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On Hammersley's minimum problem for a rolling sphere

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Abstract

The problem posed by Hammersley (1983) of finding the shortest path along which a sphere can roll from one prescribed state to another is formulated by using quaternion calculus of variations and optimal control theory. This leads to a system of coupled nonlinear differential equations with prescribed end conditions. From the resulting expression for the curvature, it is shown that the differential equation of the required path in intrinsic coordinates is the same as the equation of motion of a simple pendulum, giving a solution in terms of elliptic integrals.

1. *Introduction*

Hammersley [1] has posed the problem of finding the shortest path between two given points in a horizontal plane along which a sphere can roll about a horizontal axis without slipping so as to reach a given orientation. When the final point is not prescribed the problem is much simpler and the solution is either a straight line or an arc of a circle. Here we consider the full problem and give a formulation based on the calculus of variations and optimal control theory. This approach leads to a boundary value problem for the state and adjoint variables. From the resulting expression for the curvature we find that the differential equation of the required path in intrinsic coordinates is the same as the equation of motion of a simple pendulum, and therefore has a solution in terms of elliptic integrals.

We shall roll the sphere with constant velocity along the curved path Γ on the plane, starting from the origin $O(0, 0)$ and writing t for the arc length along Γ measured from $t = 0$ at the origin. The path Γ is to end at the point $E(x_E, y_E)$. Following Hammersley [1] we write $q = q(t)$ for the quaternion such that $\pm q$ represents the resultant rotation of the sphere at arc length t. We take $q(0) = 1$ and suppose that $q(t)$ is a differentiable function of t. We use dots to denote differentiation with respect to t. Let $\mathbf{h} = \mathbf{h}(t)$ be the unit vector along the instantaneous axis of rotation at t and suppose $\mathbf{h}$ subtends an angle $\psi(t)$ with the x axis. Then we may write

$$\mathbf{h} = h_1\mathbf{i} + h_2\mathbf{j} = \mathbf{i}\cos\psi + \mathbf{j}\sin\psi, \tag{1·1}$$

and the position vector of the centre of the sphere at time t is given by

$$\begin{aligned}\mathbf{r}(t) &= \int_0^t \mathbf{h}\mathbf{k}\,dt + \mathbf{k} \\ &= \mathbf{i}\int_0^t h_2\,dt - \mathbf{j}\int_0^t h_1\,dt + \mathbf{k},\end{aligned} \tag{1·2}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the standard triad of unit vectors.

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If T represents the length of a curve joining O and E, then the problem is to minimize

$$T = \int_0^T 1\,dt \tag{1·3}$$

subject to

$$\int_0^T h_2\,dt = x_E, \quad \int_0^T h_1\,dt = -y_E, \tag{1·4}$$

and also subject to a prescribed final orientation $q(T)$.

2. *State and adjoint equations*

As shown by Hammersley [1], the rolling condition is

$$\dot{q} = \tfrac{1}{2}\mathbf{h}q, \tag{2·1}$$

which in component form, with

$$q = \rho_0 + \rho_1\mathbf{i} + \rho_2\mathbf{j} + \rho_3\mathbf{k}, \tag{2·2}$$

gives

$$\dot{\rho}_0 = -\tfrac{1}{2}(h_1\rho_1 + h_2\rho_2), \tag{2·3}$$

$$\dot{\rho}_1 = \tfrac{1}{2}(h_1\rho_0 + h_2\rho_3), \tag{2·4}$$

$$\dot{\rho}_2 = \tfrac{1}{2}(-h_1\rho_3 + h_2\rho_0), \tag{2·5}$$

$$\dot{\rho}_3 = \tfrac{1}{2}(h_1\rho_2 - h_2\rho_1). \tag{2·6}$$

We are going to formulate the problem in 'phase space', made up of the state vector ρ_i and the adjoint vector λ_i of Lagrange multipliers (cf. Leitmann [2]). The controls in this problem are represented by h_1 and h_2, and to deal with the integral constraints in (1·4) on them we introduce two more state equations

