Combinatorics, Probability and Computing (2002) 11, 11–20. © 2002 Cambridge University Press
DOI: 10.1017/S0963548301004953 Printed in the United Kingdom

On the Lock-in Probability of Stochastic Approximation

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Received 7 August 2000; revised 3 February 2001

For a stochastic approximation-type recursion with finitely many possible limit points, we find a lower bound on the probability of converging to a prescribed point in its 'domain of attraction'. This has implications for the lock-in phenomena in the stochastic models of increasing return economics and the sample complexity of stochastic approximation algorithms in engineering.

1. Introduction

The celebrated stochastic approximation algorithm is a d-dimensional ($d \geqslant 1$) recursion of the form

$$X(n+1) = X(n) + a(n)[h(X(n)) + M(n+1)], \quad n \geqslant 0, \tag{1.1}$$

where $\{M(n)\}$ is a martingale difference sequence, that is, a family of integrable random variables satisfying

$$E[M(n+1)/M(m), X(m), m \leqslant n] = 0, \quad n \geqslant 0,$$

and the positive stepsizes $\{a(n)\}$ satisfy

$$\sum_n a(n) = \infty, \tag{1.2}$$

$$\sum_n a(n)^2 < \infty. \tag{1.3}$$

The idea is to solve the equation $h(x) = 0$ given a black box that delivers only noisy measurements of h at the chosen values of its input. Thus the expression in square brackets in (1.1) is a noisy measurement of $h(X(n))$ with $M(n+1)$ representing 'noise'.

† Work partially supported by grant no. III 5(12)/96-ET from the Department of Science and Technology, Government of India.

https://doi.org/10.1017/S0963548301004953 Published online by Cambridge University Press

While the classical approach to the analysis of (1.1) is based on martingale theory, a parallel approach that has gained much currency in the engineering community is the o.d.e. (for 'ordinary differential equations') approach due to Ljung [19]. (See [4] for an extensive account of this.) This approach treats (1.1) as a noisy discretization of the o.d.e.

$$\dot{x}(t) = h(x(t)) \tag{1.4}$$

with non-uniform stepsizes $\{a(n)\}$. Under reasonable conditions, we can then show that (1.1) tracks (1.4) in a suitable sense, so that, if $x(t) \to D \stackrel{\Delta}{=} \{x : h(x) = 0\}$, then $X(n) \to D$ almost surely (a.s.).

In recent years, there have been two important sources of stochastic dynamics of this type. These are, respectively, engineering applications such as adaptive filtering [14] and neural network training [15] on one hand, and economic models such as models of increasing returns phenomena [1] and learning by boundedly rational economic agents [24] on the other. In engineering applications, we want to have $X(n) \to x^*$ for a desired 'optimal' limit point x^* of (1.4). Often, however, we cannot rule out other possible equilibria that may be less desirable. This is especially so in simple models of increasing returns economics where convergence to an undesirable equilibrium amounts to the economic system getting 'locked into' an undesirable or non-optimal mode, such as the use of a substandard technology [1]. Of course, this presupposes that (1.4) has only isolated equilibria as possible limit points. In general, we must allow for more general attractors. In the case of finitely many isolated equilibria, it is known that, under reasonable conditions, every stable equilibrium will have a positive probability of emerging as $\lim_{n\to\infty} X(n)$ [1], while this probability is zero for unstable equilibria [10, 11, 22]. For the more general case, the best-known result seems to be that $\{X(n)\}$ converges to a set that is 'chain-recurrent' under (1.4) [2]. An extensive 'dynamical systems-theoretic' perspective of stochastic approximations appears in [3].

In this article we consider only the case of isolated equilibria. As already mentioned, convergence to an undesirable point corresponds to the 'lock-in' phenomenon in economics, a major concern in the emerging area of 'information economics' [16, 25]. Thus there is considerable interest in getting a quantitative hold on this phenomenon. Likewise, in the learning algorithms of engineering (*e.g.*, in neural network training), convergence to an undesirable equilibrium can lead to suboptimal design, a matter of serious concern. Unfortunately, little is known about the probability of lock-in. It may be recalled that the a.s. convergence of (1.1) is basically a strong law of large numbers for a class of dependent random variables. There are other associated limit theorems as well, such as a functional central limit theorem [4, Chapter 4 of Part II], the law of the iterated logarithm [21] and large deviations [12]. But they are all asymptotic in nature and do not provide any hard bounds for finite time behaviour. (See, however, [23] and [13], which characterize rate of convergence of appropriate moments.)

