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Distant Vertex Partitions of Graphs

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We consider the function $\chi(G^k)$, defined to be the smallest number of colours that can colour a graph G in such a way that no vertices of distance at most k receive the same colour. In particular we shall look at how small a value this function can take in terms of the order and diameter of G. We get general bounds for this and tight bounds for the cases $k = 2$ and $k = 3$.

1. Introduction

The kth *power* G^k of a graph G is the graph with the same vertex set and with an edge between any pair of vertices that are distance at most k away from each other in G; for example, $G^1 = G$. We shall consider $\chi(G^k)$, the chromatic number of the kth power of G. Alternatively, this is the minimum number of colours needed to colour the vertices of G so that no vertices at distance at most k have the same colour. Chen, Gyárfas and Schelp [2] called this the k-distant chromatic number, and examined the maximum number of edges for such graphs with a given value of $\chi(G^k)$. This function has also been studied by several other authors, including Kramer and Kramer [5], Speranza [9], and Wegner [12], and there is a survey of these in Gionfriddo [4]. For notation see Bollobás [1].

We shall consider graphs G with diameter $\mathrm{diam}(G) = d$. It is trivial to find graphs whose kth powers have a large chromatic number, and easy to find the upper bounds in terms of n (the order of G), k and d. However, it seems to be much harder to find graphs whose kth powers have a very small chromatic number. We shall look at how small $\chi(G^k)$ can be, as a function of n, k and d, a problem posed by Smyth [8]. We define

$$\chi(k;d,n) = \min\{\chi(G^k); |G| = n,\ \mathrm{diam}(G) = d\}.$$

The purpose of this paper is to estimate $\chi(k;d,n)$, with an interesting case being when $k = d-1$, when we are colouring the vertices in such a way that only points at maximum distance apart can have the same colour. It turns out to be natural to fix k and estimate

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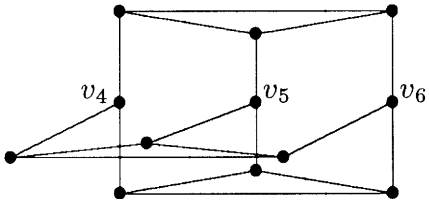


Figure 1

$\chi(k;d,n)$ for different values of d and n. The case $k = 1$ is just the same as ordinary colouring of graphs, and for any d we can find arbitrarily large graphs with diameter d that require only two colours. For example, the complete bipartite graph with a pendant path of length $d-2$ has diameter d and is two-colourable. The case $k = d$ is equally trivial, since every vertex must have a different colour, so $\chi(G^d) = |G|$. Thus we shall just consider $2 \leqslant k \leqslant d-1$, for which the value of $\chi(k;d,n)$ is not immediately apparent.

2. Elementary bounds

Let us start by finding a crude lower bound for this function. Consider a vertex $x \in V(G)$ and consider all vertices within distance $\lfloor k/2 \rfloor$ of x. There are at most $\chi(G^k)$ vertices within this distance, as they are all within k of each other and so must all take different colours. Similarly, there are at most $\chi(G^k)^2$ vertices within distance $2\lfloor k/2 \rfloor$ of x, since each of these vertices must be within $\lfloor k/2 \rfloor$ of one of the vertices within $\lfloor k/2 \rfloor$ of x. Similarly, within distance $r\lfloor k/2 \rfloor$ of x there must be at most $\chi(G^k)^r$ vertices, and therefore we have that at most $\chi^{\lceil (d/\lfloor (k/2) \rfloor) \rceil}$ vertices must lie within distance d of a particular vertex. Hence we get that

$$\chi(G^k) \geqslant n^{1/\lceil 2d/k \rceil} \quad \text{for } k \text{ even},$$

and

$$\chi(G^k) \geqslant n^{1/\lceil 2d/(k-1) \rceil} \quad \text{for } k \text{ odd}.$$

