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Invariant densities for generalized β-maps

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Abstract. We find an explicit formula for the invariant density of a generalized β-map. This allows us to also find an explicit formula for the invariant density of Chebyshev map and discuss the monotonicity of the asymptotic average for such maps. Our results are based on a generalization of works of Parry (*Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416; **15** (1964), 95–105).

1. *Introduction*

The first non-trivial piecewise expanding map for which the existence of an absolutely continuous invariant measure (ACIM) was established was the β-map

$$x \mapsto \beta x \pmod 1, \quad \beta > 1,\ 0 \le x \le 1.$$

This was done by Rényi [**14**]. (When we say absolutely continuous, we mean with respect to Lebesgue measure λ.) The explicit formula for the density was found by Parry [**12**] and independently by Gelfond [**7**]. It was also proved in [**5**]. Later, Parry [**13**] generalized the formula to maps of the form

$$x \mapsto \beta x + \alpha \pmod 1, \quad \beta > 1,\ 0 < \alpha < 1,\ 0 \le x \le 1.$$

Another class of maps for which the invariant density was explicitly known [**15**] were Chebyshev polynomials including the logistic map

$$x \mapsto 2x^2 - 1, \quad -1 \le x \le 1.$$

The problem of generalizing Parry's formulas to more general piecewise linear, piecewise expanding maps and extending known formulas for Chebyshev polynomials has been approached in a number of works. In [**1**] and [**2**] the authors studied the ergodic properties of 'skewed' tent maps. In particular, it was proved that any skewed tent map can be approximated by a Markov one for which the invariant density can be calculated. In [**4**] it was shown how to find invariant densities for Markov Chebyshev maps, generalizations

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of Chebyshev polynomials. In the meantime, a theory of number expansions using more general piecewise expanding maps was developed, see [**6**] and the list of references therein. These results are closely connected with the problem of invariant densities.

In this paper we generalize the results of Parry to the case of what we call 'generalized β-transformations' (see §§2 and 5 for exact definitions). We produce the explicit formulas for invariant densities and in §3 we study some of their properties. In §4 we derive explicit formulas for invariant densities of Chebyshev maps. In particular, we study the mean value of a typical trajectory of such maps for $1 < \beta < 2$. This seems to be of importance in engineering applications [**10**]. In §6 we characterize the admissible itineraries of 1.

The general theory of piecewise expanding maps of an interval is well developed and we often refer to its results. The classical papers are [**9**] and [**11**] among many others. There is a number of books on the subject, see, for example, [**3**] or [**8**].

2. *Generalized β-transformations*

The classical β-map, for $\beta > 1$, is the transformation $\tau_{\mathrm{cl}(\beta)} : [0, 1] \to [0, 1]$ given by

$$\tau_{\mathrm{cl}(\beta)}(x) = \beta x - [\beta x],$$

where $[t]$ denotes the integral part of t. We will also use the notation $\langle t \rangle = t - [t]$. The generalized β-map, which we will denote by τ (or sometimes by τ_β or $\tau_{\beta,E}$), is similar but allows the branches to be either increasing or decreasing. Let $E = (E(0), \ldots, E([\beta]))$ be a vector of 0's and 1's. 0's correspond to increasing branches, while 1's correspond to the decreasing ones. We first define

$$\mathrm{I}_\beta(x) = [\beta x],$$

and then

$$\tau(x) = E(\mathrm{I}_\beta(x)) + (-1)^{E(\mathrm{I}_\beta(x))} \cdot \tau_{\mathrm{cl}(\beta)}(x).$$

Obviously, when all elements of E are 0's we get the classical map $\tau_{\mathrm{cl}(\beta)}$.

Similarly to the classical one, the generalized β-map produces an expansion of numbers in [0, 1]. It is analogous to that given by generalized Lüroth series (see [**6**]). We define the vector $D = (D(0), \ldots, D([\beta]))$, where

$$D(i) = i + E(i), \quad i = 0, \ldots, [\beta].$$

For an $x \in [0, 1]$ we define 'digits' $d(j)$ and 'signs' $e(j)$, $j = 0, 1, \ldots$, as follows:

$$d(j) = D(\mathrm{I}_\beta(\tau^j(x))), \quad e(j) = (-1)^{E(\mathrm{I}_\beta(\tau^j(x)))}.$$

Then, we define 'cumulative signs' $s(j)$, $j = 0, 1, \ldots$,

$$s(0) = 1, \quad s(j+1) = s(j) \cdot e(j) \quad \text{for } j = 0, 1, \ldots.$$

PROPOSITION 1. *For any $k \geq 0$ we have*

$$x = \frac{s(0)d(0)}{\beta} + \frac{s(1)d(1)}{\beta^2} + \cdots + \frac{s(k)d(k)}{\beta^{k+1}} + \frac{s(k+1)\tau^{k+1}(x)}{\beta^{k+1}}, \tag{1}$$

and thus

$$\left| x - \left(\frac{s(0)d(0)}{\beta} + \frac{s(1)d(1)}{\beta^2} + \cdots + \frac{s(k)d(k)}{\beta^{k+1}} \right) \right| \leq \frac{1}{\beta^{k+1}}.$$

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Proof. Let $x \in [0, 1]$ and $m = \mathrm{I}_\beta(x)$. Then,

$$\tau(x) = E(m) + (-1)^{E(m)}(\beta x - m)$$

or

$$\begin{aligned} x &= (m - (-1)^{E(m)}E(m))/\beta + (-1)^{E(m)}\tau(x)/\beta \\ &= D(m)/\beta + (-1)^{E(m)}\tau(x)/\beta. \end{aligned} \tag{2}$$

Using this formula inductively we obtain formula (1) which then implies the estimate. □

COROLLARY 2. *For any $x \in [0, 1]$ we have*

$$x = \sum_{j=0}^{\infty} s(j)d(j)/\beta^{j+1}. \tag{3}$$

We will call the expansion (3), with the coefficients defined using map $\tau = \tau_{\beta,E}$, the τ-expansion or sometimes the valid τ-expansion.

COROLLARY 3. *If $x = \sum_{j=0}^{\infty} s(j)d(j)/\beta^{j+1}$ is an τ-expansion of x ($\tau = \tau_{\beta,E}$), then*

$$\tau^n(x) = s(n)\beta^n \cdot \sum_{j=n}^{\infty} s(j)d(j)/\beta^{j+1}. \tag{4}$$

We define an itinerary of point x as the sequence $\mathrm{It}_\beta(x) = \{\mathrm{It}_\beta(x, n)\}_{n\geq 0} = \{\mathrm{I}_\beta(\tau^n(x))\}_{n\geq 0}$. It shows the consecutive numbers of subintervals through which x travels. Since, for any $n \geq 0$,

$$d(n) = \mathrm{It}_\beta(x, n) + (1 - s(n + 1)s(n))/2,$$

the correspondence between expansion coefficients and itineraries is one-to-one.

