Combinatorics, Probability and Computing (2005) 14, 349–370. © 2005 Cambridge University Press
doi:10.1017/S096354830400642X Printed in the United Kingdom

The Cycle-Complete Graph Ramsey Numbers

V. NIKIFOROV

Department of Mathematical Sciences, University of Memphis,
Memphis, Tennessee, 38152, USA
(e-mail: vnikifrv@memphis.edu)

Received 3 February 2003; revised 12 January 2004

In 1978 Erdős, Faudree, Rousseau and Schelp conjectured that

$$r(C_p, K_r) = (p-1)(r-1)+1$$

for every $p \geqslant r \geqslant 3$, except for $p = q = 3$. This has been proved for $r \leqslant 6$, and for

$$p \geqslant r^2 - 2r.$$

In this note we prove the conjecture for $p \geqslant 4r + 2$.

1. Introduction

The problem of finding the Ramsey number $r(C_p, K_r)$ has attracted considerable attention in the last decades. In particular, relatively good asymptotics are known in the case of p fixed and r large (see [11] for a recent survey). In contrast, the case $p \geqslant r$ is poorly known. In [5] Bondy and Erdős proved that for $r > 3$ and $p \geqslant r^2 - 2$ the following exact result holds:

$$r(C_p, K_r) = (p-1)(r-1)+1. \tag{1.1}$$

Later, in [8], Erdős, Faudree, Rousseau and Schelp conjectured that (1.1) holds for every $p \geqslant r \geqslant 3$, except for $p = r = 3$. This has been proved for $r = 4$ in [14], for $r = 5$ in [1], and for $r = 6$ in [12]. In [12] Schiermeyer also showed that (1.1) holds for $r > 3$, $p \geqslant r^2 - 2r$.

In this note we prove that (1.1) holds for all $r \geqslant 3$ and $p \geqslant 4r + 2$.

2. Main results

Our graph-theoretic notation is standard (*e.g.*, see [2]). In particular, for any vertex u, $\Gamma(u)$ is the set of its neighbours. A graph G is said to be *H-free* if it does not contain a subgraph isomorphic to H. A path with end-vertices u and v is called a *uv-path*; $P(u, v)$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

denotes a path P that joins u to v. We write $(v_1,\dots,v_k,v_1)$ for the cycle whose edges are $(v_1,v_2),\dots,(v_{k-1},v_k),(v_k,v_1)$. The set $\{m,m+1,\dots,n\}$ is denoted by $[m,n]$ and $[n]$ is the set $\{1,\dots,n\}$. An *interval* of length l is a set of $l\geqslant 0$ consecutive integers. Given a graph G and two distinct vertices u,v of G, we let $R_G(u,v)$ denote the set of the orders of all uv-paths; we shorten $R_G(u,v)$ to $R(u,v)$ when G is implicit.

Our main goal in this note is to prove the following theorem.

Theorem 2.1. *If $r\geqslant 4$ and $p\geqslant 4r+2$ then $r(C_p,K_r)=(p-1)(r-1)+1$.*

To shorten the proof of the theorem we have distilled its main parts into several lemmas that might be of independent interest. The proofs of the lemmas are presented in Section 2.5.

Our main tool will be a particular class of graphs containing a Hamiltonian cycle together with a rich set of chords; we call such graphs *saws* and study them in Section 2.3.

2.1. General preliminary lemmas

Following Burr and Erdős [4], we call a connected graph H *r-good* if the Ramsey number $r(H,K_r)$ of the pair (H,K_r) satisfies

$$r(H,K_r)=(r-1)(|H|-1)+1.$$

The following lemma could be regarded as a general result in the theory of the r-good graphs.

Lemma 2.2. *Suppose H is a graph such that $r(K_s,H)\leqslant sp+1$ for every $s\leqslant r$. Then every H-free graph G of order $pr+1$ and with $\alpha(G)\leqslant r$ is 2-connected.*

Erdős and Gallai proved the following assertion in [7, p. 345].

Theorem 2.3. (Erdős and Gallai). *If G is a 2-connected graph with $d(w)\geqslant\delta$ for all $w\neq u,v$, then there is a uv-path of order at least $\delta+1$ in G.* □

This versatile result can be further extended in some particular cases; we consider such extensions in the following two lemmas.

Lemma 2.4. *Let G be a 2-connected graph and u,v be two vertices such that $G-u-v$ is a union of two disjoint nonempty graphs G_1 and G_2. If $d(w)\geqslant\delta$, for every $w\neq u,v$, then there is a uv-path of order at least $\delta+1$ that has no vertices in common with G_1.*

Lemma 2.5. *Let G be a 2-connected graph and $x\in V(G)$. If $d(y)\geqslant\delta$, for every $y\neq x$, then for every two vertices u and v there is a uv-path of order at least $\delta+1$.*

2.2. The chopping and the collating lemmas

Definition. Let P be a uv-path; a *reduction* of P is a uv-path Q such that all vertices of Q belong to P. A *q-reduction* of P is a reduction of order q.

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

The following two lemmas will be cornerstones in the proof of Theorem 2.1; the first one is called the Chopping Lemma.

Lemma 2.6. (Chopping Lemma). *Let G be a graph with $\alpha(G) \leqslant \alpha$, let x, y be two distinct vertices of G, and let P be a xy-path of order l. Then, for every interval I with*

$$I \subset [l], \quad |I| = 2\alpha,$$

there is a q-reduction of P for some $q \in I$.

The following lemma summarizes the main tool in the proof of Theorem 2.1; it is called the Collating Lemma.

Lemma 2.7. (Collating Lemma). *Suppose G is a graph and $V(G) = V_1 \cup V_2$ is a nontrivial partition. Let (x_1, x_2), (y_1, y_2) be two disjoint edges such that $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$. Let $G_1 = G[V_1]$, $G_2 = G[V_2]$, and a, b, k, l_1, l_2 be positive integers such that:*

(i) $b - a \geqslant k - 1$,
(ii) $[a, b] \subset R_{G_1}(x_1, y_1)$,
(iii) *for every interval $I \subset [l_1, l_2]$ with $|I| = k$, $I \cap R_{G_2}(x_2, y_2) \neq \varnothing$.*

Then for every $s \in [a + l_1 + k, b + l_2]$ there is a cycle of order s in G.

2.3. Saws

Definition. Let S be a graph with $V(S) = \{v_1, \ldots, v_{2k+1}\}$; S is called a *saw* if it contains the Hamiltonian cycle $(v_1, \ldots, v_{2k+1}, v_1)$ together with the chords (v_{2s-1}, v_{2s+1}) for every $s \in [k]$. The cycle

$$(v_1, \ldots, v_{2k+1}, v_1)$$

is called the *backbone* of S, and the value

$$d(S) = \min\{d_S(v_{2k}), d_S(v_{2k+1})\}$$

is called the *degree* of S.

We shall define saws by identifying their backbones. Although saws look quite complicated, there are simple sufficient conditions for the existence of large saws, as shown in the following lemma.

Lemma 2.8. *If G is a graph with minimal degree $\delta(G) \geqslant p$ and independence number $\alpha(G) \leqslant r$, then G has a saw of degree at least $p - r$.*

The following four lemmas show that saws contain many paths of consecutive lengths; actually, we introduce and study them exactly for that reason.

