Canad. Math. Bull. Vol. 65 (1), 2022, pp. 30–38
http://dx.doi.org/10.4153/S0008439521000047
© Canadian Mathematical Society 2021

On two conjectures about the sum of element orders

Morteza Baniasad Azad and Behrooz Khosravi

Abstract. Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. There are many results on the influence of this function on the structure of a finite group G.

In this paper, as the main result, we answer a conjecture of Tărnăuceanu. In fact, we prove that if G is a group of order n and $\psi(G) > 31\psi(C_n)/77$, where C_n is the cyclic group of order n, then G is supersolvable. Also, we prove that if G is not a supersolvable group of order n and $\psi(G) = 31\psi(C_n)/77$, then $G \cong A_4 \times C_m$, where $(m, 6) = 1$.

Finally, Herzog et al. in (2018, *J. Algebra*, 511, 215–226) posed the following conjecture: If $H \leq G$, then $\psi(G) \leqslant \psi(H)|G : H|^2$. By an example, we show that this conjecture is not satisfied in general.

1 Introduction

In this paper, all groups are finite. The cyclic group of order n is denoted by C_n. Let $\psi(G) = \sum_{g\in G} o(g)$, the sum of element orders in a group G. The function $\psi(G)$ was introduced by Amiri, Jafarian Amiri, and Isaacs in [2]. Many authors try to get some relations between the structure of the group G and $\psi(G)$ (see [5, 6, 8, 11–14, 16, 18]). It is proved that if $\psi(G) > \frac{211}{1617}\psi(C_n)$, where G is a finite group of order n, then G is solvable [4, 10].

From the observation that A_4 satisfies $\psi(A_4) = 31$ and $\psi(C_{12}) = 77$, in [19], M. Tărnăuceanu posed the following conjecture:

Conjecture 1.1 [19, Conjecture 1.5] *If G is a group of order n and $\psi(G) > \frac{31}{77}\psi(C_n)$, then G is supersolvable.*

In this paper, we prove the validity of this conjecture. We also proved that if G is a nonsupersolvable group of order n and $\psi(G) = \frac{31}{77}\psi(C_n)$, then $G \cong A_4 \times C_m$, where $(6, m) = 1$. Nevertheless, for groups of odd order, it is possible to prove a stronger result:

Theorem 1.2 *Suppose that $n = |G|$ is odd. If $\psi(G) > \frac{271}{3647}\psi(C_n)$, then G is supersolvable.*

Also, in [10], three conjectures about $\psi(G)$ are stated. The validity of the first one (Conjecture 5) was proved in [4], and the validity of the second one (Conjecture 6) was proved in [3]. Moreover, the following conjecture was posed in that paper.

Received by the editors October 29, 2020; revised January 12, 2021, accepted January 14, 2021.
Published online on Cambridge Core January 26, 2021.
AMS subject classification: 20D60, 20F16.
Keywords: Sum of element orders, supersolvable group, element orders.

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

Conjecture 1.3 [10, Conjecture 7] *If $H \leq G$, then $\psi(G) \leqslant \psi(H)|G:H|^2$.*

In this paper, we give an example, which shows that this conjecture is not true in general, and we need extra conditions on G and H to have the result of Conjecture 1.3.

We note that there are similar functions related to the orders of elements of a finite group (for example, see [7, 15]). For the proof of our results, we need the following lemmas.

Lemma 1.4 [2, Corollary B] *Let $P \in Syl_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then, $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 1.5 [10, Proposition 2.6] *Let H be a normal subgroup of the finite group G. Then, $\psi(G) \leq \psi(G/H)|H|^2$.*

Lemma 1.6 [1, Lemma 2.1] *If G and H are finite groups, then $\psi(G \times H) \leq \psi(G)\psi(H)$. Also, $\psi(G \times H) = \psi(G)\psi(H)$ if and only if $\gcd(|G|, |H|) = 1$.*

Lemma 1.7 [9, Proposition 2.5] *Let G be a finite group and suppose that there exists $x \in G$ such that $|G : \langle x\rangle| < 2p$, where p is the maximal prime divisor of $|G|$. Then, one of the following holds:*

(i) *G has a normal cyclic Sylow p-subgroup,*
(ii) *G is solvable and $\langle x\rangle$ is a maximal subgroup of G of index either p or $p+1$.*