Remark 4. The sequence $\{s(n)d(n)\}_{n\geq 0}$ lets us recover the itinerary $\mathrm{It}_\beta(x)$ of the corresponding point x. Since τ is piecewise expanding, this lets us find the point x from its expansion coefficients. Thus, the correspondence $x \leftrightarrow$ expansion coefficients is one-to-one.

If some image $\tau^n(x)$ of x equals 0 or 1 we can consider this expansion finite. In particular, we can consider expansion of the number 1. If it is finite, we call β simple. If the last branch of τ is increasing it is equivalent to the expansion of β, or rather $\langle\beta\rangle$, being finite, although it does not have to be a valid τ-expansion.

Example. Let $\beta = (3 + \sqrt{5})/2$ and $E = [0, 1, 0]$. Then,

$$\begin{aligned} 1 &= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{-2}{\beta^3} + \frac{2}{\beta^4} + \frac{2}{\beta^5} + \frac{-2}{\beta^6} + \cdots \\ \langle\beta\rangle &= \frac{2}{\beta} + \frac{-2}{\beta^2} + \frac{2}{\beta^3} + \frac{2}{\beta^4} + \frac{-2}{\beta^5} + \cdots = \frac{2}{\beta} + \frac{-2}{\beta^2} + \frac{1}{\beta^2}. \end{aligned}$$

If the last branch of τ is decreasing, then some image of $1 - \langle\beta\rangle$ falls into 0 or 1 and $\langle\beta\rangle$ again is a finite integer combination of inverse powers of β, which does not have to be a valid τ-expansion. This shows that every simple β is an algebraic number.

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If some image $\tau^n(x)$ of x equals $\tau^k(x)$ for $k < n$, then the expansion of x is ultimately periodic. If the expansion of 1 is ultimately periodic we will call the corresponding β ultimately periodic. Thus, simple is a special case of ultimately periodic. The expansion also depends on vector E so, when we say that β is simple or ultimately periodic, we mean that the pair (β, E) has this property.

From the general theory of piecewise expanding maps (see, e.g., [**3**]) it follows that β is simple if and only if the map τ is a Markov map.

A question arises: what expansions are valid expansions defined by the map τ_β? Since the correspondence between expansion coefficients and itineraries is one-to-one, we will answer it in terms of itineraries. We first introduce an order on the set of itineraries. We say that $\mathrm{It}_\beta(x) < \mathrm{It}_\beta(y)$ if and only if, for the first $n \geq 0$ such that $\mathrm{It}_\beta(x, n+1) \neq \mathrm{It}_\beta(y, n+1)$, we have

$$\begin{aligned} &\mathrm{It}_\beta(x, n+1) < \mathrm{It}_\beta(y, n+1) \quad \text{if } s(n+1) = 1, \\ &\mathrm{It}_\beta(x, n+1) > \mathrm{It}_\beta(y, n+1) \quad \text{if } s(n+1) = -1, \end{aligned} \tag{5}$$

where the relevant part of the sequence of the cumulative signs $\{s(n)\}_{n\geq 0}$ is obtained from the common beginning of both itineraries. We say that $\mathrm{It}_\beta(x) \leq \mathrm{It}_\beta(y)$ if $\mathrm{It}_\beta(x) < \mathrm{It}_\beta(y)$ or $\mathrm{It}_\beta(x) = \mathrm{It}_\beta(y)$.

PROPOSITION 5. *The itinerary* $\mathrm{It}_\beta(x)$ *is valid if and only if, for any* $k \geq 0$*, the shifted itinerary*

$$\mathrm{It}_\beta^{(k)}(x) = \mathrm{It}_\beta(\tau^k(x)) = \{\mathrm{I}_\beta(\tau^n(x))\}_{n\geq k} \leq \mathrm{It}_\beta(1).$$

In particular, for any $k \geq 0$ *the shifted itinerary of* 1 *satisfies*

$$\mathrm{It}_\beta^{(k)}(1) \leq \mathrm{It}_\beta(1). \tag{6}$$

Proof. The result follows from the general properties of the piecewise expanding maps. □

An integrable non-negative function h is a density of an absolutely continuous τ-invariant measure if and only if it satisfies the Perron–Frobenius equation

$$h(x) = \sum_{y:\tau(y)=x} h(y)/|\tau'(y)| = (P_\tau(h))(x),$$

for almost all $x \in [0, 1]$. Operator P_τ is called the Perron–Frobenius operator [**3**]. The preimages of x are $x(m) = (D(m) + (-1)^{E(m)}x)/\beta$, $m = 0, \ldots, [\beta]$, but note that the last one corresponding to $m = [\beta]$ does not always exists. In our case the Perron–Frobenius equation becomes

$$\beta \cdot h(x) = \sum_m h(x(m)). \tag{7}$$

THEOREM 6. *Let* $\beta > 1$ *and a vector* E *be fixed. Let*

$$1 = \sum_{n=0}^{\infty} s(n)d(n)/\beta^{n+1}$$

be the τ *defined expansion of* 1*. Let us define a (non-normalized) density by*

$$h = \sum_{n=0}^{\infty} s(n)\chi_{[0,\tau^n(1))}/\beta^{n+1}. \tag{8}$$

Then, h *is* τ*-invariant.*

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Proof. The proof is very similar to that of Parry [**12**]. We will show that h satisfies (7) except possibly at the images $\{\tau^n(1)\}_{n\geq 0}$ of 1. Let us fix an $x \in [0, 1]$ and let $x(m)$, $m = 0, \ldots, [\beta]$, be its preimages (with $x([\beta])$ possibly not existing). Let

$$a_{n,m} = \begin{cases} 1 & \text{if } x(m) < \tau^n(1), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a_n = \begin{cases} 1 & \text{if } x < \tau^n(1), \\ 0 & \text{otherwise.} \end{cases}$$

Let us calculate $\sum_{m\geq 0} a_{n,m}$. It is equal to the number of m's for which $x(m) < \tau^n(1)$ or $D(m) + (-1)^{E(m)}x < \beta\tau^n(1)$ or $m + E(m) + (-1)^{E(m)}x < \beta\tau^n(1)$. Let us consider two cases.

