4

Two Simple Examples

Having established a connection with classical scattering theory, we can now continue our introduction of the asymptotic method by discussing examples that are both elementary and useful. The first is the familiar case of the scattering of a charged particle by a Coulomb field and the second is the scattering by a much smoother interaction that is Gaussian in shape. The latter example will lead us to consider pairs of trajectories that contribute interfering amplitudes and we show in a third section how these paired trajectories can exhibit an interesting behavior that leads in the scattering of light by water droplets to the formation of rainbows.

4.1 Coulomb Scattering

The electrostatic potential energy of a charge e in the field of a point charge Ze at the origin ($r = 0$) is

$$V(\boldsymbol{r}) = \frac{Ze^2}{4\pi r}. \tag{4.1}$$

One of the familiar properties of this Coulomb potential is that its influence is felt so far from the origin that a wave incident upon it cannot, strictly speaking, be represented as a plane wave at any distance from the potential, however far away. The result, in the present context, is that the phase shift function $\chi(\boldsymbol{b})$ given by Eqs. (2.3) or (2.4) is logarithmically divergent. The problem presented by that behavior is removed however, by imagining that the field is electrostatically screened by an equal and opposite distribution of charge, for example, one uniformly distributed over a spherical shell, no matter at how great a distance from the origin. The screening can then be seen to have virtually no effect on the intensity of scattering at any but infinitesimal scattering angles.

We shall discuss screening in greater detail in Chapter 11, but let us first examine what is perhaps its most elementary example. The nuclear charge Ze is screened by

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a uniform spherical shell of radius R. The Coulomb potential is given by Eq. (4.1) for $r < R$, and it drops abruptly to zero for $r > R$. Then the phase shift function $\chi(\boldsymbol{b})$ is given by

$$\chi(b) = -\frac{2Ze^2}{4\pi\hbar v}\int_b^R \frac{\mathrm{d}r}{\sqrt{r^2 - b^2}} \tag{4.2}$$

$$= -2\eta \log\left(\frac{R + \sqrt{R^2 - b^2}}{b}\right) \tag{4.3}$$

where η is the Sommerfeld parameter

$$\eta = \frac{Ze^2}{4\pi\hbar v}. \tag{4.4}$$

If the screening radius is of atomic dimensions, for example, it is so much larger than the wavelengths of typical nuclear projectiles that we may well consider R large enough to permit expanding the phase shift function as

$$\chi(b) = 2\eta \log\left(\frac{b}{2R}\right) + \mathcal{O}\left(\frac{b^2}{R^2}\right). \tag{4.5}$$

Retaining the terms of order b^2/R^2 and higher, in that case will tend only to influence the scattered intensity at angles θ of order $1/kR$ and smaller [9]. For scattering angles larger than this infinitesimal value, it suffices to write

$$\chi(b) = 2\eta \log\left(\frac{b}{2R}\right), \tag{4.6}$$

and it then becomes clear from Eq. (3.4) that the screening radius R only affects the scattering amplitude through an angle-independent phase factor.

Since $\chi(b)$ is rotationally symmetric for the Coulomb field, the stationary phase point determined by Eqs. (3.11b) and (3.12) has $b_y = 0$ and b_x determined by

$$\frac{2\eta}{b_x} = q. \tag{4.7}$$

It is worth remembering that q is an intrinsically positive variable, so that the stationary point

$$b_x = \frac{2\eta}{q}, \tag{4.8}$$

has the same sign as η, that is positive for repulsive forces and negative for attractive ones ($\eta \to -\eta$). For small angles of scattering the stationary point tends to lie far from the nucleus. As the scattering angle increases it draws closer to it.

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The second derivative of the phase shift function at the stationary phase point is

$$\chi''(|b_x|) = -\frac{2\eta}{b_x^2} = -\frac{q^2}{2\eta}. \tag{4.9}$$

The scattering amplitude given by Eq. (3.23) is thus

$$f(\boldsymbol{k}', \boldsymbol{k}) = \frac{2\eta k}{q^2} \exp\left\{-2i\eta \log \frac{qR}{\eta} - 2i\eta + \frac{i\pi}{2}\right\} \tag{4.10}$$

and the differential cross section is thus given by the Rutherford formula

$$|f(\boldsymbol{k}', \boldsymbol{k})|^2 = \left(\frac{2\eta k}{q^2}\right)^2 = \frac{(2\eta k)^2}{|\boldsymbol{k} - \boldsymbol{k}'|^4}. \tag{4.11}$$