Lemma 1.8 [17] *Let A be a cyclic proper subgroup of a finite group G, and let $K = core_G(A)$. Then, $|A : K| < |G : A|$, and, in particular, if $|A| > |G : A|$, then $K > 1$.*

Lemma 1.9 [4, Lemma 2.1] *Let G be a group of order $n = {p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes. Let $\psi(G) > \frac{r}{s}\psi(C_n)$, for some integers r, s. Then, there exists a cyclic subgroup $\langle x\rangle$ such that*

$$[G : \langle x\rangle] < \frac{s}{r} \cdot \frac{p_1 + 1}{p_1} \cdots \frac{p_k + 1}{p_k}.$$

Lemma 1.10 [10, Lemmas 2.4 and 2.5] *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_r}^{\alpha_r}$ be a positive integer, where p_i are primes, $p_1 < p_2 < \cdots < p_r = p$ and $\alpha_i > 0$, for each $1 \leq i \leq r$. If $p \geq 11$, then*

$$\psi(C_n) \geq \frac{385}{96} \frac{n^2}{p+1}.$$

2 Supersolvability

If G is a finite group, then $\pi(G)$ is the set of all prime divisors of $|G|$.

2.1 Proof of Conjecture 1.1

Let G be a group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_r}^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are primes, $p_1 < p_2 < \cdots < p_r = p$, such that $\alpha_i > 0$, for each $1 \leq i \leq r$. By assumption, $\frac{31}{77}\psi(C_n) < \psi(G)$. By induction on $|\pi(G)|$, we prove that G is a supersolvable group.

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

If $|\pi(G)| = 1$, then G is a p-group and so G is supersolvable. Assume that $|\pi(G)| \geq 2$ and the theorem holds for each group H such that $|\pi(H)| < |\pi(G)|$. Now, we consider the following two cases:

Case (I). If G has a normal cyclic Sylow subgroup Q, then by Lemma 1.4 we have $\psi(G) \leq \psi(Q)\psi(G/Q)$. Using Lemma 1.6 and the assumptions, we have

$$\frac{31}{77}\psi(C_{|G/Q|})\psi(C_{|Q|}) = \frac{31}{77}\psi(C_n) < \psi(G) \leq \psi(Q)\psi(G/Q) = \psi(C_{|Q|})\psi(G/Q).$$

Therefore, $\frac{31}{77}\psi(C_{|G/Q|}) < \psi(G/Q)$ and $|\pi(G/Q)| < |\pi(G)|$. By the inductive hypothesis, G/Q is supersolvable and so G is a supersolvable group.

Case (II). Let G have no normal cyclic Sylow subgroup.

If $p \geq 11$, then by Lemma 1.10 we have

$$\psi(G) > \frac{31}{77}\psi(C_n) \geq \frac{31}{77} \cdot \frac{385}{96}\frac{n^2}{p+1}.$$

Thus, there exists $x \in G$ such that $o(x) > \frac{31}{77} \cdot \frac{385}{96}\frac{n}{p+1}$. Therefore,

$$[G : \langle x\rangle] < \frac{77}{31} \cdot \frac{96}{385}(p+1) \leq \frac{77}{31} \cdot \frac{96}{385} \cdot \frac{12}{11}p < 0.68p < p,$$

which is a contradiction, by Lemma 1.7. Therefore, $\pi(G) \subseteq \{2, 3, 5, 7\}$, where $2 \leq |\pi(G)| \leq 4$. Now, we consider the following cases:

Case 1. Let $\pi(G) = \{2, 3\}$. Then, $|G| = 2^{\alpha_1}3^{\alpha_2}$. In this case, we have

$$\psi(G) > \frac{31}{77}\psi(C_{|G|}) > \frac{31}{77} \cdot \frac{2^{2\alpha_1+1}}{2+1} \cdot \frac{3^{2\alpha_2+1}}{3+1} = \frac{31}{77} \cdot \frac{1}{2}n^2. \tag{2.1}$$

It follows that there exists $x \in G$ such that $o(x) > \frac{31}{77} \cdot \frac{1}{2}n$. We conclude that $|G : \langle x\rangle| < \frac{77}{31} \cdot 2 < 4.97$. By Lemma 1.7, we have $[G : \langle x\rangle] = 3$ or 4. Let $H = \mathrm{core}_G(\langle x\rangle)$.