Lemma 2.9. *Let $S = (v_1, \ldots, v_{2k+1}, v_1)$ be a saw, and let $P(v_i, v_j)$ be a path along the cycle $(v_1, \ldots, v_{2k+1}, v_1)$ of order l. Then:*

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

(i) *if the edge* (v_1, v_{2k+1}) *does not belong to* $P(v_i, v_j)$*, then* $P(v_i, v_j)$ *has a* q*-reduction for every*

$$q \in \left[\left\lceil \frac{l}{2} \right\rceil + 1, l\right],$$

(ii) *if the edge* (v_1, v_{2k+1}) *belongs to* $P(v_i, v_j)$*, then* $P(v_i, v_j)$ *has a* q*-reduction for every*

$$q \in \left[\left\lfloor \frac{l}{2} \right\rfloor + 2, l\right].$$

In particular, if $P(x, y)$ *is a path of order* $2k+1$ *along the cycle* $(v_1, \ldots, v_{2k+1}, v_1)$ *then* P *has* q*-reductions for every* $q \in [k+2, 2k+1]$.

Lemma 2.9 implies that if $S = (v_1, \ldots, v_{2k+1}, v_1)$ is a saw then, for every two consecutive vertices x, y along the cycle $(v_1, \ldots, v_{2k+1}, v_1)$, we have

$$[k+2, 2k+1] \subset R_S(x, y).$$

It turns out that if the degree of S is large compared to its order then, for every two vertices x, y, the set $R_S(x, y)$ contains even larger intervals. We shall distinguish three different cases of pairs $(x, y) \in V(S)$; each case is considered separately in one of the following three lemmas.

Lemma 2.10. *Let* $S = (v_1, \ldots, v_{2k+1}, v_1)$ *be a saw of degree* $d(S) \geqslant 2(2k+1)/3$*. Then*

$$[2, 2k+1] \subset R_S(v_{2k}, v_{2k+1}).$$

Lemma 2.11. *Let* $S = (v_1, \ldots, v_{2k+1}, v_1)$ *be a saw with* $d(S) = d$*, and let* x, y *be two consecutive vertices along the cycle* $(v_1, \ldots, v_{2k+1}, v_1)$*. Then*

$$[2k - d + 6, 2k+1] \subset R_S(x, y).$$

Lemma 2.12. *Let* $S = (v_1, \ldots, v_{2k+1}, v_1)$ *be a saw of degree* $d(S) = d \geqslant k$*, and let* x, y *be two distinct vertices of* S*. Then there exists some* $l > d$ *such that*

$$\left[l - \left\lceil \frac{d}{2} \right\rceil + 5, l\right] \subset R_S(x, y).$$

In the following lemma we combine path chopping with path reduction to prove the existence of cycles of consecutive lengths in every saw with bounded independence number.

Lemma 2.13. *Let* $k \geqslant 3$ *and* S *be a saw of order* $2k+1$ *with independence number* $\alpha(S) \leqslant r$*. If* $2r \leqslant k$ *then* S *contains a cycle of order* q *for every*

$$q \in [4r, 2k+1].$$

2.4. Proof of the theorem

Proof of Theorem 2.1. To simplify the notation slightly, we shall actually prove $r(K_{r+1}, C_{p+1}) = pr + 1$, under the constraints $r \geqslant 2$, $p \geqslant 4r + 5$.

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Observe that the disjoint union of r complete graphs of order p is a C_{p+1}-free graph of order rp and has no independent set on $r+1$ vertices. Thus, for every p and r we have

$$r(K_{r+1}, C_{p+1}) \geqslant pr + 1;$$

so all we have to prove is the inequality

$$r(K_{r+1}, C_{p+1}) \leqslant pr + 1. \tag{2.1}$$

We shall use induction on r; for $r \leqslant 5$, (2.1) follows from the earlier results in [14], [1], and [12]; so we assume that $r \geqslant 6$, and (2.1) holds for all $r' < r$. Assume (2.1) does not hold for r and p, and let G be a C_{p+1}-free graph of order $rp+1$ with $\alpha(G) \leqslant r$; we shall show that these assumptions lead to a contradiction.

First we shall prove that $\delta(G) \geqslant p$. Indeed, let u be a vertex of minimal degree in G and let V' be the set of vertices that are not adjacent to u and are distinct from u. Clearly,

$$\alpha(G[V']) \leqslant r - 1$$

and $G[V']$ is C_{p+1}-free; thus, by the induction hypothesis,

$$|V'| \leqslant p(r-1),$$

and therefore,

$$\delta(G) = pr - |V'| \geqslant p.$$

Applying Lemma 2.2 with $H = C_{p+1}$, we see that G is 2-connected.

Applying Lemma 2.8, we can find in G a saw $S = (v_1, \ldots, v_{2k+1}, v_1)$ with

$$d(S) = d \geqslant p - r.$$

Since G is C_{p+1}-free, Lemma 2.13 implies $2k+1 \leqslant p$, so we have

$$p - r \leqslant d < 2k + 1 \leqslant p. \tag{2.2}$$

Set $G^* = G - S$. Observe that from $\delta(G) \geqslant p$ it follows that every vertex $w \in V(S)$ has at least $p - 2k$ neighbours in G^*.

In the rest of the proof we construct a cycle of order $p+1$ under various assumptions about the connectivity of G^* and the edges of $E(V(S), V(G^*))$. To achieve this goal we combine a x_1y_1-path in S with a x_2y_2-path in G^*, where (x_1, x_2), (y_1, y_2) are disjoint edges. Depending on the location of x_1, y_1, we use Lemmas 2.10, 2.11, or 2.12, to find in S sufficiently many x_1y_1-paths of consecutive orders. On the other hand, we find in G^* a sufficiently long x_2y_2-path and use the Chopping Lemma (Lemma 2.6) to show that $R_{G^*}(x_2, y_2)$ hits any sufficiently short interval. Finally, we apply the Collating Lemma (Lemma 2.7) to show the existence of cycles of specified order, including C_{p+1}, and thus obtain a contradiction.

Let us now give the details of the proof. It is not hard to see that every graph G^* has one of the following properties:

- G^* is 2-connected;
- G^* is not connected and all its components are 2-connected;
- G^* contains a connected component that is not 2-connected (it may be G^* itself).

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Case 1: G^* is 2-connected.
First we shall prove that v_{2k} and v_{2k+1} have two different neighbours,

$$x_2 \in V(G^*) \cap \Gamma(v_{2k}), \quad y_2 \in V(G^*) \cap \Gamma(v_{2k+1}).$$

Indeed, this is clear if one of the vertices v_{2k} and v_{2k+1} has two or more neighbours in G^*. Otherwise, in view of (2.2), we see that $p = 2k+1$, and v_{2k}, v_{2k+1} are joined to every vertex of S. But then, if v_{2k} and v_{2k+1} have a neighbour in common in G^*, we immediately obtain a C_{p+1}, a contradiction. Thus, v_{2k} and v_{2k+1} have two different neighbours, $x_2, y_2 \in V(G^*)$.

Next we shall show that there is a path $P_2(x_2, y_2)$ of order at least $p-2k$ in G^*. We see this immediately if $p = 2k+1$, since G^* is connected. If $p > 2k+1$, the assertion follows from

$$\delta(G^*) \geqslant p - 2k - 1,$$

and Lemma 2.5 applied to G^*.

Let the order of $P_2(x_2, y_2)$ be $l \geqslant p-2k$. Hence, from the Chopping Lemma, for every interval I with

$$I \subset [l], \quad |I| = 2r,$$

there is a q-reduction of $P_2(x_2, y_2)$ for some $q \in I$. On the other hand, from Lemma 2.10,

$$[2, 2k+1] \subset R_S(v_{2k}, v_{2k+1}).$$

Thus, as

$$2k \geqslant p - r \geqslant 2r,$$

from the Collating Lemma, we see that G contains a cycle of order s for every

$$s \in [2r+2, l+2k+1],$$

and hence, from

$$l + 2k \geqslant p \geqslant 2r + 1,$$

G contains a C_{p+1}, a contradiction.