(i) Let the branch of τ over $\tau^n(1)$ be increasing. We are looking for the largest m_0 such that $x(m_0) < \tau^n(1)$ or $m_0 + x < \beta\tau^n(1)$. We can write this as

$$m_0 + x < [\beta\tau^n(1)] + \langle\beta\tau^n(1)\rangle = [\beta\tau^n(1)] + \tau^{n+1}(1).$$

Thus, the largest m_0 satisfying this inequality is $[\beta\tau^n(1)]$ if $x < \tau^{n+1}(1)$ and $[\beta\tau^n(1)] - 1$ otherwise. Thus, in this case

$$\sum_{m\geq 0} a_{n,m} = [\beta\tau^n(1)] + a_{n+1} = d(n) + e(n)a_{n+1}.$$

(ii) Let the branch of τ over $\tau^n(1)$ be decreasing. Then, we are looking for the largest m_0 satisfying

$$m_0 + 1 - x < \beta\tau^n(1) = [\beta\tau^n(1)] + \langle\beta\tau^n(1)\rangle = [\beta\tau^n(1)] + 1 - \tau^{n+1}(1),$$

or

$$m_0 + \tau^{n+1}(1) < [\beta\tau^n(1)] + x.$$

The largest such m_0 is $[\beta\tau^n(1)]$ if $\tau^{n+1}(1) < x$, and $[\beta\tau^n(1)] - 1$ if $x < \tau^{n+1}(1)$. We have $d(n) = [\beta\tau^n(1)] + 1$. Thus, also in this case

$$\sum_{m\geq 0} a_{n,m} = [\beta\tau^n(1)] + 1 - a_{n+1} = d(n) + e(n)a_{n+1}.$$

Using the above calculations we obtain

$$\begin{aligned} \sum_{m\geq 0} h(x(m)) &= \sum_{m\geq 0}\sum_{n=0}^{\infty} \frac{s(n)}{\beta^{n+1}} a_{n,m} = \sum_{n=0}^{\infty} \frac{s(n)}{\beta^{n+1}} \sum_{m\geq 0} a_{n,m} \\ &= \sum_{n=0}^{\infty} \frac{s(n)(d(n) + e(n)a_{n+1})}{\beta^{n+1}} = \sum_{n=0}^{\infty} \frac{s(n)d(n)}{\beta^{n+1}} + \sum_{n=0}^{\infty} \frac{s(n)e(n)a_{n+1}}{\beta^{n+1}} \\ &= 1 - s(0)a_0 + s(0)a_0 + \sum_{n=1}^{\infty} \frac{s(n)a_n}{\beta^n} = \beta \cdot h(x). \end{aligned}$$

□

Remark 7. The infinite sum in (8) reduces to a finite one if and only if we have $\tau^n(1) = \tau^m(1)$ for some $n \neq m$. This is equivalent to 1 having an ultimately periodic τ-expansion or, in other words, β being ultimately periodic.

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3. *Ergodic properties of τ and properties of the invariant density*

In this section we will discuss the ergodic properties of transformations τ for different β and vectors E. Then, we prove a few properties of the τ invariant density h for a fixed β and vector E, and the dependence of h on β and vector E.

In the following proposition we discuss the ergodic properties of the system $(\tau, h\lambda)$.

PROPOSITION 8. *For all $\beta > 1$ and all E the absolutely continuous invariant measure $h \cdot \lambda$ is unique and the system $(\tau, h\lambda)$ is ergodic. For $\beta > \sqrt{2}$ it is exact. More details about the specific cases are given in the proof.*

Proof. From general theory of piecewise expanding transformations it follows that, for $1 < \beta \leq 2$, map τ has unique ergodic absolutely continuous measure (e.g., [**3**, Theorem 8.2.2]). For $2 < \beta$ we reason as follows. From general theory we know that the support of ACIM contains an interval, say J (e.g., [**3**, Theorem 8.2.1]). Since $2 < \beta$, the lengths of the largest connected pieces of the images $\tau^n(J)$ increase until this largest piece contains two consecutive points of discontinuity or non-monotonicity of τ. Then, the next image is equal to the whole $[0, 1]$. Thus, we have established uniqueness of ACIM for all $\beta > 1$.

From the general theory we know that some power of τ with the ACIM is the union of disjoint exact systems (e.g., [**3**, Theorem 7.2.1]). From the above reasoning it follows that $(\tau, h\lambda)$ is exact for any $\beta > 2$ and any vector E. We will discuss the case of $1 < \beta \leq 2$ for each possible vector E separately.

(i) $E = [1, 0]$: For $\sqrt{2} < \beta < 2$ the map τ^2 has slope larger than 2 and has exactly two points of non-monotonicity joined by the onto branch. For $\beta = 2$ it has three such points, but both consecutive pairs are joined by onto branches. The above reasoning can be applied and we see that τ^2 is exact. Thus, τ is exact.

For $2^{1/4} < \beta \leq 2^{1/2}$ the map τ^2 separates into two systems $\tau^2_{|J_1}$ with $J_1 = [0, \tau^2(0)]$ and $\tau^2_{|J_2}$ with $J_2 = [\tau^2(1), 1]$. The first is conjugated to τ_{β^2} (with the same E) by a linear homeomorphism. The second is conjugated to τ_{β^2} (also with the same E) by a linear homeomorphism reversing orientation. Thus, both are exact.

Now, by induction, for $2^{1/2^k} < \beta \leq 2^{1/2^{k-1}}$ the map τ^{2^k} separates into 2^k systems which are conjugated to τ with the slope β^{2^k} and the same E and are thus exact. It is worth noting that the invariant density for these systems can be obtained either as a restriction of h to the appropriate interval or by applying formula (8) to τ with the slope β^{2^k} and the same E and then using the conjugation.

(ii) $E = [0, 1]$: The attracting basin for τ is inside $J = [\tau(1), 1]$ and $\tau_{|J}$ is conjugated by a linear homeomorphism reversing orientation to τ with the same slope and $E = [1, 0]$. Thus, the results from part (i) apply.

(iii) $E = [0, 0]$: This is a classical β-transformation. It is exact for any $\beta > 1$. From the general theory we know that the support of an ACIM contains an interval containing the discontinuity point $1/\beta$. Thus, its image contains 0 which is a repelling fixed point. The further images grow in length and after a finite number of steps cover the whole $[0, 1]$.

(iv) $E = [1, 1]$: The fixed point of τ in the domain of the first (onto) branch is $x_0 = 1/(1 + \beta)$. The solution of the equation $x_0 = \tau(1)$ is $\beta_0 = (1 + \sqrt{5})/2 \simeq 1.618\,0339$. For $\beta > \beta_0$, τ is exact: $\beta^2 > 2$ so for any interval contained in the support of the ACIM

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the largest connected pieces of images under τ^2 grow until they cover two consecutive discontinuity points of τ^2. The next image covers the fixed point x_0. Then, the further images grow until after a finite number of steps they cover the whole [0, 1].

The second fixed point of τ is $x_1 = 2/(1+\beta)$. The equation $x_1 = \tau^3(1)$ is $2 = (1+\beta)(1-\beta+2\beta^2-\beta^3)$ and has a solution $\beta_1 \simeq 1.324\,7179$. For $\beta_1 < \beta \leq \beta_0$, τ is also exact, but the support of h is the union of two disjoint intervals $J_0 = [0, \tau^2(1)]$ and $J_1 = [\tau(1), 1]$. This can be proved as follows. $\beta^3 > 2$, so for any interval contained in the support of the ACIM the largest connected pieces of images grow under τ^3 until they cover two consecutive discontinuity points of τ^3. Then, the next or the second next image covers the fixed point x_1. The further images grow to cover the whole J_1 and then finally the next image covers J_0. Thus, τ^3 is exact. Hence, τ is exact.

