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Induction and restriction of cellular automata

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Abstract. We analyze in detail the notions of induction and restriction for cellular automata. As a by-product we extend a few classical and recent theorems on cellular automata to uncountable groups.

1. *Introduction and statements of the results*

In group theory there are several notions of induction for group actions (standard induction, tensor induction, etc). It is worth mentioning that tensor induction plays a fundamental role in representation theory thanks to the most important, pioneering work of Frobenius.

In this paper, we investigate a notion of induction for cellular automata over groups, and present several applications.

Let G be a group and A a set, called the *alphabet*. Consider the set A^G consisting of all maps $x\colon G \to A$, equipped with the right action of G defined by $A^G \times G \ni (x, g) \mapsto x^g \in A^G$, where $x^g(g') = x(gg')$ for all $g' \in G$.

A *cellular automaton* over G on the alphabet A is a map $\tau\colon A^G \to A^G$ satisfying the following condition: there exists a finite subset $M \subset G$ and a map $\mu\colon A^M \to A$ such that

$$\tau(x)(g) = \mu(x^g|_M) \quad \text{for all } x \in A^G,\ g \in G, \tag{1.1}$$

where $x^g|_M$ denotes the restriction of x^g to M. Such a set M is called a *memory set* and μ is called a *local defining map* for τ. Note that a cellular automaton $\tau\colon A^G \to A^G$ is G-*equivariant*; that is,

$$\tau(x)^g = \tau(x^g)$$

for all $x \in A^G$ and $g \in G$. Also, we equip the space $A^G = \prod_{g\in G} A$ with the *prodiscrete topology*, i.e. the smallest topology on A^G for which the projection map $\pi_g\colon A^G \to A$, defined by $\pi_g(x) = x(g)$, is continuous for all $g \in G$, where A is equipped with the discrete topology. We recall the following characterization of cellular automata over finite alphabets ([**10**]; see also our generalization [**5**]).

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THEOREM 1.1. (Curtis–Hedlund) *Let G be a group and A a finite alphabet. For a map $\tau : A^G \to A^G$, the following are equivalent:*

(i) *τ is a cellular automaton;*
(ii) *τ is G-equivariant and continuous with respect to the prodiscrete topology on A^G.*

Now let $H \subset G$ be a subgroup.

Let $\tau : A^G \to A^G$ be a cellular automaton with memory set M and local defining map μ. Suppose that the subgroup H contains M. Then the cellular automaton $\tau_H : A^H \to A^H$, defined by $\tau_H(x)(h) = \mu(x^h|_M)$ for all $x \in A^H$ and $h \in H$, is called the *restriction* of τ to H.

Conversely, let $\sigma : A^H \to A^H$ be a cellular automaton with memory set $M \subset H$ and local defining map $\nu : A^M \to A$. Then the cellular automaton $\sigma^G : A^G \to A^G$, defined by setting $\sigma^G(x)(g) = \nu(x^g|_M)$ for all $x \in A^G$, $g \in G$, is called the *induction* of σ from H to G.

It follows immediately from the definitions that induction and restriction, of cellular automata admitting a memory set contained in the subgroup, are inverse to each other; that is, with the above notation, $(\tau_H)^G = \tau$ and $(\sigma^G)_H = \sigma$.

Our main result, which establishes that several important properties of a cellular automaton are preserved under the operations of induction and restriction, is the following.