Hereafter we shall suppose that G^* is decomposed into blocks and edges. Clearly at least one of the endblocks of G^* has independence number at most $(r+1)/2$; let B be an endblock of G^* with $\alpha(B) \leqslant (r+1)/2$. The following case will appear in several contexts, so we shall consider it separately.

Case 2: There are two consecutive vertices along S that are joined to two distinct vertices of B.
Let $x_2, y_2 \in V(B)$ and x_1, y_1 be consecutive vertices along S such that

$$x_1 \in V(S) \cap \Gamma(u_1), \quad x_2 \in V(S) \cap \Gamma(u_2), \quad x_1 \neq x_2.$$

Observe that, except possibly for the cutvertex of B, for every vertex $w \in V(B)$ we have

$$d_B(w) \geqslant d_G(w) - |S| \geqslant p - 2k - 1.$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

From Lemma 2.5, every two vertices of B are joined in B by a path $P_2(x_2, y_2)$ of order

$$l \geqslant p - 2k.$$

Set $r_1 = \alpha(B)$; from the choice of B we have

$$2r_1 \leqslant r + 1;$$

the Chopping Lemma implies that, for every interval I with

$$I \subset [l], \quad |I| = 2r_1,$$

there is a q-reduction of $P_2(x_2, y_2)$ for some $q \in I$. On the other hand, from Lemma 2.11,

$$[2k - d + 6, 2k + 1] \subset R_S(x_1, y_1).$$

As, from (2.2), we have

$$2k - d + 6 \leqslant p - d + 5 \leqslant r + 5,$$

we see that

$$[r + 5,\ 2k + 1] \subset R_S(x_1,\ y_1),$$

and from (2.2),

$$2k - r \geqslant p - 2r \geqslant r.$$

Applying the Collating Lemma, we find that G contains a cycle of order s for every

$$s \in [r + 2r_1 + 5, l + 2k + 1].$$

Hence, from

$$l + 2k \geqslant p > 2r + 6 \geqslant r + 2r_1 + 5,$$

G contains a C_{p+1}, a contradiction.

From here on we shall assume that, for every endblock B of G^* with

$$\alpha(B) \leqslant (r + 1)/2,$$

there are no consecutive vertices along S that are joined to two distinct vertices of B.

Case 3: G^* is not connected and all its components are 2-connected.

Suppose G^* is a union of disjoint connected components. Clearly, there is a component G_1 of G with $\alpha(G_1) \leqslant r/2$. Select $u_1 \in V(G_1)$ to have the maximum number of neighbours in S among the vertices of G_1; then, for every $w \in V(G_1)$, $w \neq u_1$ we have

$$d_{G_1}(w) \geqslant \frac{p}{2}.$$

Indeed, assume there is some $w \in V(G_1)$, $w \neq u_1$ with $d_{G_1}(w) < p/2$. Then

$$|\Gamma(u_1) \cap V(S)| \geqslant |\Gamma(w) \cap V(S)| = d_G(w) - d_{G_1}(w) > p - \frac{p}{2} = \frac{p}{2},$$

and since $|V(S)| = 2k + 1 \leqslant p$, we see that u_1 and w are joined to two consecutive vertices along S, a contradiction.

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Since G is 2-connected, there are vertices $x_1, y_1 \in V(S)$ and $x_2, y_2 \in V(B)$ such that (x_1, x_2) and (y_1, y_2) are disjoint edges. Applying Lemma 2.5 to G^*, we see that there is a x_2y_2-path $P_2(x_2, y_2)$ in G_1 of length $l > p/2$. Set

$$V_1 = V(S), \quad V_2 = V(G_1), \quad \alpha(G_1) = r_1.$$

In view of $r_1 \leqslant r/2$, the Chopping Lemma implies that, for every interval I with

$$I \subset [l], \quad |I| = 2r_1,$$

there is a q-reduction of $P_2(x_2, y_2)$ for some $q \in I$. On the other hand, from Lemma 2.12, there is some $l_1 > d$, such that

$$\left[l_1 - \left\lfloor \frac{d}{2} \right\rfloor + 5, l_1\right] \subset R_S(x_1, y_1).$$

Applying the Collating Lemma we find that G contains a cycle of order s for every

$$s \in \left[l_1 - \left\lfloor \frac{d}{2} \right\rfloor + 5 + 2r_1, l_1 + l\right].$$

Hence, from

$$l_1 + l > \frac{p}{2} + d \geqslant \frac{p}{2} + p - r \geqslant p$$

and

$$p \geqslant 2r + 5 \geqslant r + 2r_1 + 5,$$

G contains a C_{p+1}, a contradiction.

Case 4: G^* contains a connected component that is not 2-connected.
Clearly we can select an endblock B of G^* with

$$\alpha(B) = r_1 \leqslant \frac{r+1}{2};$$

let z be the only cutvertex of B.

Select $u_1 \in V(B - z)$ to have the maximum number of neighbours in S among the vertices of $B - z$, that is,

$$|\Gamma(u_1) \cap V(S)| = \max_{w \in V(B-z)} |\Gamma(w) \cap V(S)|. \tag{2.3}$$

We shall show that, for every $w \in V(B - z)$, $w \neq u_1$, we have

$$d_B(w) \geqslant \frac{p}{2}. \tag{2.4}$$

Indeed, assume there is some $w \in V(B)$, $w \neq u_1$ with $d_B(w) < p/2$. Then

$$|\Gamma(u_1) \cap V(S)| \geqslant |\Gamma(w) \cap V(S)| = d_G(w) - d_B(w) > p - \frac{p}{2} = \frac{p}{2},$$

and since $|V(S)| = 2k + 1 \leqslant p$, we see that u_1 and w are joined to two consecutive vertices along S, a contradiction.

Since B is 2-connected and (2.4) holds, from Theorem 2.3 we see that the vertex u_1 is joined to z by a path $P_1(u_1, z)$ of order $l > p/2$.

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Since $G^*\backslash B$ is nonempty, there is some $u_2 \in V(G^*)\backslash V(B)$ that is joined to a vertex of S - otherwise $G - z$ is not connected, contradicting the fact that G is 2-connected.

Case 4(a): There exist u_2, x_1, x_2 with

$$u_2 \in V(G^*)\backslash V(B), \quad x_1 \in V(S) \cap \Gamma(u_1), \quad x_2 \in V(S) \cap \Gamma(u_2), \quad x_1 \neq x_2.$$

Set $\alpha(G^*\backslash B) = r_2$; clearly, $r_1 + r_2 \leqslant r + 1$. Select the shortest zu_2-path $P_2(z, u_2)$; set

$$l_1 = |P_2(z, u_2)|.$$

Since, except for the vertex z, the path $P_2(z, u_2)$ is entirely in $G^*\backslash B$ and

$$\alpha(G^*\backslash B) = r_2,$$

the Chopping Lemma implies that

$$2 \leqslant l_1 \leqslant 2r_2 + 1.$$

The concatenation

$$Q = (P_1(u_1, z), P_2(z, u_2))$$

is a u_1u_2-path with

$$\frac{p}{2} + 2 < |Q| = l + l_1 - 1 \leqslant l + 2r_2.$$

Observe that the Chopping Lemma applied to the path $P_1(u_1, z)$, implies that for every interval I with

$$I \subset [l], \quad |I| = 2r_1,$$

there is a q-reduction of $P_1(u_1, z)$ for some $q \in I$. Therefore, for every interval I with

$$I \subset [l_1 + 2r_1, l + l_1 - 1], \quad |I| = 2r_1,$$

there is a q-reduction of $Q(u_1, u_2)$ for some $q \in I$.