We now try to obtain an upper bound on the function $\chi(k;d,n)$ by considering some examples. In the smallest case, $d = 3$ and $k = 2$. A very natural example for this is the following graph. Let G_1 consist of $2r$ vertices $v_1,\ldots,v_{2r}$, such that $v_1,\ldots,v_r$ induce a complete graph (in G_1), $v_iv_{r+i} \in E(G_1)$ for all $1 \leqslant i \leqslant r$, and there are no other edges in G_1. Now we take r copies of G_1 and identify the r different copies of v_{r+1}, and similarly identify all the other vertices $v_{r+2},\ldots,v_{2r}$, to form G. Now $G^2 \supset K_{r+1}$, so $\chi(G^2) \geqslant r+1$. But $r+1$ colours will suffice to colour G^2 since one colour can be used for $v_{r+1},\ldots,v_{2r}$, and the other r colours used for the rest; we use colours $1,\ldots,r$ respectively for $v_1,\ldots,v_r$ in the first copy of G_1, then $2,\ldots,r,1$ for the second copy, and so on. Hence $\chi(G^2) = r+1$. Also G^2 has $r(r+1)$ vertices. This gives us $\chi(G^2) \approx \sqrt{n}$, and seems quite a good example in that all the colours are being used equally often. Figure 1 shows the graph for $r = 3$.

In a similar way we can find examples for all d and k. For k even and $d = lk+1$, we form a graph in the following way. Take r paths of length $k/2$ with vertices $v_{i0}, v_{i1},\ldots,v_{i(k/2)}$, $1 \leqslant i \leqslant r$ and identify $v_{10}, v_{20},\ldots,v_{r0}$. The kth power of this graph needs $rk/2+1$ colours.

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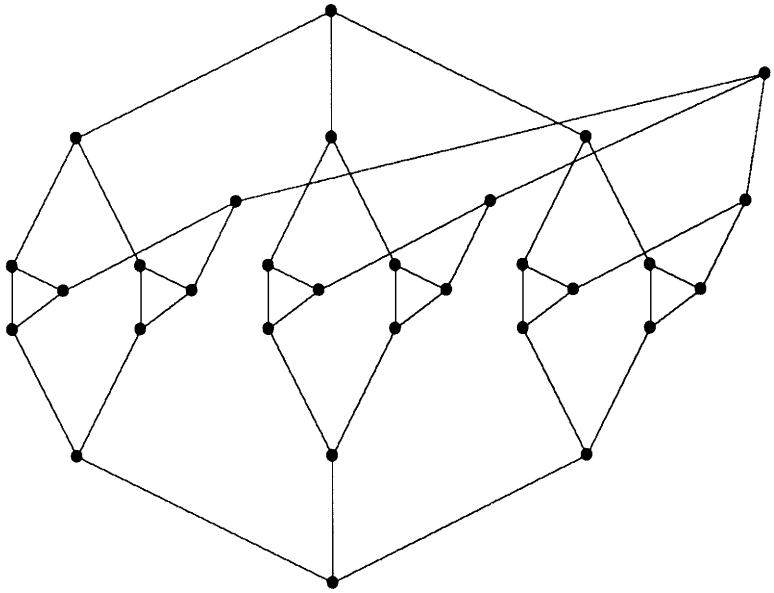


Figure 2

Now, for each i take $r-1$ paths of length $k/2$, $u_{j0},\dots,u_{j(k/2)}$, $1 \leqslant j \leqslant r-1$ and identify $u_{10},\dots,u_{(r-1)0}$ with $v_{i(k/2)}$. Letting $d(x,y)$ be the distance between x and y in the usual way, we notice that, for the path attached to $v_{r(k/2)}$,

$$d(u_{js}, v_{j(k/2+1-s)}) = k+1;$$

thus u_{js} can be given the same colour as $v_{j(k/2+1-s)}$, so the new graph still only requires $rk/2+1$ colours. Now to each end-vertex of this graph we attach another $r-1$ paths of length $k/2$ and do this a total of $l-1$ times, forming a graph G' which requires no extra colours. (Thus G' is a subdivision of a truncated r-regular tree.) Now take r copies of G' and add all edges in between corresponding end-vertices (so the corresponding end-vertices form complete subgraphs), forming a new graph G. We show the case $r=3$, $k=2$, and $d=5$ in Figure 2 (and if we subdivide all the edges not in one of the triangles, that is, replace them by paths of length $k/2$, we would get a more general case with $d=2k+1$).