With this backdrop, the present work aims to obtain a quantitative bound on the lock-in probability, in the spirit of sample complexity results in statistical learning theory [26]. We shall do so based on the local analysis of the algorithm in the neighbourhood of a stable equilibrium of (1.4). Thus we start with the hypothesis that $X(n)$ is in the

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domain of attraction of such an equilibrium point x^* and find a bound on the probability that it does not converge to x^*. In particular, we show that the latter is $o(b(n))$ where $b(n) = \sum_{m=n}^{\infty} a(m)^2$. (Note that $b(n) \to 0$ by virtue of (1.3).) While an estimate in this spirit is available in [4, p. 236], we get a much stronger result by using the so-called 'concentration inequality'.

Our analysis will be based upon a variation of the o.d.e. approach developed in [7], [6] and [9]. This is recalled in the next section in some detail because these very details form the backdrop of our analysis of the lock-in probability. The latter analysis appears in the final section, along with some relevant remarks.

We conclude this section with two examples.

Example 1: Stochastic gradient method. Consider the case $h(\cdot) = -\nabla f(\cdot)$ for a continuously differentiable $f : \mathscr{R}^d \to \mathscr{R}$. Equilibria of (1.4) are now the critical points of f, the stable equilibria corresponding to local minima. Many neural network training algorithms have this form [15].

Example 2: Nonlinear urn. Consider adding one ball at a time, either red or black, to an initially empty urn. Let $X(n)$ denote the fraction of red balls after n such additions and suppose that the conditional probability that the $(n+1)$st ball added is red is given by $p(X(n))$ for a continuous function $p : [0,1] \to [0,1]$. Letting $Y(n+1) = I\{(n+1)\text{st ball is red}\}$, simple algebra leads to

$$X(n+1) = X(n) + \frac{1}{n+1}(Y(n+1) - X(n)),$$

which is of the form (1.1) for $h(x) = p(x) - x$, $a(n) = \frac{1}{n+1}$ and $M(n+1) = Y(n+1) - p(X(n))$. The set $D = \{x : p(x) = x\}$ of equilibria of the corresponding equation (1.4) is necessarily nonempty, by Brouwer's fixed point theorem, and can contain more than one point. Since this is a scalar o.d.e. with bounded trajectories in $[0,1]$, convergence to one of these equilibria is guaranteed (see, *e.g.*, [1]). This is the simplest model for increasing returns in economics [1].

2. Convergence analysis

We shall assume that the iterates (1.1) remain bounded with probability one, that is,

$$\sup_n \|X(n)\| < \infty \quad \text{a.s.} \tag{2.1}$$

We shall also assume that h is Lipschitz, that is,

$$K \stackrel{\Delta}{=} \sup_{x \neq y} \frac{\|h(x) - h(y)\|}{\|x - y\|} < \infty. \tag{2.2}$$

This ensures that (1.4) is well-posed. We shall relax these conditions later. Finally, we assume that

$$\|M(n+1)\| \leqslant K_1(1 + \|X(n)\|) \tag{2.3}$$

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for some constant $K_1 < \infty$. As an example, consider the case

$$X(n+1) = X(n) + a(n)f(X(n), \xi_{n+1}),$$

where $\{\xi_n\}$ are i.i.d. This can be cast in the format of (1.1) by setting $h(x) = E[f(x, \xi_1)]$ and $M(n+1) = f(X(n), \xi_{n+1}) - h(X(n)),\ n \geq 0$. This is, in fact, often the manner in which (1.1) arises in practice. Suppose that f satisfies $\|f(x,u)\| < C(1 + \|x\|)$, where the constant C is independent of u. Then (2.3) holds. This is the case, for example, in the reinforcement learning algorithms studied in [9]. We shall comment more on this condition later.

Let $x^* \in D$ be an isolated locally asymptotically stable equilibrium of (1.4) with domain of attraction O, and A a bounded open set satisfying $x^* \in A \subset \bar{A} \subset O$. By [17, Theorem 5.3, p. 31] (see also [4, p. 233]), there exists a C^2 map $V : O \to R$ such that $V(x^*) = 0 < V(x)$ for $x^* \neq x \in O$, $\nabla V(x).h(x) < 0$ for $x^* \neq x \in O$ and $V(x) \to \infty$ as $x \to \partial O$ or $\|x\| \to \infty$. The main result of this section is as follows (see also [4, Section 1.8 of Part II], [18, Section 5.8]).