While this result could hardly be more familiar, the phase factor in the scattering amplitude, Eq. (4.10), may be less so. The precise form assumed for the shape of the screening function may easily be seen to alter the scattering amplitude by an angle-independent phase factor. We have taken a discontinuous model here to describe screening, but the result derived in Ref. [10] for exponential screening, for example,

$$\frac{2\eta k}{q^2} \exp\{-2i\eta \log qR + 2i \arg \Gamma(1 + i\eta)\}, \tag{4.12}$$

differs, after expanding $\arg \Gamma(1 + i\eta)$ to leading order in η, from the result of Eq. (4.10) only by the constant phase factor

$$\exp\{2i\eta \log \eta\}. \tag{4.13}$$

The above result in Eq. (4.11) for Coulomb scattering has been shown [14] to apply also to the case of gravitational interactions in small-angle scattering, where the effective strength however is energy-dependent.

4.2 Paired Trajectories

One of the many unusual features of scattering by the Coulomb field is that there is only one classical trajectory for each angle of scattering. For other sorts of interactions that number may be quite different. Sometimes two or more particle trajectories may contribute, occasionally there are even none that contribute in this way. To see a simple example of the latter sort of behavior let us consider a real-valued potential that decreases smoothly from some finite value to zero within a finite range. An example might be the Gaussian function $V(r) = V_0 \exp\{-r^2/R^2\}$. We shall discuss this example in detail in Chapters 7 and 11, but all we need for the moment is to let the phase shift function $\chi(\boldsymbol{b})$ be rotationally symmetric, and to take $\chi(b_x)$ to be an even-valued function with a similar shape, perhaps like that shown in Fig. 4.1(a).

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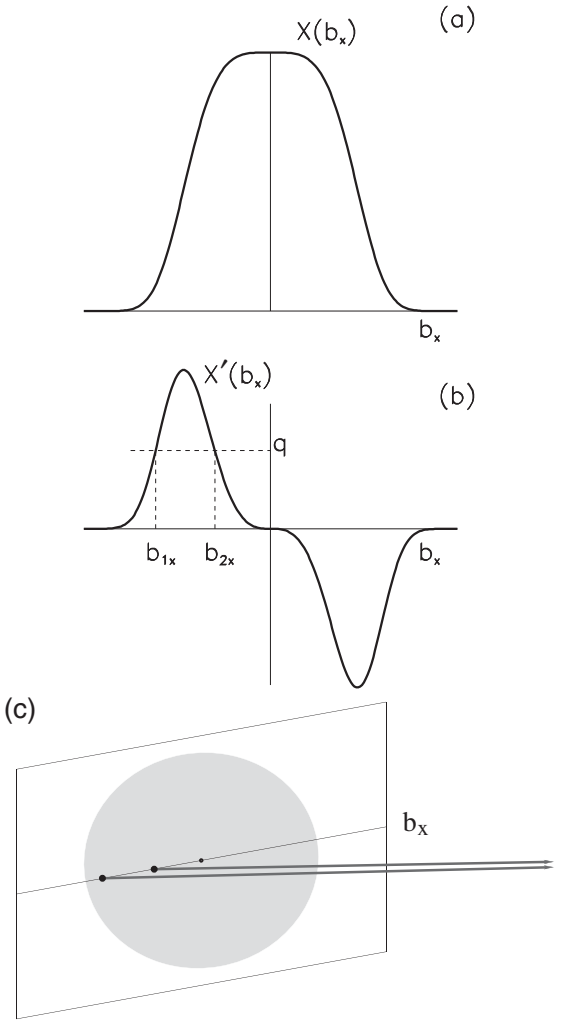


Figure 4.1 (a) A simple monotonic phase shift function, $\chi(b_x)$. (b) The equation $\chi'(b_x) = q$ has two solutions, b_{1x} and b_{2x}. (c) Schematic depiction of the trajectories corresponding to impact parameters b_{1x} and b_{2x}.

It follows then that $\chi'(b_x)$ is an odd function of b_x. The roots of the equation

$$\chi'(b_x) = q, \tag{4.14}$$

which is equivalent to Eq. (3.12), can immediately be seen from Fig. 4.1(b) to occur in pairs. For the case shown, which corresponds to an attractive potential, there are two stationary phase points at negative values of b_x, that are labelled b_{1x} and b_{2x}. The corresponding trajectories, shown in Fig. 4.1(c), both lead to the same momentum transfer $\hbar\boldsymbol{q}$.