Notice that G satisfies

$$|G| = \frac{r^2k}{2}(1+(r-1)+\cdots+(r-1)^{l-1})+r \approx kr^{l+1}/2$$

and

$$\chi(G^k) = \frac{rk}{2}+1.$$

For $d=kl+2$ we add an extra vertex to each end-vertex of G' (together with an edge joining them), then take r copies of G', but this time identify the corresponding end-vertices. Similarly, we can find graphs for the case $d=lk+i$, for all $2 \leqslant i \leqslant k$. For odd k we follow a similar procedure but let our paths be of length $(k+1)/2$, and need to

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find l so that $d \geqslant l(k+1)$. Thus, to sum up, we have found a class of easily describable graphs which can be arbitrarily large such that

$$\chi(G^k) \approx (k/2)^{l/(l+1)} n^{1/(l+1)} \quad \text{where} \quad l = \lfloor (d-1)/k \rfloor, \quad \text{for } k \text{ even},$$

and

$$\chi(G^k) \approx ((k-1)/2)^{l/(l+1)} n^{1/(l+1)} \quad \text{where} \quad l = \lfloor d/(k+1) \rfloor, \quad \text{for } k \text{ odd}.$$

All these examples share the property that every colour is being used almost exactly the same number of times.

Notice that the exponents here are about twice of those in our lower bounds, so we shall set about trying to construct better examples, but first we summarize these results in the following theorem.

Theorem 2.1. *For any k and d, there are constants $c_1(k,d)$ and $c_2(k,d)$ such that, for all n,*

$$n^{1/\lceil 2d/k \rceil} \leqslant \chi(k;d,n) \leqslant c_1 n^{1/(\lfloor (d-1)/k \rfloor + 1)}, \quad \textit{for } k \textit{ even},$$

and

$$n^{1/\lceil 2d/(k-1) \rceil} \leqslant \chi(k;d,n) \leqslant c_2 n^{1/(\lfloor d/(k+1) \rfloor + 1)}, \quad \textit{for } k \textit{ odd}. \qquad \square$$

It turns out that the exponent in the lower bound of this theorem is correct for $k = 2$ and $k = 3$. We shall construct better examples to show this in the next two sections. It is worth noting that any vertex of the graph we construct must have at most one neighbour from any given colour class, and therefore the graph induced by two colour classes is an independent set of edges. Hence it would be natural to investigate graphs such that any two colour classes have a 1-factor between them. For a graph H with vertex set $\{v_1, \ldots, v_r\}$ we shall define $R(H)$ to be the set of all graphs consisting of r classes $C_1, \ldots, C_r$ of independent vertices each of size c, with a 1-factor between C_i and C_j if $v_i v_j \in E(H)$. We shall use probabilistic arguments to show that if we choose the right H there is a graph in $R(H)$ which shows that the exponent in the lower bound in Theorem 2.1 is in fact correct. To do this, we shall need to choose H to satisfy two constraints. First, H needs to have girth (*i.e.*, the length of the shortest cycle) at least $k+1$, which ensures that we never have two vertices within one class of a graph in $R(H)$ that are at distance at most k. Secondly, we shall want H to have lots of walks of length d between any two pairs of vertices, so that graphs in $R(H)$ will have diameter at most d with high probability.

3. The chromatic number of squares and cubes

In this section we shall consider the cases $k = 2$ and $k = 3$. We call G^2 the *square* of G, and G^3 the *cube* of G; thus we are considering the chromatic numbers of the square and cube of G. We start by considering the case when $d = 3$ and $k = 2$ more carefully. We already know that $n^{1/3} \leqslant \chi(2;3,n) \leqslant c_1 n^{1/2}$. It is natural to choose H to be the complete graph of order r, since this has girth three (thus ensuring that no pair of vertices within one class have distance less than three), and the maximum number of edges, thus giving us the best possible chance of having diameter at most three. If we construct a graph G

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in this way that has diameter three, then it will have rc vertices, so if $r \leqslant c$ then we have a better example than the one constructed for Theorem 2.1. In fact we can achieve r of order $c^{1/2}$. The following lemma uses a standard probabilistic argument.

Lemma 3.1. *$\chi(2;3,n) \leqslant n^{1/3}[(2+o(1))\log n]^{1/3}$, provided n is large enough.*

Proof. Let $\delta > 0$ be fixed. We shall construct a graph G randomly so that G has diameter three and $\chi(G^2) \leqslant [(2+\delta)n\log n]^{1/3}$. Let $r = \lfloor[(2+\delta)n\log n]^{1/3}\rfloor$. In the case that n is divisible by r, let $c = n/r$. Let $C_1,\ldots,C_r$ be r disjoint sets of c vertices, and put in random 1-factors between each pair of classes. (By this we mean that, if $C_1 = \{u_1,\ldots,u_c\}$, $C_2 = \{v_1,\ldots,v_c\}$, then we pick a vertex at random from C_2, say v_i, and join it to u_1. Next we pick another vertex at random from $C_2\backslash\{v_i\}$ and join it to u_2, and so on.) This guarantees that $|\Gamma(x)\cap C_i| \leqslant 1$, for all $x \in V(G)$, for all i, so we just need to check whether the graph has diameter three.