$$\dot{\rho}_4 = \tfrac{1}{2}h_1, \tag{2·7}$$

$$\dot{\rho}_5 = \tfrac{1}{2}h_2. \tag{2·8}$$

Corresponding to the integral in (1·3) with integrand 1, we add the equation

$$\dot{\rho}_{-1} = 1. \tag{2·9}$$

The full state equations are therefore given by equations (2·3) to (2·9) and have the form

$$\dot{\boldsymbol{\rho}} = \partial H/\partial\boldsymbol{\lambda}, \tag{2·10}$$

where the Hamiltonian H is defined by

$$\begin{aligned} H = {} & \lambda_{-1} - \tfrac{1}{2}\lambda_0(h_1\rho_1 + h_2\rho_2) + \tfrac{1}{2}\lambda_1(h_1\rho_0 + h_2\rho_3) \\ & + \tfrac{1}{2}\lambda_2(-h_1\rho_3 + h_2\rho_0) + \tfrac{1}{2}\lambda_3(h_1\rho_2 - h_2\rho_1) \\ & + \tfrac{1}{2}\lambda_4 h_1 + \tfrac{1}{2}\lambda_5 h_2. \end{aligned} \tag{2·11}$$

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Since we may take $\rho_{-1} = t$, equation (2·10) consists essentially of six coupled differential equations with end conditions

$$\left.\begin{aligned} \rho_0(0) &= 1, \quad \rho_0(T) = \alpha_0, \\ \rho_1(0) &= 0, \quad \rho_1(T) = \alpha_1, \\ \rho_2(0) &= 0, \quad \rho_2(T) = \alpha_2, \\ \rho_3(0) &= 0, \quad \rho_3(T) = \alpha_3, \\ \rho_4(0) &= 0, \quad \rho_4(T) = -\tfrac{1}{2}y_E, \\ \rho_5(0) &= 0, \quad \rho_5(T) = \tfrac{1}{2}x_E, \end{aligned}\right\} \tag{2·12}$$

where α_0, α_1, α_2, α_3 are prescribed by the final orientation.

To complete the description we now add to the state equations (2·10) the adjoint equations

$$\dot{\boldsymbol{\lambda}} = -\partial H/\partial \boldsymbol{\rho}, \tag{2·13}$$

that is,

$$\left.\begin{aligned} \dot{\lambda}_0 &= -\partial H/\partial \rho_0 = -\tfrac{1}{2}(h_1\lambda_1 + h_2\lambda_2), \\ \dot{\lambda}_1 &= -\partial H/\partial \rho_1 = \tfrac{1}{2}(h_1\lambda_0 + h_2\lambda_3), \\ \dot{\lambda}_2 &= -\partial H/\partial \rho_2 = \tfrac{1}{2}(-h_1\lambda_3 + h_2\lambda_0), \\ \dot{\lambda}_3 &= -\partial H/\partial \rho_3 = \tfrac{1}{2}(h_1\lambda_2 - h_2\lambda_1), \\ \dot{\lambda}_4 &= -\partial H/\partial \rho_4 = 0, \\ \dot{\lambda}_5 &= -\partial H/\partial \rho_5 = 0, \\ \dot{\lambda}_{-1} &= -\partial H/\partial \rho_{-1} = 0. \end{aligned}\right\} \tag{2·14}$$

Thus λ_4, λ_5 and λ_{-1} are constants, and (2·14) therefore provides a further four coupled nonlinear differential equations.

By the Pontryagin maximum principle the optimal path is found by solving the equations (2·10) and (2·13) for the variables ρ_i, λ_i subject to the prescribed end conditions, the Hamiltonian H being maximized with respect to the controls h_1 and h_2. Also, the final time T being free, we have

$$\max_{h_1, h_2} H = 0. \tag{2·15}$$

Now, by (2·11), the Hamiltonian is

$$\begin{aligned} H = \lambda_{-1} &+ \tfrac{1}{2}h_1(\lambda_4 - \lambda_0\rho_1 + \lambda_1\rho_0 - \lambda_2\rho_3 + \lambda_3\rho_2) \\ &+ \tfrac{1}{2}h_2(\lambda_5 - \lambda_0\rho_2 + \lambda_1\rho_3 + \lambda_2\rho_0 - \lambda_3\rho_1), \end{aligned} \tag{2·16}$$