THEOREM 1.2. *Let G be a group, $H \subset G$ a subgroup and A a set. Let $\tau : A^G \to A^G$ be a cellular automaton with memory set M, and suppose that H contains M. Then the following hold:*

(i) *τ is injective if and only if τ_H is injective;*
(ii) *τ is pre-injective if and only if τ_H is pre-injective;*
(iii) *τ is surjective if and only if τ_H is surjective;*
(iv) *τ is bijective if and only if τ_H is bijective;*
(v) *τ is invertible if and only if τ_H is invertible;*
(vi) *$\tau(A^G)$ is closed in A^G for the prodiscrete topology on A^G if and only if $\tau_H(A^H)$ is closed in A^H for the prodiscrete topology on A^H.*

A few words about the terminology. Two maps $x, x' : G \to A$ are called almost equal if the set $\{g \in G : x(g) \neq x'(g)\}$ is finite. Then, a cellular automaton $\tau : A^G \to A^G$ is said to be *pre-injective* [**11**] if $\tau(x) = \tau(x')$ with $x, x' \in A^G$ almost equal implies $x = x'$. A cellular automaton $\tau : A^G \to A^G$ is said to be *invertible* provided it is bijective and the inverse map $\tau^{-1} : A^G \to A^G$ is also a cellular automaton.

The interest of the notions of induction and restriction lies in several applications, as we show in the following.

Moore and Myhill [**13, 15**] proved that a cellular automaton $\tau : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ with a finite alphabet A is surjective if and only if it is pre-injective. This result was extended to (finitely generated) groups of sub-exponential growth by Machì and Mignosi [**12**], and to countable amenable groups by Ceccherini-Silberstein *et al* [**7**]. Here, as an application of the previous theorem, we extend it to all amenable groups.

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COROLLARY 1.3. (The Garden of Eden theorem) *Let A be a finite alphabet, G an amenable group and $\tau\colon A^G \to A^G$ a cellular automaton. Then the following are equivalent:*

(a) *τ is pre-injective;*

(b) *τ is surjective.*

Suppose now that the alphabet is a vector space V over a field $\mathbb{K}$. We say that a map $\tau\colon V^G \to V^G$ is a *linear cellular automaton* if it is a cellular automaton with alphabet V and it is $\mathbb{K}$-linear with respect to the vector space structure on V^G. If $\dim_{\mathbb{K}} V < \infty$ we say that a linear cellular automaton $\tau\colon V^G \to V^G$ is *finite-dimensional*. In [**2**] we proved an analogue of the Garden of Eden theorem for finite-dimensional linear cellular automata over countable amenable groups; we can now prove its extension to all amenable groups.

COROLLARY 1.4. (The linear Garden of Eden theorem) *Let V be a finite-dimensional vector space over a field $\mathbb{K}$, let G be an amenable group, and let $\tau\colon V^G \to V^G$ be a linear cellular automaton. Then the following are equivalent:*

(a) *τ is pre-injective;*

(b) *τ is surjective.*

A group G is called *surjunctive*, a terminology due to Gottschalk [**9**], if the following holds: for any finite set A, every injective cellular automaton $\tau\colon A^G \to A^G$ is surjective. It is a well-known result of Lawton [**9**] that residually finite groups are surjunctive. From the Garden of Eden theorem one immediately deduces that amenable groups are surjunctive as well. These results were generalized by Gromov [**11**] and Weiss [**16**] who showed that the class of *sofic* groups (a class containing all residually amenable groups, and therefore all residually finite groups and all amenable groups) is contained in the class of surjunctive groups. It is an open problem to determine whether all groups are surjunctive or not. From Theorem 1.2 we deduce the following result (recall that if $\mathcal{P}$ is a property of groups, then a group G is called *locally $\mathcal{P}$* if all finitely generated subgroups of G satisfy $\mathcal{P}$).

COROLLARY 1.5. *A group is surjunctive if and only if it is locally surjunctive.*

Let A be a finite set and $\tau\colon A^G \to A^G$ a cellular automaton. Then the space A^G is a Hausdorff compact space by Tychonoff's theorem and, by continuity of τ (Theorem 1.1), $\tau(A^G)$ is a closed subset of A^G with respect to the prodiscrete topology on A^G. In [**2**] we proved the analogue of this fact for finite-dimensional linear cellular automata over countable groups. Here we are able to extend it to all groups.