On the other hand, from Lemma 2.12 there is some $l_2 > d$ such that

$$\left[l_2 - \left\lfloor \frac{d}{2} \right\rfloor + 5, l_2\right] \subset R_S(x_1, x_2).$$

Set

$$V_1 = V(S), \quad V_2 = V(G^*);$$

applying the Collating Lemma with the partition $V(G) = V_1 \cup V_2$, we find that G contains a cycle of order s for every

$$s \in \left[l_1 + l_2 + 2r_1 - \left\lfloor \frac{d}{2} \right\rfloor + 5, l_1 + l_2 + l - 1\right].$$

Now, from

$$l_1 \leqslant 2r_2 + 1, \quad l_2 \leqslant p - 1, \quad \left\lfloor \frac{d}{2} \right\rfloor \geqslant \frac{p - r - 1}{2}, \quad r_1 + r_2 \leqslant r + 1,$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

it follows that

$$l_1 + l_2 + 2r_1 - \left\lfloor \frac{d}{2} \right\rfloor + 5 \leqslant p + 2r_1 + 2r_2 - \frac{p-r-1}{2} + 5$$
$$\leqslant p + r - \frac{p-r-1}{2} + 6 \leqslant p + 1.$$

On the other hand, from

$$l_1 \geqslant 2, \quad l_2 > d \geqslant p - r, \quad l > \frac{p}{2},$$

it follows that

$$l_1 + l_2 + l - 1 > p - r + \frac{p}{2} + 1 > p.$$

Hence, G contains a C_{p+1}, a contradiction.

Case 4(b): There are no u_2, x_1, x_2 with

$$u_2 \in V(G^*)\backslash V(B), \quad x_1 \in V(S) \cap \Gamma(u_1), \quad x_2 \in V(S) \cap \Gamma(u_2), \quad x_1 \neq x_2.$$

Clearly the assumption implies that u_1 has exactly one neighbour,

$$x_1 \in V(S) \cap \Gamma(u_1),$$

and for every $w \in V(G^*)\backslash V(B)$, either w has no neighbours in $V(S)$, or is joined exactly to x_1. Observe that the choice of u_1 implies that every $w \in V(B - z)$ has at most one neighbour in $V(S)$. Therefore, for every $w \in V(B - z)$,

$$d_B(w) \geqslant p - 1,$$

and from Lemma 2.5, every two vertices of B are joined by a path of order at least p.

On the other hand, there must be some $u_2 \in V(B)$, distinct from u_1 and having a neighbour

$$x_2 \in V(S) \cap \Gamma(u_2), \quad x_2 \neq x_1,$$

otherwise $G - x_1$ is not connected, contradicting the fact that G is 2-connected. Select

$$u_2 \in V(B), \quad x_2 \in V(S) \cap \Gamma(u_2), \quad x_2 \neq x_1.$$

We know that there is a path $P_1(u_1, u_2)$ in B of order $l \geqslant p$; The Chopping Lemma implies that, for every interval I with

$$I \subset [l], \quad |I| = 2r_1,$$

there is a q-reduction of $P_1(u_1, z)$ for some $q \in I$.

On the other hand, from Lemma 2.12 there is some $l_2 > d$ such that

$$\left[l_2 - \left\lfloor \frac{d}{2} \right\rfloor + 5, l_2 \right] \subset R_S(x_1, x_2).$$

Exactly as in the previous case, we see that G contains a C_{p+1}, a contradiction, completing the proof. □

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

2.5. Proofs of the lemmas

Proof of Lemma 2.2. Assume first that G is disconnected; say G is the union of two disjoint nonempty graphs G_1 and G_2. Since both G_1 and G_2 are H-free and

$$\alpha(G) = \alpha(G_1) + \alpha(G_2),$$

we have

$$v(G) = v(G_1) + v(G_2) \leqslant \alpha(G_1)p + \alpha(G_2)p \leqslant pr,$$

a contradiction. Thus, G is connected.

Assume now that G is not 2-connected and let u be a cutvertex of G. Then $G - u$ is the union of two disjoint graphs G_1 and G_2. Since both G_1 and G_2 are H-free, and

$$\alpha(G_1) + \alpha(G_2) \leqslant \alpha(G),$$

we have

$$pr + 1 = v(G) = v(G_1) + v(G_2) + 1 \leqslant \alpha(G_1)p + \alpha(G_2)p + 1 \leqslant pr + 1.$$

Hence,

$$v(G_1) = \alpha(G_1)p, \quad v(G_2) = \alpha(G_2)p$$

and

$$\alpha(G) = \alpha(G_1) + \alpha(G_2).$$

By the condition of the lemma,

$$\alpha(G_1 + u) \geqslant \alpha(G_1) + 1, \quad \alpha(G_2 + u) \geqslant \alpha(G_2) + 1;$$

thus,

$$\alpha(G) \geqslant \alpha(G_1 + u) + \alpha(G_2 + u) - 1 \geqslant r + 1,$$

a contradiction. Hence, G is 2-connected. □

Proof of Lemma 2.4. Remove G_1, take a second copy of the remaining graph and identify the vertices u and v in both copies. The resulting graph satisfies the hypothesis of Theorem 2.3, and therefore, contains a uv-path of length at least $\delta + 1$. This path is contained entirely in G and consequently has no vertices in common with G_1. □

Proof of Lemma 2.5. If $u = x$ or $v = x$ the assertion follows from Theorem 2.3, so assume $u \neq x$ and $v \neq x$. Set $G^* = G - x$; clearly we have $\delta(G^*) \geqslant \delta - 1$. From the 2-connectivity of G it follows that G^* is connected.

Case 1: G^* is not 2-connected.

Let y be a vertex such that $G^* - y$ is a union of two disjoint nonempty graphs G_1 and G_2. Then, from Lemma 2.4, there exist two paths $P(x, y)$ and $Q(x, y)$ such that

$$P \cap V(G_1) = \varnothing, \quad Q \cap V(G_2) = \varnothing, \quad |P| \geqslant \delta + 1, \quad |Q| \geqslant \delta + 1.$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Clearly, the concatenation of P and Q is a cycle C of order at least 2δ. Since G is 2-connected, there are two vertices u_1 and v_1 of C and two disjoint paths $P_u(u,u_1)$ and $P_v(v,v_1)$, possibly of order 1. Select $Q(u_1,v_1)$ to be the path that u_1 and v_1 cut from C with $|Q|\geqslant |C|/2$; the concatenation (P_u,Q,P_v) is a uv-path of order at least $\delta+1$.

Case 2: G^* is 2-connected.
From a theorem of Dirac, since G^* is 2-connected and $\delta(H)\geqslant\delta-1$, there is a cycle of order at least $2\delta-2$ unless $v(G^*)<2\delta-2$. Consider first the latter case.

Case 2(a): $v(G^*)<2\delta-2$.
As Bondy proved (attributing the result to Erdős and Gallai) in [3], Corollary 2.13, the assumption $v(H)<2\delta-2$, together with $\delta(G^*)\geqslant\delta-1$, implies that G^* is Hamilton-connected, *i.e.*, every two vertices are connected by a Hamiltonian path. Hence, we obtain a uv-path of order at least $\delta+1$, unless G^* is a complete graph of order δ. In the latter case x must be adjacent to every vertex of G^*, as, otherwise, there is some $w\in V(G^*)$ with $d_G(w)=\delta-1$, a contradiction. Now we trivially obtain a uv-path of order $\delta+1$.