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If the scattering amplitudes that correspond to the two trajectories are $F(b_{1x})$ and $F(b_{2x})$ respectively, then the classical differential cross section is the sum of the two corresponding intensities,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{class.}} = |F(b_{1x})|^2 + |F(b_{2x})|^2. \tag{4.15}$$

The quantum mechanical differential cross section, on the other hand, will be given, as long as the two roots b_{1x} and b_{2x} remain distinct, by the intensity

$$\frac{d\sigma}{d\Omega} = |f(\boldsymbol{k}', \boldsymbol{k})|^2 = |F(b_{1x}) + F(b_{2x})|^2 \tag{4.16}$$

that results from superposing the two amplitudes before squaring. Because there is no way of knowing which of the two trajectories the particle has followed the cross section will contain interference terms, and these, as we shall see, go a long way toward explaining the oscillations present in cross sections like those of Fig. 1.1.

4.3 Rainbows

It is clear from Fig. 4.1(a) that as we increase the value of q the two roots b_{1x} and b_{2x} will approach each other and for a certain value of q will coalesce. For larger values of q there are no roots, which is to say there is no classical trajectory at all leading to that momentum transfer. This disappearance of the classical trajectories or "rays" is quite familiar in the theory of the formation of the rainbow. The principal difference is that for scattering by raindrops it takes place at a rather backward angle of about 139°, rather than outside a forward diffraction cone.

We may pause for a moment to note that this element of the theory of the rainbow was one of the most remarkable insights of René Descartes. Descartes seems to have discovered the law of refraction, perhaps independently of Snell, during a lengthy stay in Holland. He realized that the angular distribution of light rays refracted and reflected by spherical raindrops would be independent of their radii, which he knew to vary considerably. He found the angles of refraction and reflection of the variously impinging rays, and showed that for the combination of two refractions and one internal reflection, the rays have a minimum angle of deflection (or scattering) of about 139°*, relative to the forward direction. There are two possible ray paths that lead to scattering through angles still larger than that. What is perhaps most impressive was Descartes' realization, in 1636, long before the invention of calculus, that the extremal character of this* 139° *deflection would lead to a concentration of the scattered light there. He had indeed explained most of the geometry of the rainbow and, by including the possibility of a second internal reflection, was even able to explain the corresponding features of the secondary*

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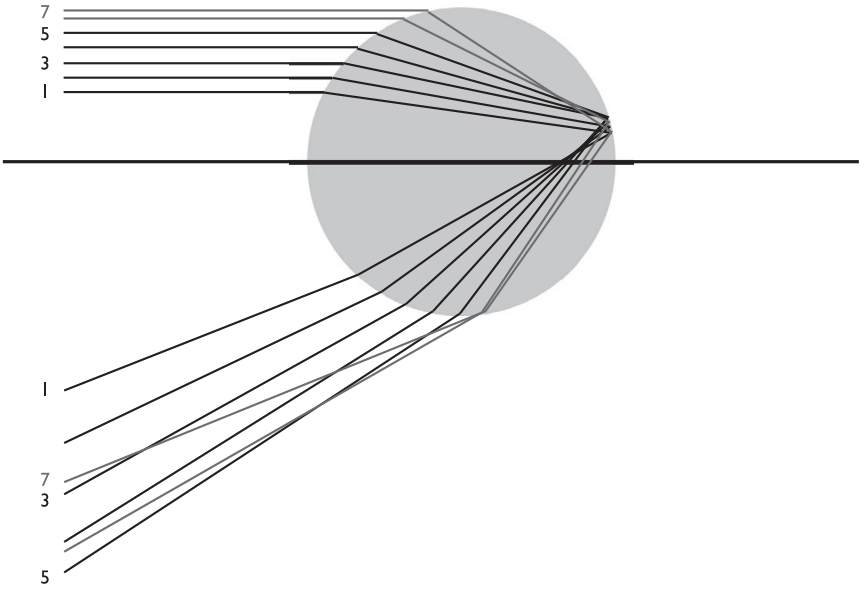


Figure 4.2 Parallel rays of light entering from top left and undergoing rainbow scattering. Only every second one is labeled for lack of space. The rays 6 and 7 hit at an impact parameter beyond the rainbow point and get scattered to angles inside the rainbow.

rainbow. But because the discovery of dispersion by Newton was still many years off, Descartes' rainbows lacked altogether any spectral separation of colors which is their principal hallmark. They were simply bright arcs of light just as white as direct sunlight.

