11 Amplitudes in dimensions $D \neq 4$

Just in case it slipped your mind, our discussion up to now has focused on scattering amplitudes in $D=4$ spacetime dimensions. There is a good reason for this: for one, this review was written in 3+1 dimensions (as far as we know) and this is where our particle physics experiments take place. And secondly, the power of the $D=4$ spinor helicity formalism and its extensions to twistors and momentum twistors allows us to explore the rich and exciting mathematical structure of 4d scattering amplitudes, especially those in planar $\mathcal{N}=4$ SYM. However, there are also interesting quantum field theories in other dimensions; in this chapter we take a look at their scattering amplitudes. We discuss $D=6$ briefly, but otherwise our eyes are on $D=3$, particularly on the $\mathcal{N}=8$ and $\mathcal{N}=6$ superconformal theories known as BLG and ABJM.

11.1 Helicity formalism in $D \neq 4$

We have often emphasized that the modern on-shell approach relies heavily on having a "good" set of variables to parameterize the on-shell degrees of freedom: "good" typically means that the variables trivialize the kinematic constraints and transform linearly under the global symmetries of the theory. This is realized strikingly by the supertwistors and momentum supertwistors of planar $\mathcal{N}=4$ SYM, but the trivialization of the massless on-shell condition $p_i^2=0$ in the spinor helicity formalism with $|i\rangle$ and $|i]$ was our starting point. So this is also where we begin for $D \neq 4$.

To parameterize massless kinematics in D-dimensions, consider bosonic spinors that carry a spinor index A of the Lorentz group Spin$(1, D-1)$ and a fundamental index a of the little group $SO(D-2)$:

$$\lambda_{i\mathsf{a}}^{\substack{A \leftarrow \text{Lorentz} \\ \mathsf{a} \leftarrow \text{little grp}}} . \tag{11.1}$$

As per usual, $i=1,2,\ldots,n$ is a particle label. The spinor-type (Weyl, Majorana, etc.) will be specified when we specialize to a given dimension D. If the spinors are complex, there will be a conjugate spinor $\tilde{\lambda}$ whose A and a indices are in the appropriate conjugate representations.

The 4d Lorentz group is Spin(1,3) $= SL(2,\mathbb{C})$ and the little group is $SO(2)=U(1)$. The spinors

$$D=4: \qquad \tilde{\lambda}_{i-}^{\dot{a}} = |i\rangle^{\dot{a}} , \qquad \lambda_{i+}^{a} = [i|^{a} , \tag{11.2}$$

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are Weyl-spinors, so the index A is the familiar $SL(2,\mathbb{C})$ indices $a, \dot{a}$. The little group index a is $+$ or $-$ depending on how the spinors transform under the $SO(2) = U(1)$ little group transformations. The $D = 4$ light-like momentum is written as the bi-spinor as the familiar relation

$$D = 4: \qquad p_i^{\dot{a}a} = -\tilde{\lambda}_{i-}^{a}\, \lambda_{i+}^{\dot{a}} = -|i\rangle^{\dot{a}}[i|^{a}\,. \tag{11.3}$$

As discussed in Section 2.2, p_i^{μ} is real when the spinors (11.2) are complex conjugate.

The relation (11.3) implies that the 2×2 matrix $p_i^{\dot{a}a}$ has rank 1 and therefore solves the $D = 4$ massless constraint $p_i^2 = -\det(p_i) = 0$. To see if a similar construction could be available in D dimensions, we count degrees of freedom. A real light-like vector has $D-1$ degrees of freedom with the -1 from the condition $p_i^2 = 0$.[1] So the strategy is to find a Spin$(1,D-1)$ spinor representation that allows forming a little group invariant bi-spinor with $D-1$ degrees of freedom. Here is how the counting works in $D = 4$. The complex 2-component spinor $\lambda_{i+}^{a} = [i|^{a}$ has four real degrees of freedom, and when combined with its complex conjugate $\tilde{\lambda}_{i-}^{\dot{a}} = |i\rangle^{\dot{a}}$, the resulting bi-spinor (11.3) is invariant under the $U(1)$ little group rotation. Thus subtracting out the $U(1)$ redundancy, we indeed have $4 - 1 = 3$ degrees of freedom, matching that of a real light-like vector in 4d. Now let us look at how the counting works in other dimensions.

▶ Exercise 11.1

Do the above counting for the case where the spacetime signature is $(--++)$.

For $D = 3$, the Lorentz group is Spin(1,2)$=SL(2,\mathbb{R})$ and the minimal spinor representation is a 2-component Majorana spinor λ_i^{a}, where $a = 1, 2$ is an $SL(2,\mathbb{R})$ index. The null momentum is given by

$$D = 3: \qquad p_i^{ab} = \lambda_i^{a}\lambda_i^{b}\,. \tag{11.4}$$

For p_i real, the spinors λ_i^{a} may be either real or purely imaginary. Either way, they encode two real degrees of freedom. So the RHS of (11.4) has two degrees of freedom, the correct count for a 3d light-like vector. Note that no little group index was included on the spinors λ_i^{a} because the little group $\mathbb{Z}_2$ is discrete. It acts as $\lambda_i^{a} \to -\lambda_i^{a}$, indeed leaving the momentum (11.4) invariant.

For $D = 6$, we have Spin(1,5) $=SU^*(4)$ and the little group is $SO(4) = SU(2) \times SU(2)$. The $*$ on the $SU^*(4)$ indicates that it is pseudo-real.[2] We pick a chiral spinor $\lambda_{i\mathsf{a}}^{A}$ in the fundamental of $SU^*(4)$, so $A = 1, 2, 3, 4$. The spinor is chiral, as opposed to anti-chiral, because it is in the fundamental, not anti-fundamental, representation of $SU^*(4)$. The two $SU(2)$-factors of the little group belong to the chiral and anti-chiral spinors, respectively, so $\lambda_{i\mathsf{a}}^{A}$ carries a little group index $\mathsf{a} = 1, 2$ of the chiral $SU(2)$ factor. A candidate for the

[1] An on-shell massive momentum has $p_i^2 = -m_i^2$, but we view the constraint as imposed on $D + 1$ degrees of freedom, p_i^{μ} and m_i.

[2] Pseudo-real means that for each group element g, the complex conjugate g^* is related to g via a similarity transformation $g = \Omega g^* \Omega^{-1}$, where Ω is an antisymmetric matrix. (If Ω is symmetric, then the representation is a real.)

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light-like momentum can now be formed as the little group invariant bi-spinor

$$D = 6: \qquad p_i^{AB} = \lambda_i^{A\mathsf{a}} \lambda_{i\mathsf{a}}^{B} \,. \tag{11.5}$$

This works to give the right number of degrees of freedom, namely five, for a massless momentum in 6d: the spinor $\lambda_i^{A\mathsf{a}}$ has 4×2 degrees of freedom, but we have to mod out by the little group $SU(2)$-factor, giving $4 \times 2 - 3 = 5$.

The results for $D = 3, 4, 6$ can be summarized as follows:

	Spin$(1, D-1)$	little group	$p^2 = 0$
$D = 3$	$SL(2, \mathbb{R})$	$\mathbb{Z}_2$	$p_i^{ab} = \lambda_i^a \lambda_i^b$
$D = 4$	$SL(2, \mathbb{C})$	$SO(2) = U(1)$	$p_i^{\dot{a}a} = -\lambda_i^a \tilde{\lambda}_i^{\dot{a}}$
$D = 6$	$SU^*(4)$	$SO(4) = SU(2) \times SU(2)$	$p_i^{AB} = \lambda_i^{A\mathsf{a}} \lambda_{i\mathsf{a}}^{B}$

(11.6)

How about general D dimensions? The strategy is to introduce a bosonic spinor $\lambda_{i\mathsf{a}}^{A}$ (where a transforms under the little group, or a subgroup as in the 6d example) and use it (and possibly its conjugate spinor) to form a (real) light-like vector as a little group invariant bi-spinor, e.g. $\lambda_{i\mathsf{a}}^{A} \tilde{\lambda}_i^{\mathsf{a}B}$. However, for this to encode a null momentum, the number of real degrees of freedom of the bi-spinor, modulo the number of little group generators, has to match that of a light-like vector:

$$\mathrm{DOF}\big[\lambda_{i\mathsf{a}}^{A} \tilde{\lambda}_i^{\mathsf{a}B}\big] - \#(\text{little group generators}) = D - 1 \,. \tag{11.7}$$

This is a non-trivial constraint because the bi-spinor typically has more than $D - 1$ degrees of freedom. One has to find a minimal spinor representation with maximal little group redundancy; this was particularly clear in the 6d example above. Indeed, we know solutions to these constraints only for $D = 3, 4, 6$.[3]

▶ **Exercise 11.2**

What is the smallest possible number of degrees of freedom for a little group invariant bi-spinor in $D = 10$? [Hint: In $D = 10$, the minimum spinor representation is a Majorana–Weyl spinor; it has 16 real components.]