- Let $[G : \langle x\rangle] = 3$. By Lemma 1.8, $[\langle x\rangle : H] < [G : \langle x\rangle] = 3$. Therefore, G/H is supersolvable and so G is a supersolvable group.
- Let $[G : \langle x\rangle] = 4$. By Lemma 1.8, $[\langle x\rangle : H] < [G : \langle x\rangle] = 4$. If G/H is a supersolvable group, then we get the result. Let G/H be nonsupersolvable. Therefore, $[\langle x\rangle : H] = 3$, $|G/H| = 12$, and so $G/H \cong A_4$.
  - If 2 divides $|H|$, then there exists a characteristic subgroup M in H such that $|H : M| = 2$. Thus, G/M is a nonsupersolvable group of order 24. Therefore, using the list of such groups $(\mathrm{SL}(2, 3), S_4, C_2 \times A_4)$ and their ψ-values $(99, 67, 87)$, we have $\psi(G/M) \leq 99$. By Lemma 1.5, we have $\psi(G) \leq \psi(G/M)|M|^2 \leq 99(n/24)^2$. Using (2.1), $\psi(G) > \frac{31}{77} \cdot \frac{1}{2}n^2$. Therefore,

$$\frac{31}{77} \cdot \frac{1}{2}n^2 < 99\frac{n^2}{24^2},$$

    which is a contradiction.
  - If $2 \nmid |H|$, then $|G| = 2^2 3^\beta$. Let $\beta \geq 2$. Then, there exists a characteristic subgroup M in H such that $|H : M| = 3$. Therefore, $M \trianglelefteq G$ and $|G/M| = 36$. Thus, G/M is a nonsupersolvable group of order 36. Therefore, using the list of such groups $((C_2 \times C_2) : C_9, (C_3 \times C_3) : C_4, C_3 \times A_4)$ and their ψ-values $(265, 115, 121)$,

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

we have $\psi(G/M) \leq 265$. By Lemma 1.5, we have $\psi(G) \leq \psi(G/M)|M|^2 \leq 265(n/36)^2 = 265 \cdot 3^{2\beta-4}$. On the other hand, we have $\psi(G) > \frac{31}{77}\psi(C_n)$. Thus,

$$\psi(G) > \frac{31}{77}\psi(C_4)\psi(C_{3^\beta}) = \frac{31}{77} \cdot 11 \cdot \frac{3^{2\beta+1}+1}{4}.$$

Therefore,

$$\frac{31}{77} \cdot 11 \cdot \frac{3^{2\beta+1}+1}{4} < 265 \cdot 3^{2\beta-4} \Rightarrow \frac{31}{28} \cdot 3^{2\beta+1} < 265 \cdot 3^{2\beta-4}$$
$$\Rightarrow 31 \cdot 3^5 < 28 \cdot 265,$$

which is a contradiction. Thus, $\beta = 1$ and so $|H| = 1$ and $G \cong A_4$, which is a contradiction, because $\psi(A_4) = 31$.

Case 2. Let $\pi(G) = \{2, 3, 5\}$. Then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x \rangle] < \frac{77}{31} \cdot \frac{12}{5} < 5.97$. By Lemma 1.7, we have $[G : \langle x \rangle] = 5$. By Lemma 1.8, $[\langle x \rangle : \mathrm{core}_G(\langle x \rangle)] < [G : \langle x \rangle] = 5$. We have $|G/\mathrm{core}_G(\langle x \rangle)| = 5, 10, 15,$ or 20. Therefore, $G/\mathrm{core}_G(\langle x \rangle)$ is supersolvable and so G is a supersolvable group.

Case 3. Let $\pi(G) = \{2, 3, 5, 7\}$. Then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x \rangle] < \frac{77}{31} \cdot \frac{96}{35} < 6.9 < 7$, which is impossible.

Case 4. Let $\pi(G) = \{2, 5\}$ or $\pi(G) = \{3, 5\}$. Then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x \rangle] < 5$, which is a contradiction.

Case 5. Let $\pi(G) = \{2, 7\}$, $\pi(G) = \{3, 7\}$, $\pi(G) = \{5, 7\}$, $\pi(G) = \{2, 3, 7\}$, or $\pi(G) = \{2, 5, 7\}$. Then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x \rangle] < 7$ and we get a contradiction.

The proof is now complete.