COROLLARY 1.6. (Closure lemma) *Let G be a group, V a finite-dimensional vector space over a field $\mathbb{K}$, and $\tau\colon V^G \to V^G$ a linear cellular automaton. Then $\tau(V^G)$ is closed in V^G with respect to the prodiscrete topology on V^G.*

Since every bijective continuous self-map of a Hausdorff compact space is a homeomorphism, it follows again from the Curtis–Hedlund characterization of cellular automata (Theorem 1.1) that when A is a finite set, every bijective cellular automaton $\tau\colon A^G \to A^G$ is invertible. In [**3**] we proved the analogue of this fact for finite-dimensional linear cellular automata over countable groups. We can now prove it in the following general form.

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COROLLARY 1.7. (Invertibility of bijective finite-dimensional linear cellular automata) *Let G be a group, V a finite-dimensional vector space over a field $\mathbb{K}$, and $\tau\colon V^G \to V^G$ a bijective linear cellular automaton. Then τ is invertible.*

The paper is structured as follows. In the next section we present the proofs of our statements. In §3, as a further application of Theorem 1.2, we discuss the validity of the Garden of Eden theorem (Corollaries 1.3 and 1.4) in the non-amenable case.

2. *Proofs*

Let G be a group, A an alphabet, and $\tau\colon A^G \to A^G$ a cellular automaton with memory set $M \subset G$ and local defining map $\mu\colon A^M \to A$.

First, note that any finite set $M' \subset G$ containing M is also a memory set for τ, with local defining map $\mu'\colon A^{M'} \to A$ given by $\mu' = \mu \circ \pi_{M',M}$, where $\pi_{M',M}\colon A^{M'} \to A^M$ is the restriction map.

PROPOSITION 2.1. *Let $\tau\colon A^G \to A^G$ be a cellular automaton. Suppose that $M_1 \subset G$ and $M_2 \subset G$ are memory sets for τ. Then $M_1 \cap M_2$ is a memory set for τ.*

Proof. Let $x \in A^G$. Let us show that $\tau(x)(1_G)$ only depends on the restriction of x to $M_1 \cap M_2$. To see this, consider an element $y \in A^G$ such that $x|_{M_1\cap M_2} = y|_{M_1\cap M_2}$. Choose an element $z \in A^G$ such that $z|_{M_1} = x|_{M_1}$ and $z|_{M_2} = y|_{M_2}$. We have $\tau(z)(1_G) = \tau(x)(1_G)$ since x and z coincide on the memory set M_1. On the other hand, we have $\tau(z)(1_G) = \tau(y)(1_G)$ since z and y coincide on the memory set M_2. This implies that $\tau(x)(1_G) = \tau(y)(1_G)$. Hence, there is a map $\mu\colon A^{M_1\cap M_2} \to A$ such that

$$\tau(x)(1_G) = \mu(x|_{M_1\cap M_2})$$

for all $x \in A^G$. Since τ is G-equivariant, we have

$$\tau(x)(g) = \tau(x^g)(1_G) = \mu(x^g|_{M_1\cap M_2})$$

for all $x \in A^G$ and $g \in G$. Thus $M_1 \cap M_2$ is a memory set for τ. □

COROLLARY 2.2. *Let $\tau\colon A^G \to A^G$ be a cellular automaton. Then there is a unique memory set $M_0 \subset G$ of minimal cardinality.*

Proof. Let M_1 and M_2 be memory sets of minimal cardinality. Since $M_1 \cap M_2$ is a memory set, we have $|M_1 \cap M_2| = |M_1| = |M_2|$. Thus $M_1 \cap M_2 = M_1 = M_2$. □

PROPOSITION 2.3. *Let G be a group, A an alphabet, and $\tau_1\colon A^G \to A^G$ and $\tau_2\colon A^G \to A^G$ two cellular automata with memory sets M_1 and M_2, respectively. Then $\tau_1 \circ \tau_2$ is a cellular automaton with memory set $M_1M_2 = \{m_1m_2 : m_1 \in M_1, m_2 \in M_2\}$.*

