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Introduction

Kernel-based methods represent well-established tools of modern numerical analysis and machine learning. Kernel theory has attracted ongoing interest throughout the twentieth century, with significant contributions by mathematicians such as Mercer (1909), Bochner (1933), and Aronszajn (1950), and has experienced renewed interest since the 1990s. From an approximation theory perspective, this interest was driven by the growing availability of scattered and unstructured data in scientific and engineering applications, which required meshfree approximation techniques. From another viewpoint, the development of support vector machines (SVMs) and other kernel-based learning tools demonstrated the effectiveness of kernel methods in machine learning. The strength of kernel methods lies in their flexibility and elegant theoretical grounding, which allows the construction of adaptive approximation spaces that naturally "shape" themselves to the data at hand, without requiring predetermined grids or meshes. This property makes kernel methods particularly suitable for high-dimensional problems and scattered data interpolation.

The book is organized into nineteen chapters that guide the reader from foundational concepts to recent research topics. The first part, namely Chapters 2–5, concerns fundamental notions in kernel approximation: positive-definite kernels and functions, reproducing kernel Hilbert spaces (RKHSs), polynomial precision, and error estimates. Examples of classical families of radial basis functions (RBFs) are also provided, such as Gaussians, Matérn functions, and Wendland's compactly supported functions. Chapters 6–8 explore advanced computational strategies to improve efficiency and stability: rational kernels, partition of unity (PU) methods, and efficient greedy/knot-removal algorithms. These techniques address practical challenges arising from large datasets or ill-conditioned systems.

Chapters 9–11 deal with variably scaled (discontinuous) kernels, deepening the theoretical analysis of the discontinuous setting. This novel approach enables

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the effective reconstruction of functions with jumps, overcoming the classical Gibbs phenomenon that plagues standard methods. Applications to magnetic particle imaging (MPI) and astronomical imaging demonstrate the practical relevance of these theoretical developments.

Then, Chapters 12–16 shift the focus to machine learning, covering SVMs and data-driven optimization techniques. In particular, we present recent research on approximation methods empowered by learning tools such as data-driven greedy approximation, discontinuous neural networks (δNNs) for learning optimal scaling functions in the variably scaled kernel framework and machine learning approaches enriched by approximation techniques such as PU-based SVMs, learning with Variably Scaled Kernels (VSKs).

Chapter 17 concerns concepts of topological data analysis and related kernels that address the output of persistent homology.

An outline of helpful information and links to available software packages related to the topics discussed throughout the book is provided in Chapter 18, where the reader can find guidance on tools for implementing kernel-based methods, performing numerical experiments, and exploring further resources.

Finally, Chapter 19 offers concluding remarks, summarizing the book's main contributions and commenting on several relevant topics that, while closely connected to kernel methods, have been intentionally omitted or only briefly mentioned because they fall outside the scope of these notes or are already thoroughly covered in other specialized references.

The book assumes familiarity with fundamental functional analysis, linear algebra, and numerical methods. Knowledge of approximation theory fundamentals is helpful, but not strictly necessary, as key concepts are introduced when needed.

Kernel methods remain a vibrant area of research with many open questions. The interplay between approximation theory, machine learning, and computational mathematics continues to generate new insights and techniques. This book provides both a solid foundation in established results and a window into current research trends, stimulating interested readers to contribute to future developments in this exciting field.

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