It is possible to reduce the number of independent spinor degrees of freedom further by imposing the equations of motion, i.e. the zero-mass Dirac equation. Now you may be puzzled, because back in Chapter 2 we set up the $D = 4$ spinor helicity formalism by requiring at the starting point that the spinors $|i\rangle$ and $|i]$ satisfied the Dirac–Weyl equation. For $D = 3, 4, 6$, this approach is equivalent: the Lorentz contraction of the $(D-1)$-component bi-spinor with one of its spinors gives zero, so the momentum space form of the massless Dirac equation is automatic.

For $D \neq 3, 4, 6$, setting up a spinor helicity formalism is possible but the resulting spinors are constrained in the sense that the Dirac equation is imposed as a non-trivial condition [135–137]. Constrained spinors are more difficult to work with, especially if one wants to construct symmetry generators in order to study symmetries of the D-dimensional

[3] $D = 3, 4, 6$ are precisely the dimensions in which twistor constructions that describe conformal symmetry are known; see Section II.C.5 of Siegel's "Fields" [57].

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amplitudes. For this reason, we focus on $D = 3, 6$ in this chapter: we describe $D = 6$ briefly, then offer more details about the interesting structure of $D = 3$ amplitudes.

11.2 Scattering amplitudes in $D = 6$

Oh, who cares about 6d scattering amplitudes!!? Don't we live in 4d? Well, the 6d massless condition

$$-p_0^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^3 = 0 \tag{11.8}$$

can be viewed from 4d spacetime as the on-shell condition for a massive 4d momentum vector: take $p_4^2 + p_5^2 = m^2$ (or $= m\tilde{m}$ if you are willing to accept complex masses). Then $p_{4\mathrm{d}}^2 = -m^2$ follows from (11.8) with $p_{4\mathrm{d}}$ denoting the first four components of the 6d momentum. This makes the 6d formalism very useful for studies of 4d amplitudes with massive particles. An example of an explicit application to Higgs production processes can be found in [138]. See [87, 139, 140] for other examples. So let's care about 6d scattering amplitudes, at least for one section.

The 6d spinor helicity formalism was first developed by Cheung and O'Connell [141] and its supersymmetrization was carried out in [142]. It has been applied to tree- and loop-level scattering amplitudes in maximal super Yang–Mills theory in 6d [139, 143, 144] and also used in other 6d theories [145–148].

In 4d, we used $(\sigma^\mu)_{a\dot{b}}$ and $(\bar{\sigma}^\mu)^{\dot{a}b}$ to define the 2×2 matrices $p_{a\dot{b}} = p_\mu(\sigma^\mu)_{a\dot{b}}$ and $p^{\dot{a}b} = p_\mu(\bar{\sigma}^\mu)^{\dot{a}b}$. Similarly, the 6d Lorentz group $SO(1,5) \sim SU^*(4)$ has antisymmetric 4×4 matrices $(\sigma^\mu)_{AB}$ and $(\tilde{\sigma}^\mu)^{AB}$, $A, B = 1, 2, 3, 4$, that allow us to define

$$p_{AB} = p_\mu\,(\sigma^\mu)_{AB}\,, \qquad p^{AB} = p_\mu\,(\tilde{\sigma}^\mu)^{AB}\,. \tag{11.9}$$

The explicit form of the $(\sigma, \tilde{\sigma})$ matrices as well as their relation to the 6d 8×8 γ-matrices can be found in Appendix A of [141].

The pseudo-real property of $SU^*(4)$, implies that p^{AB} and p_{AB} are related as

$$p^{AB} = \frac{1}{2}\epsilon^{ABCD}p_{CD}\,. \tag{11.10}$$

In this notation, the $SO(1,5)$ invariant product $p^\mu p_\mu$ can be written as the manifestly $SU^*(4)$-invariant contraction

$$p^\mu p_\mu = -\frac{1}{4}p^{AB}p_{AB} = -\frac{1}{8}\epsilon_{ABCD}p^{AB}p^{CD}\,. \tag{11.11}$$

Now, in momentum space, the 6d Dirac equation for massless spinors is

$$p_{AB}\lambda_i^{B\mathsf{a}} = 0\,, \qquad p^{AB}\,\tilde{\lambda}_{iB\dot{\mathsf{a}}} = 0\,, \tag{11.12}$$

where $\lambda^{B\mathsf{a}}$ and $\tilde{\lambda}_{B\dot{\mathsf{a}}}$ are chiral and anti-chiral spinors, and $\mathsf{a} = 1, 2$ and $\dot{\mathsf{a}} = 1, 2$ are fundamental indices of the two $SU(2)$s of the little group $SO(4) = SU(2) \times SU(2)$. The two pairs of Weyl spinors

$$\lambda_i^{A\mathsf{a}} = \langle i^{\mathsf{a}}|^A = {}^A|i^{\mathsf{a}}\rangle \qquad \text{and} \qquad \tilde{\lambda}_{iB\dot{\mathsf{a}}} = [i_{\dot{\mathsf{a}}}|_B = {}_B|i_{\dot{\mathsf{a}}}] \tag{11.13}$$

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are the building blocks of the 6d spinor helicity formalism. There is no distinction between bras and kets because there is no raising or lowering of the $SU^*(4)$ indices.

The little group indices can be raised/lowered using the $SU(2)$ Levi-Civita symbol as $\lambda_{\mathsf{a}} = \epsilon_{\mathsf{ab}}\lambda^{\mathsf{b}}$ and $\tilde{\lambda}^{\dot{\mathsf{a}}} = \epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}}\tilde{\lambda}_{\dot{\mathsf{b}}}$. This allows us to form little group invariants, as in the bi-spinor construction (11.5). Indeed, the massless momentum is given as

$$p_i^{AB} = \lambda_i^{A\mathsf{a}}\lambda_{i\mathsf{a}}^{B}\,, \qquad p_{iAB} = \tilde{\lambda}_{iA\dot{\mathsf{a}}}\tilde{\lambda}_{iB}{}^{\dot{\mathsf{a}}}\,. \tag{11.14}$$

Due to the antisymmetric contraction of the $SU(2)$ indices, the bi-spinors in (11.14) are automatically antisymmetric in the $SU^*(4)$ indices A and B. By (11.14), the 4×4 matrix p_i^{AB} has rank 2, so $p_i^2 \sim \epsilon_{ABCD}\, p_i^{AB} p_i^{CD}$ is zero. Hence the massless on-shell condition $p_i^2 = 0$ is satisfied. Thus this realizes the construction (11.5).

Reverting the momentum in (11.14) from matrix form to vector form, we have

$$p_i^{\mu} = -\frac{1}{4}\langle i^a|\sigma^{\mu}|i_a\rangle = -\frac{1}{4}[i_{\dot{\mathsf{a}}}|\tilde{\sigma}^{\mu}|i^{\dot{\mathsf{a}}}]\,. \tag{11.15}$$

These expressions are the 6d versions of the 4d relation $k^{\mu} = \frac{1}{2}\langle k|\gamma^{\mu}|k]$ you derived in Exercise 2.4.

The Dirac equation (11.12) implies that $\lambda_i^{A\mathsf{a}}\,\tilde{\lambda}_{iA\dot{\mathsf{a}}} = 0$, so the chiral and anti-chiral spinors are related. Construction of symmetry generators using these variables must take such constraints into account. However, if only chiral spinors are needed, we can still work with unconstrained variables.

To get a better feeling for the 6d 4×2 spinors – and to facilitate reduction to 4d – consider the embedding of our good old 4d spinors in the new 6d spinors. Choosing $\mu = 0, 1, 2, 3$ to be the 4d subspace and setting $p_4 = p_5 = 0$, the 4d spinors appear in the 6d ones as

$$\lambda_{i\mathsf{a}}^{A} = \begin{pmatrix} 0 & \langle i|_{\dot{a}} \\ [i|^a & 0 \end{pmatrix}\,, \qquad \tilde{\lambda}_{iA\dot{\mathsf{a}}} = \begin{pmatrix} 0 & |i\rangle^{\dot{a}} \\ -|i]_a & 0 \end{pmatrix}\,. \tag{11.16}$$

Thus the constraint $\lambda_i^{A\mathsf{a}}\tilde{\lambda}_{iA\dot{\mathsf{a}}} = 0$ becomes nothing but the familiar $\langle i\, i\rangle = [i\, i] = 0$.