It seems that in case (iv) τ is exact for all $\beta > 1$ although the further partition of the support of h occurs. We were not able to find a proof of this conjecture. □

PROPOSITION 9. *Let $\beta > 1$ and vector $E = (E(0), \ldots, E([\beta]))$ be fixed. Then, the τ invariant density h is continuous from the right. It is also continuous except at images of* 1. *h reduces to a finite sum of characteristic functions of intervals if and only if β is periodic.*

Proof. h is continuous on the right as a uniform limit of functions with this property. The second property can be proved by splitting h into an arbitrarily small tail and a finite sum of functions discontinuous only at images of 1. The last statement follows trivially from the definitions. □

LEMMA 10. *If the τ-invariant density is unique it is non-negative.*

Proof. Let h be the τ-invariant density, i.e. the fixed point of Perron–Frobenius operator (7): $P_\tau h = h$. If h is non-positive we consider $-h$. Let us represent h as the sum of its positive and negative parts: $h = h^+ - h^-$. Since P_τ is a positive operator $h^+ = (P_\tau h)^+ = P_\tau h^+$ and $h^- = (P_\tau h)^- = P_\tau h^-$. Thus, h^+ is also a fixed point of P_τ. By uniqueness, $h = h^+$. □

PROPOSITION 11. *For all $\beta > 1$ and all E the invariant density h is non-negative. Furthermore, there exists a constant $a > 0$ such that $h(x) \geq a$ on the support of h, except possibly at a countable number of points. h is bounded by $1/(\beta(1-\beta))$.*

Proof. The first statement follows from the uniqueness proved in Proposition 8 and Lemma 10. The second one is a general property of piecewise expanding maps. The last one follows directly from the formula (8). □

PROPOSITION 12. *For any $\beta_0 > 1$ and any vector E there exist simple β, i.e. such that some image $\tau_\beta^n(1)$ of* 1 *is equal to* 0 *or* 1, *arbitrarily close to β_0.*

Proof. We can assume that β_0 is not simple and in particular not an integer. As β changes from $m = [\beta_0]$ to $m+1$ the image $\tau_\beta(1)$ changes linearly from 0 to 1, or from 1 to 0. For each $\beta \in (m, m+1)$ the preimages of $\{0, 1\}$, $\{\tau_\beta^{-n}(\{0, 1\})\}_{n\geq 1}$ are dense in the support of the ACIM, since τ_β is piecewise expanding.

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$\tau_\beta(1)$ is in the support of the ACIM. If $\beta > 2$, it holds since this support is the whole $[0, 1]$. For $1 < \beta < 2$ the support contains the non-monotonicity point x_1. If $\tau_\beta(x_1) = 1$ we are done. If $\tau_\beta(x_1) = 0$, there are two possibilities: either $\tau_\beta(0) = 1$ and again we are done or $\tau_\beta(0) = 0$ and the support of the ACIM is again the whole $[0, 1]$.

Let $x_n(\beta)$ be an nth preimage of 1 or 0. As β changes from m to $m + 1$, $x_n(\beta)$ changes smoothly from its value for $\beta = m$, s/m^n for some $0 \le s \le m^n$, to its value for $\beta = m + 1$, $t/(m + 1)^n$ for some $0 \le t \le (m + 1)^n$. Not all preimages of 0 or 1 belong to such a 'complete' curve, but we consider only those which do. There is enough of them, since, from any point s/m^n, $0 \le s \le m^n$, starts a curve of preimages which is defined for all $m \le \beta \le m + 1$.

We will show that the slopes of curves $x_n(\beta)$, as functions of β, converge to 0 as n increases. Let $x_n(\beta)$ be one of these curves. Its preimage using the mth branch of τ is $x_{n+1}(\beta) = (D(m) + e(m)x_n(\beta))/\beta$ and its derivative is

$$\frac{d}{d\beta}x_{n+1}(\beta) = \frac{e(m)\,dx_n(\beta)\,d\beta - (D(m) + e(m)x_n(\beta))/\beta}{\beta}.$$

Thus, we have

$$\left|\frac{d}{d\beta}x_{n+1}(\beta)\right| \le \frac{1}{\beta}\left(\left|\frac{d}{d\beta}x_n(\beta)\right| + 1\right).$$

Applying this inductively we obtain the fact that slopes of curves $x_n(\beta)$ diminish to 0 as n increases, at least in some neighborhood of β_0 (in the case of $\beta_0 < 2$ and on the whole interval $[m, m + 1]$ otherwise). Let us fix an $\varepsilon > 0$. We can find N such that some preimage $x_N(\beta_0)$ of 1 or 0 is within ε of $\tau_{\beta_0}(1)$ and the slope of the curve $x_N(\beta)$ containing $x_N(\beta_0)$ is within $(-\delta, \delta)$ for some $\delta > 0$. The curve $\tau_\beta(1)$ is the diagonal or the antidiagonal so it intersects $x_N(\beta)$ at a point β_1 in the interval $[\beta_0 - \varepsilon/(1 - \delta), \beta_0 + \varepsilon/(1 - \delta)]$. The Nth image $\tau^N_{\beta_1}(1)$ is 0 or 1, so β_1 is simple. □

Let $h_{\beta,E}$ denote the invariant density for $\tau_{\beta,E}$ and let $F(\beta, E) = \int_0^1 h_{\beta,E}\,d\lambda$ be its normalizing factor.

PROPOSITION 13. *Let β_0 be not simple, i.e. images $\tau^n_{\beta_0,E}(1)$ never fall into 0 or 1. Then, $h_{\beta,E} \to h_{\beta_0,E}$ in L^1 as $\beta \to \beta_0$.*

Proof. The tail in the expansion of $h_{\beta,E}$ can be made uniformly small in some neighborhood of β_0. The remaining finite number of images $\tau^n_{\beta,E}(1)$ vary continuously for β in some possibly smaller neighborhood of β_0. This gives L^1 continuity of $h_{\beta,E}$ around β_0. □

COROLLARY 14. *If β_0 is not simple, then the normalizing factor $F(\beta, E)$ is continuous at β_0.*

Remark 15. It is worth noting that it follows from the general theory (e.g., [**3**, Theorem 10.2.1] describing conditions for the stability of the invariant density) that for

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non-integer β_0 the normalized density $h_{\beta,E}/F(\beta, E)$ converges to $h_{\beta_0,E}/F(\beta_0, E)$ in L^1 as $\beta \to \beta_0$.