Case 2(b): $v(G^*)\geqslant 2\delta-2$.
Hence, G^* has a cycle C of order at least $2\delta-2$. From the 2-connectivity of G^* it follows that there are two disjoint paths $P_1(u,u_1)$ and $P_2(v_1,v)$, where $u_1,v_1\in C$. Select $Q(u_1,v_1)$ to be the path that u_1 and v_1 cut from C with $|Q|\geqslant|C|/2$; the concatenation

$$(P_1(u,u_1),Q(u_1,v_1),P_2(v_1,v))$$

is a uv-path of order at least $\delta+1$, unless the order of C is precisely $2\delta-2$, $u,v\in C$ and the distance along C between u and v is $\delta-1$. Let

$$P_1=(u,u_1,\dots,u_{\delta-2},v),\quad P_2=(v,v_1,\dots,v_{\delta-2},u)$$

be the two paths joining u and v along C, *i.e.*, $C=(P_1,P_2)$. Consider first the case of connected $G-u-v$.

Case 2(b(i)): $G-u-v$ is connected.
Hence, there is a path $P(u_i,v_j)$ joining, say, $u_i\in\{u_1,\dots,u_{\delta-2}\}$ to $v_j\in\{v_1,\dots,v_{\delta-2}\}$ and such that $P(u_1,v_1)$ has no internal vertices in common with C. Let $Q(v_j,u_i)$ be the path $P(u_i,v_j)$ taken in reverse order; we easily see that at least one of the paths

$$(u,u_1,\dots,u_{i-1},P(u_i,v_j),v_{j+1},,\dots,v_{\delta-2},v),$$
$$(v,v_1,\dots,v_{j-1},Q(v_j,u_i),u_{i+1},,\dots,u_{\delta-2},u)$$

has order at least $\delta+1$.

Case 2(b(ii)): $G-u-v$ is disconnected.
Let $G-u-v$ be the union of two disjoint nonempty graphs G_1 and G_2. Without loss of generality we may suppose $x\in G_2$. Since the graph $G^*=G-G_2$ is 2-connected and $d_{G^*}(w)\geqslant\delta$ for all $w\neq u,v$ then, from Theorem 2.3, there is a uv-path of order at least $\delta+1$ in G^* and the proof is completed. □

Proof of Lemma 2.6. If $l\leqslant 2\alpha$ the assertion is trivially true, so suppose $l\geqslant 2\alpha+1$. Observe that G has no induced path on $2\alpha+1$ vertices; otherwise, choosing every other

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

vertex along such a path, we obtain an independent set on $r+1$ vertices. Hence, the first $2\alpha+1$ vertices of P induce a chord and there is a q-reduction P_1 of P for some

$$q \in [l-2\alpha, l-1].$$

Setting $P_0 = P$ and repeating the same argument as long as possible, we obtain a sequence

$$P_0, P_1, \ldots, P_s$$

of reductions of P such that, for every $i = 0, \ldots, s-1$,

$$|P_i| \geqslant 2\alpha+1, \quad |P_i| - 2\alpha \leqslant |P_{i+1}| < |P_i|$$

and $|P_s| \leqslant 2\alpha$. Clearly, every interval I of length 2α in $[l]$ contains the order of some $|P_i|$ and the proof is completed. □

Proof of Lemma 2.7. Let $\{s_1, s_2, \ldots, s_t\} = R_{G_2}(x_2, y_2) \cap [l_1, l_2]$, and suppose

$$l_1 = s_1 < s_2 < \cdots < s_t = l_2.$$

From (iii) we see that $s_{i+1} - s_i \leqslant k \leqslant b - a + 1$. Combining a fixed path $P(x_2, y_2)$ of order s_i in G_2 with a path $Q(y_1, x_1)$ in G_1 of order q for every $q \in [a, b]$ we obtain a cycle of order s for every $s \in [s_i + a, s_i + b]$. Since for every $i \in [t-1]$ we have

$$s_{i+1} - s_i \leqslant b - a + 1,$$

the intervals

$$[s_i + a, s_i + b], \quad [s_{i+1} + a, s_{i+1} + b]$$

are contiguous or overlap, and the assertion follows. □

Proof of Lemma 2.8. Let $P = (v_1, \ldots, v_{2t+1})$ be the longest path in G such that the chord (v_{2s-1}, v_{2s+1}) exists for every $s = 1, \ldots, t$. The vertex v_{2t+1} is joined to at least $p-r$ of the vertices $v_1, \ldots, v_{2t}$; otherwise the set

$$N = \Gamma(v_{2t+1}) \backslash \{v_1, \ldots, v_{2t}\}$$

contains at least

$$d(v_{2t+1}) - (p-r) + 1 \geqslant r+1$$

vertices, and thus N induces an edge that extends P by two more vertices. By symmetry, v_{2t} is joined to at least $p-r$ of the vertices $v_1, \ldots, v_{2t-1}, v_{2t+1}$. Let i be the minimal index such that $v_i \in P$ is joined to either v_{2t} or v_{2t+1}; without loss of generality we may assume that v_i is joined to v_{2t+1}. If i is odd then the graph induced by

$$\{v_i, v_{i+1}, \ldots, v_{2t+1}\}$$

is a saw of degree at least $p-r$. If i is even then the graph induced by

$$\{v_i, v_{i-1}, v_{i+1}, \ldots, v_{2t+1}\}$$

is a saw of degree at least $p-r$. □

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Proof of Lemma 2.9. Observe that every 3-path $(v_{2s-1}, v_{2s}, v_{2s+1})$ along $P(v_i, v_j)$ can be replaced by the chord (v_{2s-1}, v_{2s+1}) shortening P by 1. Such a replacement can be done as many times as there are 3-paths $(v_{2s-1}, v_{2s}, v_{2s+1})$ along $P(v_i, v_j)$, so, all we have to do is to estimate their number.

(i) The edge (v_1, v_{2k+1}) does not belong to $P(v_i, v_j)$.
In this case we have $j - i = l - 1$ and

$$P(v_i, v_j) = (v_i, , v_{i+1}, \dots, v_{i+l-1}).$$

The number of the 3-paths $(v_{2s-1}, v_{2s}, v_{2s+1})$ along $P(v_i, v_j)$ is exactly the number of all s such that

$$i \leqslant 2s - 1 < i + l - 1,$$

and it is at least $\lfloor l/2 \rfloor - 1$. Hence the assertion follows.

(ii) The edge (v_1, v_{2k+1}) belongs to $P(v_i, v_j)$.
In this case we have $j - i = l - 2k - 2$, and

$$P(v_i, v_j) = (v_i, \dots, v_{2k+1}, v_1, \dots, v_{i+l-2k-2}).$$

The number of the 3-paths $(v_{2s-1}, v_{2s}, v_{2s+1})$ along the path $(v_i, \dots, v_{2k+1})$ is

$$\left\lfloor \frac{2k + 1 - i}{2} \right\rfloor,$$

and the number of the 3-paths $(v_{2s-1}, v_{2s}, v_{2s+1})$ along $(v_1, \dots, v_{i+l-2k-2})$ is

$$\left\lfloor \frac{i + l - 2k - 3}{2} \right\rfloor.$$

Thus, the number of the 3-paths $(v_{2s-1}, v_{2s}, v_{2s+1})$ along $P(v_i, v_j)$ is

$$\left\lfloor \frac{2k + 1 - i}{2} \right\rfloor + \left\lfloor \frac{i + l - 2k - 3}{2} \right\rfloor \geqslant \left\lfloor \frac{l}{2} \right\rfloor - 2$$

and the assertion follows. □

Proof of Lemma 2.10. From Lemma 2.9 we have

$$[k + 2, 2k + 1] \subset R_S(v_{2k}, v_{2k+1}),$$

so we need to prove only

$$[2, k + 2] \subset R_S(v_{2k}, v_{2k+1}).$$

In fact we shall prove the following more general assertion, implying the required result: *Let G be a Hamiltonian graph of order $n \geqslant 5$ and let $(v_1, \dots, v_n, v_1)$ be a Hamiltonian cycle in G. If*