Theorem 2.1. *On the set $\{X(n) \in A$ infinitely often$\}$, $X(n) \to x^*$ almost surely.*

We start with some preliminary lemmas.

Let $t(0) = 0,\ t(n) = \sum_{m=1}^{n} a(m)$. By (1.2), $t(n) \to \infty$. Define $\bar{x}(t), t \geq 0$, by $\bar{x}(t(n)) = X(n)$, $\bar{x}(\cdot)$ continuous and linear on each interval $[t(n), t(n+1)]$. Similarly, define $\tilde{x}(t), t \geq 0$, by: $\tilde{x}(t) = X(n),\ t \in [t(n), t(n+1))$. Fix $T > 0$ and set $T_0 = 0,\ T_{n+1} = \min\{t(m) : t(m) \geq T_n + T\}$. Let $a^* = \sup_n a(n)$. Then $T_n + T \leq T_{n+1} \leq T_n + T + a^*,\ n \geq 1$, in particular, $T_n \to \infty$. In fact, $T_n = t(m(n))$ for suitable $m(n) \to \infty$. Finally, let $I_n \stackrel{\Delta}{=} [T_n, T_{n+1}]$ and $x^n(t),\ t \in I_n$, denote the trajectories of (1.4) with $x^n(T_n) = X(m(n))$, for $n \geq 0$.

Lemma 2.2. $\zeta(n) \stackrel{\Delta}{=} \sum_{m=0}^{n-1} a(m)M(m+1),\ n \geq 0$, *converges almost surely.*

Proof. For $\mathcal{F}_n = \sigma(X(i), M(i), i \leq n)$, $\{\zeta(n)\}$ is an $\{\mathcal{F}_n\}$-martingale. In view of (2.2), (2.3),

$$E[\|X(n+1)\|^2] \leq \hat{K}(1 + E[\|X(n)\|^2]),\ n \geq 0.$$

for some $\hat{K} < \infty$. Assume without any loss of generality that $E[\|X(0)\|^2] < \infty$. (If not, we can argue 'conditioned on $X(0) = x$'.) Then it follows inductively that $E[\|X(n)\|^2] < \infty$ for all n. In view of (2.3), this implies that $\{\zeta(n), \mathcal{F}_n\}$ is a square-integrable martingale. Using (2.3), we verify that its quadratic variation process $\langle\zeta\rangle(n),\ n \geq 0$, satisfies

$$\begin{aligned}
\langle\zeta\rangle(n) &\leq K_1 \sum_{m=0}^{n} a(m)^2(1 + \|X(m)\|^2) \\
&\leq K_1 \left(\sum_m a(m)^2 \right) \left(1 + \sup_m \|X(m)\|^2 \right) \\
&< \infty,
\end{aligned}$$

in view of (1.3), (2.1). Thus it converges almost surely. Hence, by [5, Theorem 3.3.4, p. 53], so does $\{\zeta(n)\}$. □

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Lemma 2.3. $\sup_{t\in I_n} \|\bar{x}(t) - x^n(t)\| \to 0$ *almost surely.*

Proof. Fix n. For $m(n) \leqslant k < m(n+1)$, we have

$$\bar{x}(t(k+1)) = \bar{x}(t(k)) + \int_{t(k)}^{t(k+1)} h(\tilde{x}(t))dt + \zeta(k+1) - \zeta(m(n)).$$

Thus

$$\begin{aligned}\bar{x}(t(k+1)) &= \bar{x}(T_n) + \int_{T_n}^{t(k+1)} h(\bar{x}(t))dt \\ &\quad + \int_{T_n}^{t(k+1)} (h(\tilde{x}(t)) - h(\bar{x}(t)))dt + \zeta(k+1) - \zeta(m(n)).\end{aligned}$$

Also,

$$x^n(t(k+1)) = \bar{x}(T_n) + \int_{T_n}^{t(k+1)} h(x^n(t))dt.$$

Subtracting and taking the absolute value,

$$\begin{aligned}\|\bar{x}(t(k+1)) - x^n(t(k+1))\| &\leqslant \int_{T_n}^{t(k+1)} K\|\bar{x}(t) - x^n(t)\|dt \\ &\quad + \int_{T_n}^{t(k+1)} K\|\bar{x}(t) - \tilde{x}(t)\|dt + \|\zeta(k+1) - \zeta(m(n))\|.\end{aligned} \tag{2.4}$$