2.2 Equality condition of Conjecture 1.1

Lemma 2.1 *Let G be a nonsupersolvable group of order n and p is a prime divisor of n. Furthermore, suppose that $\psi(G) = \frac{31}{77}\psi(C_n)$ and $P \in Syl_p(G)$ is cyclic and normal in G. Then, P has a normal p-complement K in G, where $\psi(K) = \frac{31}{77}\psi(C_{|K|})$.*

Proof By Lemma 1.4, we have

$$\psi(P)\psi(G/P) \geq \psi(G) = \frac{31}{77}\psi(C_n) = \frac{31}{77}\psi(C_{|P|})\psi(C_{|G/P|}). \tag{2.2}$$

Therefore,

$$\psi(G/P) \geq \frac{31}{77}\psi(C_{|G/P|}).$$

Now, by the nonsupersolvability of G and the above result, we conclude that

$$\psi(G/P) = \frac{31}{77}\psi(C_{|G/P|}).$$

It follows that equality holds in (2.2). Thus, $P \leq Z(G)$, by Lemma 1.4. Therefore, by Burnside's normal p-complement theorem, there exists $K \trianglelefteq G$ such that $G = P \times K$. ∎

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

Theorem 2.2 *Suppose that G is a nonsupersolvable group of order n such that $\psi(G) = \frac{31}{77}\psi(C_n)$. Then, $G = A_4 \times C_m$, where $(6, m) = 1$.*

Proof Let p be the largest prime divisor of n. We prove the result by induction on p. By the assumption, $|\pi(G)| \geq 2$ and so $p \geq 3$.

(1) Let $p = 3$.
We note that G has no normal cyclic Sylow subgroup. There exists $x \in G$ such that $|G : \langle x\rangle| < \frac{77}{31} \cdot 2 < 4.97$. By Lemma 1.7, we have $[G : \langle x\rangle] = 3$ or 4. If $[G : \langle x\rangle] = 3$, then G is a supersolvable group, which is a contradiction. Therefore, $[G : \langle x\rangle] = 4$. We have $G/\mathrm{core}_G(\langle x\rangle) \cong A_4$, because G is nonsupersolvable. If $\mathrm{core}_G(\langle x\rangle) \neq 1$, then, similarly to Case 1 in the proof of Conjecture 1.1, we get a contradiction. Therefore, $\mathrm{core}_G(\langle x\rangle) = 1$ and so $G \cong A_4$.

(2) Let $p = 5$.
(a) Let $\pi(G) = \{2, 3, 5\}$.
Then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x\rangle] < \frac{77}{31} \cdot \frac{12}{5} < 5.97 < 10$. By Lemma 1.7, G has a normal cyclic Sylow 5-subgroup or $[G : \langle x\rangle] = 5$.
- Let G have a normal cyclic Sylow 5-subgroup, say P_5. Using Lemma 2.1, there exists $K \trianglelefteq G$ such that $G = P_5 \times K$ and $\psi(K) = \frac{31}{77}\psi(C_{|K|})$. We note that K is nonsupersolvable and $\pi(K) = \{2, 3\}$. By the above case, we have $K = A_4$ and $G = A_4 \times P_5$.
- Let $[G : \langle x\rangle] = 5$. Then, $|G/\mathrm{core}_G(\langle x\rangle)| = 5, 10, 15,$ or 20. Therefore, $G/\mathrm{core}_G(\langle x\rangle)$ is supersolvable and so G is a supersolvable group, which is a contradiction.

(b) Let $\pi(G) = \{2, 5\}$ or $\pi(G) = \{3, 5\}$. Then, there exists $x \in G$ such that $[G : \langle x\rangle] < 5$. By Lemma 1.7, G has a normal cyclic Sylow 5-subgroup. Thus, G is a supersolvable group, which is a contradiction.

(3) Let $p = 7$.
Then, there exists $x \in G$ such that $[G : \langle x\rangle] < 7$. By Lemma 1.7, G has a normal cyclic Sylow 7-subgroup. Using Lemma 2.1, there exists $K \trianglelefteq G$ such that $G = P_7 \times K$ and $\psi(K) = \frac{31}{77}\psi(C_{|K|})$. We note that K is nonsupersolvable and $\pi(K) = \{2, 3\}$ or $\{2, 3, 5\}$. Therefore, by the above cases, we have $G \cong P_7 \times A_4$ or $P_7 \times P_5 \times A_4$, respectively, and we get the result.