Proof. Denote by $\mu_1\colon A^{M_1} \to A$ and by $\mu_2\colon A^{M_2} \to A$ the local defining maps corresponding to M_1 and M_2, respectively. For $\phi \in A^{M_1M_2}$ and $m_1 \in M_1$, define $\phi_{m_1} \in A^{M_2}$ by setting $\phi_{m_1}(m_2) = \phi(m_1m_2)$ for all $m_2 \in M_2$. Also, denote by $\phi' \in A^{M_1}$ the map defined by $\phi'(m_1) = \mu_2(\phi_{m_1})$ for all $m_1 \in M_1$. Finally, we define the map $\mu\colon A^{M_1M_2} \to A$ by setting

$$\mu(\phi) = \mu_1(\phi') \tag{2.1}$$

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for all $\phi \in A^{M_1M_2}$. Let us show that μ is a local defining map for $\tau_1 \circ \tau_2$ with memory set M_1M_2. Let $x \in A^G$, $g \in G$, $m_1 \in M_1$ and $m_2 \in M_2$. We then have

$$\begin{aligned}(x^{gm_1}|_{M_2})(m_2) &= x^{gm_1}(m_2)\\ &= x(gm_1m_2)\\ &= x^g(m_1m_2)\\ &= x^g|_{M_1M_2}(m_1m_2)\\ &= (x^g|_{M_1M_2})_{m_1}(m_2).\end{aligned}$$

This shows that

$$x^{gm_1}|_{M_2} = (x^g|_{M_1M_2})_{m_1}$$

and therefore

$$\tau_2(x^g)(m_1) = \mu_2(x^{gm_1}|_{M_2}) = \mu_2((x^g|_{M_1M_2})_{m_1}) = (x^g|_{M_1M_2})'(m_1).$$

As a consequence,

$$\tau_2(x^g)|_{M_1} = (x^g|_{M_1M_2})' \tag{2.2}$$

and, finally, one has

$$\begin{aligned}(\tau_1 \circ \tau_2)(x)(g) = &\quad \tau_1(\tau_2(x))(g)\\ = &\quad \mu_1(\tau_2(x)^g|_{M_1})\\ = &\quad \mu_1(\tau_2(x^g)|_{M_1})\\ =_* &\quad \mu_1((x^g|_{M_1M_2})')\\ =_{**} &\quad \mu(x^g|_{M_1M_2})\end{aligned} \tag{2.3}$$

where $=_*$ follows from (2.2) and $=_{**}$ follows from (2.1). □

COROLLARY 2.4. *Let G be a group, A an alphabet, and $\tau_1 : A^G \to A^G$ and $\tau_2 : A^G \to A^G$ two cellular automata. Let H be a subgroup of G containing the minimal memory sets of both τ_1 and τ_2. Then H contains the minimal memory set of $\tau_1 \circ \tau_2$, and $(\tau_1 \circ \tau_2)_H = (\tau_1)_H \circ (\tau_2)_H$.*

Proof of Theorem 1.2. Let H be a subgroup of G. Fix a set T of representatives for the left cosets of H in G, so that $G = \coprod_{t\in T} tH$. For our convenience, we suppose that $1_G \in T$. For $t \in T$, denote by $\pi_t : A^G \to A^{tH}$ the restriction map defined by $\pi_t(x) = x|_{tH}$ for all $x \in A^G$, and denote by $\phi_t : A^{tH} \to A^H$ the bijection induced by the map $h \mapsto th$, for $h \in H$. When $t = 1_G$, we write π_H instead of π_t. Thus, $A^G = \prod_{t\in T} A^{tH}$ and, setting $x_t = \pi_t(x) \in A^{tH}$ for all $x \in A^G$, one writes $x = (x_t)_{t\in T}$. Note that, in particular, we have

$$\begin{aligned}\phi_t(\pi_t(x))(h) &= \pi_t(x)(th)\\ &= x|_{tH}(th)\\ &= x(th)\\ &= x^t(h)\end{aligned} \tag{2.4}$$

for all $x \in A^G$, $t \in T$ and $h \in H$.