Fix a sample path for which (2.1), (2.3) and Lemma 2.2 hold. Let

$$C_1 = \sup_n \|X(n)\|, \quad C_2 = \sup_{\|x\|\leqslant C_1} \|h(x)\|.$$

Then, for $t \in [t(k), t(k+1)]$,

$$\|\bar{x}(t) - \tilde{x}(t)\| \leqslant (C_2 + (1 + C_1)K_1)a(k),$$

and therefore

$$\int_{t(k)}^{t(k+1)} \|\bar{x}(t) - \tilde{x}(t)\|dt \leqslant (C_2 + (1 + C_1)K_1)a(k)^2.$$

Thus (2.4) leads to

$$\begin{aligned}\|\bar{x}(t(k+1)) - x^n(t(k+1))\| &\leqslant K\int_{T_n}^{t(k+1)} \|\bar{x}(t) - x^n(t)\|dt \\ &\quad + (C_2 + (1 + C_1)K_1)\left(\sum_{i=m(n)}^{m(n+1)} a(i)^2\right) \\ &\quad + \sup_{m(n)\leqslant k\leqslant m(n+1)} \|\zeta(k) - \zeta(m(n))\|.\end{aligned}$$

By the Gronwall lemma,

$$\begin{aligned}\sup_{t\in I_n} \|\bar{x}(t) - x^n(t)\| &\leqslant \exp\big(K(T + a^*)\big)(C_2 + (1 + C_1)K_1)\left(\sum_{i=m(n)}^{m(n+1)} a(i)^2\right) \\ &\quad + \sup_{m(n)\leqslant k\leqslant m(n+1)} \|\zeta(k) - \zeta(m(n))\|).\end{aligned} \tag{2.5}$$

By Lemma 2.2 and the fact that $b(n) \to 0$, the r.h.s. tends to zero a.s. as $n \to 0$. □

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Proof of Theorem 2.1. We may suppose that $P(X(n) \in A \text{ infinitely often}) > 0$, otherwise there is nothing to prove. We shall continue to work with a fixed sample path as above, taking in particular a sample path for which $X(n) \in A$ infinitely often. Without loss of generality, we assume that $A = \{x \in O : V(x) \leqslant \gamma\}$ for some $\gamma > 0$. Pick $\gamma > \epsilon > 0$ and let $B(\epsilon) = \{x : V(x) < \epsilon\}$. Then $\overline{B(\epsilon)} \subset A$. Let $X(x,t),\ t \geqslant 0$, denote the trajectory of (1.4) with $X(x,0) = x$, and define

$$\Delta(\epsilon) = \min\{V(x) - V(X(x,T)) : x \in \bar{A} - B(\epsilon)\}.$$

By our hypotheses on $V(\cdot)$, $\Delta(\epsilon) > 0$. Pick $\eta > 0$ such that, for $x \in \bar{A}$, $\|x - y\| < \eta$ implies $|V(x) - V(y)| < \frac{\Delta(\epsilon)}{4}$. Let $n_0 \geqslant 1$ be such that, for $n \geqslant n_0$, $\sup_{t \in I_n} \|\bar{x}(t) - x^n(t)\| < \eta$ for $n \geqslant n_0$. Let $n_1 \geqslant n_0$ be such that $X(n_1) \in A$. Without loss of generality, we may take n_1 such that $m(n_1) = T_n$ for some n. (If not, we can simply shift the time origin appropriately.) Then, if $x^n(T_n) = \bar{x}(T_n) \in A - B(\epsilon)$,