In 6d massless kinematics, the basic Lorentz invariant spinor products are:

- $\langle i^{\mathsf{a}}|j_{\dot{\mathsf{b}}}] \equiv \lambda_i^{A\mathsf{a}}\tilde{\lambda}_{jA\dot{\mathsf{b}}} = [j_{\dot{\mathsf{b}}}|i^{\mathsf{a}}\rangle\,,$
- $\langle i^{\mathsf{a}}\, j^{\mathsf{b}}\, k^{\mathsf{c}}\, l^{\mathsf{d}}\rangle \equiv \epsilon_{ABCD}\lambda_i^{A\mathsf{a}}\lambda_j^{B\mathsf{b}}\lambda_k^{C\mathsf{c}}\lambda_l^{D\mathsf{d}}\,,$
- $[i_{\dot{\mathsf{a}}}\, j_{\dot{\mathsf{b}}}\, k_{\dot{\mathsf{c}}}\, l_{\dot{\mathsf{d}}}] \equiv \epsilon^{ABCD}\tilde{\lambda}_{iA\dot{\mathsf{a}}}\tilde{\lambda}_{jB\dot{\mathsf{b}}}\tilde{\lambda}_{kC\dot{\mathsf{c}}}\tilde{\lambda}_{lD\dot{\mathsf{d}}}\,.$

In particular, the Mandelstam variable $s_{ij} = -(p_i + p_j)^2$ is

$$s_{ij} = -\frac{1}{2}\epsilon^{\mathsf{ab}}\epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}}\langle i_{\mathsf{a}}|j_{\dot{\mathsf{a}}}]\langle i_{\mathsf{b}}|j_{\dot{\mathsf{b}}}] = -\det\langle i_{\mathsf{a}}|j_{\dot{\mathsf{a}}}]\,. \tag{11.17}$$

We have outlined the 6d spinor helicity formalism, so now it is time to apply it to amplitudes. Let us begin with 3-point amplitudes; this involves ***special 3-particle kinematics*** because all s_{ij} vanish. In 4d, we got around this by working with complex kinematics such that $\langle ji\rangle \neq ([ij])^*$ and that allowed us to choose either all the angle- or the square-brackets to be non-vanishing, but not both. In 6d 3-particle kinematics, the only Lorentz invariants

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available for s_{ij} are the brackets $\langle i_{\mathsf{a}}|j_{\dot{\mathsf{a}}}]$. But since $0 = s_{ij} = -\det\langle i_{\mathsf{a}}|j_{\dot{\mathsf{a}}}]$, the 2×2 matrix $\langle i_{\mathsf{a}}|j_{\dot{\mathsf{a}}}]$ must be rank 1. We have encountered 2×2 matrices of rank 1 before, namely the 4d massless $p_{a\dot{a}}$, and by now it should be a simple reflex to introduce two 2-component spinors, u_{ia} and $\tilde{u}_{j\dot{a}}$ such that $\langle i_a|j_{\dot{a}}] = u_{ia}\tilde{u}_{j\dot{a}}$ [141], so the 3-point amplitudes in 6d are written in terms of these "auxiliary" 2-component spinors.

Just as in 4d, the 6d ***3-point amplitudes*** are highly constrained by the little group and Lorentz invariance. For example, one finds that the 3-vector amplitudes only come in two types, one is generated by the $AA\partial A$ vertex of the Yang–Mills action while the other is generated by the operator $F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu$ [141]. A wide class of possible 3-point interactions was categorized in [146]. In particular, for 6d self-dual antisymmetric tensors – which are part of the (2,0) supermultiplet that describes the degrees of freedom of M5-membranes in M-theory – one can demonstrate [146] that a 3-point amplitude cannot be both Lorentz invariant and carry the correct little group indices to describe scattering of three self-dual tensors; so it does not exist.

The 6d ***4-point Yang–Mills amplitude*** is given by:

$$A_4(1,2,3,4) = \frac{\langle 1^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}4^{\mathsf{d}}\rangle[1_{\dot{\mathsf{a}}}2_{\dot{\mathsf{b}}}3_{\dot{\mathsf{c}}}4_{\dot{\mathsf{d}}}]}{s\,u}\,. \tag{11.18}$$

The 6d gluons are not labeled by the 4d concept of helicity: instead, a 6d massless spin-1 particle has four physical states labeled by the little group indices ${}^{\mathsf{a}}{}_{\dot{\mathsf{a}}}$.

▶ **Exercise 11.3**

Use the map in (11.16) to reduce the 6d amplitude (11.18) to 4d. You should find the usual suspect, the MHV gluon amplitude. But that is not all: identify the other possibilities and describe their origin.

In maximal SYM in 6d, the 4-point superamplitude takes the simple form

$$\mathcal{A}_4(1,2,3,4) = \delta^6\big(P\big)\,\delta^{(4)}\big(Q\big)\,\delta^{(4)}\big(\tilde{Q}\big)\,\frac{1}{y_{13}^2 y_{24}^2}\,. \tag{11.19}$$

We have used dual space to write $s = -y_{13}^2$ and $u = -y_{24}^2$. The supermomentum delta functions are defined in [142]. If we write the n-point superamplitude as $\mathcal{A}_n = \delta^6(P)\,\delta^{(4)}(Q)\,\delta^{(4)}(\tilde{Q})\,f_n$, we note from (11.19) that $I[f_4] = y_1^2 y_2^2 y_3^2 y_4^2\,f_4$ under dual conformal inversion (5.37), i.e. f_4 inverts in exactly the same way as the 4d 4-point superamplitude of $\mathcal{N} = 4$ SYM. Using a 6d version super-BCFW recursion, it was proven [144] for all n that

$$I[f_n] = \left[\prod_{i=1}^{n} y_i^2\right] f_n\,. \tag{11.20}$$

In 4d, it was essential for dual superconformal symmetry of planar superamplitudes in $\mathcal{N} = 4$ SYM that the inversion weights of the bosonic and fermionic delta functions canceled, as shown in (5.38). This, however, does not happen in 6d maximal SYM: $\delta^6(P)$ inverts with weight 6, while the Grassmann delta functions have weight $-(4+4)/2$. Therefore,

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the planar superamplitudes of 6d maximal SYM do not have uniform inversion weight. Nonetheless, as is often the case with scattering amplitudes, even if a symmetry is not exact, it is still useful if it is broken in a predetermined fashion, as is the case here. Remarkably, using generalized unitarity methods it has been shown [144] that the planar L-loop integrands of 6d maximal SYM theory have the same dual conformal inversion weight as in 4d. (A similar result was found for 10d SYM [136].) While the origin of this form of dual superconformal symmetry is not clear (and the 6d and 10d SYM theories are *not* (super)conformal), it has non-trivial implications in 4d for the structure of (super)amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM [140, 149].

11.3 Scattering amplitudes in $D = 3$

Scattering amplitudes in $D = 3$ turn out to have very interesting properties. After introducing the necessary kinematic tools and basic examples of amplitudes, we focus on scattering processes in the 3d $\mathcal{N} = 8$ and $\mathcal{N} = 6$ superconformal theories called BLG and ABJM.

11.3.1 $D = 3$ kinematics

We construct 3d kinematics by reduction from 4d using the fact that the 4d massless condition, $-p_0^2 + p_1^2 + p_2^2 + p_3^2 = 0$, is equivalent to a 3d massive constraint. It is convenient to identify the p_2-component with the 3d mass as $p_2^2 = m^2$ so that the 3d momentum p^μ with $\mu = 0, 1, 3$ satisfies $p^\mu p_\mu = -m^2$.

Recall that in 4d, the momentum can be given as

$$D = 4: \qquad p_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} . \tag{11.21}$$

We restrict this to 3d by removing p_2 and writing

$$D = 3: \qquad p_{ab} = \begin{pmatrix} -p^0 + p^3 & p^1 \\ p^1 & -p^0 - p^3 \end{pmatrix} . \tag{11.22}$$

Then $\det p_{ab} = -(-p_0^2 + p_1^2 + p_3^2) = m^2$.

The 2×2 matrix p_{ab} is symmetric. If the 3d momentum p^μ, $\mu = 0, 1, 3$, is real, p_{ab} is also real.[4] A generic real symmetric 2×2 matrix can be written as [150][5]

$$m \neq 0: \qquad p_{ab} = \lambda_a \bar{\lambda}_b + \lambda_b \bar{\lambda}_a , \tag{11.23}$$

where $\bar{\lambda}_a = (\lambda_a)^*$ when p^μ is real.

[4] This contrasts with the 4d case, where $p_{a\dot{b}}$ is complex valued, and it reflects the different Lorentz groups, $SO(1, 2) = SL(2, \mathbb{R})$ in 3d and $SL(2, \mathbb{C})$ in 4d.

[5] We could also have written $p_{ab} = \lambda_a \lambda_b + \mu_a \mu_b$, where μ_a is some real 2-component spinor not parallel to λ_a, but this is equivalent to (11.23) by a linear redefinition.

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▶ **Exercise 11.4**

If p^μ is complex, we take λ_a and $\bar{\lambda}_a$ to be independent. For each case, p^μ real or complex, count the number of degrees of freedom on each side of (11.23).