Given $x, y \in V(G)$, we wish to show that with very high probability $d(x,y) \leqslant 3$. Without loss of generality we may assume that $x, y \in C_1 \cup C_2$. Now x and y each have $r-2$ neighbours in $C_3 \cup \ldots \cup C_r$. Let x_i be the vertex in $\Gamma(x)\cap C_i$, $3 \leqslant i \leqslant r$, and let y_i be the vertex in $\Gamma(y)\cap C_i$, $3 \leqslant i \leqslant r$. Consider a particular pair of vertices, x_i and y_j. The probability that this pair is joined by an edge is $1/c$ if $i \neq j$, and they are the same vertex with probability $1/c$ if $i = j$. These events are independent of the possible connections between other neighbours of x and y. Hence

$$P(d(x,y) > 3) \leqslant \left(1 - \frac{1}{c}\right)^{\binom{r-2}{2}} \leqslant e^{-(r-3)^2/2c}.$$

Thus the expected number of pairs of vertices $\{x,y\}$ that are not joined by a path of length at most three is less than $r^2c^2e^{-(r-3)^2/2c}$. Hence the probability that G has diameter greater than three is at most $r^2c^2e^{-(r-2)^2/c} \leqslant n^2e^{-(r-3)^3/2n} \to 0$. Thus (since $\chi(G^2) \leqslant r$), we can find graphs G of order n with diameter three and $\chi(G^2) \leqslant n^{1/3}[(2+o(1))\log n]^{1/3}$.

If n is not a multiple of r, we modify the construction to allow the classes to have order c or $c-1$, and then put in random maximal independent sets between pairs of classes. For each pair of dissimilarly sized classes we nominate a vertex in the larger class which will not meet an edge. Since c is much larger than r, this can be done in such a way that any vertex has degree at least $r-2$. We can work through the arguments in a similar way and get the same result. □

For $k = 2$ and general d, we know from Theorem 2.1 that $n^{1/d} \leqslant \chi(2;d,n) \leqslant [(2+o(1))n\log n]^{1/\lfloor(d+1)/2\rfloor}$. To improve these bounds we can use the same construction as in the previous theorem. However, for the sake of brevity we shall prove a bound with a worse constant, by combining the proof with that for the case $k = 3$.

We will again consider a graph in $R(H)$, and the obvious candidate for H is the complete bipartite graph, as this has the maximum number of edges for any graph with girth four. However, if the graph $G \in R(H)$ is to have diameter four, then for at least half the pairs x, y of vertices we will have $d(x,y) \leqslant 3$, namely, those pairs for which $x \in C_i$ and $y \in C_j$ with $1 \leqslant i \leqslant r/2$, $r/2 < j \leqslant r$. For this reason the construction turns out not to

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be very efficient, so to get round the parity problem whilst preserving essentially the same structure, we take H to be a blow-up of a pentagon for the next theorem (*i.e.*, replace each vertex of a 5-cycle by a class of c independent vertices, and join every pair of vertices from distinct classes). This has comparable size to a complete bipartite graph but has the advantage that between any two vertices there are many walks of length four. Note that the pentagonal structure has no significance other than that of avoiding the parity problem we mentioned earlier.

Theorem 3.2. *Given $d \geqslant 4$, there exists $n_0(d)$ such that, for all $n \geqslant n_0(d)$,*

$$n^{1/d} \leqslant \chi(2;d,n) \leqslant \chi(3;d,n) \leqslant 5(n\log n)^{1/d}.$$