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Let $\tau\colon A^G \to A^G$ be a cellular automaton with memory set $M \subset H$. Denote by $\tau_H\colon A^H \to A^H$ its restriction to H. For all $t \in T$, set $\tau_t = \phi_t^{-1} \circ \tau_H \circ \phi_t\colon A^{tH} \to A^{tH}$; note that for $t = 1_G$ one has $\tau_t = \tau_H$. Then, for all $x \in A^G$, one has

$$\tau(x) = (\tau_t(x_t))_{t\in T}, \tag{2.5}$$

i.e. τ may be expressed as the product map

$$\tau = \prod_{t\in T} \tau_t. \tag{2.6}$$

Indeed, if $x \in A^G$, $t \in T$ and $h \in H$, then one has

$$\begin{aligned}
\tau_t(x_t)(th) &= \phi_t^{-1}(\tau_H(\phi_t(\pi_t(x))))(th) \\
&= \tau_H(\phi_t(\pi_t(x)))(h) \\
&= \mu((\phi_t(\pi_t(x)))^h|_M) \\
&=_* \mu((x^t)^h|_M) \\
&= \mu(x^{th}|_M) \\
&= \tau(x)(th)
\end{aligned}$$

where $=_*$ follows from (2.4). Note that the factorization (2.6) does not depend on the particular choice of T; that is, if $T' \subset G$ is another set of representatives for the left cosets of H in G, then for $t' \in T'$ one has $\tau_{t'} = \tau_t$ where t is the (unique) element in T such that $t'H = tH$.

We now proceed to the proof of our statements. It immediately follows from (2.6) that τ is injective (respectively surjective) if and only if τ_t is injective (respectively surjective) for all $t \in T$. As the map ϕ_t is a bijection, it follows that τ_t is injective (respectively surjective) for some $t \in T$ and thus, equivalently, for all $t \in T$ if and only if $\tau_{1_G} = \tau_H$ is injective (respectively surjective). This shows (i) and (iii).

(ii) This is more delicate. Suppose first that τ is pre-injective. Suppose that $x', y' \in A^H$ are almost equal and that $\tau_H(x') = \tau_H(y')$. Extend x' and y' to x and y in A^G by setting $x(th) = y(th) = x'(h)$ for all $h \in H$ and $t \in T \setminus \{1_G\}$. Note that x and y are almost equal: indeed, $\{g \in G : x(g) \neq y(g)\} = \{h \in H : x'(h) \neq y'(h)\}$. From (2.5) one deduces $\tau(x) = \tau(y)$. It follows that $x = y$, by pre-injectivity of τ. But then $x' = x|_H$ equals $y|_H = y'$. This shows that τ_H is pre-injective.

Conversely, suppose that τ_H is pre-injective. Suppose that $x, y \in A^G$ are almost equal and that $\tau(x) = \tau(y)$. For each $t \in T$, define $x^t, y^t \in A^H$ by setting $x^t = \phi_t \circ \pi_t(x)$ and $y^t = \phi_t \circ \pi_t(y)$ (i.e. $x^t(h) = x(th)$ and $y^t(h) = y(th)$ for all $h \in H$). Note that, *a fortiori*, x^t and y^t are almost equal in A^H for all $t \in T$. Also, we have

$$\tau_H(x^t)(h) = \mu((x^t)^h|_M) = \mu(x^{th}|_M) = \tau(x)(th)$$

and, similarly, $\tau_H(y^t)(h) = \tau(y)(th)$. It follows that $\tau_H(x^t) = \tau_H(y^t)$ for all $t \in T$. Thus $x^t = y^t$ for all t, by pre-injectivity of τ_H. Composing with the bijections $\phi_t^{-1}\colon A^H \to A^{tH}$, we deduce that $x_t = y_t$ for all $t \in T$. By (2.6), this yields $x = y$, and pre-injectivity of τ follows.