$$V(x^n(T_{n+1})) \leqslant V(x^n(T_n)) - \Delta(\epsilon),$$

and hence

$$V(\bar{x}(T_{n+1})) \leqslant V(\bar{x}(T_n)) - \frac{\Delta(\epsilon)}{2},$$

implying in particular that $\bar{x}(T_{n+1}) \in A$. Since $V(\bar{x}(T_n))$ cannot go on decreasing at this rate indefinitely (recall that $V(\cdot) \geqslant 0$ in A), we have $V(\bar{x}(T_m)) \in B(\epsilon)$ for some $m \geqslant n$. Then $x^m(t), t \in I_m$, remains in $B(\epsilon)^\eta \stackrel{\Delta}{=} \{x : \|x - y\| < \eta \text{ for some } y \in B(\epsilon)\}$. We are free to make η smaller if necessary (which may increase n_0). Thus let η be such that $|V(x)| < \epsilon + \frac{\Delta(\epsilon)}{2}$ for $x \in B(\epsilon)^\eta - B(\epsilon)$. Then either $\bar{x}(T_{n+1}) \in B(\epsilon)$ or $\bar{x}(T_{n+1}) = x^{n+1}(T_{n+1}) \in B(\epsilon)^\eta - B(\epsilon)$, in which case $\bar{x}(T_{n+2}) \in B(\epsilon)$. It follows that $\bar{x}(T_n) \in B(\epsilon)^\eta$ from some n on. Since $\epsilon,\ \eta > 0$ can be made arbitrarily small, it follows that $\bar{x}(T_n) = x^n(T_n) \to x^*$. It then follows from standard o.d.e. arguments that $x^n(T_n + t) \to x^*$ uniformly in $t \in [0, T + a^*]$. Then by Lemma 2.3, $\bar{x}(t) \to x^*$. In particular, it follows that $X(n) = \bar{x}(t(n)) \to x^*$. □

Remark. Some of the assumptions above can be relaxed. For example, note that we require the convergence claim of Lemma 2.3 to hold only along the subsequence of $\{n\}$ for which $X(n) \in A$, thus (2.1) is not needed. Also, (2.2) and (2.3) need to be true only in A.

3. Lock-in probability

We now consider a fixed stable equilibrium x^* of (1.4) and a bounded open set A containing x^* such that $x \in \bar{A}$ implies that $X(x,t) \in \bar{A}, t \geqslant 0$, and $X(x,t) \to x^*$. Pick $T, \delta > 0$ such that $x \in \bar{A}$ implies that $\inf_{y \in \partial A} \|X(x,T) - y\| > 2\delta$, where ∂A is the boundary of A. (Note that, for a given δ, a sufficiently large T will always ensure this. In turn, δ can be any number of the form αr where $0 < \alpha < 1$ and $r =$ the distance of x^* from ∂A.) Suppose that $X(n_0) = x_0 \in A$ with $\inf_{y \in \partial A} \|x_0 - y\| > \delta$, for some $n_0 \geqslant 1$. We shall consider $X(n),\ n \geqslant n_0$, conditioned on $X(n_0) = x_0$ and find conditions under which $X(n) \in A$ infinitely often and almost surely. In view of Theorem 2.1, this implies that $X(n) \to x^*$ almost surely.

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Define $t(0) = T_0 = 0$ and $t(m) = \sum_{i=n_0+1}^{m} a(i),\ m \geqslant 0$. $\bar{x}(\cdot), \{T_i\}$ are then defined as before. Let $x^n(\cdot)$ solve (1.4) on I_n with $x^n(T_n) = \bar{x}(T_n)$. In view of our choice of δ and T, $\bar{x}(T_n) \in A$ a.s. (and therefore $X(n) \to x^*$ a.s.) on the set $\{\sup_{t\in I_n} \|\bar{x}(t) - x^n(t)\| \leqslant \delta,\ n \geqslant n_0\}$. We shall estimate the probability of this event. Our main result then is as follows.

Theorem 3.1. $P(X(n) \to x^*) \geqslant P(\sup_{t\in I_n} \|x^n(t) - \bar{x}(t)\| \leqslant \delta,\ n \geqslant n_0) \geqslant 1 - o(b(n_0))$.

In fact, we shall obtain a much stronger explicit bound. We need the following lemma.

Lemma 3.2. *There exists a finite constant K_2 such that*

$$\|\bar{x}(t)\| \leqslant K_2(1 + \|\bar{x}(T_n)\|), \quad t \in I_n. \tag{3.1}$$

Proof. From (1.1), (2.2) and (2.3), it follows that, for a suitable $K' > 0$,

$$\|X(n+1)\| \leqslant \|X(n)\| + a(n)K'(1 + \|X(n)\|).$$

The claim follows easily from this and the discrete Gronwall inequality. □

Proof of Theorem 3.1. Note that

$$P\left(\sup_{t\in I_n} \|x^n(t) - \bar{x}(t)\| > \delta \text{ for some } n \geqslant n_0\right) \tag{3.2}$$
$$= \sum_{n\geqslant n_0} P\left(\sup_{t\in I_n} \|x^n(t) - \bar{x}(t)\| > \delta \;\middle|\; \sup_{t\in I_m} \|x^m(t) - \bar{x}(t)\| \leqslant \delta,\ n_0 \leqslant m < n\right).$$