Let us write q_{R} for the extremal value of q, which defines the "rainbow" angle of deflection for waves of a particular wavelength λ and let $b_{x,\mathrm{R}}$ be the value at which the roots $b_{1\,x}$ and $b_{2\,x}$ coalesce. It is quite instructive to discuss the behavior of the asymptotic approximation for q-values in the neighborhood of q_{R}. We can assume that in the immediate neighborhood of its maximum the function $\chi'(b_x)$, as shown in Fig. 4.3, takes the form

$$\chi'(b_x) \simeq q_{\mathrm{R}} - C(b_x - b_{x,\mathrm{R}})^2 \tag{4.17}$$

where C is some positive constant. Then for $q < q_{\mathrm{R}}$ the roots of Eq. (4.14) are given by

$$b_{1\,x} = b_{x,\mathrm{R}} - \sqrt{\frac{q_{\mathrm{R}} - q}{C}} \tag{4.18a}$$

$$b_{2\,x} = b_{x,\mathrm{R}} + \sqrt{\frac{q_{\mathrm{R}} - q}{C}}, \tag{4.18b}$$

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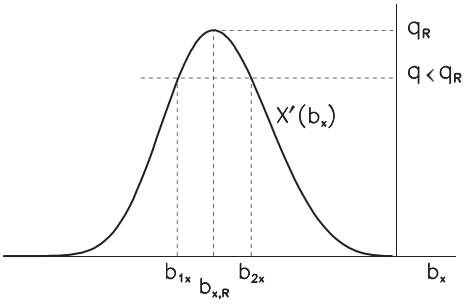


Figure 4.3 The occurrence of a rainbow at the maximum of the deflection function $\chi'(b_x)$.

and the corresponding values of χ'' are

$$\chi''(b_{1x}) = 2\sqrt{C(q_R - q)} \tag{4.19a}$$

$$\chi''(b_{2x}) = -2\sqrt{C(q_R - q)}. \tag{4.19b}$$

The two stationary points contribute, according to Eq. (3.23), the scattering amplitudes

$$F_1(b_{1x}) = k\left\{\frac{b_{1x}}{2q\sqrt{C(q_R - q)}}\right\}^{\frac{1}{2}} e^{i\{-qb_{1x}+\chi(b_{1x})\}}, \tag{4.20a}$$

$$F_2(b_{2x}) = k\left\{\frac{-b_{2x}}{2q\sqrt{C(q_R - q)}}\right\}^{\frac{1}{2}} e^{i\{-qb_{2x}+\chi(b_{2x})\}}. \tag{4.20b}$$

Both of these amplitudes become singular as $(q_R - q)^{-\frac{1}{4}}$ as q approaches the "rainbow" value q_R, and that indeed is the essence of Descartes' early observation. In the classical limit of zero wavelength, which corresponds to the limit of ray optics, the scattered intensity rises as $(q_R - q)^{-\frac{1}{2}}$ as q approaches q_R and then drops abruptly to zero. There is no scattering for larger momentum transfers, no illumination outside the rainbow cone.

In quantum mechanics, however, as in wave optics, the picture is somewhat different. The intensity may reach a strong maximum at the rainbow point $q = q_R$, but it is not truly singular there. Furthermore scattering to larger momentum transfers is not completely forbidden. It can indeed take place with easily observed intensities as a kind of barrier penetration phenomenon.

We shall discuss in Chapter 7 a simple way of correcting the singular behavior of scattering amplitudes like Eq. (4.20). These singularities are so localized in their occurrence and their effects, however, that the problem of regularizing them is not

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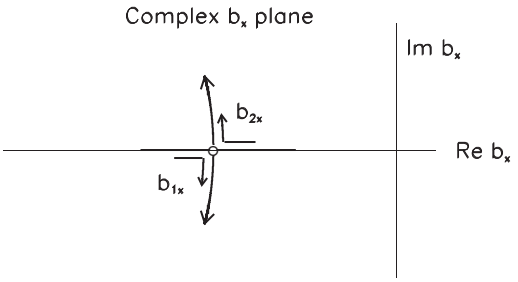


Figure 4.4 The impact parameters $b_{1\,x}$ and $b_{2\,x}$ in the region of the rainbow point $b_{x,\mathrm{R}}$ (denoted by the circle). For $q > q_{\mathrm{R}}$ they become complex.

often an important one. What is far more important is to gain some understanding of the non-classical scattering that is present beyond the rainbow angle. To do that let us begin by noting that for $q > q_{\mathrm{R}}$ the stationary points do not disappear altogether. They simply move off the real axis into the complex plane of b_x-values. For $q > q_{\mathrm{R}}$ we can thus write

$$b_{1\,x} = b_{x,\mathrm{R}} - i\sqrt{\frac{q - q_{\mathrm{R}}}{C}} \tag{4.21a}$$

$$b_{2\,x} = b_{x,\mathrm{R}} + i\sqrt{\frac{q - q_{\mathrm{R}}}{C}}. \tag{4.21b}$$

This shift of the roots into the complex plane of b_x is illustrated in Fig. 4.4.