(iv) This follows immediately from (i) and (iii). Also note that, in fact, a product map $\tau = \prod_{t\in T} \tau_t$ is bijective if and only if τ_t is bijective for all $t \in T$ and that, if this were the case, one would have

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$$\tau^{-1} = \prod_{t\in T} (\tau_t)^{-1}. \tag{2.7}$$

(v) Suppose that τ is invertible. This implies, on the one hand, that τ is bijective, and therefore so is τ_H; on the other hand, it also implies that the inverse map τ^{-1} is a cellular automaton. From (2.7) it then follows that

$$(\tau^{-1})_H = (\tau_H)^{-1}, \tag{2.8}$$

which is the restriction of a cellular automaton, is a cellular automaton as well. This shows that τ_H is invertible.

Conversely, suppose that τ_H is invertible, so that $(\tau_H)^{-1}\colon A^H \to A^H$ is a cellular automaton. By (iv) we have that τ is bijective; we denote by $\tau^{-1}\colon A^G \to A^G$ its inverse map. Clearly, (2.7) holds and, for all $t \in T$,

$$(\tau_t)^{-1} = (\phi_t^{-1} \circ \tau_H \circ \phi_t)^{-1} = \phi_t^{-1} \circ (\tau_H)^{-1} \circ \phi_t. \tag{2.9}$$

This shows that τ^{-1}, being the induction of the cellular automaton $(\tau_H)^{-1}\colon A^H \to A^H$, is a cellular automaton. Note that from (2.9) we then have $(\tau_t)^{-1} = (\tau^{-1})_t$ for all $t \in T$.

(vi) Suppose that $\tau(A^G)$ is closed in A^G for the prodiscrete topology on A^G. Let $y \in A^H$ be such that for all finite subsets $\Omega \subset H$, there exists $x_\Omega \in A^H$ such that $\tau_H(x_\Omega)|_\Omega = y|_\Omega$. Let us show that there exists $x \in A^H$ such that $\tau_H(x) = y$.

Extend y to the element $y' \in A^G$ defined by setting $y'(th) = y(h)$ for all $t \in T$ and $h \in H$. For all $\Omega \subset G$, set $\Omega_H = \Omega \cap H$ and denote by $x'_\Omega \in A^G$ the extension of $x_{\Omega_H} \in A^H$ defined by $x'_\Omega(th) = x_{\Omega_H}(h)$ for all $t \in T$ and $h \in H$. We have

$$\tau(x'_\Omega)|_\Omega = y'|_\Omega$$

for all finite subsets $\Omega \subset G$. As $\tau(A^G)$ is closed in A^G, there exists $x' \in A^G$ such that $\tau(x') = y'$. Then $x = x'|_H \in A^H$ satisfies $\tau_H(x) = \tau(x')|_H = y'|_H = y$. It follows that $\tau_H(A^H)$ is closed in A^H for the prodiscrete topology in A^H.

Conversely, if $\tau_H(A^H)$ is closed in A^H, then as the maps $\phi_t\colon A^{tH} \to A^H$ are homeomorphisms, it follows that $\tau_t(A^{tH}) = \phi_t^{-1}(\tau_H(A^H))$ is closed in A^{tH} for all $t \in T$. Therefore $\tau(A^G) = \prod_{t\in T} \tau_t(A^{tH})$, being the product of closed spaces, is also closed in A^G. □

Remark. (1) For the statements (i) and (iii), we also refer to the paper by Glebsky and Gordon [**8**] (Lemmas 2 and 1, respectively), where a similar notion of induction for a generalization of cellular automata on groups, called group-like configuration systems, is considered.