(4) Let $p \geq 11$. Then, there exists $x \in G$ such that

$$[G : \langle x\rangle] < \frac{77}{31} \cdot \frac{96}{385}(p+1) \leq \frac{77}{31} \cdot \frac{96}{385} \cdot \frac{12}{11}p < 0.68p < p.$$

By Lemma 1.7, G has a normal cyclic Sylow p-subgroup, say P. Using Lemma 2.1, there exists $K \trianglelefteq G$ such that $G = P \times K$ and $\psi(K) = \frac{31}{77}\psi(C_{|K|})$. Then, K is nonsupersolvable and $\pi(K) = \pi(G) \setminus \{p\}$. Therefore, using the above discussion and the induction hypothesis, we get the result. ∎

2.3 Supersolvability for groups of odd order

Notation 2.3 *Let $\{q_1, q_2, q_3, \ldots\}$ be the set of* ***all*** *primes in increasing order: $2 = q_1 < q_2 < q_3 < \cdots$. Let also $q_0 = 1$. If r and $s \geq 2$ are two positive integers, we define the*

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

functions $f'(r)$ and $h'(s)$ as follows:

$$f'(1) = 1,\ f'(r) = \prod_{i=2}^{r} \frac{q_i}{q_i + 1},\ \textit{where } r \geq 2;$$
$$h'(2) = 3,\ h'(s) = f'(s-1)q_s,\ \textit{where } s \geq 3.$$

In the sequel, let $A := \mathrm{SmallGroup}(75, 2) = (C_5 \times C_5) : C_3$. We know that $\psi(A) = 271$ and $\psi(C_{75}) = 3647$. Similarly to the proofs of Lemmas 2.4 and 2.5 in [10] we get that:

Lemma 2.4 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_r}^{\alpha_r}$ be a positive integer, where $p_1, p_2, \ldots, p_r$ are primes, $2 < p_1 < p_2 < \cdots < p_r = p$ and $\alpha_i > 0$, for each $1 \leq i \leq r$. If $p \geq 37 = q_{12}$, then*

$$\psi(C_n) \geq h'(12)\frac{n^2}{p+1}.$$

The proof of Theorem 1.2 is very similar to the proof of Conjecture 1.1, and so we omit the details of the proof.

Proof of Theorem 1.2 We prove the result by induction on $|\pi(G)|$. The assertion is trivial if G is a p-group, and suppose that the result holds for each group H, where $|\pi(H)| < |\pi(G)|$ (exactly similar to the proof of Conjecture 1.1).

We consider the following two cases:

Case(I). If G has a normal cyclic Sylow subgroup Q, then we get the result.

Case (II). Let G have no normal cyclic Sylow subgroup.

Let p be the largest prime divisor of $|G|$. If $p \geq 37$, then by Lemma 2.4 we have

$$\psi(G) > \frac{271}{3647}\psi(C_n) \geq \frac{271}{3647} \cdot h'(12)\frac{n^2}{p+1}.$$

Thus, there exists $x \in G$ such that $o(x) > \frac{271}{3647} \cdot h'(12)\frac{n}{p+1}$. Therefore,

$$[G : \langle x\rangle] < \frac{3647}{271} \cdot \frac{1}{h'(12)}(p+1) \leq \frac{3647}{271} \cdot \frac{1}{h'(12)} \cdot \frac{38}{37}p < p,$$

which is a contradiction.

Therefore, $\pi(G) \subseteq \{3, 5, 7, \ldots, 31\}$ and so $2 \leq |\pi(G)| \leq 10$. Now, we consider the following cases:

Case 1. Let $p = 31$, i.e., $\{31\} \subseteq \pi(G) \subseteq \{3, 5, 7, \ldots, 31\}$.

Using Lemma 1.9, there exists $x \in G$ such that $[G : \langle x\rangle] < 2p = 62$. By Lemma 1.7, we have $[G : \langle x\rangle] = 31$. Using Lemma 1.8, $[\langle x\rangle : \mathrm{core}_G(\langle x\rangle)] < [G : \langle x\rangle] = 31$. We see that $G/\mathrm{core}_G(\langle x\rangle)$ is supersolvable and so G is a supersolvable group.

