and $m < 2^N$, we have $m < 2^n$ and $r_n \psi_m = \psi_{m+2^n}$. For such n,

$$f(\cdot,t)^\wedge(\gamma_n) \sim \sum_{m=0}^{2^N-1} w(\gamma_n)\, d_m r_n(t)\, \psi_m(t)$$
$$= \sum_{m=0}^{2^N-1} w(\gamma_n)\, d_m \psi_{m+2^n}(t)$$
$$= \sum_{m=2^n}^{2^n+2^N-1} w(\gamma_n)\, d_{m-2^n} \psi_m(t). \tag{5}$$

All that really matters is that, since $n \geqslant N$, the above expansion only involves indices m with $2^n \leqslant m < 2^{n+1}$.

Factor
$$f(x,t) = g(x,t)\,\bar{h}(x,t)$$

with $\|g\|_2 = \|h\|_2 = \|f\|_1^{\frac{1}{2}}$. We shall apply the Lemma of section 2 with $H = L^2(G\times[0,1])$. For all k in H and all $n \geqslant 0$ set

$$A_n k = \gamma_n r_n k = \gamma_n \psi_{2^n} k.$$

The A_n are unitary operators on H since

$$|\gamma_n(x)\, r_n(t)| = 1 \quad \text{p.p. on} \quad G\times[0,1].$$

The reason for this choice of A_n is that

$$\langle g, A_n h\rangle = \int_0^1 \int_G g(x,t)\,\bar{h}(x,t)\,\bar{\gamma}_n(x)\,\psi_{2^n}(t)\,dx\,dt$$
$$= \int_0^1 \psi_{2^n}(t) \int_G f(x,t)\,\bar{\gamma}_n(x)\,dx\,dt$$
$$= \int_0^1 \psi_{2^n}(t) f(\cdot,t)^\wedge(\gamma_n)\,dt$$
$$= w(\gamma_n)\, d_0, \quad \text{for all} \quad n \geqslant N,$$

and the lemma will provide us with an estimate of $\left(\sum_N^\infty |w(\gamma_n)|^2\right)^{\frac{1}{2}}$.

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For each $n > 0$, let M_n be the closed subspace in H generated by

$$\{\gamma\psi_m h | \gamma_1 - \gamma_n \leqslant \gamma < 0, \quad m < 2^{n+1}\}.$$

The total ordering of E guarantees that

$$M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots.$$

The rules established in section 3 for multiplying Walsh–Paley functions by Rademacher functions show that $A_n M_n$ is the closed linear span of

$$\{\gamma\psi_m h | \gamma_1 \leqslant \gamma < \gamma_n, m < 2^{n+1}\}.$$

So,
$$A_1 M_1 \subset A_2 M_2 \subset \ldots.$$

Also it is clear that for each $n \geqslant 0$

$$A_n h = \gamma_n \psi_{2^n} h \in A_{n+1} M_{n+1}.$$

Finally, $A_{n+1} M_n$ is generated by

$$\{\gamma\psi_m h | \gamma_1 + \gamma_{n+1} - \gamma_n \leqslant \gamma < \gamma_{n+1}, 2^{n+1} \leqslant m < 2^{n+2}\}.$$

We want the inner product

$$\begin{aligned} \langle g, \gamma\psi_m h\rangle &= \int_0^1 \int_G g(x,t)\, \overline{h}(x,t)\, \overline{\gamma}(x)\, \psi_m(t)\, dx\, dt \\ &= \int_0^1 f(\cdot,t)^{\wedge}(\gamma)\, \psi_m(t)\, dt \end{aligned} \tag{6}$$

to be 0 for every generator $\gamma\psi_m h$ of $A_{n+1} M_n$. Now any γ restricted as above must be in I since

$$\gamma_1 < \gamma_1 + \gamma_{n+1} - \gamma_n \leqslant \gamma < \gamma_{n+1}$$

and I is a convex set containing γ_1 and γ_{n+1}. A glance at the various expansions (3), (4) and (5) for $f(\cdot,t)\,(\gamma)^{\wedge}$ when $\gamma \in I$ shows that for any fixed m the integral (6) is non-zero for at most one choice of γ, namely

$$\gamma = \gamma_{n'} \quad \text{with} \quad n' \geqslant N \quad \text{and} \quad 2^{n'} \leqslant m < 2^{n'+1}. \tag{7}$$

For generators $\gamma\psi_m h$ of $A_{n+1} M_n$ we have $2^{n+1} \leqslant m < 2^{n+2}$, so that (7) would require $n' = n+1$ and $\gamma = \gamma_{n+1}$, contrary to $\gamma < \gamma_{n+1}$. Therefore (7) never arises for generators of $A_{n+1} M_n$, $\langle g, \gamma\psi_m h\rangle = 0$ for every such generator, and g is orthogonal to $A_{n+1} M_n$ for all $n \geqslant 1$.