Proof. The first inequality is immediate from Theorem 2.1, and clearly $\chi(2;d,n) \leqslant \chi(3;d,n)$, so we just need to prove that $\chi(3;d,n) \leqslant 5(n\log n)^{1/d}$. Set r to be the biggest multiple of five which is at most $\lfloor 5(n\log n)^{1/d} \rfloor$. Fix $\lambda < 1/2$, and let δ be chosen so that $(1-\lambda) = (1-\delta)^{d-1} - 2(1-\delta)^{\lceil (d-1)/2 \rceil}/r$. We shall construct a graph G of order n and diameter d with $\chi(G^3) \leqslant 5(n\log n)^{1/d}$. As before we shall assume that n is divisible by r, although the same modification as that used in Lemma 3.1 will give the result for all n. G will be a member of $R(H)$ where H is a blow-up of the pentagon. That is, let $C_1,\ldots,C_r$ be classes of vertices of G with $|C_i| = c$, for all $1 \leqslant i \leqslant r$. Furthermore, let $S_i = \{C_{1+(i-1)r/5},\ldots,C_{r/5+(i-1)r/5}\}$ for $1 \leqslant i \leqslant 5$. We shall form a graph G from these vertices by putting a random 1-factor between $C_s \in S_i$ and $C_t \in S_j$ if $i \equiv j \pm 1 \pmod 5$. This construction will ensure that no two vertices in a particular class are at distance less than four, and hence that $\chi(G^3) \leqslant r$.

Consider a vertex $x \in G$. We shall view G as being constructed step by step according to how far a vertex is from x. However, for simplicity we shall only do this for one particular sequence from the set $\{S_1,\ldots,S_5\}$. Without loss of generality we shall suppose that x is in a class that is in S_2. We then fix a sequence (s_i) of length $\lceil (d-1)/2 \rceil$ from the set $\{1,2,3,4,5\}$ such that $s_0 = 2$, and $s_i = s_{i-1} \pm 1 \pmod 5$. Thus, for example, $2,3,4,3,4,5$ would be one possible sequence of length five. We shall only consider vertices at distance i from x for which there is a path $v_1,\ldots,v_i$ such that $v_j \in S_{s_j}$ for $1 \leqslant j \leqslant i$. We shall call such a path an *allowable* path. (Thus, for any given sequence there can be many allowable paths of the same length between two vertices.) Notice that there are at most $(r/5)^i$ vertices at distance i from x (along allowable paths). Let $t = r/5$.

Given a vertex z in G, then z is *fecund* (*with respect to* x) if there is exactly one x–z allowable path of length $d(x,z)$, where here we take distance between x and z to be the minimum length of any allowable path between x and z. (Hence, crucially, if z is fecund then it has $t-1$ or t neighbours all of which are at length at least $d(x,z)+1$ from x.) A vertex z is a *failure* if it is fecund, $d(x,z) < \lceil (d-1)/2 \rceil$, and z has fewer than $(1-\delta)t$ fecund (with respect to x) neighbours at distance $1+d(x,z)$ from x (on allowable paths). We also call a vertex z a failure if $d(x,z) = \lceil (d-1)/2 \rceil$ and z has fewer than $(1-\delta)t$ neighbours not on allowable paths from x. Our first aim is to show that with high probability G has no failures.

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Fix $z \in G$. We shall estimate the probability that this vertex is a failure. Let $i = d(x, z)$. If z is not a fecund vertex then it is certainly not a failure. If z is fecund then it has $t-1$ or t edges to other vertices in G that are at least $i+1$ away (on any allowable path) from x. The total number of vertices in one class that are within distance $i+1$ of x is at most $(1+t+\cdots+t^i) \leqslant 2t^i$. Thus the probability of one of the edges from z leading to a vertex which is not both fecund and has distance $i+1$ from x, is at most $2t^i/c$. Therefore the probability that z has fewer than $(1-\delta)t$ fecund neighbours at distance $i+1$ from x, that is, the probability that z is a failure (given that z is fecund), can be estimated as follows. We know that $i \leqslant \lceil (d-1)/2 \rceil$, $rc = n$, and $\binom{a}{b} \leqslant (ea/b)^b$, hence the probability is crudely at most

$$\binom{t}{\delta t}\left(\frac{2t^i}{c}\right)^{\delta t} < \left(\frac{2e}{\delta}\frac{t^i}{c}\right)^{\delta t} < \left(\frac{2e}{\delta}\frac{t^{i+1}}{n}\right)^{\delta t} < \left(\frac{1}{n^{1/2d}}\right)^{\delta t} < \frac{1}{n^4},$$

for large enough n.

Thus the probability of there being at least one failure in G is at most $1/n^3$. If G has no failures then it has at least $(1-\delta)^i t^i$ fecund vertices at distance i from x, and in particular $[(1-\delta)t]^{\lceil (d-1)/2 \rceil}$ at distance $\lceil (d-1)/2 \rceil$.