We can represent the normalizing factor in the following form:

$$\begin{aligned} F(\beta, E) &= \int_0^1 h_{\beta,E}\, d\lambda = \sum_{n=0}^{\infty} \frac{s(n)}{\beta^{n+1}} \tau^n(1) \\ &= \sum_{n=0}^{\infty} \frac{s(n)}{\beta^{n+1}} \left(s(n)\beta^n \cdot \sum_{j=n}^{\infty} s(j)d(j)/\beta^{j+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(n+1)s(n)d(n)}{\beta^{n+2}}. \end{aligned} \tag{9}$$

PROPOSITION 16. *For any $\beta > 1$ and any vector E we can approximate $\tau_{\beta,E}$ arbitrarily close by a Markov map $\tau_{\beta_1,E}$ with the same vector E. The invariant density h_{β_1} approximates h_β in L^1.*

Proof. The result follows from Propositions 12 and 13. □

4. *Chebyshev maps*

In this section we consider the family of Chebyshev maps

$$T_\beta(x) = \cos(\beta \arccos x), \quad -1 \le x \le 1,$$

where $\beta > 1$. For integer $\beta = n > 1$ the map T_n is the famous Chebyshev polynomial of nth order. Properties of these polynomials have been extensively studied and are well known. In particular, it is known (see, e.g., [**15**]) that T_n preserves absolutely continuous invariant measure with the density

$$g(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

For non-integer β the ACIMs of T_β were found only in special cases, when the map was Markov [**4**]. This result was based on the observation that T_β is differentiably conjugated to a generalized β transformation. The following proposition was proved in [**4**].

PROPOSITION 17. *Let $\beta > 1$ and let $\phi(x) = \cos(\pi x) : [0, 1] \to [-1, 1]$. Then,*

$$\tau_{\beta,E} = \phi^{-1} \circ T_\beta \circ \phi,$$

where vector $E = (E(0), E(1), \ldots, E([\beta]))$ with $E(k) = 0$ for even k and $E(k) = 1$ for odd k.

Using known (see, e.g., [**3**]) formula for the invariant density of a conjugated map, we obtain the following.

PROPOSITION 18. *Let $\beta > 1$. Then, the T_β-invariant density is*

$$g_\beta(x) = h_{\beta,E}(\phi^{-1}(x))|(\phi^{-1})'(x)| = h_{\beta,E}\left(\frac{1}{\pi}\arccos x\right)\frac{1}{\pi\sqrt{1-x^2}},$$

where $h_{\beta,E}$ is the $\tau_{\beta,E}$-invariant density for vector $E = (E(0), E(1), \ldots, E([\beta]))$ with $E(k) = 0$ for even k and $E(k) = 1$ for odd k. Thus, for $T = T_\beta$ we have

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$$g_\beta = \frac{1}{\pi\sqrt{1-x^2}} \sum_{n=0}^{\infty} s_T(n)\chi_{(T^n(-1),1]}/\beta^{n+1}, \tag{10}$$

where $s_T(n)$ is obtained as follows. Let $E_T = (E_T(0), E_T(1), \ldots, E_T([\beta]))$ and $E_T(j) = E([\beta] - j)$, i.e. $E_T(j)$ is 0 if the jth branch (counting from the left) of T is increasing and $E_T(j)$ is 1 if the jth branch of T is decreasing. We define

$$s_T(0) = 1, \quad s_T(n+1) = s_T(n) \cdot (-1)^{E(j(T^n(-1)))}, \quad n = 0, 1, \ldots,$$

where the point $T^n(-1)$ belongs to the domain of the branch with number $j(T^n(-1))$.

Proof. We need to prove only the second part. We have

$$\chi_{[0,\tau^n(1))}(\phi^{-1}(x)) = \chi_{(\phi(\tau^n(1)),\phi(0)]}(x) = \chi_{(T^n(-1),1]}(x),$$

since $\phi(\tau^n(1)) = \phi(\tau^n(\phi^{-1}(\phi(1)))) = T^n(-1)$ as ϕ also conjugates τ^n with T^n. Since τ and T are conjugated, the trajectories $\tau^n(1)$ and $T^n(-1)$ visit the same intervals of monotonicity of τ and T respectively, but since ϕ reverses orientation we have to reverse the numeration. The method described in the proposition gives $s_T(n) = s(n)$ for all $n \geq 0$. □

The analogues of results proved in the previous section hold also for Chebyshev maps. In particular, the normalized invariant densities $\bar{g}_\beta = g_\beta/F(\beta)$ behave continuously in L^1 for non-integer β. Since $\int_0^1 h_\beta\, d\lambda = \int_{-1}^1 g_\beta\, d\lambda$ the normalizing factors are the same.

In [**10**] the authors use the experimentally established monotonicity of expected values of a random variable with normalized densities $\bar{g}_\beta$, for $1 < \beta < 2$. Using formula (10) we can write

$$\begin{aligned}\mathcal{E}(\beta) = \mathcal{E}(\bar{g}_\beta) &= \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{s_T(n)}{\beta^{n+1}} \int_{-1}^{1} x \frac{\chi_{(T^n(-1),1](x)}}{\pi\sqrt{1-x^2}}\, dx \\ &= \frac{1}{\pi F(\beta)} \sum_{n=0}^{\infty} (s_T(n)/\beta^{n+1})\sqrt{1-(T^n(-1))^2} \\ &= \frac{\sum_{n=0}^{\infty}(s_T(n)/\beta^{n+1})\sqrt{1-(T^n(-1))^2}}{\sum_{n=0}^{\infty}(s_T(n)/\beta^{n+1})\arccos(T^n(-1))}. \end{aligned} \tag{11}$$

We know that $\mathcal{E}(\beta)$ is continuous for non-integer β.

We were not able to prove that $\mathcal{E}(\beta)$ is actually monotonic for $1 < \beta < 2$. Our formula gives a much better way to computationally verify this conjecture than the experimental method based on calculating averages of trajectories. Using it, we were able to numerically confirm the conjecture at least for $1.2 < \beta < 2$. For smaller β the calculations were still not very reliable.

5. *Further generalization*

In [**13**] the results for β-maps were generalized to maps of the form

$$x \mapsto \beta x + \alpha \pmod 1. \tag{12}$$

We also tried to generalize the results of §2 to maps which generalize (12). Although our results are only partial, we report them in this section.

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Let $\beta > 1$ and $0 < \alpha < 1$. We define the classical β, α map as

$$\tau_{\mathrm{cl}(\beta,\alpha)}(x) = \beta x + \alpha - [\beta x + \alpha], \quad 0 \le x \le 1.$$

The first branch of this map is defined on $[0, x_1)$, where $x_1 = (1-\alpha)/\beta$, and consecutive branches are defined on intervals $[x_1 + k/\beta, x_1 + (k+1)/\beta)$. If $\beta + \alpha < [\beta] + 1$, then $k = 0, \ldots, [\beta] - 2$ and the last interval is $[x_1 + ([\beta]-1)/\beta, 1]$. If $\beta + \alpha = [\beta] + 1$, i.e. $\beta + \alpha$ is an integer, then again $k = 0, \ldots, [\beta] - 1$ but the last of these intervals is closed on the right at 1. If $\beta + \alpha > [\beta] + 1$, then $k = 0, \ldots, [\beta] - 1$ and the last interval is $[x_1 + [\beta]/\beta, 1]$.