and since $\mathbf{h}$ is a unit vector we also have the constraint

$$\Psi \equiv h_1^2 + h_2^2 - 1 = 0. \tag{2·17}$$

The solution of the original minimizing problem is obtained by choosing (cf. Leitmann [2])

$$\lambda_{-1} = -1, \tag{2·18}$$

and then maximizing H subject to (2·17) and imposing (2·15) .We find that

$$\left.\begin{aligned} 2h_1 &= \lambda_4 - \lambda_0\rho_1 + \lambda_1\rho_0 - \lambda_2\rho_3 + \lambda_3\rho_2, \\ 2h_2 &= \lambda_5 - \lambda_0\rho_2 + \lambda_1\rho_3 + \lambda_2\rho_0 - \lambda_3\rho_1. \end{aligned}\right\} \tag{2·19}$$

These are the expressions for h_1 and h_2 that are used in the state and adjoint equations (2·10) and (2·13).

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From the first four state equations, namely (2·3) to (2·6), and the first four adjoint equations in (2·14), we find that

$$\frac{d}{dt}(\rho_0^2+\rho_1^2+\rho_2^2+\rho_3^2) = 0,$$

$$\frac{d}{dt}(\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2) = 0,$$

$$\frac{d}{dt}(\lambda_0\rho_0+\lambda_1\rho_1+\lambda_2\rho_2+\lambda_3\rho_3) = 0,$$

and so

$$\left.\begin{aligned}&\rho_0^2+\rho_1^2+\rho_2^2+\rho_3^2 = 1,\\&\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2 = \text{const.},\\&\lambda_0\rho_0+\lambda_1\rho_1+\lambda_2\rho_2+\lambda_3\rho_3 = \lambda_0(0),\end{aligned}\right\} \tag{2·20}$$

where we have used $\rho_0 = 1$, $\rho_1 = 0$, $\rho_2 = 0$, $\rho_3 = 0$ at $t = 0$.

If we differentiate the expressions in (2·19) and use the first four state and adjoint equations, we find that

$$2\dot{h}_1 = h_2(-\lambda_0\rho_3-\lambda_1\rho_2+\lambda_2\rho_1+\lambda_3\rho_0)$$

and

$$2\dot{h}_2 = -h_1(-\lambda_0\rho_3-\lambda_1\rho_2+\lambda_2\rho_1+\lambda_3\rho_0). \tag{2·21}$$

Now

$$h_1 = \cos\psi(t), \quad h_2 = \sin\psi(t)$$

so that

$$\dot{h}_1 = -h_2\dot{\psi}, \quad \dot{h}_2 = h_1\dot{\psi}. \tag{2·22}$$

Also, the curvature of Γ is given by

$$\kappa = \dot{\psi}. \tag{2·23}$$

Hence from (2·21) we have

$$\kappa = \tfrac{1}{2}(\lambda_0\rho_3+\lambda_1\rho_2-\lambda_2\rho_1-\lambda_3\rho_0). \tag{2·24}$$

If we now differentiate equation (2·24) and use the first four state and adjoint equations, we obtain

$$2\dot{\kappa} = h_1(\lambda_0\rho_2-\lambda_1\rho_3-\lambda_2\rho_0+\lambda_3\rho_1)+h_2(-\lambda_0\rho_1+\lambda_1\rho_0-\lambda_2\rho_3+\lambda_3\rho_2),$$

which, on using (2·19), leads to

$$2\dot{\kappa} = \lambda_5 h_1-\lambda_4 h_2. \tag{2·25}$$

Integrating again, we obtain an alternative to (2·24), namely

$$\kappa(t) = \lambda_5\rho_4(t)-\lambda_4\rho_5(t)+\kappa(0). \tag{2·26}$$

At $t = 0$, the expressions (2·19), (2·20) and (2·24) provide the following initial conditions:

$$\left.\begin{aligned}&\lambda_0(0) \text{ arbitrary},\\&\lambda_1(0) = 2\cos\psi(0)-\lambda_4,\\&\lambda_2(0) = 2\sin\psi(0)-\lambda_5,\\&\lambda_3(0) = -2\psi(0).\end{aligned}\right\} \tag{2·27}$$