Now let x and y be two given vertices of G, and (s_i) and (u_i) corresponding sequences for x and y such that $s_{\lceil (d-1)/2 \rceil} = u_{\lceil (d-1)/2 \rceil} \pm 1 \pmod 5$. Notice that these sequences can always be chosen for $d \geqslant 4$. We shall estimate the probability that $d(x, y) > d$. We form G initially by growing edges out of x in the manner just described, until we have reached vertices at distance $\lceil (d-1)/2 \rceil$ from x. We then start to grow edges out of y until either we reach a vertex also reached from x or we reach all vertices at distance $\lfloor (d-1)/2 \rfloor$ from y. In the first case, $d(x, y) \leqslant d-1$. In the second case, with probability $1 - 2/n^3$ we obtain at least $[(1-\delta)t]^{\lceil (d-1)/2 \rceil}$ fecund vertices with respect to x at distance $\lceil (d-1)/2 \rceil$ from x, which we shall call A, and similarly at least $[(1-\delta)t]^{\lfloor (d-1)/2 \rfloor}$ fecund vertices with respect to y at distance $\lfloor (d-1)/2 \rfloor$ from y, which we shall call B. Let $a = |A|$ and $b = |B|$. Note that, at this stage of forming G, every vertex in $A \cup B$ has a unique neighbour.

Given that the sets A and B exist as described, we wish to show that there is a very high probability that at least one A–B edge is in G. Let $A = \{x_1, \ldots, x_a\}$. Suppose without loss of generality that $A \subset S_2$, and $B \subset S_1$. We also suppose that $x_j \in C_l$. Since x_j is fecund, it has a neighbour in C_m, say, and we know that there are exactly $t-1$ edges to vertices in S_1 which still need to be added to G from x_j. Let b_{ij} be the number of vertices in $B \cap C_i$ which could still be joined to x_j. This will be zero if $i = m$. Similarly, a vertex in $B \cap C_i$ whose neighbour is in C_l will not contribute towards b_{ij}. However, since there are at most $t^{\lfloor (d-1)/2 \rfloor - 2}$ vertices in C_l at distance $\lfloor (d-1)/2 \rfloor - 1$ from y, there are at most $t^{\lfloor (d-1)/2 \rfloor - 1}$ vertices in B whose neighbour is in C_l. Therefore, if we let $b_j = \sum_i b_{ij}$, then

$$\begin{aligned} b_j &= \sum_i b_{ij} = b - |B \cap C_m| - |\Gamma(C_l) \cap B| \\ &\geqslant (1-\delta)^{\lfloor (d-1)/2 \rfloor} r^{\lfloor (d-1)/2 \rfloor} - 2r^{\lfloor (d-1)/2 \rfloor - 1} \\ &\geqslant \left[(1-\delta)^{\lfloor (d-1)/2 \rfloor} - \frac{2}{r}\right] r^{\lfloor (d-1)/2 \rfloor}. \end{aligned}$$

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Now continue to build up G by inserting edges first at x_1, then at x_2, and so on. The conditional probability that x_j fails to meet $B \cap C_i$, given that none of $x_1, \ldots, x_{j-1}$ have met B is at most $1 - b_{ij}/c$. Thus the probability of there being no A–B edge is at most

$$\prod_{\substack{1 \leqslant j \leqslant a \\ 1 \leqslant i \leqslant t}} \left(1 - \frac{b_{ij}}{c}\right) \leqslant \exp\left\{-\frac{1}{c}\sum_{i,j} b_{ij}\right\} \leqslant \exp\left\{-\frac{a}{c}\left[(1-\delta)^{\lfloor (d-1)/2 \rfloor} - \frac{2}{r}\right] r^{\lfloor (d-1)/2 \rfloor}\right\}.$$

Now $a \geqslant (1-\delta)^{\lceil (d-1)/2 \rceil} t^{\lceil (d-1)/2 \rceil}$, so, recalling our definition of λ, the probability that $d(x, y) > d$ is less than

$$\frac{2}{n^3} + \exp\left\{-(1-\lambda)t^{d-1}/c\right\}.$$

Putting in $r = 5t$ and $n = rc$, we have the probability that $\operatorname{diam}(G) > d$ is at most

$$\frac{2}{n} + n^2 \exp\left\{-5(1-\lambda)\left(\frac{r}{5}\right)^d \Big/ n\right\} \to 0, \qquad \text{as } n \to \infty.$$

Hence we can find graphs G with diameter d, and with $\chi(G^3) \leqslant 5(n \log n)^{1/d}$, as required. □

It is clear that the constant in the previous theorem could be improved somewhat by tightening up the arguments. (For example, we only made use of one sequence for each vertex in the previous theorem.) In fact, it would seem very likely that $\chi(3; d, n) \leqslant [(2 + o(1))n \log n]^{1/d}$.