(2) Note that if $\tau\colon A^G \to A^G$ is an invertible cellular automaton with memory set M, then although there is no control on the size or the position inside G of the minimal memory set M' for the inverse cellular automaton τ^{-1}, from the proof of (v) in the previous theorem and, more precisely, from (2.8), we can deduce that M' is contained in the subgroup H generated by M. If M is also minimal, then, by symmetry, the subgroups generated by M and M' do in fact coincide.

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Proof of the Garden of Eden theorem (Corollary 1.3). Let G be a group, A a finite set and $\tau\colon A^G \to A^G$ a cellular automaton. Let M be a memory set for τ, and denote by H the (countable) subgroup generated by M.

Suppose that τ is pre-injective. By Theorem 1.2(ii) we have that the restriction $\tau_H\colon A^H \to A^H$ is also pre-injective. By the Garden of Eden theorem for countable groups in [**7**], we have that τ_H is surjective. Then, by Theorem 1.2(iii), we have that τ is surjective as well.

Conversely, suppose that τ is surjective. By Theorem 1.2(iii) we have that τ_H is surjective. By the Garden of Eden theorem for countable groups in [**7**], we have that τ_H is pre-injective. By Theorem 1.2(ii), we finally have that τ is also pre-injective. □

Proof of the linear Garden of Eden theorem (Corollary 1.4). Let G be a group, V a finite-dimensional vector space over a field $\mathbb{K}$ and $\tau\colon V^G \to V^G$ a linear cellular automaton. Then, the same argument as in the proof of the Garden of Eden theorem (Corollary 1.3), with the phrase 'Garden of Eden theorem for countable groups in [**7**]' replaced by '[**2**, Theorem 1.2]', yields the equivalence between the pre-injectivity and the surjectivity of τ. □

Proof of Corollary 1.5. We first show that if G is a surjunctive group, then all of its subgroups are surjunctive. Let G be a surjunctive group. Suppose that H is a subgroup of G. Let A be a finite set and let $\tau\colon A^H \to A^H$ be an injective cellular automaton over H. Consider the cellular automaton $\tau^G\colon A^G \to A^G$ over G obtained from τ by induction. The fact that τ is injective implies that τ^G is injective by Theorem 1.2(i). Since G is surjunctive, it follows that τ^G is surjective. By applying Theorem 1.2(iii), we deduce that τ is surjective.

Conversely, we now show that if G is a locally surjunctive group, then it is surjunctive. Let G be a locally surjunctive group and let A be a finite set. Consider an injective cellular automaton $\tau\colon A^G \to A^G$ with memory set M. Let H denote the subgroup of G generated by M. Consider the cellular automaton $\tau_H\colon A^H \to A^H$ obtained by restriction of τ. The fact that τ is injective implies that τ_H is injective, by Theorem 1.2(i). As H is surjunctive by our hypothesis on G, we deduce that τ_H is surjective. By applying Theorem 1.2(iii), it follows that τ is also surjective. □

Remark. We refer again to [**8**, Proposition 3 and Corollary 1] for a similar statement on surjunctivity for group-like configuration systems.

Proof of the Closure lemma (Lemma 1.6). Let G be a group, V a finite-dimensional vector space over a field $\mathbb{K}$ and $\tau\colon V^G \to V^G$ a linear cellular automaton. Let M be a memory set for τ, and denote by H the (countable) subgroup generated by M. By [**2**, Lemma 3.1] we have that $\tau_H(V^H)$ is closed for the prodiscrete topology on V^H. By Theorem 1.2(vi) we have that $\tau(V^G)$ is closed for the prodiscrete topology on V^G. □

Proof of Corollary 1.7. Let G be a group, V a finite-dimensional vector space over a field $\mathbb{K}$ and $\tau\colon V^G \to V^G$ a bijective linear cellular automaton. Let M be a memory set for τ, and denote by H the (countable) subgroup generated by M. By Theorem 1.2(iv) we have