$$d(v_1) + d(v_n) \geqslant (4n - 1)/3$$

then

$$[2, \lceil n/2 \rceil + 2] \subset R(v_1, v_n).$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Indeed, choose some $q \in [2, \lceil n/2 \rceil]$. Our first goal is to find two vertices

$$v_i, v_{i+q} \in \{v_2, \ldots, v_{n-1}\}$$

such that

$$e(\{v_i, v_{i+q}\}, \{v_1, v_n\}) \geqslant 3.$$

Assume this assertion is not true and consider first the case $q > (n-2)/3$. The pairs

$$(v_2, v_{q+2}), (v_3, v_{q+3}), \ldots, (v_{q+1}, v_{2q+1})$$

are disjoint and their union is the set $\{v_2, \ldots, v_{2q+1}\}$. Hence, we have

$$e(\{v_1, v_n\}, \{v_2, \ldots, v_{2q+1}\}) \leqslant 2q,$$

and thus

$$\begin{aligned} d(v_1) + d(v_n) &= e(\{v_1, v_n\}, \{v_2, \ldots, v_{2q+1}\}) + e(\{v_1, v_n\}, \{v_{2q+2}, \ldots, v_{n-1}\}) + 2 \\ &\leqslant 2q + 2(n-2-2q) + 2 = 2n - 2q - 2 \\ &< 2n - 2 - \frac{2(n-2)}{3} = \frac{4n-2}{3}, \end{aligned}$$

a contradiction.

Now let $q \leqslant (n-2)/3$ and suppose

$$n - 2 = qs + t, (0 \leqslant t \leqslant q-1).$$

It is not hard to find a set of $\lceil qs/2 \rceil$ disjoint pairs of vertices $\{v_i, v_{i+q}\}$ in $\{v_2, \ldots, v_{n-1}\}$. Since for every pair $\{v_i, v_{i+q}\}$ we have by assumption

$$e(\{v_i, v_{i+q}\}, \{v_1, v_n\}) \leqslant 2,$$

we find that

$$\begin{aligned} d(v_1) + d(v_n) &\leqslant 2\left\lceil \frac{qs}{2} \right\rceil + 2\left(n - 2 - 2\left\lceil \frac{qs}{2} \right\rceil\right) + 2 = 2n - 2 - 2\left\lceil \frac{qs}{2} \right\rceil \\ &\leqslant 2n - 2 - (qs - 1) = n + (t+1) \leqslant n + q \leqslant \frac{4n-2}{3}, \end{aligned}$$

a contradiction.

Therefore, there are two vertices $v_i, v_{i+q} \in \{v_2, \ldots, v_{n-1}\}$ such that

$$e(\{v_i, v_{i+q}\}, \{v_1, v_n\}) \geqslant 3.$$

Hence, either the edges $(v_1, v_i), (v_n, v_{i+q})$, or the edges $(v_1, v_{i+q}), (v_n, v_i)$ exist. So, either the path

$$(v_1, v_i, v_{i+1}, \ldots, v_{i+q}, v_n)$$

or the path

$$(v_1, v_{i+q}, v_{i+q-1}, \ldots, v_i, v_n)$$

exists, and we see that $q + 2 \in R(v_1, v_n)$. Hence,

$$[4, \lceil n/2 \rceil + 2] \subset R(v_1, v_n),$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

and since, obviously

$$2 \in R(v_1, v_n), \quad 3 \in R(v_1, v_n),$$

the proof is completed. □

Proof of Lemma 2.11. Set

$$t = 2k - d \tag{2.5}$$

and observe that the set

$$M = \{v_1, \dots, v_{2k}\} \setminus \Gamma(v_{2k+1})$$

has at most t members. We assume that $\{x, y\} \neq \{v_{2k}, v_{2k+1}\}$ since the case $\{x, y\} = \{v_{2k}, v_{2k+1}\}$ is covered by Lemma 2.10. Thus, up to labelling, there are only two different cases,

$$\{x, y\} \subset \{v_1, \dots, v_{2k}\},$$

and

$$x = v_1, \quad y = v_{2k+1}.$$

Case 1: $\{x, y\} \subset \{v_1, \dots, v_{2k}\}$.
Let $x = v_j$, $y = v_{j+1}$ and $PR(j)$ be the set of all pairs of vertices (v_i, v_l) such that

$$v_i, v_l \in \Gamma(v_{2k+1}), \quad 1 \leqslant i \leqslant j < l \leqslant 2k. \tag{2.6}$$

For every $(v_i, v_l) \in PR(j)$ the value $(l - i)$ is called its *span*. Observe that if $(v_i, v_l) \in PR(j)$ then the sequence

$$(v_{j+1}, \dots, v_l, v_{2k+1}, v_i, \dots, v_j)$$

is a $v_j v_{j+1}$-path of order $(l - i + 2)$ and this motivates the investigation of $PR(j)$ that follows.

Suppose $(v_h, v_m), (v_i, v_l) \in PR(j)$ are distinct; we write

$$(v_h, v_m) \succ (v_i, v_l)$$

if

$$i \geqslant h, \quad \text{and} \quad l \leqslant m.$$

We shall construct a sequence of $(v_{i_h}, v_{l_h}) \in PR(j)$ in the following way. Note first that v_{2k+1} is joined to both v_1 and v_{2k} and thus $PR(j) \neq \varnothing$. Set $i_1 = 1$, $l_1 = 2k$.

It turns out that if $(v_{i_h}, v_{l_h}) \in PR(j)$ has a large span then there exists $(v_{i_{h+1}}, v_{l_{h+1}}) \in PR(j)$ such that

$$(v_{i_h}, v_{l_h}) \succ (v_{i_{h+1}}, v_{l_{h+1}}),$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

and whose span is not much smaller than that of (v_{i_1}, v_{l_1}). Indeed, let $(v_{i_h}, v_{l_h}) \in PR(j)$ be with

$$l_h - i_h \geqslant 2t + 5. \tag{2.7}$$

The set of all pairs (v_i, v_l) such that

$$(v_i, v_l) \in PR(j), \quad (v_{i_h}, v_{l_h}) \succ (v_i, v_l) \tag{2.8}$$

is not empty; otherwise no vertex of $\{v_{i_h+1}, \ldots, v_{l_h-1}\}$ is joined to v_{2k+1} and hence

$$l_h - i_h + 1 \leqslant |M| \leqslant t,$$

a contradiction to (2.7). Choose a pair $(v_{i_{h+1}}, v_{l_{h+1}})$ satisfying (2.8) with maximal span; thus, no vertex of

$$\{v_{i_h+1}, \ldots, v_{i_{h+1}-1}\} \cup \{v_{l_{h+1}+1}, \ldots, v_{l_h-1}\}$$

is joined to v_{2k+1}. Hence, we find that

$$(l_h - i_h - 1) - (l_{h+1} - i_{h+1} + 1) \leqslant |M| \leqslant t,$$

and thus,

$$l_{h+1} - i_{h+1} \geqslant l_h - i_h - (t + 2).$$

Repeating the same argument we construct a sequence $(v_{i_h}, v_{l_h}) \in PR(j)$, $h = 1, \ldots, m$ such that, for every $h = 1, \ldots, m-1$,

$$\begin{aligned} &(v_{i_h}, v_{l_h}) \succ (v_{i_{h+1}}, v_{l_{h+1}}), \quad l_h - i_h \geqslant 2t + 5 \\ &l_h - i_h > l_{h+1} - i_{h+1} \geqslant l_h - i_h - (t + 2), \end{aligned} \tag{2.9}$$

and

$$l_m - i_m \leqslant 2t + 4. \tag{2.10}$$

Select some $h \in [m-1]$ and observe there are at least $\lceil (l_h - i_h)/2 \rceil - 1$ paths of the type $(v_{2s-1}, v_{2s}, v_{2s+1})$ along the path $P = (v_{i_h}, \ldots, v_{l_h})$. One of these paths contains the edge (v_j, v_{j+1}) and each of the remaining can be replaced independently by the chord joining its ends, thus shortening P by 1. In this way we see that, for every

$$q \in \left[\left\lfloor \frac{l_h - i_h}{2} \right\rfloor + 3, l_h - i_h + 1 \right],$$

there are q-reductions of P that contain the edge (v_j, v_{j+1}). Since the ends of P, and thus the ends of each of its reductions, are joined to v_{2k+1}, it follows that

$$\left[\left\lfloor \frac{l_h - i_h}{2} \right\rfloor + 4, l_h - i_h + 2 \right] \subset L_S(v_j, v_{j+1}).$$