4. Higher powers

Having got good bounds for colouring squares and cubes of graphs it is obvious to wonder whether this can be generalized to find bounds for the chromatic number of kth powers in general. The problem is to find an appropriate sequence of graphs H with large enough girth whilst still having enough trails of length at most d between any two pairs of vertices. In order for H to work, we want H to have large minimum degree $\delta(H)$, and girth $g = k + 1$. It is clear from a standard argument that $\delta(H)^{\lfloor (g-1)/2 \rfloor} \leqslant |H|$. If we could find graphs H with $|H| = \delta(H)^{\lfloor (g-1)/2 \rfloor}$ we would in fact have every chance of proving that the lower bound in Theorem 2.1 is the correct bound. The best that was known in general was that there are graphs H with $|H| = [2\delta(H)]^g$ (due to Sauer [7], being a slight improvement on earlier work of Erdős and Sachs [3], and Walther [10] and [11]), and even if we assumed that all paths spread exactly as we would wish, so giving lots of walks of length d in H, we wouldn't get any better bound than that already given in Theorem 2.1. New results by Lazebnik, Ustimenko and Woldar [6] show that there are graphs H with order approximately $[2\delta(H)]^{3g/4}$, and this gives an improvement on the upper bound, but is still not close to the lower bound.

However, for the case $k = 4$ and $k = 5$ we can do better. The bipartite graphs corresponding to projective planes of order p (which exist for p a prime power) have girth six, and have order smaller than $2\delta^{\lfloor (g-1)/2 \rfloor}$. Thus we shall use these for H in the next theorem; all that will be necessary is to insert 1-factors in such a way as to achieve

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the diameter desired. However, because these graphs are bipartite, our comment before Theorem 3.2 explains why we do not quite achieve the lower bound of Theorem 2.1.

Theorem 4.1. *For $k = 4$ and $k = 5$, there exists an $n_0(d)$ such that, for all $n \geqslant n_0(d)$, for all $d \geqslant 6$, and for a fixed $\epsilon > 0$,*

$$\begin{aligned} n^{2/d} &\leqslant \chi(k; d, n) \leqslant (2+\epsilon)(2n\log n)^{2/(d-2)}, \quad \textit{for } d \textit{ even}, \\ n^{2/d+1} &\leqslant \chi(k; d, n) \leqslant (2+\epsilon)(2n\log n)^{2/(d-1)}, \quad \textit{for } d \textit{ odd}. \end{aligned}$$

Proof. The lower bounds are immediate from Theorem 2.1. We will need to consider the cases when d is even separately from those when d is odd; however, in both cases we shall construct a graph $G \in R(H)$ of order n and diameter d, whose kth power has appropriate chromatic number.

Let q be a prime power, and $t = q^2 + q + 1$. Let H be the bipartite graph made up from two sets V_1 and V_2, where V_1 corresponds to the set of points in the projective plane of order t, and V_2 the lines. There is an edge between $u \in V_1$ and $v \in V_2$ if and only if the point u lies on the line v. Then H is $(q+1)$-regular. Notice that there are no 4-cycles in H, as that would necessitate two lines meeting at two different points. Thus, given $x \in V_1$ with neighbours $y, z \in V_2$, we know that $\Gamma(y) \cap \Gamma(z) = \{x\}$. Thus, since there are $(q+1)q$ paths of length two from x, there is exactly one path of length two to each of the other vertices in V_1; similarly for V_2. Consequently the graph $G^2[V_1]$ is exactly the same as a member of $R(K_t)$.