The assumptions of the lemma are satisfied and we have:

$$\begin{aligned} \left(\sum_0^\infty |w(\gamma_n)\, d_0|^2\right)^{\frac{1}{2}} &= \left(\sum_N^\infty |w(\gamma_n)\, d_0|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_N^\infty |\langle g, A_n h\rangle|^2\right)^{\frac{1}{2}} \\ &\leqslant 2\|g\|_2 . \|h\|_2 \\ &= 2\|f\|_1 \\ &\leqslant 2\left[C + (\tfrac{1}{4})\left(\sum_0^\infty |w(\gamma_n)|^2\right)^{\frac{1}{2}}\right] d_0. \end{aligned}$$

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Since $d_0 > 0$ we can divide by it and transpose the term involving w to obtain

$$\left(\sum_0^\infty |w(\gamma_n)|^2\right)^{\frac{1}{2}} \leqslant 4C.$$

Now we remove the condition $w(\gamma_n) = 0$ for all $n < N$, but still assume that $\hat{\mu}_t$ is supported by the finite set L for all t in D. Define a new sequence v on E by

$$v(\gamma_n) = 0 \qquad \text{for} \quad n < N$$

and

$$v(\gamma_n) = w(\gamma_n) \quad \text{for} \quad n \geqslant N.$$

Define a new measure ν_t for each t in D by

$$d\nu_t(x) = d\mu_t(x) - \left[\sum_{n=0}^{N-1} w(\gamma_n)\, r_n(t)\, \gamma_n(x)\right] dx.$$

Then the restriction of ν_t to I is 0 off E and equal to $v(\gamma_n)\, r_n(t)$ on E. Moreover, for all t in D,

$$\begin{aligned}\|\nu_t\| &\leqslant \int_G d|\mu_t|(x) + \int_G \left|\sum_0^{N-1} w(\gamma_n)\, r_n(t)\, \gamma_n(x)\right| dx \\ &\leqslant C + \left(\int_G \left|\sum_0^{N-1} w(\gamma_n)\, r_n(t)\, \gamma_n(x)\right|^2 dx\right)^{\frac{1}{2}} \\ &= C + \left(\sum_0^{N-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Our previous assumptions apply to v and we conclude that

$$\left(\sum_0^\infty |v(\gamma_n)|^2\right)^{\frac{1}{2}} \leqslant 4\left[C + \left(\sum_0^{N-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}}\right].$$

But for any t in D

$$|w(\gamma_n)| = |\hat{\mu}_t(\gamma_n)| \leqslant C$$

so that,

$$\left(\sum_0^{N-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}} \leqslant C\sqrt{N}.$$

Therefore

$$\begin{aligned}\left(\sum_0^\infty |w(\gamma_n)|^2\right)^{\frac{1}{2}} &\leqslant \left(\sum_0^{N-1} |w(\gamma_n)|^2\right)^{\frac{1}{2}} + \left(\sum_N^\infty |v(\gamma_n)|^2\right)^{\frac{1}{2}} \\ &\leqslant C\sqrt{N} + 4[C + C\sqrt{N}] \\ &= (4 + 5\sqrt{N})\, C.\end{aligned}$$

Finally, suppose that there is no finite set supporting all $\hat{\mu}_t$. Let k be a trigonometric polynomial on G and let L be the support of $\hat{k}$. Define a new sequence v by

$$v(\gamma_n) = w(\gamma_n)\, \hat{k}(\gamma_n)$$

for all n, and new measures ν_t by $\nu_t = \mu_t * k$ for all t in D. Then $\|\nu_t\| \leqslant C\|k\|_1$ for all t in D, and the restriction of $\hat{\nu}_t$ to I is 0 off E and equal to $v(\gamma_n)$ on E. Moreover, $\hat{\nu}_t$ is supported by L for all t in D. The previous work applies to v and yields

$$\begin{aligned}\left(\sum_0^\infty |w(\gamma_n)\, \hat{k}(\gamma_n)|^2\right)^{\frac{1}{2}} &= \left(\sum_0^\infty |v(\gamma_n)|^2\right)^{\frac{1}{2}} \\ &\leqslant (4 + 5\sqrt{N})\, C\, .\, \|k\|_1.\end{aligned}$$