We shall show that the shortest of these reductions of P has order at most $l_{h+1} - i_{h+1} + 2$. Indeed, assume

$$\left\lfloor \frac{l_h - i_h}{2} \right\rfloor + 3 \geqslant l_{h+1} - i_{h+1} + 2.$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Hence, from (2.9), we see that

$$\left\lfloor \frac{l_h - i_h}{2} \right\rfloor + 3 \geqslant l_{h+1} - i_{h+1} + 3 \geqslant l_h - i_h - t + 1,$$

and after simple calculations we obtain $2t + 4 \geqslant l_h - i_h$, a contradiction to (2.7).

Therefore, for $h = 1, \ldots, m-1$, the intervals

$$\left[\left\lfloor \frac{l_h - i_h}{2} \right\rfloor + 4, l_h - i_h + 2\right], \quad \left[\left\lfloor \frac{l_{h+1} - i_{h+1}}{2} \right\rfloor + 4, l_{h+1} - i_{h+1} + 2\right]$$

are contiguous or overlap, and thus their union is also an interval. From (2.5) and (2.10) we obtain

$$[2k - d + 6, 2k + 1] \subset R_S(v_j, v_{j+1}),$$

as required.

Case 2: $x = v_1$, $y = v_{2k+1}$.

Observe that in the proof of the previous case we have shown that, for every

$$\{v_j, v_{j+1}\} \subset \{v_1, \ldots, v_{2k}\},$$

and for every

$$q \in [2k - d + 6, 2k + 1], \tag{2.11}$$

there is a cycle of order q of the form

$$(v_{2k+1}, v_i, \ldots, v_j, v_{j+1}, \ldots, v_{i+q-2}, v_{2k+1}).$$

Applying this assertion to $(v_j, v_{j+1}) = (v_1, v_2)$ we see that for every q satisfying (2.11), there is a cycle of order q of the form

$$(v_{2k+1}, v_1, v_2, \ldots, v_{q-1}, v_{2k+1})$$

and therefore,

$$[2k - d + 6, 2k + 1] \subset R_S(v_1, v_{2k+1}). \qquad \square$$

Proof of Lemma 2.12. Denote by C the cycle $(v_1, \ldots, v_{2k+1}, v_1)$. Set $t = 2k - d$ and observe that from $d_S(v_{2k}) \geqslant d$ and $d_S(v_{2k+1}) \geqslant d$, we have

$$|\{v_1, \ldots, v_{2k}\} \setminus \Gamma(v_{2k+1})| \leqslant t, \tag{2.12}$$

$$|\{v_1, \ldots, v_{2k-1}, v_{2k+1}\} \setminus \Gamma(v_{2k})| \leqslant t. \tag{2.13}$$

Suppose that the distance between x and y along C is at most $t + 2$ and let $P(x, y)$ be the longer xy-path along C. Clearly,

$$|P(x, y)| \geqslant 2k + 1 - t = d + 1.$$

Hence, setting $l = |P(x, y)|$ and applying Lemma 2.9(i) we find that

$$\left[\left\lfloor \frac{l}{2} \right\rfloor + 2, l\right] \subset R_S(x, y),$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

and since for $l \geqslant d+2$ we have

$$\left\lfloor \frac{l}{2} \right\rfloor + 1 \leqslant l - \left\lceil \frac{d}{2} \right\rceil,$$

the assertion is proved in this case. So we shall hereafter assume that the distance between x and y along the cycle $(v_1, \ldots, v_{2k+1}, v_1)$ is at least $t+2$.

Case 1: $\{x, y\} \subset \{v_1, \ldots, v_{2k-1}\}$.
Let $x = v_i$, $y = v_j$; without loss of generality we assume $i < j$; hence,

$$j - i \geqslant t + 2.$$

Our first goal is to show that there exist two vertices v_p, v_q such that

$$i < p, \quad p + (j - i - t - 2) \leqslant q < j, \tag{2.14}$$

and either the edges (v_{2k}, v_p), (v_{2k+1}, v_q) or the edges (v_{2k}, v_q), (v_{2k+1}, v_p) exist. Indeed, observe that the set $\{v_{i+1}, \ldots, v_{j-1}\}$ has at least $t+1$ members; therefore,

$$\{v_{i+1}, \ldots, v_{j-1}\} \cap \Gamma(v_{2k}) \neq \varnothing,$$

and

$$\{v_{i+1}, \ldots, v_{j-1}\} \cap \Gamma(v_{2k+1}) \neq \varnothing.$$

Among the vertices

$$\{v_{i+1}, \ldots, v_{j-1}\} \cap (\Gamma(v_{2k}) \cup \Gamma(v_{2k+1}))$$

let v_p be the one with minimal index; assume without loss of generality that $v_p \in \Gamma(v_{2k+1})$. Among the vertices

$$\{v_{i+1}, \ldots, v_{j-1}\} \cap \Gamma(v_{2k}),$$

let v_q be the one having the maximal index. By our choice the edges (v_{2k}, v_q), (v_{2k+1}, v_p) exist. Clearly

$$(\{v_{i+1}, \ldots, v_{p-1}\} \cup \{v_{q+1}, \ldots, v_{j-1}\}) \cap \Gamma(v_{2k}) = \varnothing,$$

implying (2.14).

Now consider the paths

$$P_1 = (v_i, v_{i-1}, \ldots, v_1, v_{2k+1}),$$
$$P_2 = (v_p, v_{p+1}, \ldots, v_q),$$
$$P_3 = (v_{2k}, v_{2k-1}, \ldots, v_j),$$

and set $l_i = |P_i|$, $i = 1, 2, 3$. The concatenation $Q = (P_1, P_2, P_3)$ is a $v_i v_j$-path with

$$|Q| = l_1 + l_2 + l_3 + 2 = 2k + 1 - (j - i) + (q - p) + 2.$$

Set $l = |Q|$; from (2.14) we obtain

$$l \geqslant 2k + 1 - (j - i) + (q - p) + 2 \geqslant 2k + 1 - t = d + 1.$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

Applying Lemma 2.9(i) to each of the paths P_1, P_2, P_3, we see that Q has a q-reduction for every

$$q \in \left[l - \left\lfloor \frac{l_1}{2} \right\rfloor - \left\lfloor \frac{l_2}{2} \right\rfloor - \left\lfloor \frac{l_3}{2} \right\rfloor + 3, l \right].$$

In view of

$$\left\lfloor \frac{l_1}{2} \right\rfloor + \left\lfloor \frac{l_2}{2} \right\rfloor + \left\lfloor \frac{l_3}{2} \right\rfloor \geqslant \left\lfloor \frac{l_1 + l_2 + l_3}{2} \right\rfloor - 1 = \left\lfloor \frac{l}{2} \right\rfloor - 2 \geqslant \left\lceil \frac{d}{2} \right\rceil - 2,$$

the assertion follows.