Now, if $\sup_{t\in I_m} \|x^m(t) - \bar{x}(t)\| \leqslant \delta$ for $m < n$, then $\bar{x}(T_n) \in A$. Letting $R = \max\{\|x\| : x \in A\}$, (3.1) leads to

$$\|\bar{x}(t)\| \leqslant (R+1)K_2,\ t \in I_n.$$

In (2.5), we may then take $C_1 = (R+1)K_2$, $C_2 = \max\{\|h(x)\| : \|x\| \leqslant C_1\}$. Take n_0 sufficiently large such that, for this choice of C_1, C_2, we have

$$\exp\big(K(T + a^*)\big)(C_2 + (1 + C_1)K_1)b(n_0) \leqslant \frac{\delta}{2}.$$

Then

$$\sup_{t\in I_n} \|x^n(t) - \bar{x}(t)\| \leqslant \frac{\delta}{2} + \sup_{m(n)\leqslant k\leqslant m(n+1)} \|\zeta(k) - \zeta(m(n))\| \exp\big(K(T + a^*)\big).$$

For $\delta_0 = \exp(-K(T + a^*))\delta$,

$$P\left(\sup_{t\in I_n} \|x^n(t) - \bar{x}(t)\| > \delta \;\middle|\; \sup_{t\in I_i} \|x^i(t) - \bar{x}(t)\| \leqslant \delta,\ n_0 \leqslant i < n\right)$$
$$\leqslant P\left(\sup_{m(n)\leqslant k\leqslant m(n+1)} \|\zeta(k) - \zeta(m(n))\| > \frac{\delta_0}{2} \;\middle|\; \sup_{t\in I_i} \|x^i(t) - \bar{x}(t)\| \leqslant \delta,\ n_0 \leqslant i < n\right).$$

By (2.3) and (3.1), conditioned on $\sup_{t\in I_i} \|x^i(t) - \bar{x}(t)\| < \delta,\ n_0 \leqslant i < n$, we have, for $m(n) \leqslant k < m(n+1)$,

$$\|\zeta(k+1) - \zeta(k)\| \leqslant a(k)K_1(1 + (R+1)K_2) \text{ a.s.}$$

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Then, by [20, (3.30), p. 227],

$$P\left(\sup_{m(n)\leqslant k<m(n+1)}\|\zeta(k+1)-\zeta(m(n))\|>\frac{\delta_0}{2}\ \Big|\ \sup_{t\in I_m}\|x^m(t)-\bar{x}(t)\|\leqslant\delta,\ n_0\leqslant m<n\right)$$
$$\leqslant 2d\exp\left(-\frac{\delta_0^2}{2(b(m(n))-b(m(n+1)))\bar{K}^2}\right),$$

where $\bar{K}=K_1(1+(R+1)K_2)$. Thus

$$\sum_{n\geqslant n_0}P\left(\sup_{t\in I_n}\|x^n(t)-\bar{x}(t)\|>\delta\ \Big|\ \sup_{t\in I_m}\|x^m(t)-\bar{x}(t)\|\leqslant\delta,\ n_0\leqslant m<n\right)$$
$$\leqslant 2d\sum_{n=n_0}^{\infty}\exp\left(-\frac{\delta_0^2}{2\bar{K}^2(b(m(n))-b(m(n+1)))}\right),$$

which is $o(b(m(n_0)))$ and therefore $o(b(n_0))$. (Recall that $m(n)\geqslant n$.) This completes the proof of the theorem. □

Remark. One can bound the r.h.s. of the last inequality above by a more convenient expression, using the fact that the function $\frac{\exp(-\frac{C}{x})}{x}$ is increasing in x for $0<x<C$. Thus, for sufficiently large n_0, the r.h.s. of the above inequality is bounded by (for $C=\delta_0^2/(2\bar{K}^2)$)

$$2d\sum_{n\geqslant n_0}\exp\left(-\frac{C}{b(m(n))-b(m(n+1))}\right)$$
$$\leqslant\ 2db(m(n_0))\frac{\exp\left(-\frac{C}{b(m(n_0))}\right)}{b(m(n_0))}$$
$$=\ 2d\exp\left(-\frac{C}{b(m(n_0))}\right).$$