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Now let k run through an approximate identity, with $\lim \|k\|_1 = 1$, and take limits on both sides of the above to obtain

$$\left(\sum_0^\infty |w(\gamma_n)|^2 \right)^{\frac{1}{2}} \leqslant (4 + 5\sqrt{N})\, C. \qquad \blacksquare$$

Remark 1. The restriction to countable E is necessary only because the Rademacher system on $(0, 1]$ is countable. To deal with uncountable sets E, one can devise a system of $\pm$ valued functions r_γ, as follows: Consider the strong direct product $X = \prod_E \{-1, 1\}_\gamma$ of copies of the multiplicative group $\{-1, 1\}$, indexed by E. If $t \in X$ and $\gamma \in E$ then $r_\gamma(t)$ is simply the γth entry t_γ. With the product topology, X is a compact Abelian group under termwise multiplication and carries a Haar measure dt. The system $r_\gamma(t)$ is still orthonormal and can be completed by adding all finite products of r_γ's, including the empty product. If E happens to be countable this construction yields a system which is equivalent to the usual Rademacher and Walsh–Paley system ((1), pp. 376–377).

With $[0, 1]$ replaced by X and the r_n replaced by the r_γ, Theorems 1 and 2 remain true for uncountable sets E. Of course, the conclusion $w \in l^2(E)$ implies that w has countable support after all.

Remark 2. In Theorem 1 it is possible to apply the 0–1 law of probability ((4), p. 37, Théorème 1) to conclude that there are measures μ_t, with the desired properties, for almost all t. For the $r_n(t)$ are independent random variables and for each t the existence of a suitable μ_t does not depend on the initial values of the sequence

$$\left\{ w(\gamma_n)\, r_n(t) \right\}_{n=0}^\infty .$$

Since the set D of t for which such μ_t exist has positive Lebesgue measure, the 0–1 law says that D has measure 1.

In our proof, however, there seems to be no advantage in knowing that D is of full measure for we immediately go to the smaller set D_1 on which the norms $\|\mu_t\|$ stay bounded. The conclusion of our theorem does guarantee that measures μ_t can be found whose norms stay bounded for all t, but we have no direct proof of this fact. Such a direct proof could be combined with the method of ((2), section 3, Theorem 3) to shorten the proof that $w \in l^2(E)$.

5. *The L^p case.* The following theorem may not be new but it provides an interesting contrast with the situation already discussed.

THEOREM 3. *Let G be a compact Abelian group with dual group Γ and let E, with elements $\gamma_0, \gamma_1, \gamma_2, \ldots$, be a countable subset of Γ. Suppose that w is a sequence defined on E with the property that for every t in a subset D of $[0, 1]$, having positive Lebesgue measure, there is an index $p(t) > 1$ so that the sequence $\{r_n(t)\, w(\gamma_n)\}_{n=0}^\infty$ is the restriction to E of the Fourier transform of a function in $L^{p(t)}(G)$. Then $w \in l^2(E)$.*

Proof. If $w \notin l^2(E)$ choose sequence $v \in l^2(E)$ with

$$v(\gamma)\, w(\gamma) \geqslant 0 \quad \text{for all} \quad \gamma \quad \text{in} \quad E$$

and

$$\sum_0^\infty v(\gamma_n)\, w(\gamma_n) = \infty.$$

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Since $v \in l^2(E)$, the series $\sum_0^\infty r_n(t)\, v(\gamma_n)\, \gamma_n(x)$

is for almost all t and all $q < \infty$ the Fourier series of a function $g_t(x)$ in $L^q(G)$. [The proof of this is identical to the one given for the circle group in (6), p. 215, Theorem 8·16.] In particular there is a t in D for which $g_t \in L^q$ for all q and $|r_n(t)| = 1$ for all n. Let $p = p(t) > 1$ be an index for which there is a function $f_t \in L^p(G)$ with

$$\hat{f}_t(\gamma_n) = r_n(t)\, w_n \quad \text{for all} \quad n.$$

Now $$\int_G |f_t(x)\, g_t(-x)|dx \leqslant \|f_t\|_p \cdot \|g_t\|_{p'} < \infty$$

where p' is the index conjugate to p. Moreover, the inequality remains valid if g_t is replaced by $g_t * k$, where k is any trigonometric polynomial with $\|k\|_1 = 1$.

