NOTE ON AN IDENTITY CONNECTED WITH DIOPHANTINE APPROXIMATION

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In discussing Hecke's function

$$J(s,\theta) = \sum_1^\infty \{n\theta\} n^{-s},$$

where $\{t\} = t - [t] - \frac{1}{2}$, Hardy and Littlewood* made use of the identity

$$J(s,\theta) = \theta\zeta(s-1) - \tfrac{1}{2}\zeta(s) + \sum_{\nu=1}^\infty (-1)^\nu \zeta_2(s, q_{\nu-1}+q_\nu, q_{\nu-1}, q_\nu). \tag{1}$$

Here

$$\zeta_2(s,a,w,w') = \sum_0^\infty \sum_0^\infty (a+mw+nw')^{-s},$$

θ is an irrational number between 0 and 1, and p_ν/q_ν is the νth convergent to θ (beginning with $p_0 = 0$, $q_0 = 1$). The proof of (1) given by Hardy and Littlewood is of an analytic character.

The identity (1) is obviously equivalent to

$$[n\theta] = \sum_{\nu=1}^\infty (-1)^{\nu-1} n_\nu, \tag{2}$$

where n_ν denotes the number of representations of n as $mq_{\nu-1} + m'q_\nu$ with $m \geqslant 1$, $m' \geqslant 1$.

To prove (2), it is only necessary to observe that, since

$$p_\nu q_{\nu-1} - p_{\nu-1} q_\nu = (-1)^{\nu-1},$$

the general solution of $n = mq_{\nu-1} + m'q_\nu$ is

$$m = (-1)^{\nu-1}(np_\nu - jq_\nu), \quad m' = (-1)^\nu (np_{\nu-1} - jq_{\nu-1}).$$

Hence n_ν is the number of integers strictly between $np_{\nu-1}/q_{\nu-1}$ and np_ν/q_ν. Since $np_0/q_0 = 0$, and np_ν/q_ν is alternately less than and greater than $n\theta$ and tends to $n\theta$ as $\nu \to \infty$, we have the result.

The power-series identity corresponding to (1) is obviously

$$\sum_{n=1}^\infty [n\theta] x^n = \sum_{\nu=1}^\infty (-1)^{\nu-1} \frac{x^{q_{\nu-1}+q_\nu}}{(1-x^{q_{\nu-1}})(1-x^{q_\nu})}.$$

* *Hamburg Abh.* 3 (1923), 57–68.

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