Consider the case d even. We choose a prime power q as large as possible subject to $r = 2(q^2+q+1)$ being less than $(2+\epsilon)(2n\log n)^{2/(d-2)}$. Since there is a prime between N and $N + N^{1-\epsilon'}$ for some $\epsilon' > 0$ and large enough N, then for large enough n we have $r \geqslant 2(2n\log n)^{2/(d-2)}$. Take $\lambda < 1/2$, and define δ as in the proof of Theorem 3.2. We want the probability that $d(x,y) > d$ for all $x, y \in G$. If $x, y \in V_1$ or $x, y \in V_2$, then we know that $d(x,y)$ is even. By our remarks above the probability of an $x - y$ path of length at most d is exactly the same as that of an $x - y$ path of length $d/2$ in the case when $H = K_r$. Thus, by the proof of Theorem 3.2 the probability that there is a pair $x, y \in V_1$ with $d(x,y) \geqslant d$ is at most

$$\frac{2}{n} + n^2 \exp\left\{-2(1-\lambda)\left(\frac{r}{2}\right)^{d/2} \Big/ n\right\} \to 0.$$

To join $x \in V_1$ to $y \in V_2$ we might try to form sets $A \subset V_1$ and $B \subset V_2$ of size at least $[(1-\delta)t]^{(d-2)/2}$ at distance $(d-2)/2$ from x or y, and then hope to join A to B. However, the low degree of H makes this process inefficient, and in fact no better than the following much simpler argument: simply select a neighbour z of y in V_1 and then join by a path of length $d-2$ to x. As before, this corresponds to a path of length $(d-2)/2$ in Theorem 3.2, and hence the probability that there is a pair of vertices $x \in V_1$ and $y \in V_2$ with $d(x,y) > d-1$ is at most

$$\frac{2}{n} + n^2 \exp\left\{-2(1-\lambda)\left(\frac{r}{2}\right)^{(d-2)/2} \Big/ n\right\} \to 0.$$

For the case d odd, we proceed in a similar way, and get that the probability of a pair

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of vertices $x, y \in G$ with $d(x, y) > d$ is at most

$$\frac{2}{n} + n^2 \exp\left\{-2(1-\lambda)\left(\frac{r}{2}\right)^{(d-1)/2} \Big/ n\right\} \to 0.$$

Hence for all values of $d \geqslant 6$ we can find a graph $G \in R(H)$ as required. □

The case when $k = d - 1$ is an interesting one, and we note that $n^{1/3} \leqslant \chi(2; 3, n) \leqslant [(2 + o(1))n \log n]^{1/3}$, and that $n^{1/4} \leqslant \chi(3; 4, n) \leqslant 5(n \log n)^{1/4}$. In general, for all $d \geqslant 5$ we have that $n^{1/3} \leqslant \chi(d-1; d, n) \leqslant c_1 n^{1/2}$. Unfortunately the previous theorem doesn't in fact help us for the case $d = 5$. We would suppose that maybe $\chi(d-1; d, n) \approx n^{1/3}$ for all $d \neq 4$. The anomaly for $d = 4$ may seem strange, but that is probably because the value of the parameter k is more significant than that of d.

References

[1] Bollobás, B. (1978) *Extremal Graph Theory*, Academic Press.
[2] Chen, G., Gyárfas, A. and Schelp, R. H. (1998) Vertex colorings with a distance restriction. *Discrete Math.* **191** 65–82.
[3] Erdős, P. and Sachs, H. (1963) Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. *Wiss. Z. Univ. Halle Martin Luther Univ. Halle-Wittenberg Math.-Natur. Reine* **12** 251–257.
[4] Gionfriddo, M. (1987) A short survey on some generalized colourings of graphs. *Ars Combinatorica* **24B** 155–163.
[5] Kramer, F. and Kramer, H. (1986) On the generalized chromatic number. *Ann. Discrete Math.* **30** 275–284.
[6] Lazebnik, F., Ustimenko, V. A. and Woldar, A. J. New upper bounds on the order of cages. Preprint.
[7] Sauer, N. (1967) Extremaleigenschaften regulärer Graphen gegebener Taillenweite, I and II, *Sitzungsberichte Österreich. Acad. Wiss. Math. Natur. Kl., S-B II* **176** 9–25 and 27–43.
[8] Smyth, W. F. (1993) at the P. Erdős 80th birthday conference, Keszthély, Hungary.
[9] Speranza, F. (1975) Colorazioni di specie superiore d'un grafo. *Boll. Un. Matem. Ital.* **12, suppl. fasc. 3** 53–62.
[10] Walther, H. (1965) Eigenschaften von regulären Graphen gegebener Taillenweite und minimaler Knotenzahl. *Wiss. Z. Ilmenau* **11** 167–168.
[11] Walther, H. (1965) Über reguläre Graphen gegebener Taillenweite und minimaler Knotenzahl. *Wiss. Z. Ilmenau* **11** 93–96.
[12] Wegner, G. Graphs with given diameter and a coloring problem. Dortmund 1976, preprint.

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