By direct calculation of the determinant of (11.23), we find that $m^2 = \det p_{ab} = -\langle\lambda\bar{\lambda}\rangle^2$, where $\langle\lambda\bar{\lambda}\rangle = \lambda^a\bar{\lambda}_a$ and spinor indices are raised and lowered with the 2-index Levi-Civita of the $SL(2,\mathbb{R})$ Lorentz group. For 3d massless kinematics, $m = 0$, we must therefore have $\langle\lambda\bar{\lambda}\rangle = 0$, implying that $\bar{\lambda} \propto \lambda$. Thus, we can write

$$m = 0: \qquad p_{ab} = \lambda_a\lambda_b = \langle p|_a \langle p|_b \,, \tag{11.24}$$

where $\lambda = \langle p|$ was rescaled such that the prefactor is just 1. Note that $\langle p|$ must be either purely real or purely imaginary for p_{ab} to be real.

It follows from (11.24) that in *3d massless kinematics, all Lorentz invariants are built out of one kind of angle brackets, namely* $\langle ij\rangle = \lambda_i^a\lambda_{j\,a}$. For example, since $2p_i.p_j = -\langle ij\rangle^2$, the Mandelstams s_{ij} are

$$D = 3: \qquad s_{ij} = -(p_i + p_j)^2 = \langle ij\rangle^2 \,. \tag{11.25}$$

Momentum conservation $\sum_{i=1}^n p_i^\mu = 0$ can be written

$$D = 3: \qquad \sum_{i=1}^{n} |i\rangle\langle i| = 0 \,. \tag{11.26}$$

Our 3d kinematics is ready, so let us see some amplitudes. As it is our style, we start with ***3-particle amplitudes***. These are particularly easy in 3d, because 3-particle kinematics requires all $s_{ij} = \langle ij\rangle^2 = 0$ and hence there are no Lorentz invariants available for a massless 3-point amplitude. *Thus for massless kinematics, there are no 3-point on-shell amplitudes in 3d.*

The ***little group*** for massless kinematics in 3d is the discrete group $\mathbb{Z}_2$; it acts on the spinor variables as $|i\rangle \to -|i\rangle$. The homogeneous scaling of the scattering amplitudes distinguishes only two types of particles in 3d: *scalar particles* scale with $+1$ and *fermions* scale with -1. And spin-1 vector particles? A massless vector in D-dimensions has $D-2$ degrees of freedom, so in $D = 3$ this is just 1, the same as a scalar.

Tree-level scattering amplitudes of 3d super Yang–Mills theory can be obtained directly from 4d ones using dimensional reduction. For example, the dimensional reduction of the 4-point gluon amplitude $A_4[1^-2^+3^-4^+]$ of 4d Yang–Mills theory gives

$$A_4[1^-2^+3^-4^+] = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \xrightarrow{4\text{d}\to 3\text{d}} -\frac{\langle 13\rangle^4}{\langle 12\rangle^2\langle 23\rangle^2} \,. \tag{11.27}$$

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We have used 3d momentum conservation $\langle 34\rangle\langle 41\rangle = -\langle 32\rangle\langle 21\rangle$ to simplify the result. The two helicity states of the 4d gluon become two degrees of freedom in 3d that we can organize as a 3d "gauge boson" and a scalar.

11.3.2 3d SYM and Chern–Simons theory

The 3-dimensional Yang–Mills action

$$\mathcal{L}_{\rm YM} = \frac{1}{g^2}\int d^3x \ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{11.28}$$

has a coupling g^2 of mass-dimension (mass)^1. We are particularly interested in theories with extra symmetry (after all, we keep getting marathonic mileage out of planar $\mathcal{N}=4$ SYM), but a superconformal theory needs dimensionless couplings.

In 3d, the gauge field can be introduced with a dimensionless coupling via the ***Chern–Simons Lagrangian***

$$\mathcal{L}_{\rm CS} = \frac{\kappa_{\rm CS}}{4\pi}\,\epsilon^{\mu\nu\rho}\ \mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right). \tag{11.29}$$

The coupling $\kappa_{\rm CS}$ is an integer and is called the ***Chern–Simons level***.

The equation of motion derived from varying $\mathcal{L}_{\rm CS}$ with respect to the gauge field is

$$\partial_{[\mu}A_{\nu]} + i[A_\mu, A_\nu] = F_{\mu\nu} = 0\,. \tag{11.30}$$

The solution to this equation is simply $A_\mu = g\partial_\mu g^{-1}$, where g is an arbitrary element in the gauge group. This means that the gauge field is pure gauge, or a flat connection. For us, the relevant implication is that the Chern–Simons gauge field does not carry any physical degrees of freedom, since one can always choose a gauge such that $A_\mu = 0$. This is an important difference between a gauge field whose dynamics is governed by $\mathcal{L}_{\rm CS}$ versus the usual Yang–Mills Lagrangian $\mathcal{L}_{\rm YM}$: the gauge boson scattering amplitudes of 3d Yang–Mills theory are non-trivial, but for a theory with just a Chern–Simons term the scattering amplitudes are trivially zero because there are no physical states to scatter.

There can be non-trivial scattering amplitudes for Chern–Simons theory provided matter fields are introduced. The ***Chern–Simons matter Lagrangian*** is typically written

$$\mathcal{L} = \mathcal{L}_{\rm CS} + \mathcal{L}_{\phi\psi}\,, \tag{11.31}$$

where the matter Lagrangian $\mathcal{L}_{\phi\psi}$ encodes the interactions of scalars ϕ and fermions ψ with the Chern–Simons gauge field as well as their mutual interactions. In 3d, the (complex) scalar- and fermion-interactions with dimensionless couplings are of the form $\phi^3\bar{\phi}^3$ and $\bar{\psi}\psi\bar{\phi}\phi$. Thus for superconformal theories, $\mathcal{L}_{\phi\psi}$ takes the form

$$\mathcal{L}_{\phi\psi} = -D^\mu\bar{\phi}D_\mu\phi + i\bar{\psi}\not{D}\psi + V_{\psi\bar{\psi}\phi\bar{\phi}} + V_{\phi^3\bar{\phi}^3}\,, \tag{11.32}$$

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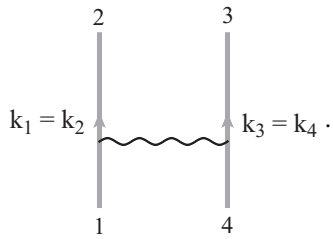
where $V_{\psi\bar{\psi}\phi\bar{\phi}}$ and $V_{\phi^3\bar{\phi}^3}$ are quartic and sextic interaction terms. The explicit form of these terms depends on the theory; we will show you two examples, namely the $\mathcal{N} = 8$ and $\mathcal{N} = 6$ superconformal 3d theories (Sections 11.3.5 and 11.3.6). But let us first explore the properties of amplitudes in 3d a little further.

11.3.3 Special kinematics and poles in amplitudes

The are 3-particle interaction terms in the Lagrangians discussed in Section 11.3.2, but we have learned in Section 11.3.1 that all on-shell 3-point amplitudes vanish in 3d. Nonetheless, the 3-particle vertices still make their presence felt by hiding in special kinematic limits of higher-point amplitudes. As an example of this, consider the limit $s_{12} = \langle 12\rangle^2 \to 0$ of a 4-point amplitude. In this limit, $|1\rangle$ becomes proportional to $|2\rangle$, so $|1\rangle = \alpha|2\rangle$ for some α. Further, we must have $(1+\alpha^2)s_{23} = 0$, since

$$0 = p_4^2 = (p_1 + p_2 + p_3)^2 \quad \xrightarrow{s_{12}\to 0} \quad 0 = s_{13} + s_{23} = (1+\alpha^2)s_{23}\,. \tag{11.33}$$

There are two types of solutions to this constraint. For generic α, s_{23} must be zero and one can conclude that all Lorentz invariants vanish. This is in line with our previous discussion that there are no Lorentz invariants for on-shell 3-point kinematics. However, the constraint $(1+\alpha^2)s_{23} = 0$ also admits a solution that allows non-trivial Lorentz invariants: $\alpha = \pm i$. For $\alpha = \pm i$, we have $p_1 = -p_2$ and similarly $p_3 = -p_4$. Thus this corresponds to the kinematic configuration where two particles are traveling in straight lines:



(11.34)

From momentum conservation, any exchange between the two particle lines must have zero momentum, so when we approach the $s_{12} \to 0$ limit the amplitude should develop a singularity associated with the propagator of an exchanged soft particle.

To see this in an explicit example, consider the 3d YM gluon amplitude (11.27)

$$A_4[1234] = -\frac{\langle 13\rangle^4}{\langle 12\rangle^2\langle 23\rangle^2}\,. \tag{11.35}$$

Taking the limit $|1\rangle \to i\,|2\rangle$, the amplitude indeed develops a non-trivial $1/s_{12}$ singularity with a non-vanishing residue:

$$A_4[1^-2^+3^-4^+]\Big|_{|1\rangle \to i\,|2\rangle} = -\frac{s_{23}}{s_{12}}\,. \tag{11.36}$$

The $1/s_{12}$ singularity reflects the $1/p^2$ behavior of the gluon propagator.