For values of q inside the rainbow angle, $q < q_{\mathrm{R}}$, the integration over all values of b_x required in Eq. (3.4) actually passes through two stationary points, $b_{1\,x}$ and $b_{2\,x}$. When q exceeds q_{R}, those points are complex, and the path of integration of b_x, along the real axis from $-\infty$ to ∞ passes through neither of them. We can easily shift the path of integration into the complex plane by adding an imaginary constant to b_x. That could pass the path through one of the stationary points, but which one should we choose?

It is important to note at this stage that the second derivative of the phase function $\chi(b_x)$ for $q > q_{\mathrm{R}}$ takes the values

$$\chi''(b_{1\,x}) = 2i\sqrt{C(q - q_{\mathrm{R}})} \tag{4.22a}$$

$$\chi''(b_{2\,x}) = -2i\sqrt{C(q - q_{\mathrm{R}})} \tag{4.22b}$$

at the stationary points, while $1/b_x$ (cf. Eq. (3.18a)) takes the values

$$\frac{1}{b_{1\,x}} = \frac{1}{b_{x,\mathrm{R}} - i\beta(q)} \tag{4.23a}$$

$$\frac{1}{b_{2\,x}} = \frac{1}{b_{x,\mathrm{R}} + i\beta(q)} \tag{4.23b}$$

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where

$$\beta(q) \equiv \sqrt{\frac{q - q_{\mathrm{R}}}{C}}. \tag{4.24}$$

For the roots we have designated as b_{1x}, both $1/b_{1x}$ and χ'' have positive imaginary parts. By referring back to Eqs. (3.18a), (3.18b), (3.21a), and (3.21b) we see that the b_x- and b_y-integrations, when carried out through this point will both converge as the integral of a Gaussian function, which is to say quite rapidly. Integrating through the point b_{2x} on the other hand would furnish divergent results for both. The most natural procedure therefore is to shift the integration path in the negative imaginary direction of the b_x-plane by introducing the variable

$$b'_x = b_x - i\beta(q) \tag{4.25}$$

and then integrating over real values of b'_x from $-\infty$ to ∞. That procedure, it is easy to show, leads to no change in the value of the integral Eq. (3.21b) as long as the imaginary part of the parameter we have called α_x is positive.

In this way we see that for $q > q_{\mathrm{R}}$ only the stationary point b_{1x} contributes to the scattering. It furnishes, according to Eq. (3.23), the scattering amplitude

$$F(b_{1x}) = k\left\{\frac{b_{1x}}{2iCq\beta(q)}\right\}^{\frac{1}{2}} \exp\{-q\beta(q) - iqb_{x,\mathrm{R}} + i\chi(b_{1x})\}. \tag{4.26}$$

Certain qualitative features of the differential cross section this expression provides are worth noting immediately. First of all, the interference effects present when two or more stationary points contribute are absent in this case, so the differential cross section may be expected to show less oscillation outside the rainbow cone. Secondly, if the stationary point b_{1x} were fixed at some value and its slow dependence on q neglected, the most important remaining dependence on q would be $F \sim \exp\{-q\beta\}$. This is a result reminiscent of the single-slit model we discussed in the introduction, cf. Eq. (1.5), and we have thus made a first step towards it. In this sense the displacement of our stationary phase point from the real axis yields an amplitude that corresponds to our earlier model of a slit of width β. The accuracy of this correspondence depends, of course, on the degree to which the stationary point moves as q varies. In the examples of interactions of finite range that we shall consider presently, the stationary point usually shifts less rapidly for large q than it does in the present simplified example. The result is that the square root factor and the factor $\exp\{i\chi(b_{1x})\}$ in the expression in Eq. (4.24) vary much more slowly than the factor $\exp\{-q(\beta + ib_{x,\mathrm{R}})\}$, which, as we have noted, could be obtained from a model of diffraction by a single slit, one that is slightly refracting and has diffusely absorbing edges.

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