We define the function

$$\mathrm{I}_{\beta,\alpha}(x) = [\beta x + \alpha],$$

which gives the number of the branch in which domain the x is (starting from 0). Now, we define a generalized map $\tau = \tau_{\beta,\alpha,E}$ as follows: let E be a vector of 0's and 1's, of length $[\beta] + 1$ or $[\beta] + 2$, which will indicate whether a branch should be increasing or decreasing. Let

$$\tau_{\beta,\alpha,E}(x) = E(\mathrm{I}_{\beta,\alpha}(x)) + (-1)^{E(\mathrm{I}_{\beta,\alpha}(x))}\tau_{\mathrm{cl}(\beta,\alpha)}(x). \tag{13}$$

Similarly as in §2, we define the vector D, $D(i) = i + E(i)$, and for any point $x \in [0, 1]$ the sequence of signs $\{e(n) = (-1)^{E(\mathrm{I}_{\beta,\alpha}(\tau^n(x)))}\}_{n\ge 0}$, digits $\{d(n) = D(\mathrm{I}_{\beta,\alpha}(\tau^n(x))) - \alpha\}_{n\ge 0}$ and cumulative signs $\{s(n)\}_{n\ge 0}$. Again, we have the expansion

$$x = \sum_{n=0}^{\infty} s(n)d(n)/\beta^{n+1},$$

which has the properties analogous to that of expansion in §2.

The following theorem generalizes the result of [**13**].

THEOREM 19. *Let $\{s_0(n)\}_{n\ge 0}$ and $\{s_1(n)\}_{n\ge 0}$ denote the sequences of cumulative signs for points 0 and 1, respectively. If $s_0(n) = s_1(n)$ for all $n \ge 0$, then a version of $\tau_{\beta,\alpha,E}$-invariant density is*

$$h = \sum_{n=0}^{\infty} \frac{s_1(n)}{\beta^{n+1}} \chi_{[0,\tau^n(1))} - \sum_{n=0}^{\infty} \frac{s_0(n)}{\beta^{n+1}} \chi_{[0,\tau^n(0))}. \tag{14}$$

In particular, this holds in the following cases:

(i) all branches of $\tau_{\beta,\alpha,E}$ are increasing (classical case);
(ii) all branches of $\tau_{\beta,\alpha,E}$ are decreasing;
(iii) for some n_0 we have $\tau^{n_0}(0) = \tau^{n_0}(1)$ and cumulative signs satisfy $s_0(n) = s_1(n)$ for $n = 0, \ldots, n_0$.

Proof. The proof is similar to the proof of Theorem 6. The only difference is that for map $\tau_{\beta,\alpha,E}$ the first (or rather 0) preimage may exist or not for different x. If the cumulative signs of $s_0(n)$ and $s_1(n)$ coincide, then contributions of these preimages (or lack of them) to both sums in (14) cancel out and h is still invariant. □

Example 20. If β is an integer and the first and the last branches are both decreasing or both increasing, then $\tau(0) = \tau(1)$ and Lebesgue measure λ is τ-invariant.

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Example 21. Let β satisfy $1 - \langle\beta\rangle = 1/\beta$, $\beta \simeq 5.828\,4271$ and $\alpha \in (0.88, 0.99)$. For either $E(0) = E(5) = E(6) = 0$ or $E(0) = E(1) = E(2) = E(6) = 1$ we have $\tau^2(0) = \tau^2(1)$ and the density (14) is τ-invariant.

The following example describes one more situation in which Theorem 19 applies.

Example 22. Let the first and the last branches of τ be both increasing or both decreasing. Let images $\tau(0)$ and $\tau(1)$ be different fixed points of τ belonging to the domains of branches with the same character of monotonicity. Then, the cumulative signs of trajectories of 0 and of 1 are the same and the density (14) is τ-invariant.

This happens, for example, when $\alpha = 2 - \sqrt{3}$, $\beta = 6 - 2\alpha$ and $E = [0, 1, a, b, 1, 0]$, with $a, b \in \{0, 1\}$.

We now describe some cases where Theorem 19 does not apply but an invariant density can still be found.

PROPOSITION 23. *If $\alpha = [\beta] + 1 - \beta$, then the map $\tau_{\beta,\alpha,E}$ is conjugated by $\phi(x) = 1 - x$ to $\tau_{\beta,\bar{E}}$, where $\bar{E}(i) = E([\beta] - j)$, $j = 0, \ldots, [\beta]$. The invariant density can be found using conjugation or directly by the formula*

$$h = \sum_{n=0}^{\infty} \frac{s_0(n)}{\beta^{n+1}} \chi_{(\tau^n(0),1]}, \tag{15}$$

where $\tau = \tau_{\beta,\alpha,E}$ and cumulative signs $\{s_0(n)\}_{n\geq 0}$ are calculated using τ.

PROPOSITION 24. *Let $1 < \beta < 2$, $\alpha > 2 - \beta$ and $E = [1, 0, 1]$. The system $\tau_{\beta,\alpha,E}$ separates into two systems each conjugated to τ_β with vector $\bar{E} = [0, 1]$. The invariant density can be found for each subsystem using conjugation or formula (14) can be used to obtain both densities at the same time. Another method to obtain these densities separately is to use formulas analogous to (15).*

For the same β and E, if $\alpha = 2 - \beta$, then the system is conjugated to τ_β with vector $\bar{E} = [0, 1]$. The invariant density can be found by conjugation or by formula (15).

Again, for the same β and E, if $\alpha < 2 - \beta$, the system has an attracting basin $[0, 1 - \alpha]$ and $\tau_{|[0,1-\alpha]}$ is conjugated to τ_β with vector $\bar{E} = [1, 0]$. The invariant density can be found by conjugation or by the formula

$$h = \sum_{n=0}^{\infty} \frac{s_{1-\alpha}(n)}{\beta^{n+1}} \chi_{[0,\tau^n(1-\alpha)]},$$

where $\{s_{1-\alpha}(n)\}_{n\geq 0}$ are cumulative signs for the trajectory of $1 - \alpha$.

Most results of §3 can be generalized for the maps $\tau_{\beta,\alpha,E}$. It is easy to prove that if $\beta \geq 2$, then $\tau_{\beta,\alpha,E}$ is exact. Study of the case $\beta < 2$ requires considering a number of cases. One of the cases is described in Proposition 24.

6. *Itineraries of 1*

In this section we try to describe the sequences which are the itineraries of 1 for some $\beta > 1$ and a vector E. In the classical case it was done by Parry [**12**].

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Let $a_0 \geq 1$ be an integer and let a vector $E = [E(0), \ldots, E(a_0)]$ be fixed. We define a_0-admissible itinerary It as a sequence of non-negative integers satisfying:

(1) $\mathrm{It}_0 = a_0$ and $\mathrm{It}_n \leq a_0$ for all $n \geq 1$;

(2) It satisfies the condition (6), where the sequence $\{s(n)\}_{n\geq 0}$ of cumulative signs is obtained from the itinerary It and the vector E.