Case 2. Let $p \in \{17, 19, 23, 29\}$. Similarly to Case 1, we obtain that G is a supersolvable group.

Therefore, $\pi(G) \subseteq \{3, 5, 7, 11, 13\}$, where $2 \leq |\pi(G)| \leq 5$.

Case 3. Let $p = 13$. If $5 \nmid |G|$, then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x\rangle] < 25$. Thus, by Lemma 1.7, we have $[G : \langle x\rangle] = 13$, and we get the result similarly to the

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

above. If $5 \in \pi(G)$, then by Lemma 1.9 there exists $x \in G$ such that $[G : \langle x \rangle] < 29$, and by Lemma 1.7 we have $[G : \langle x \rangle] = 13$ or 27.

If $[G : \langle x \rangle] = 13$, then by Lemma 1.8 we have $|G/\mathrm{core}_G(\langle x \rangle)| = 13m$, where $m \in \{1, 3, 5, 7, 9, 11\}$. Therefore, $G/\mathrm{core}_G(\langle x \rangle)$ is a supersolvable group and so G is a supersolvable group.

If $[G : \langle x \rangle] = 27$, then by Lemma 1.8 we have $|\langle x \rangle : \mathrm{core}_G(\langle x \rangle)| < 27$. If $13 \mid |\langle x \rangle : \mathrm{core}_G(\langle x \rangle)|$, then $|G/\mathrm{core}_G(\langle x \rangle)| = 13 \cdot 27$. Let $P_5 \in \mathrm{Syl}_5(\mathrm{core}_G(\langle x \rangle))$. We have $P_5 \in \mathrm{Syl}_5(G)$ and $P_5 \trianglelefteq G$, which is a contradiction. Therefore, $13 \nmid |\langle x \rangle : \mathrm{core}_G(\langle x \rangle)|$. Let $P_{13} \in \mathrm{Syl}_{13}(\mathrm{core}_G(\langle x \rangle))$. Hence, $P_{13} \in \mathrm{Syl}_{13}(G)$ and $P_{13} \trianglelefteq G$, which is a contradiction.

Case 4. Let $p = 11$.

- If $\pi(G) = \{3, 5, 7, 11\}$, then there exists $x \in G$ such that $[G : \langle x \rangle] < 26$. Therefore, $[G : \langle x \rangle] = 11$ or 25.
  - If $[G : \langle x \rangle] = 11$, then we have the result.
  - Let $[G : \langle x \rangle] = 25$. Then, $|G/\mathrm{core}_G(\langle x \rangle)| = 25m$, where $m \in \{1, 3, 5, 7, 9, 11, 15, 21\}$. If $m = 11$, then we get the result. Otherwise, G has a Sylow 11-subgroup, which is cyclic and normal, and this is impossible.
- If $\pi(G) \subsetneq \{3, 5, 7, 11\}$, then there exists $x \in G$ such that $[G : \langle x \rangle] < 24$. Therefore, $[G : \langle x \rangle] = 11$ and so we have the result.

Case 5. Let $p = 7$.

- If $\pi(G) = \{3, 7\}$ or $\{5, 7\}$, then easily we can see that G is supersolvable.
- If $\pi(G) = \{3, 5, 7\}$, then there exists $x \in G$ such that $[G : \langle x \rangle] < 25$. Therefore, $[G : \langle x \rangle] = 7, 15$, or 21.
  - If $[G : \langle x \rangle] = 7$, then we get the result.
  - Let $[G : \langle x \rangle] = 15$. Then, $|G/\mathrm{core}_G(\langle x \rangle)| = 15m$, where $m \in \{1, 3, 5, 7, 9\}$. If $m = 7$, then we get the result. Otherwise, the Sylow 7-subgroup of G is cyclic and normal, which is impossible.
  - Let $[G : \langle x \rangle] = 21$. Then, $|G/\mathrm{core}_G(\langle x \rangle)| = 21m$, where $m \in \{1, 3, 5, 7, 9, 15\}$. If $m \in \{5, 15\}$, then we get the result. Otherwise, the Sylow 5-subgroup of G is cyclic and normal, which is impossible.