Case 2: $x = v_{2k+1}$, $y \in \{v_1, \dots, v_{2k-1}\}$.
Let $y = v_j$; since the xy-distance along C is at least $t+2$, we have $j \geqslant t+2$, and thus

$$M = \{v_1, \dots, v_{j-1}\} \cap \Gamma(v_{2k}) \neq \varnothing,$$

Among the vertices of M, let v_q be the one having the maximal index. Clearly,

$$\{v_{q+1}, \dots, v_{j-1}\} \cap \Gamma(v_{2k}) = \varnothing,$$

and thus

$$j - q - 1 \leqslant t. \tag{2.15}$$

Now consider the paths

$$P_1 = (v_{2k+1}, v_1, \dots, v_q),$$
$$P_2 = (v_{2k}, v_{2k-1}, \dots, v_j),$$

and set $l_i = |P_i|$, $i = 1, 2$. The concatenation $Q = (P_1, P_2)$ is a $v_{2k+1}v_j$-path with

$$|Q| = l_1 + l_2 + 1 = 2k + 1 - j + q + 1.$$

Set $l = |Q|$; from (2.15) we obtain

$$l \geqslant 2k + 1 - j + q + 1 \geqslant 2k + 1 - t = d + 1.$$

Applying Lemma 2.9(i) to each of the paths P_1, P_2, we see that Q has a q-reduction for every

$$q \in \left[l - \left\lfloor \frac{l_1}{2} \right\rfloor - \left\lfloor \frac{l_2}{2} \right\rfloor + 2, l \right].$$

In view of

$$\left\lfloor \frac{l_1}{2} \right\rfloor + \left\lfloor \frac{l_2}{2} \right\rfloor \geqslant \left\lfloor \frac{l_1 + l_2}{2} \right\rfloor - 1 = \left\lfloor \frac{l-1}{2} \right\rfloor - 1 \geqslant \left\lceil \frac{d}{2} \right\rceil - 2,$$

the assertion follows.

Case 3: $y = v_{2k}$, $x \in \{v_1, \dots, v_{2k-1}\}$.
This case is symmetric to the previous. Setting $x = v_j$, we find a vertex

$$v_q \in \{v_{j+1}, \dots, v_{2k}\} \cap \Gamma(v_{2k+1}),$$

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

then consider the paths

$$P_1 = (v_{2k}, v_{2k-1}, \dots, v_q),$$
$$P_2 = (v_{2k+1}, v_1, v_2, \dots, v_j),$$

and find xy-paths of proper order among the reductions of the concatenation (P_1, P_2). □

Proof of Lemma 2.13. Applying Lemma 2.9(i) to the path

$$P = (v_1, \dots, v_{2k+1}),$$

we see that S contains cycles of order q for every $s \in [k+1, 2k+1]$ and the proof is completed under the assumption $k+1 \leqslant 4r$.

Assume now that $k \geqslant 4r$; let

$$S_1 = \{v_1, \dots, v_{4r-1}\}, \quad S_2 = \{v_{4r}, \dots, v_{2k+1}\}.$$

We have shown that for every $q \in [2r, 4r-1]$ there is a q-reduction of the path $(v_1, \dots, v_{4r-1})$. On the other hand the order of the path

$$P = (v_{4r}, \dots, v_{2k+1})$$

is $(2k - 4r + 2)$, and applying the Chopping Lemma, we see that for every interval

$$I \subset [2k - 2r + 1], \quad |I| = 2r,$$

there is a q-reduction of P for some $q \in I$. Applying the Collating Lemma to the graph S with the partition

$$V(S) = S_1 \cup S_2$$

and the edges (v_1, v_{2k+1}) and (v_{4r+1}, v_{4r+2}), it follows that S contains cycles of order q for every

$$q \in [4r, 2k+1].$$ □

2.6. Concluding remarks and open problems

There is a much simpler proof of (1.1) under the assumption $p \geqslant 8r + 7$. Actually, except for Lemma 2.13, our methods are good enough to prove (1.1) for $p \geqslant 3r + 9$, and it seems that with some additional refinement it is possible to prove (1.1) for

$$p \geqslant 2r + o(r).$$

The following conjecture, however, looks more challenging.

Conjecture 2.14. *For every k there exists $r_0 = r_0(k)$ such that, for $r > r_0$ and $p > r^{1/k}$,*

$$r(C_p, K_r) = (p-1)(r-1) + 1.$$

There are known Ramsey numbers $r(C_p, K_r)$ for $p < r$: Jayawardene and Rousseau found that $r(C_4, K_6) = 18$ in [9] and $r(C_5, K_6) = 21$ in [10]; Schiermeyer found that

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press

$r(C_5, K_7) = 25$ in [13]. These values, although very few, give some hope that the conjecture might be true.

Acknowledgement

The author is grateful to Cecil Rousseau and to Dick Schelp for the many delightful hours spent discussing Ramsey problems and the above problem in particular. The help and advice of Cecil Rousseau were in every respect invaluable. Finally, Béla Bollobás suggested many corrections and improvements to the manuscript.

References

[1] Bollobás, B., Jayawardene, C., Yang, J., Huang, Y., Rousseau, C. C. and Zhang, K. (2000) On a conjecture involving cycle-complete graph Ramsey numbers. *Australas. J. Combin.* **22** 63–71.
[2] Bollobás, B. (1998) *Modern Graph Theory*, Vol. 184 of *Graduate Texts in Mathematics*, Springer, New York.
[3] Bondy, J. A. (1995) Basic graph theory: Paths and circuits. *Handbook of Combinatorics*, Vol. **1**, Elsevier, Amsterdam, pp. 3–110.
[4] Burr, S., Erdős, P., Faudree, R. J., Rousseau, C. C. and Schelp, R. H. (1980) An extremal problem in generalized Ramsey theory. *Ars Combinatoria* **10** 193–203.
[5] Bondy, J. A. and Erdős, P. (1973) Ramsey numbers for cycles in graphs. *J. Combin. Theory Ser. B* **14** 46–54.
[6] Dirac, G. A. (1952) Some theorems on abstract graphs. *Proc. London Math. Soc.* **2** 69–81.
[7] Erdős, P. and Gallai, T. (1959) On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.* **10** 337–356.
[8] Erdős, P., Faudree, R. J., Rousseau, C. C. and Schelp, R. H. (1978) On cycle-complete graph Ramsey numbers. *J. Graph Theory* **2** 53–64.
[9] Jayawardene, C. and Rousseau, C. C. (1997) The Ramsey number for a quadrilateral vs. a complete graph on six vertices. *Congr. Numer.* **123** 97–108.
[10] Jayawardene, C. and Rousseau, C. C. (2000) The Ramsey number for a cycle of length five vs. a complete graph of order six. *J. Graph Theory* **35** 99–108.
[11] Rousseau, C. C. Asymptotic bounds for Ramsey numbers. Preprint.
[12] Schiermeyer, I. (2003) All cycle-complete graph Ramsey numbers $R(C_m, K_6)$. *J. Graph Theory* **44** 251–260.
[13] Schiermeyer, I. (2003) The cycle-complete graph Ramsey numbers $R(C_5, K_7)$. Preprint.
[14] Yang, J., Huang, Y. and Zhang, K. (1999) The value of the Ramsey number $R(C_n, K_4)$ is $3(n-1)+1$ $(n \geqslant 4)$. *Australas. J. Combin.* **20** 205–206.

https://doi.org/10.1017/S096354830400642X Published online by Cambridge University Press