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Thus, although there are no massless 3-point amplitudes in 3d, the 4-point amplitude still develops a non-trivial "soft" pole. The origin of this singularity comes from the exchange of a soft particle between two particles going in straight lines. Note that the exchanged particle has momentum $p^\mu \to 0$, so it is not strictly going on-shell. This is also reflected in the observation that amplitude (11.36) does not factorize into two 3-point amplitudes.

Importantly, the precise behavior of the singularity is dictated by the propagator of the intermediate particle. If the exchanged particle is an ordinary Yang–Mills gluon, then we should observe a $1/p^2$ singularity. That is what happened in the example (11.36). However, if it is a fermion or a Chern–Simons gauge boson, one should find a $1/\sqrt{p^2}$ singularity.[6] In the case of a Chern–Simons boson, it comes from the propagator of the gauge field in the Lagrangian $\mathcal{L}_{\rm CS}$ of (11.29); in Landau gauge it is

$$\langle A^\mu(p) A^\nu(-p)\rangle = \frac{\epsilon^{\mu\rho\nu} p_\rho}{p^2}\,. \tag{11.37}$$

We are going to use information about poles in this special kinematic limit to constrain the possible 4-point amplitudes in Section 11.3.5.

11.3.4 $D = 3$ superconformal algebra

We stated in Section 11.1 that the minimal spinors in 3d are 2-component Majorana spinors. They satisfy the Majorana reality condition, and so do the supersymmetry charges. Thus for $\mathcal{N}$-fold supersymmetry in 3d, we have $\mathcal{N}$ real supercharges and the R-symmetry group is $SO(\mathcal{N})$. Since the 3d theories discussed in this chapter have $\mathcal{N} = 8$ or $\mathcal{N} = 6$ supersymmetry, we focus on $\mathcal{N}$=even in the following. When $\mathcal{N} = 2M$, the real supercharges can be grouped into $M = \mathcal{N}/2$ complex spinors Q^{aA} and their complex conjugates $\widetilde{Q}^a_A$. Here $A = 1, \ldots, M$ is the index of the reduced $SU(M)$ R-symmetry.

We introduce M on-shell superspace coordinates η_i^A for each external leg. The supercharge can now be written as

$$\widetilde{Q}^a_A = \sum_i |i\rangle^a \eta_{iA}\,, \qquad Q^{aA} = |i\rangle^a \frac{\partial}{\partial \eta_{iA}}\,. \tag{11.38}$$

You can quickly see that $\{\widetilde{Q}^a_A, Q^{bB}\} = \delta_A{}^B P^{ab}$.

The generators (11.38) are part of a larger symmetry group: the $OSp(\mathcal{N}|4)$ superconformal group. The notation $OSp(\mathcal{N}|4)$ means that the bosonic generators include the $SO(\mathcal{N})$

[6] Our reasoning here is valid only for $n = 4$. For example, the n-point Parke–Taylor amplitude with $n > 4$ has only $1/\langle i, i+1\rangle$ poles that do not exhibit the $1/p^2$ of the Yang–Mills gluon propagator. A careful inspection of the 3-point gluon vertex reveals that the only non-vanishing term is proportional to $k_2^\mu(\epsilon_1 \cdot \epsilon_2)$ in the limit where legs 1, 2 are the two gluons propagating in a straight line. This is dotted into the remaining Feynman diagram which for $n = 4$ is simply another 3-point vertex that in this limit contributes just one term proportional to $k_3^\mu(\epsilon_3 \cdot \epsilon_4)$. Hence, on this soft pole, the residue is simply given by the product of the two 3-point vertices. For $n > 4$, the remaining Feynman diagrams give multiple contributions, and thus the residue of this soft pole contains several terms that can cancel to leave behind a milder singularity. Therefore, our discussion of soft-pole structure is only valid for $n = 4$.

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R-symmetry as well as the $Sp(4)$ conformal symmetry generators. More precisely, the generators are:

$$
\begin{array}{lll}
 & P^{ab} = \sum_i |i\rangle^a |i\rangle^b & \\
\widetilde{Q}^a_A = \sum_i |i\rangle^a \eta_{iA} & & Q^{aA} = \sum_i |i\rangle^a \partial_{\eta_{iA}} \\
M_{ab} = \sum_i \langle i|_{(a} \partial_{|i\rangle^{b)}} & & D = \sum_i \left(\frac{1}{2}|i\rangle^a \partial_{|i\rangle^a} + \frac{1}{2}\right) \\
R_{AB} = \sum_i \eta_{iA}\eta_{iB} & R_A{}^B = \sum_i \left(\eta_{iA}\partial_{\eta_{iB}} - \frac{1}{2}\delta^A{}_B\right) & R^{AB} = \sum_i \partial_{\eta_{iA}}\partial_{\eta_{iB}} \\
\tilde{S}_{aA} = \sum_i \partial_{|i\rangle^a}\eta_{iA} & & S^A_a = \sum_i \partial_{|i\rangle^a}\partial_{\eta_{iA}} \\
 & K_{ab} = \sum_i \partial_{|i\rangle^a}\partial_{|i\rangle^b} . &
\end{array}
\tag{11.39}
$$

The $SO(\mathcal{N})$ R-symmetry generators are separated into $U(\mathcal{N}/2)$ generators $R_A{}^B$ and coset generators R_{AB} and R^{AB} of $SO(\mathcal{N})/U(\mathcal{N}/2)$.

As an important application for these generators, let us explore what kind of constraint the $U(1)$ piece of the $U(\mathcal{N}/2)$ R-symmetry imposes on the superamplitudes in a 3d $\mathcal{N} = 2M$ superconformal theory. The $U(1)$ piece is given by

$$
R_C{}^C = \sum_i \left(\eta_{iC}\partial_{\eta_{iC}} - \frac{M}{2}\right) . \tag{11.40}
$$

The $R_C{}^C$ generator annihilates the superamplitude, $R_C{}^C \mathcal{A}_n = 0$, if

$$
\sum_i \eta_{iC}\partial_{\eta_{iC}} \mathcal{A}_n = n\frac{M}{2}\mathcal{A}_n . \tag{11.41}
$$

The LHS simply counts the Grassmann degree of η_As in $\mathcal{A}_n$. Since one cannot have fractional degrees of η in $\mathcal{A}_n$, the equation (11.41) can only hold for odd M if $n =$ even. So we learn that *only even-multiplicity scattering amplitudes can be non-vanishing for a superconformal theory with M odd*.

The *same is also true for M even*. This is because superconformal theories generally require the presence of a gauge field whose self-interaction is described by the Chern–Simons Lagrangian (11.29). The form of the Chern–Simons matter Lagrangian (11.31), (11.32) implies that any odd-multiplicity Feynman diagram has at least one external leg associated with a gauge field. As discussed previously, the Chern–Simons gauge field does not carry any physical degrees of freedom, so this means that the odd-multiplicity scattering amplitudes must vanish.

So we learn that in a 3d superconformal theory, the Grassmann degree of the superamplitudes, i.e. the N^KMHV-level, is rigidly tied to the number of external particles, contrary to its freer life in 4d. For example for $\mathcal{N} = 8$, an MHV superamplitude has Grassmann degree 8 and by (11.41) it exists only for $n = 4$ external particles in a 3d superconformal theory; a 6-point superamplitude on the other hand must have Grassmann degree 12, so it has to be NMHV. Thus in a 3d $\mathcal{N} = 8$ superconformal theory, there is no tower of MHV superamplitudes, no equivalent of the n-gluon Parke–Taylor amplitude. Similarly, in a 3d $\mathcal{N} = 6$ superconformal theory, the 4-point superamplitude must have Grassmann degree 6.

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11.3.5 $\mathcal{N} = 8$ superconformal theory: BLG

A 3d superconformal theory with $\mathcal{N}=8$-fold supersymmetry has an on-shell spectrum with eight scalars $(\phi, \phi^{AB}, \bar{\phi})$ and eight fermions $(\psi^A, \bar{\psi}_A)$. Just as in $\mathcal{N} = 4$ SYM in 4d, it is convenient to encode the degrees of freedom in an on-shell superfield,

$$\Phi = \phi + \eta_A \psi^A - \frac{1}{2}\eta_A\eta_B \phi^{AB} - \frac{1}{3!}\epsilon^{ABCD}\eta_A\eta_B\eta_C \bar{\psi}_D + \eta_1\eta_2\eta_3\eta_4 \bar{\phi}\,. \tag{11.42}$$

We have $\eta_A \to -\eta_A$ under little group transformations, so the superfield Φ is inert. This means that the superamplitudes are also invariant under little group transformations.