We have therefore formulated the problem as a boundary value problem in which we solve

$$\left.\begin{aligned}&\text{six differential equations (2·3)–(2·8) subject to (2·12)}\\\text{coupled with}\quad&\text{four differential equations of (2·14) subject to (2·27).}\end{aligned}\right\} \tag{2·28}$$

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It is convenient to take $\lambda_0(0) = 0$. Then the prescribed numbers are $\alpha_0, \alpha_1, \alpha_2, \alpha_3, x_E, y_E$, five being independent since $\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2 = 1$, and there are five parameters at our disposal, namely $\psi(0)$, $\dot{\psi}(0)$, λ_4, λ_5 and T.

3. *Curvature equation*

The curvature κ of the optimal curve Γ satisfies equation (2·25), that is

$$\dot{\kappa} = \tfrac{1}{2}(\lambda_5 h_1 - \lambda_4 h_2). \tag{3·1}$$

For the simpler problem in which the final point $E(x_E, y_E)$ is not prescribed we do not have the constraints in (1·4) and so in this case

$$\lambda_4 = 0, \quad \lambda_5 = 0. \tag{3·2}$$

It therefore follows in this case from (3·1) that

$$\kappa = \text{const.} \tag{3·3}$$

and hence Γ is a circular arc, as found by Hammersley [1].

For the more general problem we see from (3·1) that the angle ψ satisfies

$$\ddot{\psi} = \lambda \cos(\psi + \epsilon), \tag{3·4}$$

where $\lambda = \tfrac{1}{2}(\lambda_4^2+\lambda_5^2)^{\frac{1}{2}}$ and $\tan\epsilon = \lambda_4/\lambda_5$. Hence the differential equation of the required path in intrinsic coordinates is the same as the equation of motion of a simple pendulum. Since ψ and t are intrinsic coordinates we have

$$\frac{dx}{dt} = \sin\psi, \quad \frac{dy}{dt} = -\cos\psi,$$

where x, y are the cartesian coordinates of a point on the path Γ. If we choose the origin of ψ suitably we can suppose that

$$\frac{dx}{dt} = \sin(\psi+\epsilon), \quad \frac{dy}{dt} = -\cos(\psi+\epsilon). \tag{3·5}$$

Then (3·4) gives on integration

$$\frac{d\psi}{dt} = -\lambda y + B$$

or

$$\frac{y''}{[1+(y')^2]^{\frac{3}{2}}} = -\lambda y + B \quad (y' = dy/dx), \tag{3·6}$$

where B is a constant. Integrating this twice we obtain

$$x = \int_0^v \left\{\frac{(A+Bv-\frac{1}{2}\lambda v^2)^2}{1-(A+Bv-\frac{1}{2}\lambda v^2)^2}\right\}^{\frac{1}{2}} dv, \tag{3·7}$$

where A is a further constant. This gives an equation for the coordinates of the required optimal path, which can be written in terms of elliptic integrals of the first and third kinds, where the constants A, B, ϵ and λ are to be determined by the final orientation of the sphere and the condition that (x_E, y_E) satisfies (3·7). For a given problem the numerical determination of the optimal path by (3·7) involves therefore an inverse interpolation problem which appears to be just as formidable as the problem of solving the equivalent boundary value problem stated in (2·28).

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4. *Concluding remarks*

The aim of this paper has been to provide a formulation of the Hammersley rolling-sphere problem. This has been done by use of the calculus of variations and optimal control theory, and has led to the boundary value problem in (2·28) or equivalently to the inverse interpolation problem for (3·7). These problems remain to be investigated numerically in specific cases.

Dr Hammersley first brought this problem to our attention and we are much indebted to him for a most helpful correspondence. After this work was completed, an equation for the intrinsic coordinates which is equivalent to (3·4) was derived independently by our colleague Dr A. Sudbery using a non-quaternion approach.

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