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that τ_H is bijective. From [**3**, Theorem 3.1] we deduce that τ_H is invertible (i.e. $(\tau_H)^{-1}$ is also a cellular automaton). By Theorem 1.2(v) we have that τ is also invertible. □

3. *Final remarks*

Examples of cellular automata showing that the Garden of Eden theorem (Corollary 1.3) fails to hold, in general, for non-amenable groups were first given by Muller [**14**] and then by Machì and Mignosi [**12**]. In the case where $G = F_2$, the free group of rank two, similar examples may be constructed as follows.

Example. [**4**] Let $G = F_2$ be the free group on two generators a and b. Let $\mathbb{K}$ be a field and set $V = \mathbb{K}^2$. Let $p_1, p_2 \colon V \to V$ be the $\mathbb{K}$-linear maps defined, respectively, by $p_1(h_1, h_2) = (h_1, 0)$ and $p_2(h_1, h_2) = (h_2, 0)$ for all $(h_1, h_2) \in V$. Consider the map $\tau \colon V^G \to V^G$ given by

$$\tau(x)(g) = p_1(x(ga)) + p_1(x(ga^{-1})) + p_2(x(gb)) + p_2(x(gb^{-1})) \tag{3.1}$$

for all $x \in V^G$ and $g \in G$. It is clear that τ is a (finite-dimensional) linear cellular automaton over the group G (having finite alphabet if $\mathbb{K}$ is finite) with memory set $S = \{a, b, a^{-1}, b^{-1}\}$. In [**4**] it is proved that τ is pre-injective but not surjective. This shows the failure of the implication (a) $\Rightarrow$ (b) in Corollary 1.4 (as well as in Corollary 1.3, provided that $\mathbb{K}$ is finite) for $G = F_2$.

Example. [**4**] Let $G = F_2$ be the free group on two generators a, b. Let $\mathbb{K}$ be a field and set $V = \mathbb{K}^2$. Consider the $\mathbb{K}$-linear maps $q_1, q_2 \colon V \to V$ defined by $q_1(k_1, k_2) = (k_1, 0)$ and $q_2(k_1, k_2) = (0, k_1)$ for all $k_1, k_2 \in \mathbb{K}$. Let $\tau \colon V^G \to V^G$ be the linear cellular automaton, with memory set $S = \{a, b, a^{-1}, b^{-1}\}$, given by

$$\tau(x)(g) = q_1(x(ga)) + q_1(x(ga^{-1})) + q_2(x(gb)) + q_2(x(gb^{-1}))$$

for all $x \in V^G$ and $g \in G$. Note that τ is finite-dimensional and has finite alphabet, provided that $\mathbb{K}$ is finite. In [**4**] it is proved that τ is surjective but not pre-injective. This shows the failure of the implication (b) $\Rightarrow$ (a) in Corollary 1.4 (and in Corollary 1.3, provided that $\mathbb{K}$ is finite) for $G = F_2$.

COROLLARY 3.1. *Let G be a group containing the free group F_2. Then both the implications of the Garden of Eden theorem (Corollary 1.3) and of the linear Garden of Eden theorem (Corollary 1.4) fail to hold.*

Using a characterization of amenability due to Tarski, based on the notion of paradoxical decompositions [**6**], Bartholdi [**1**] very recently proved that if G is non-amenable, then there exists a finite set A and a surjective cellular automaton $\tau \colon A^G \to A^G$ which is not pre-injective. This, combined with the implication (b) $\Rightarrow$ (a) in the Garden of Eden theorem (Corollary 1.3), gives a new characterization of amenability in terms of cellular automata. We mention that Bartholdi's construction works, in particular, also for non-amenable torsion groups, therefore with no free subgroups, to which Corollary 3.1 cannot be applied.

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