Since there are no massless 3-point amplitudes, let us consider the most general 4-point tree superamplitude that enjoys $\mathcal{N} = 8$ 3d superconformal symmetry. To start with, invariance under $\mathcal{N} = 8$ supersymmetry implies that an n-point superamplitude takes the form

$$\mathcal{A}_n \;=\; \delta^3\big(P\big)\,\delta^{(8)}\big(\widetilde{Q}\big)\;f_n\big(|i\rangle, \eta_i\big)\,, \tag{11.43}$$

where $\delta^{(8)}(\widetilde{Q}) = \prod_{A=1}^{4}(\frac{1}{2}\widetilde{Q}^a_A\widetilde{Q}_{aA})$. The function f_n is constrained further by the superconformal generators (11.39).

As noted at the end of Section 11.3.4, the $U(1)$ generator (11.41) requires the 4-point superamplitude to have degree 8. Since the supermomentum delta function is already degree 8 in the η_is, we infer that f_4 can only depend on the bosonic variables $|i\rangle$.

Next, annihilation of the superamplitude by the dilatation operator D in (11.39) implies

$$D\mathcal{A}_n = 0 \quad\to\quad \sum_i\Big(\tfrac{1}{2}|i\rangle^a\partial_{|i\rangle^a}\Big)\mathcal{A}_n = -\frac{n}{2}\mathcal{A}_n\,. \tag{11.44}$$

As in 4d (see Exercise 5.2), the operator $\sum_i(\frac{1}{2}|i\rangle^a\partial_{|i\rangle^a})$ counts the mass dimension when acting on a function of spinor brackets. It also acts on the delta functions in $\mathcal{A}_n$, giving a factor of -3 on $\delta^3(P)$ and 4 on $\delta^{(8)}(\widetilde{Q})$. Thus, by (11.44), dilatation invariance requires the mass-dimension of f_4 to be $-\frac{4}{2} - (-3+4) = -3$.

► **Exercise 11.5**

Use the example around equation (5.4) to show that $\sum_i(\frac{1}{2}|i\rangle^a\partial_{|i\rangle^a})\,\delta^3(P) = -3\delta^3(P)$.

To summarize, from invariance under $U(1)$ R-symmetry and dilatation, we conclude that f_4 is a purely bosonic function of mass-dimension -3. Finally, taking into account that the superamplitude must be little group invariant, we can write the 4-point superamplitude [148] as

$$\mathcal{A}_4(\Phi_1\Phi_2\Phi_3\Phi_4\big) = \delta^3\big(P\big)\,\delta^{(8)}\big(\widetilde{Q}\big)\;\frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,. \tag{11.45}$$

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We can now project out the 4-scalar amplitude $A_4(\phi\phi\bar{\phi}\bar{\phi})$ using (11.42): the Grassmann delta function produces a factor of $\langle 34\rangle^4$, so we get (with the help of momentum conservation)

$$A_4(\phi\phi\bar{\phi}\bar{\phi}) = \frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 13\rangle} = -\frac{\langle 34\rangle^3}{\langle 24\rangle\langle 23\rangle}\,, \tag{11.46}$$

The astute reader should object: multiplying the solution (11.45) by an arbitrary function of $\frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}$ still satisfies all previous criteria. This is indeed a valid objection; however, such a function would change the pole structure of component amplitudes, such as (11.46), generated by $\mathcal{A}_4$. We have imposed in (11.45) that the amplitudes only have $1/\sqrt{p^2}$ poles. Why? Well, since the only scalar-fermion interactions are of the form $\phi^3\bar{\phi}^3$ and $\bar{\psi}\psi\bar{\phi}\phi$, poles in the tree-level amplitude $A_4(\phi\phi\bar{\phi}\bar{\phi})$ cannot arise from scalar or fermion propagators. Hence the only option is that they come from gauge boson exchanges. Since we are considering a 3d superconformal theory, the gauge boson self-coupling must be dimensionless; this rules out 3d Yang–Mills theory and rules in Chern–Simons gauge theory. Hence all poles in $A_4(\phi\phi\bar{\phi}\bar{\phi})$ must be $1/\sqrt{p^2}$ and this fixes the 4-point tree superamplitude in an $\mathcal{N} = 8$ superconformal 3d theory to be (11.45).

The result (11.45) for the superamplitude has a very important property: it is antisymmetric under the exchange of any two external particles. This property is inherited by the component amplitude $A_4(\phi\phi\bar{\phi}\bar{\phi})$ in (11.46) and that contradicts the expected Bose symmetry. We encountered something similar in Section 2.6 when we wrote down the 3-point gluon amplitudes in 4d from just little group scaling and dimensional analysis. The resolution was to include the antisymmetric structure constants f^{abc} of the Yang–Mills gauge group.

At the superamplitude level, the same issue arises: the physical degrees of freedom are contained in the bosonic superfield Φ, so $\mathcal{A}_4(\Phi_1\Phi_2\Phi_3\Phi_4)$ should be Bose symmetric under the exchange of any two external legs. But – as you see from (11.45) – it is fully antisymmetric. We could avoid this contradiction if the amplitudes were color-ordered. However, the presence of the $1/\langle 24\rangle$ pole in for example (11.46) invalidates this interpretation. Instead, the contradiction can be resolved if we include more than one supermultiplet, each one with a label a_i. Then we can introduce a new 4-index "coupling constant" $f^{a_1a_2a_3a_4}$ that is completely antisymmetric in all four indices. Using this we write

$$\mathcal{A}_4\big(\Phi_1^{a_1}\Phi_2^{a_2}\Phi_3^{a_3}\Phi_4^{a_4}\big) \;=\; \delta^3\big(P\big)\,\delta^{(8)}\big(\widetilde{Q}\big)\,\frac{f^{a_1a_2a_3a_4}}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,. \tag{11.47}$$

Now Bose symmetry is respected. Thus *by requiring $\mathcal{N} = 8$ superconformal symmetry in 3d, the 4-point superamplitude forces us to introduce a completely* ***antisymmetric 4-index coupling constant***.

This new coupling constant looks similar to the totally antisymmetric 3-index structure constant of Yang–Mills theory $f^{a_1a_2a_3}$. This resemblance is not a coincidence. In the search for an $\mathcal{N} = 8$ super Chern–Simons matter theory, Bagger, Lambert, and Gustavsson (BLG) [151, 152] found a Lagrangian whose gauge symmetry is built on a ***Lie 3-algebra***.

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This algebra is defined through a triple product

$$[T^a, T^b, T^c] = f^{abc}{}_d\, T^d\,. \tag{11.48}$$

The gauge indices are raised/lowered with $h^{ab} = \mathrm{Tr}\, T^a T^b$ and its inverse. The structure constants $f^{abcd} = f^{abc}{}_e h^{ed}$ are totally antisymmetric. Just as the structure constants of the usual gauge Lie 2-algebra satisfy the Jacobi identity (2.69), the 3-algebra structure constants are required to satisfy a four-term "fundamental identity":

$$f^{fgd}{}_e f^{abce} - f^{fga}{}_e f^{bcde} + f^{fgb}{}_e f^{cdae} - f^{fgc}{}_e f^{dabe} = 0\,. \tag{11.49}$$

The fields in ***BLG theory*** [151, 152] consist of eight scalars $X_a^{I_v}$ with $I_v = 1, \ldots, 8$ transforming as a vector of $SO(8)$, eight real spinors $\Psi_a^{I_c}$ with $I_c = 1, \ldots, 8$ transforming as a chiral spinor of $SO(8)$, and a Chern–Simons gauge field A_μ^{ab}. The BLG Lagrangian is [153]

$$\begin{aligned}\frac{1}{\kappa_{\mathrm{CS}}}\mathcal{L}_{\mathrm{BLG}} = {}& \frac{1}{48}\epsilon^{\mu\nu\rho}\left(\frac{1}{2}f^{abcd}A_{\mu ab}\partial_\nu A_{\rho cd} + \frac{1}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\rho ef}\right)\\ & - \frac{1}{2}D^\mu X_a^{I_v} D_\mu X_a^{I_v} + \frac{i}{2}\bar{\Psi}_a^{I_c}\not{D}\Psi_a^{I_c} + i3\, f^{abcd}\bar{\Psi}_a\Gamma^{I_v J_v}\Psi_b X_c^{I_v} X_d^{J_v}\\ & - 12\, f^{abcd} f_a{}^{efg}(X_b^{I_v} X_c^{J_v} X_d^{K_v})(X_e^{I_v} X_f^{J_v} X_g^{K_v})\,.\end{aligned} \tag{11.50}$$

In the Lagrangian construction [151, 152], the need for the antisymmetric 4-index structure constant comes from the requirement that the supersymmetry transformations on the fields close into the correct algebra. Linear combinations of the eight scalars and fermions can be identified as the $(\phi, \phi_{AB}, \bar{\phi})$ and $(\psi_A, \bar{\psi}^A)$ components of our superfield (11.42). Indeed, the 4-point amplitudes computed from the Lagrangian (11.50) match [148] the component amplitudes of the 4-point superamplitude (11.47).