Consider the finite series

$$\sum_0^\infty v(\gamma_n)\, w(\gamma_n)\, \hat{k}(\gamma_n) = \sum_0^\infty \hat{f}_t(\gamma_n)\, \hat{g}_t(\gamma_n)\, \hat{k}(\gamma_n)$$

$$= \int_G f_t(x)\, g_t * k(-x)\, dx.$$

If $\hat{k}(\gamma) \geqslant 0$ for all γ, we have

$$\sum_0^\infty v(\gamma_n)\, w(\gamma_n)\, \hat{k}(\gamma_n) \leqslant \|f_t\|_p \cdot \|g_t\|_{p'}.$$

Taking limits as k runs through on approximate identity, we get

$$\sum_0^\infty v(\gamma_n)\, w(\gamma_n) \leqslant \|f_t\|_p \cdot \|g_t\|_{p'} < \infty$$

contrary to our choice of v.

Therefore $w \in l^2(E)$. ∎

COROLLARY. *Let $E \subset \Gamma$ be a Sidon set and suppose that $f \in L^p(G)$ for some $p > 1$. Then $\sum_E |\hat{f}(\gamma)|^2 < \infty$.*

Proof. Let $\epsilon(\gamma)$ be any ± 1 valued function on E. By ((5), p. 121, Theorem 5·7·3·(*d*)), there is a measure μ with $\hat{\mu}(\gamma) = \epsilon(\gamma)$ for all $\gamma \in E$. This means that $g = \mu * f$ is an L^p function with $\hat{g}(\gamma) = \epsilon(\gamma)\hat{f}(\gamma)$ for all γ in E.

Applying Theorem 3 to each countable subset E' of E, we obtain

$$\sum_{E'} |\hat{f}(\gamma)|^2 < \infty.$$

Therefore $$\sum_E |\hat{f}(\gamma)|^2 < \infty.$$ ∎

The analogous theorem and corollary are both false when L^p is replaced by L^1. For if E is a Sidon set and w is any sequence tending to 0 on E then for every choice of $\pm$ signs the function $\pm w(\gamma)$ is the restriction to E of the Fourier transform of an L^1 function, ((5), p. 121, Theorem 5·7·4·(*e*)).

So the assumption, in Theorem 1, that there is a convex set I, containing E, with $\hat{\mu}_t(\gamma) = 0$ for all $\gamma \in I \sim E$, is to some extent necessary. On the other hand, all that our

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proof really uses about the relation $\leqslant$ is that it is transitive and compatible with the group operation. We can allow loops of the form $\gamma \leqslant \gamma' \leqslant \gamma$ but $\gamma \neq \gamma'$, which arise if S is a semigroup with 0 containing both γ and $-\gamma$ for some $\gamma \neq 0$. The only changes needed in the proof for this case are the choices of A_n and M_n. We should take

$$A_n k = \gamma_{N+n} r_{N+n} k \quad \text{for all} \quad k \in L^2(G \times (0,1])$$

and M_n to be the closed linear span of

$$\{\gamma \psi_m h | \gamma_N - \gamma_{N+n} \leqslant \gamma \leqslant 0, \quad m < 2^{N+n+1}, \qquad \text{and when} \qquad \gamma = 0, \quad m \geqslant 2^N\}.$$

This weakening of the partial order is forced on us if we wish to discuss the situation when S contains non-zero elements of finite order. It also helps us to cast Theorem 1 in a more familiar form. If we take $I = S = \Gamma$ then the condition that E be totally ordered is no restriction at all. Our theorem then reads: If for all t in D there are measures μ_t with $\hat{\mu}_t(\gamma) = 0$ for all γ not in E and $\hat{\mu}_t(\gamma_n) = r_n(t) w_n$ for all n, then $\sum_0^\infty |w_n|^2 < \infty$. This version of the theorem is well known, at least for the circle group ((4), p. 78, Proposition 12).

Theorem 1 tells us that for some sets E it is enough to know the behaviour of $\hat{\mu}_t$ on a subset I of Γ. I is the convex hull of E with respect to some partial order chosen to make E totally ordered. In some cases I will just be a coset containing E and the theorem reduces to the one discussed above. For many sets E, however, I is a proper subset of the smallest coset containing E. For instance, if $\Gamma = Z \times Z$ and $E = \left\{\gamma_n\right\}_{n=0}^\infty$ is a subset of the first quadrant S, with γ_{n+1} lying above and to the right of γ_n for all n, then I can be taken to be S or even some subsets of S.

Remark 3. A short proof of the corollary to Theorem 3 appears in ((5), p. 130). This result can be used in a proof of Theorem 3, following the lines of the proof of Theorems 1 and 2 up to the computation of $F(\cdot, t)^\wedge(\gamma_n)$. At that point, one can use the fact that the set of products $r_n \gamma_n$ plays the role of a Sidon set and conclude that

$$\sum_{n=0}^\infty |d_0 w(\gamma_n)|^2 < \infty.$$

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