Case 6. Let $\pi(G) = \{3, 5\}$. Then, $|G| = 3^{\alpha_1}5^{\alpha_2}$. In this case, we have

$$\psi(G) > \frac{271}{3647}\psi(C_{|G|}) > \frac{271}{3647} \cdot \frac{3^{2\alpha_1+1}}{3+1} \cdot \frac{5^{2\alpha_2+1}}{5+1} = \frac{271}{3647} \cdot \frac{5}{8}n^2. \tag{2.3}$$

Hence, there exists $x \in G$ such that $|G : \langle x \rangle| < 22$. By Lemma 1.7, we have $[G : \langle x \rangle] = 5$ or 15. If $[G : \langle x \rangle] = 5$, then we have the result. Let $[G : \langle x \rangle] = 15$ and $H = \mathrm{core}_G(\langle x \rangle)$. By Lemma 1.8, $[\langle x \rangle : H] < [G : \langle x \rangle] = 15$. If G/H is a supersolvable group, then we get the result. Let G/H be nonsupersolvable. Therefore, $[\langle x \rangle : H] = 5$ and $|G/H| = 75$. Thus, $G/H \cong A = (C_5 \times C_5) : C_3$.

- If 3 divides $|H|$, then there exists a characteristic subgroup M in H such that $|H : M| = 3$. Therefore, $M \trianglelefteq G$ and $|G/M| = 225$. Thus, G/M is a nonsupersolvable group of order 225. Using the list of such groups $((C_5 \times C_5) : C_9, C_3 \times ((C_5 \times C_5) : C_3))$ and their ψ-values $(2197, 1297)$, we have $\psi(G/M) \leq 2197$. By Lemma 1.5, we have $\psi(G) \leq \psi(G/M)|M|^2 \leq 2197(n/225)^2$. Using (2.3), $\psi(G) > \frac{271}{3647} \cdot \frac{5}{8}n^2$.

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

Therefore,

$$\frac{271}{3647}\cdot\frac{5}{8}n^2 < 2197\left(\frac{n}{225}\right)^2,$$

which is a contradiction.

- If $3 \nmid |H|$, then $|G| = 3\cdot 5^\beta$. Let $\beta > 2$. Then, there exists a characteristic subgroup M in H such that $|H:M| = 5$. Therefore, $M \trianglelefteq G$ and $|G/M| = 375$. Thus, G/M is a nonsupersolvable group of order 375. Using the list of such groups $(((C_5 \times C_5) : C_5) : C_3, C_5 \times ((C_5 \times C_5) : C_3))$ and their ψ-values $(3771, 3771)$, we have $\psi(G/M) = 3771$. By Lemma 1.5, we have $\psi(G) \le \psi(G/M)|M|^2 \le 3771(n/375)^2$. Using (2.3), we have

$$\frac{271}{3647}\cdot\frac{5}{8}n^2 < 3771\left(\frac{n}{375}\right)^2,$$

which is a contradiction. Thus, $\beta = 2$ and so $G \cong A = (C_5 \times C_5) : C_3$, which is a contradiction. ∎

3 Counterexample for Conjecture 1.3

As a subgroup of the symmetric group S_{32}, let $L := \langle A, B, C, D\rangle$ and $T := \langle A, B, C\rangle$, where

$$\begin{aligned}
A :=& (1,9)(2,10)(3,12)(4,11)(5,13)(6,14)(7,16)(8,15)\\
&(17,25)(18,26)(19,28)(20,27)(21,29)(22,30)(23,32)(24,31),\\
B :=& (1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16)\\
&(17,21)(18,22)(19,23)(20,24)(25,29)(26,30)(27,31)(28,32),\\
C :=& (1,3,2,4)(5,7,6,8)(9,11,10,12)(13,15,14,16)\\
&(17,19,18,20)(21,23,22,24)(25,27,26,28)(29,31,30,32),\\
D :=& (1,17,3,19,2,18,4,20)(5,22,7,24,6,21,8,23)\\
&(9,29,11,31,10,30,12,32)(13,26,15,28,14,25,16,27).
\end{aligned}$$

Then, using the `StructureDescription` command in Groups, Algorithms, Programming (GAP), we see that $L \cong ((C_8 : C_2) : C_2) = \texttt{SmallGroup}(32,7)$ and $T \cong C_2 \times D_8$. Also, using GAP, we get that $|L:T| = 2$, $\psi(L) = 167$ and $\psi(T) = 39$. Therefore,

$$167 = \psi(L) > \psi(T)|L:T|^2 = 156,$$

which is a counterexample for Conjecture 1.3. Also, if m is an odd number and we suppose that $G = L \times C_m$ and $H = T \times C_m$, then $\psi(G) > \psi(H)|G:H|^2$.