It is quite non-trivial for an antisymmetric f^{abcd} to satisfy (11.49) and currently the only known example is if a is an index of $SO(4)$ and $f^{abcd} \sim \epsilon^{abcd}$. In search for other examples, there were many attempts to relax the symmetry properties of the 4-index structure constant [154–156]. However, as we have shown from the on-shell analysis, $\mathcal{N} = 8$ superconformal symmetry only allows for a totally antisymmetric structure constant. Indeed, all known examples of Lie 3-algebras with f^{abcd} not totally antisymmetric correspond to Chern–Simons matter theories with $\mathcal{N} < 8$ supersymmetries.

11.3.6 $\mathcal{N} = 6$ superconformal theory: ABJM

Let us now consider a 3d superconformal theory with $\mathcal{N} = 6$ supersymmetry. The R-symmetry is $SO(6) = SU(4)$ and the physical degrees of freedom are four complex scalars X_{A} and four complex fermions $\psi^{\mathsf{A}a}$ as well as their complex conjugates $\bar{X}^{\mathsf{A}}$ and $\bar{\psi}_{\mathsf{A}a}$. They transform in the fundamental or anti-fundamental of $SU(4)$ and $\mathsf{A} = 1, 2, 3, 4$. To arrange

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these states in on-shell superspace, we introduce three anticommuting variables η_A and write

$$\begin{aligned}\Phi &= X_4 + \eta_A\,\psi^A - \frac{1}{2}\epsilon^{ABC}\,\eta_A\eta_B\,X_C - \eta_1\eta_2\eta_3\,\psi^4\,,\\ \bar\Psi &= \bar\psi_4 + \eta_A\bar X^A - \frac{1}{2}\epsilon^{ABC}\,\eta_A\eta_B\,\bar\psi_C - \eta_1\eta_2\eta_3\,\bar X^4\,.\end{aligned} \tag{11.51}$$

We have split the fields as $X_{\mathsf{A}} \to (X_4, X_A)$ and $\psi^{\mathsf{A}} \to (\psi^4, \psi^A)$, and similarly for $\bar X^{\mathsf{A}}$ and $\bar\psi_{\mathsf{A}}$. So only an $SU(3)$ subgroup of the $SU(4)$ is manifest in this on-shell superspace formalism.

The on-shell superspace representation (11.51) involves a bosonic superfield Φ and a fermionic superfield $\bar\Psi$. Having two superfields is standard for superamplitudes in theories with less-than-maximal supersymmetry. For example in 4d $\mathcal{N} < 4$ SYM, the spectrum is not CPT self-conjugate and therefore a superfield is needed for each of the CPT conjugate supermultiplets; details and applications of the formalism can be found in [87]. In 3d, the need for two superfields comes from R-symmetry. Just as in 4d $\mathcal{N} < 4$ SYM, where the two superfields have states that are parity-conjugate with respect to each other, in 3d the two superfields contain states that are conjugate to each other under R-symmetry.

Since fermions transform with a minus under 3d little group transformations, the superamplitude must be odd under $|i\rangle \to -|i\rangle$ and $\eta_i \to -\eta_i$ if i is a $\bar\Psi$ state. Following the same steps as for the $\mathcal{N} = 8$ BLG theory in Section 11.3.5, we then find that the 4-point superamplitude in a 3d $\mathcal{N} = 6$ superconformal theory is fixed up to a multiplicative constant to be [157]

$$\mathcal{A}_4\big[\bar\Psi_1\Phi_2\bar\Psi_3\Phi_4\big] = \delta^3\big(P\big)\,\delta^{(6)}\big(\widetilde{Q}\big)\,\frac{1}{\langle 14\rangle\langle 43\rangle}\,. \tag{11.52}$$

The 4-point superamplitude (11.52) precisely encodes the color-ordered 4-point amplitudes of an $\mathcal{N} = 6$ Chern–Simons matter theory that was constructed by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [158]. The theory, known as the ***ABJM theory***, contains two gauge fields $A^{\mathsf{a}}{}_{\mathsf{b}}$ and $\hat A^{\dot{\mathsf{a}}}{}_{\dot{\mathsf{b}}}$ with gauge group $U(N) \times U(N)$. The matter fields are in the bi-fundamental, meaning that they transform in the fundamental of one $U(N)$ gauge group and the anti-fundamental of the other $U(N)$. Hence, the index structures of the matter fields are $(X_{\mathsf{A}})^{\dot{\mathsf{a}}}{}_{\mathsf{a}}$, $(\bar X^{\mathsf{A}})^{\mathsf{a}}{}_{\dot{\mathsf{a}}}$, $(\psi^{\mathsf{A}})^{\dot{\mathsf{a}}}{}_{\mathsf{a}}$, and $(\bar\psi_{\mathsf{A}})^{\mathsf{a}}{}_{\dot{\mathsf{a}}}$. The Lagrangian is [159, 160]

$$\begin{aligned}\mathcal{L}_{\text{ABJM}} = \frac{k}{2\pi}\Bigg[&\frac{1}{2}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho - \hat A_\mu\partial_\nu\hat A_\rho - \frac{2i}{3}\hat A_\mu\hat A_\nu\hat A_\rho\right)\\ &- (D^\mu X_{\mathsf{A}})^\dagger D_\mu X_{\mathsf{A}} + i\bar\psi_{\mathsf{A}}\not{D}\psi^{\mathsf{A}} + \mathcal{L}_4 + \mathcal{L}_6\Bigg]\,,\end{aligned} \tag{11.53}$$

where the covariant derivatives for the bi-fundamental fields are

$$\begin{aligned}D^\mu X_{\mathsf{A}} &\equiv \partial_\mu X_{\mathsf{A}} + i\,\hat A_\mu X_{\mathsf{A}} - i\,X_{\mathsf{A}} A_\mu\,,\\ (D^\mu X_{\mathsf{A}})^\dagger &\equiv \partial_\mu \bar X^{\mathsf{A}} + i\,A_\mu\bar X^{\mathsf{A}} - i\,\bar X^{\mathsf{A}}\hat A_\mu\,,\end{aligned} \tag{11.54}$$

with the same definitions for ψ^{A} and $\bar\psi_{\mathsf{A}}$.

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The quartic and sextic interaction terms in (11.53) are

$$\mathcal{L}_4 = i\,\mathrm{Tr}\Big(\bar{X}^{\mathsf{B}} X_{\mathsf{B}} \bar{\psi}_{\mathsf{A}} \psi^{\mathsf{A}} - X_{\mathsf{B}} \bar{X}^{\mathsf{B}} \psi^{\mathsf{A}} \bar{\psi}_{\mathsf{A}} + 2 X_{\mathsf{A}} \bar{X}^{\mathsf{B}} \psi^{\mathsf{A}} \bar{\psi}_{\mathsf{B}} - 2 \bar{X}^{\mathsf{A}} X_{\mathsf{B}} \bar{\psi}_{\mathsf{A}} \psi^{\mathsf{B}} - \epsilon_{\mathsf{ABCD}} \bar{X}^{\mathsf{A}} \psi^{\mathsf{B}} \bar{X}^{\mathsf{C}} \psi^{\mathsf{D}} + \epsilon^{\mathsf{ABCD}} X_{\mathsf{A}} \bar{\psi}_{\mathsf{B}} X_{\mathsf{C}} \bar{\psi}_{\mathsf{D}}\Big), \tag{11.55}$$

$$\mathcal{L}_6 = \frac{1}{3}\,\mathrm{Tr}\Big(X_{\mathsf{A}} \bar{X}^{\mathsf{A}} X_{\mathsf{B}} \bar{X}^{\mathsf{B}} X_{\mathsf{C}} \bar{X}^{\mathsf{C}} + \bar{X}^{\mathsf{A}} X_{\mathsf{A}} \bar{X}^{\mathsf{B}} X_{\mathsf{B}} \bar{X}^{\mathsf{C}} X_{\mathsf{C}} + 4 \bar{X}^{\mathsf{A}} X_{\mathsf{B}} \bar{X}^{\mathsf{C}} X_{\mathsf{A}} \bar{X}^{\mathsf{B}} X_{\mathsf{C}} - 6 X_{\mathsf{A}} \bar{X}^{\mathsf{B}} X_{\mathsf{B}} \bar{X}^{\mathsf{A}} X_{\mathsf{C}} \bar{X}^{\mathsf{C}}\Big). \tag{11.56}$$