THEOREM 25. *If $a_0 \geq 2$ and* It *is an a_0-admissible itinerary, then there exists a unique $a_0 \leq \beta \leq a_0 + 1$ such that* $\mathrm{It} = \mathrm{It}_\beta(1)$.

This β satisfies the equation

$$1 = \frac{s(0)d(0)}{\beta} + \frac{s(1)d(1)}{\beta^2} + \frac{s(2)d(2)}{\beta^3} + \cdots, \tag{16}$$

where the sequence $\{d(n)\}_{n\geq 0}$ of 'digits' is obtained from the itinerary It *and the vector E.*

Moreover, the order on the itineraries corresponds to the order on $[a_0, a_0 + 1]$, i.e. $\mathrm{It}^{(1)} \leq \mathrm{It}^{(2)}$ *if and only if the corresponding β satisfy $\beta_1 \leq \beta_2$.*

For $a_0 = 1$ this is not always true. Let $f(\beta)$ denote the right-hand side of equation (16).

Example 26. Let $E = [1, 0]$, $a_0 = 1$ and let $\mathrm{It} = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots)$. Then

$$f(\beta) = \left(\frac{1}{\beta} + \frac{1}{\beta^2} - \frac{1}{\beta^3} + \frac{1}{\beta^4}\right)\frac{1}{1 + 1/\beta^4},$$

and equation (16) has $\beta = 1$ as a unique solution in $[1, 2]$. This It satisfies condition (6) but does not correspond to an itinerary of 1 for any $\beta \in [1, 2]$.

Example 27. Let $E = [0, 1]$, $a_0 = 1$ and let $\mathrm{It} = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots)$. Then

$$f(\beta) = \frac{2}{\beta(1 + 1/\beta^2)},$$

and equation (16) has $\beta = 1$ as a unique solution in $[1, 2]$. Again, this It satisfies condition (6) but does not correspond to an itinerary of 1 for any $\beta \in [1, 2]$.

Proof of Theorem 25. The proof is geometric. We will describe a partition of the interval $[a_0, a_0 + 1]$ into subintervals corresponding to the beginnings of admissible itineraries. For any finite admissible sequence we denote

$$s((\mathrm{It}_{i_1}, \mathrm{It}_{i_2}, \mathrm{It}_{i_3}, \ldots, \mathrm{It}_{i_k})) = (-1)^{E(\mathrm{It}_{i_1})+E(\mathrm{It}_{i_2})+E(\mathrm{It}_{i_3})+\cdots+E(\mathrm{It}_{i_k})},$$

which agrees with the definition of cumulative signs for infinite itineraries. Let $I_\beta^{(j)}$ denote the domain of the jth branch of τ_β, $j = 0, 1 \ldots, a_0$. Here, we count branches as $0, 1, \ldots, a_0$.

We assume that $s((a_0)) = (-1)^{E(a_0)} = 1$. The proof for the other case is similar.

The function $\tau_\beta(1)$ for $\beta \in [a_0, a_0 + 1]$ is increasing and the intersection of its graph with the curves $\gamma_k(\beta) = k/\beta$, $k = 0, 1, \ldots, a_0$, partition $[a_0, a_0 + 1]$ into subintervals corresponding to β with itineraries $(a_0, 0), (a_0, 1), \ldots, (a_0, a_0)$. We will denote them by $J_{(a_0,0)}, J_{(a_0,1)}, \ldots, J_{(a_0,a_0)}$, correspondingly. Note that since $\tau_\beta(1)$ is increasing we have $\beta_1 \leq \beta_2$ if and only if the corresponding finite itineraries satisfy $(a_0, k_1) \leq (a_0, k_2)$.

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For $\beta \in J_{(a_0,k)}$, the values $\tau_\beta(1)$ are in $I_\beta^{(k)}$, the domain of kth branch of τ_β. Thus, $\tau_\beta^2(1)$ on $J_{(a_0,k)}$ is increasing if $s((a_0, k)) = 1$ and decreasing if $s((a_0, k)) = -1$. The intersections of the graph of $\tau_\beta^2(1)$ with the curves $\gamma_k(\beta)$ further partition intervals $J_{(a_0,\mathrm{It}_1)}$, $\mathrm{It}_1 = 0, 1, \ldots, a_0$, into subintervals $J_{(a_0,\mathrm{It}_1,\mathrm{It}_2)}$, $\mathrm{It}_2 = 0, 1, \ldots, a_0$. Again, $\beta_1 \le \beta_2$ if and only if the corresponding finite itineraries satisfy $(a_0, \mathrm{It}_1, \mathrm{It}_2)_{\beta_1} \le (a_0, \mathrm{It}_1, \mathrm{It}_2)_{\beta_2}$.

This process continues inductively with some complication whenever the element a_0 occurs in the itineraries.

Let us consider an interval $J = J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k)}$, such that $\mathrm{It}_i < a_0$ for all $i = 1, \ldots, k$. For $\beta \in J$ we have $\tau_\beta^k(1) \in I_\beta^{(\mathrm{It}_k)}$, the domain of the It_k-th branch of τ_β. The values $\tau_\beta^k(1)$ are increasing if $s((a_0, \mathrm{It}_1, \ldots, \mathrm{It}_{k-1})) = 1$ and decreasing if $s((a_0, \mathrm{It}_1, \ldots, \mathrm{It}_{k-1})) = -1$. Thus, the function $\tau_\beta^{k+1}(1)$ on J is increasing or decreasing depending on the sign of

$$s((a_0, \mathrm{It}_1, \ldots, \mathrm{It}_k)) = s((a_0, \mathrm{It}_1, \ldots, \mathrm{It}_{k-1})) \cdot (-1)^{E(\mathrm{It}_k)}.$$

The intersections of the graph of $\tau_\beta^{k+1}(1)$ with the curves $\gamma_k(\beta)$ further partition interval J into subintervals $J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k,\mathrm{It}_{k+1})}$, $\mathrm{It}_{k+1} = 0, 1, \ldots, a_0$. Because of the way the character of monotonicity of $\tau_\beta^{k+1}(1)$ is established, we again have

$$\beta_1 \le \beta_2 \Longleftrightarrow (a_0, \mathrm{It}_1, \ldots, \mathrm{It}_k, \mathrm{It}_{k+1})_{\beta_1} \le (a_0, \mathrm{It}_1, \ldots, \mathrm{It}_k, \mathrm{It}_{k+1})_{\beta_2}.$$

When we consider an interval $J = J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k)}$ such that some of the It are equal to a_0, the process of subdivision is similar except for the fact that $\tau_\beta^{k+1}(1)$ may not be onto $[0, 1]$ so we obtain a smaller number of subintervals of the next generation.