Therefore, this example shows that Conjecture 1.3 is not satisfied in general. A natural question that arises is under which extra conditions Conjecture 1.3 holds.

Acknowledgment We would like to express our sincere gratitude for the referee for the useful comments and suggestions.

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press

References

[1] H. Amiri and S. M. Jafarian Amiri, *Sum of element orders on finite groups of the same order*. J. Algebra Appl. **10**(2011), no. 2, 187–190.
[2] H. Amiri, S. M. Jafarian Amiri, and I. M. Isaacs, *Sums of element orders in finite groups*. Comm. Algebra **37**(2009), no. 9, 2978–2980.
[3] A. Bahri, B. Khosravi, and Z. Akhlaghi, *A result on the sum of element orders of a finite group*. Arch. Math. **114**(2020), no. 1, 3–12.
[4] M. Baniasad Azad and B. Khosravi, *A criterion for solvability of a finite group by the sum of element orders*. J. Algebra **516**(2018), 115–124.
[5] M. Baniasad Azad and B. Khosravi, *On the sum of element orders of PSL* $(2, p)$ *for some p*. Ital. J. Pure Appl. Math. **42**(2019), 18–24.
[6] M. Baniasad Azad and B. Khosravi, *A criterion for p-nilpotency and p-closedness by the sum of element orders*. Comm. Algebra **48**(2020), no. 12, 5391–5395. https://doi.org/10.1080/00927872.2020.1788571
[7] M. Baniasad Azad and B. Khosravi, *Properties of finite groups determined by the product of their element orders*. Bull. Aust. Math. Soc. (2020). https://doi.org/10.1017/S000497272000043X
[8] M. Baniasad Azad, B. Khosravi, and M. Jafarpour, *An answer to a conjecture on the sum of element orders*. J. Algebra Appl. https://doi.org/10.1142/S0219498822500670
[9] M. Herzog, P. Longobardi, and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*. J. Pure Appl. Algebra **222**(2018), no. 7, 1628–1642.
[10] M. Herzog, P. Longobardi, and M. Maj, *Two new criteria for solvability of finite groups*. J. Algebra **511**(2018), 215–226.
[11] M. Herzog, P. Longobardi, and M. Maj, *Properties of finite and periodic groups determined by their element orders (a survey)*. In: N. Sastry and M. Yadav (eds.), Group theory and computation, Springer, Berlin, 2018, pp. 59–90.
[12] M. Herzog, P. Longobardi, and M. Maj, *Sums of element orders in groups of order* $2m$ *with* m *odd*. Comm. Algebra **47**(2019), no. 5, 2035–2048.
[13] M. Herzog, P. Longobardi, and M. Maj, *The second maximal groups with respect to the sum of element orders*. J. Pure Appl. Algebra **225**(2021), no. 3, 106531. https://doi.org/10.1016/j.jpaa.2020.106531
[14] S. M. Jafarian Amiri and M. Amiri, *Second maximum sum of element orders on finite groups*. J. Pure Appl. Algebra **218**(2014), no. 3, 531–539.
[15] B. Khosravi and M. Baniasad Azad, *Recognition by the product element orders*. Bull. Malays. Math. Sci. Soc. **43**(2020), no. 2, 1183–1193.
[16] M. Lazorec and M. Tărnăuceanu, *A density result on the sum of element orders of a finite group*. Arch. Math. **114**(2020), 601–607.
[17] A. Lucchini, *On the order of transitive permutation groups with cyclic point-stabilizer*. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **9**(1998), no. 4, 241–243.
[18] M. Tărnăuceanu, *Detecting structural properties of finite groups by the sum of element orders*. Israel J. Math. **238**(2020), no. 2, 629–637. https://doi.org/10.1007/s11856-020-2033-9
[19] M. Tărnăuceanu, *A criterion for nilpotency of a finite group by the sum of element orders*. Comm. Algebra. https://doi.org/10.1080/00927872.2020.1840575

Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Avenue, Tehran 15914, Iran
e-mail: baniasad84@gmail.com khosravibbb@yahoo.com

https://doi.org/10.4153/S0008439521000047 Published online by Cambridge University Press