Recall ϵ,η of the preceding section. In view of our proof of Theorem 2.1, on the set

$$G(\eta)\stackrel{\Delta}{=}\left\{\sup_{t\in I_n}\|\bar{x}(t)-x^n(t)\|<\eta,\ n\geqslant n_0\right\},$$

we have $\bar{x}(t)\in B(\epsilon)^\eta$ for $t>t(n_0)+\tau$, with

$$\tau\stackrel{\Delta}{=}\frac{2\big(\max\{V(x):x\in\bar{A}\}-\min\{V(x):x\in\partial B(\epsilon)\}\big)T}{\Delta(\epsilon)}.$$

Without loss of generality, let $\eta<\delta$. Then $G(\eta)\subset G(\delta)$. We can repeat the arguments above to show that $P(G(\eta))>1-o(b(n_0))$. Thus we have the following result.

Corollary 3.3. *With probability exceeding* $1-o(b(n_0))$, $X(n)$ *is in* $B(\epsilon)^\eta$ *for all* $n\geqslant n_0+N_0$, *where* $N_0=\min\{n:t(n)-t(n_0)\geqslant\tau\}$.

4. Conclusions

The main contribution of this article is to get a useful lower bound on the probability that a stochastic approximation algorithm will converge to a stable equilibrium of the

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associated o.d.e. given that it is in the domain of attraction of the latter at a certain stage of iteration. We conclude with some remarks.

Remarks. **(i)** Condition (2.3) was used to get a 'hard' bound on the martingale difference sequence $\zeta(k+1)-\zeta(k), k \geqslant 0$, so that the 'concentration inequality' of [20] could be invoked to give a Gaussian-like bound on tail probabilities associated with the corresponding martingale. It is possible to have such a bound in the absence of (2.3), for example when $\{M(n)\}$ are i.i.d. Gaussian. More generally, the foregoing developments should be looked upon as a general recipe for estimating lock-in probability, which may be tweaked to suit available hypotheses on the noise sequence $\{M(n)\}$, the main requirement being the availability of an appropriate 'concentration inequality'.

(ii) If (1.4) has a unique globally asymptotically stable equilibrium x^*, we have $X(n) \to x^*$ almost surely, the ideal situation in algorithmic applications. In such a case, the above results (notably Corollary 3.3) may be viewed as sample complexity results, giving the number of samples (or iterates) required to be within a desired neighbourhood of x^* with a prescribed probability. We can, in fact, write an explicit bound for the latter probability as in the remark following the proof of Theorem 3.1. There is, however, a subtle difference from traditional sample complexity results. What we present here is not the number of samples needed to get within a prescribed accuracy, with a prescribed probability, starting from time zero, but starting from time n_0, and the bound depends on the position at time n_0 via its dependence on the set A that we are able to choose.

As an example, consider the situation when $h(x) = g(x) - x$, where $g(\cdot)$ is a contraction, so that $\|g(x)-g(y)\| < a\|x-y\|$ for some $a \in (0,1)$. Let x^* be the unique fixed point of $g(\cdot)$, guaranteed by the contraction mapping theorem. Straightforward calculation shows that $V(x) = \|x-x^*\|$ will satisfy our requirements, with $\|X(x,t)-x^*\| < \exp(-at)\|X(x,0)-x^*\|$. Let the iteration be at a point $\bar{x}$ at time n_0, with $\|\bar{x} - x^*\| = b$ (say). For $\epsilon, T > 0$ as above, we have $\Delta(\epsilon) = \epsilon(1 - \exp(-aT))$. We then need

$$N_0 = \min\left\{ n : \sum_{i=n_0+1}^{n} a(i) \geqslant \frac{2Tb}{\epsilon(1-\exp(-aT))} \right\}$$

more iterates to get within 2ϵ of x^*, with probability exceeding

$$1 - 2d\exp\left(-\frac{c\epsilon^2}{b(m(n_0))}\right) = 1 - o(b(n_0))$$

for a suitable constant c depending on T, among other things. (We set '$\eta = \epsilon$' previously.) See [8] for details of a specific application of these ideas. Note that $T > 0$ is a free parameter affecting both the expression for N_0 and that for the probability, and may be chosen to optimize the estimates in some sense.

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