Let $J = J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k)}$ with $\mathrm{It}_k = a_0$. For $\beta \in J$ we have $\tau_\beta^k(1) \in I_\beta^{(a_0)}$, the domain of the last branch of τ_β. These β also have the property that $\tau_\beta(1) \in I_\beta^{(\mathrm{It}_1)}$ so the image of the last branch is a subset of $I_\beta^{(0)} \cup I_\beta^{(1)} \cup \cdots \cup I_\beta^{(\mathrm{It}_1)}$ (we have assumed that $E(a_0) = 0$). This means that the graph of τ_β^{k+1} only intersects curves $\gamma_s(\beta)$ for $s = 0, 1, \ldots, \mathrm{It}_1$ and J is subdivided into $J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k,s)}$ with s in this range. This situation is reflected in condition (6).

Similarly, let $J = J_{(a_0,\mathrm{It}_1,\mathrm{It}_2,\ldots,\mathrm{It}_k)}$ with $\mathrm{It}_{k-1} = a_0$ and $\mathrm{It}_1 = \mathrm{It}_k$. If $\beta \in J$, then $\tau_\beta^{k-1}(1) \in I_\beta^{(a_0)}$ and $\tau_\beta^k(1) \in I_\beta^{(\mathrm{It}_1)}$. This means that $\tau_\beta^k(1)$ covers only part of the domain $I_\beta^{(\mathrm{It}_1)}$ such that the τ_β image of it covers $I_\beta^{(0)} \cup I_\beta^{(1)} \cup \cdots \cup I_\beta^{(\mathrm{It}_2)}$, if $s((a_0, \mathrm{It}_1)) = 1$, or $I_\beta^{(\mathrm{It}_2)} \cup I_\beta^{(\mathrm{It}_2+1)} \cup \cdots \cup I_\beta^{(a_0)}$, if $s((a_0, \mathrm{It}_1)) = -1$. Thus, the interval J is subdivided into corresponding subintervals $J_{(a_0,\mathrm{It}_1,\mathrm{It}_2,\ldots,\mathrm{It}_k,s)}$ with $s = 0, 1, \ldots, \mathrm{It}_2$ or $s = \mathrm{It}_2, \mathrm{It}_2 + 1, \ldots, a_0$. Again, this situation is reflected in condition (6).

The process of subdivision continues inductively. To show that for each itinerary It we obtain exactly one β we will show that

$$\max_{(a_0,\mathrm{It}_1,\mathrm{It}_2,\ldots,\mathrm{It}_k)} \operatorname{diam}(J_{(a_0,\mathrm{It}_1,\mathrm{It}_2,\ldots,\mathrm{It}_k)}) \to 0, \tag{17}$$

as $k \to +\infty$. Let $f_k(\beta) = \tau_\beta^k(1)$, $k \ge 1$. We have $f_1(\beta) = \tau_\beta(1) = \beta - a_0$ and $f_1'(\beta) = 1$. We also have $f_k(\beta) = \pm\beta f_{k-1}(\beta) \pm$ constant, where the signs and constants are different on different subintervals. In particular, $f_2(\beta) = \pm\beta(\beta - a_0) \pm$ constant with $|f_2'(\beta)| = |2\beta - a_0| \ge a_0$. Then, $f_k'(\beta) = \pm\beta \cdot f_{k-1}'(\beta) \pm f_{k-1}(\beta)$. Since $f_{k-1}(\beta) \in [0, 1]$ we obtain

$$|f_3'| \ge a_0^2 - 1, \ldots, |f_k'| \ge a_0^{k-1} - (a_0^{k-3} + a_0^{k-4} + \cdots + 1),$$

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i.e. the minimum of the modulus of the derivative of $\tau_\beta^k(1)$ grows exponentially. This means that the diameters in (17) converge exponentially to 0.

If the itinerary of 1 under τ_β is $\mathrm{It} = \mathrm{It}_\beta(1)$, then, under assumption $E(a_0) = 0$, we have

$$\langle\beta\rangle = s(1) \sum_{n=1}^{\infty} \frac{s(n)d(n)}{\beta^n}, \tag{18}$$

which is equivalent to equation (16). □

For $1 < \beta \leq 2$ we have the following theorem which proves the similar claims under an additional assumption.

THEOREM 28. *Let us fix* $1 < \beta_0 < 2$. *If* It *is a* 1*-admissible itinerary such that* $\mathrm{It}_{\beta_0}(1) \leq \mathrm{It}$, *then there exists unique* $\beta_0 \leq \beta \leq 2$ *such that* $\mathrm{It} = \mathrm{It}_\beta(1)$.

All other claims of Theorem 25 also hold.

Proof. The proof is exactly the same as that of Theorem 25, except for the part where we estimate the derivatives of $\tau_\beta^k(1)$. There, we have to use $\beta_0 \leq \beta$ instead of $a_0 \leq \beta$. □

Theorem 25 provides another proof of Proposition 12. For a given β with itinerary of 1 equal to $\mathrm{It}_\beta(1)$, we find an admissible simple itinerary which coincides with $\mathrm{It}_\beta(1)$ for a sufficiently large number k of places at the beginning. Theorem 25 gives us a corresponding β_1, which is simple. Then, β and β_1 belong to the same interval $J_{(a_0,\mathrm{It}_1,\ldots,\mathrm{It}_k)}$ whose diameter converges to 0, as $k \to +\infty$.

7. *Open problems*

As usual, work on any problem creates more questions than answers. We state the following open problems.

(i) Is it possible to define a reasonable arithmetic on the expansion coefficients $\{s(n)d(n)\}_{n\geq 0}$? In particular, is it possible to find the expansion of $1 - x$ knowing the expansion of x?

(ii) Let $1 < \beta < 2$ and $E = [1, 1]$. Is $\tau_{\beta,E}$ exact? Partial results are described in part (iv) of the proof of Proposition 8.

(iii) Is the normalizing factor $F(\beta, E)$ continuous on the right as in the classical case?

(iv) Is the mean $\mathcal{E}(\beta)$ of Chebyshev map increasing for $1 < \beta < 2$?

(v) What is a general formula for invariant density of the map $\tau_{\beta,\alpha,E}$?

(vi) What is the ergodic behavior of $\tau_{\beta,\alpha,E}$ for $1 < \beta < 2$ and different vectors E? Partial answers are given in Example 20 and Proposition 24.

(vii) Describe valid itineraries of 1 for $1 < \beta < 2$. Theorem 28 gives a sufficient condition. Is there a nice necessary condition?

(viii) Let $f(\beta)$ denote the right-hand side of equation (16) with the sequences $\{s(n)\}$ and $\{d(n)\}$ corresponding to a fixed a_0-admissible itinerary. Is $f(\beta)$ a decreasing function of β on $[a_0, a_0 + 1]$? This could give an alternative proof of Theorem 25.

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While working on this project the author used the computer program Maple 9.5 extensively. The programs with examples and illustrations, as well as their pdf printouts, are available at http://www.mathstat.concordia.ca/pg/beta.

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