For theories whose external states are bi-fundamental matter fields, the color structure of the amplitude is given in terms of a product of Kronecker deltas. In particular, with $n = 2m$ the full color-dressed amplitude is [157]

$$\sum_{\sigma \in S_m,\ \bar{\sigma} \in \bar{S}_{m-1}} \mathcal{A}_n(\bar{1}, \sigma_1, \bar{\sigma}_1, \ldots, \bar{\sigma}_{m-1}, \sigma_m)\, \delta^{\dot{\mathsf{a}}_{\sigma_1}}_{\dot{\mathsf{a}}_{\bar{1}}} \cdots \delta^{\dot{\mathsf{a}}_{\sigma_m}}_{\dot{\mathsf{a}}_{\bar{\sigma}_{m-1}}}\, \delta^{\mathsf{a}_{\bar{\sigma}_1}}_{\mathsf{a}_{\sigma_1}} \cdots \delta^{\mathsf{a}_{\bar{1}}}_{\mathsf{a}_{\sigma_m}}, \tag{11.57}$$

where the sums are over all distinct permutations of m even sites and $m-1$ odd sites. Each partial amplitude $\mathcal{A}_n$ is multiplied by a product of Kronecker deltas, and this naturally defines an ordering, very similar to Yang–Mills amplitudes. However, since the on-shell degrees of freedom are contained in two distinct supermultiplets, the color-ordered superamplitude is not cyclically invariant, but invariant up to a sign under cyclic rotation of *two* sites:

$$\mathcal{A}_{n=2m}\big[\bar{\Psi}_1 \Phi_2 \ldots \Phi_{2m}\big] = (-1)^{m-1} \mathcal{A}_{n=2m}\big[\bar{\Psi}_3 \Phi_4 \ldots \Phi_{2m} \bar{\Psi}_1 \Phi_2\big]; \tag{11.58}$$

the minus signs come from the exchanges of $\bar{\Psi}$s. For the superamplitude (11.52), the 2-site cyclic property (11.58) is ensured by momentum conservation.

After having seen a Lie 3-algebra appear in the $\mathcal{N} = 8$ superconformal BLG theory in Section 11.3.5, you may wonder if the above Lagrangian can also be rewritten in terms of a 3-algebra. Indeed it can! In fact, we can read off the properties of the 4-index structure constants from the 4-point superamplitude (11.52). It is symmetric under the exchange of the legs that correspond to the fermionic supermultiplet $\bar{\Psi}$, while it is antisymmetric under the exchange of the bosonic multiplets Φ. This is opposite from the expected symmetry properties of $\mathcal{A}_4(\bar{\Psi}_1 \Phi_2 \bar{\Psi}_3 \Phi_4)$, and therefore one can consider dressing the superamplitude with a 4-index structure constant $f^{a_2 a_4 \bar{a}_1 \bar{a}_3}$ that is antisymmetric with respect to the exchange of barred or unbarred indices, respectively. The color-dressed superamplitude is then[7]

$$\mathcal{A}_4\big(\bar{\Psi}_1^{\bar{a}_1} \Phi_2^{a_2} \bar{\Psi}_3^{\bar{a}_3} \Phi_4^{a_4}\big) = \delta^3\big(P\big)\, \delta^{(6)}\big(\widetilde{Q}\big)\, \frac{f^{a_2 a_4 \bar{a}_1 \bar{a}_3}}{\langle 14 \rangle \langle 43 \rangle}. \tag{11.59}$$

[7] You might wonder why this issue did not come up when we stated that the amplitude in (11.52) matched that derived from the Lagrangian (11.53). The reason is that it matched in the context of a color-ordered amplitude where the exchange of external lines is not a symmetry. In contrast, here we are considering a fully color-dressed amplitude. In other words, we are asking what properties the color factor should have such that the amplitude can be considered as a color-dressed amplitude.

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It has been shown [161] that the Lagrangian (11.53) is completely equivalent to an alternative one where the matter fields carry the 3-algebra indices indicated in (11.59).

11.3.7 BCFW recursion in 3d

We argued in Section 11.3.4 that only even-point amplitudes are non-vanishing in 3d superconformal theories. This means that the 4-point superamplitudes are the building blocks of higher-point amplitudes in these theories. Conveniently, we found that the 4-point tree-level superamplitudes in $\mathcal{N} = 8$ and $\mathcal{N} = 6$ theories in Sections 11.3.5 and 11.3.6 are completely determined by the requirements of symmetries and pole structure. Now is the time to go to higher-point, and of course our favorite tool is BCFW recursion.

To get started, we have to set up a BCFW recursion relation in 3d. And 3d is different from all other $D > 3$ in terms of defining a BCFW deformation. To see this, recall from Chapter 3 that we shift two external momenta i and j linearly,

$$p_i \to p_i + zq, \quad p_j \to p_j - zq\,, \tag{11.60}$$

with a vector q that satisfies

$$q \cdot p_i = q \cdot p_j = q^2 = 0\,. \tag{11.61}$$

This ensures that the shifted momenta remain on-shell and that invariants $\hat{P}^2_{ij\ldots k}$ are linear in z, so that each propagator going on-shell corresponds to a unique simple pole in the z-plane.

Unfortunately (or, very interestingly, if that is how you like it), in 3d the only q that satisfies these constraints is $q = 0$. The reason is this. A 3d vector q with $q^2 = 0$ can be written as a bi-spinor $q = |q\rangle^a |q\rangle^b$. The $|q\rangle$ is a 2-component spinor so it cannot be linearly independent from the $|i\rangle$ and $|j\rangle$ of the two light-like momenta we are shifting. Hence

$$|q\rangle = \alpha|i\rangle + \beta|j\rangle \tag{11.62}$$

for some numbers α and β. Solving for α and β subject to the constraints $q \cdot p_i = q \cdot p_j = 0$ in (11.61) gives $\alpha = \beta = 0$ and hence $q = 0$.

So in order to make progress, we need to relax some of the constraints imposed on the shifted momenta. We cannot give up on momentum conservation and on-shellness for the shifted momenta. Instead, we can either shift three or more external momenta or give up on the property that the momenta shift linearly in z. The former is similar to the shift associated with CSW (Section 3.4) and comes at the price of involving many diagrams for the superamplitudes. Opting for the solution with fewer diagrams, we choose to give up on linearity in z and consider the following general 2-line "deformation" [162]

$$\begin{pmatrix} |\hat{i}\rangle \\ |\hat{j}\rangle \end{pmatrix} = R(z) \begin{pmatrix} |i\rangle \\ |j\rangle \end{pmatrix}\,. \tag{11.63}$$

$R(z)$ is a 2×2 matrix that depends on z. Since we want the shift to respect momentum conservation, the matrix R must satisfy:

$$R(z)^T\, R(z) = I\,. \tag{11.64}$$

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Thus $R(z)$ is an orthogonal matrix and we can parameterize it as

$$R(z) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} . \tag{11.65}$$

If we define the deformation on the fermionic variables η_i and η_j in the same fashion, supermomentum conservation is also preserved by the shift:

$$\hat{\tilde{q}}_{iA} + \hat{\tilde{q}}_{jA} = \left(|\hat{i}\rangle , |\hat{j}\rangle \right) \begin{pmatrix} \hat{\eta}_{iA} \\ \hat{\eta}_{jA} \end{pmatrix} = \left(|i\rangle , |j\rangle \right) R^T(z) R(z) \begin{pmatrix} \eta_{iA} \\ \eta_{jA} \end{pmatrix} = \tilde{q}_{iA} + \tilde{q}_{jA} . \tag{11.66}$$

The deformation matrix (11.65) becomes the identity when $z = 1$, so unshifted kinematics corresponds to $z = 1$ and not $z = 0$. This leads to the following contour integral representation of the unshifted tree-level amplitude:

$$\mathcal{A}_n = \frac{1}{2\pi i} \oint_{z=1} \frac{\hat{\mathcal{A}}_n(z)}{z-1} , \tag{11.67}$$

where the contour wraps just the pole at $z = 1$. If the deformed superamplitude $\hat{\mathcal{A}}_n(z)$ vanishes as $z \to \infty$,[8] one can perform a contour-deformation and evaluate the amplitude as a sum of the residues at finite $z \neq 0, 1$.

Just as in 4d, the poles at finite $z \neq 0, 1$ correspond to propagators going on-shell. Let us take a closer look at what the singularities look like. Without loss of generality, we choose 1 and n as the deformed momenta:

$$\begin{aligned} \hat{p}_1^{ab} &= \frac{1}{2}(p_1^{ab} + p_n^{ab}) + z^2 q^{ab} + z^{-2} \tilde{q}^{ab} , \\ \hat{p}_n^{ab} &= \frac{1}{2}(p_1^{ab} + p_n^{ab}) - z^2 q^{ab} - z^{-2} \tilde{q}^{ab} . \end{aligned} \tag{11.68}$$

Here q and $\tilde{q}$ are given by

$$q^{ab} = \frac{1}{4}(|1\rangle + i|n\rangle)^a (|1\rangle + i|n\rangle)^b , \qquad \tilde{q}^{ab} = \frac{1}{4}(|1\rangle - i|n\rangle)^a (|1\rangle - i|n\rangle)^b . \tag{11.69}$$

Defining $P_{12\ldots i}^{ab} = p_1^{ab} + p_2^{ab} + \cdots + p_i^{ab}$, the on-shell condition for the shifted propagator $\hat{P}_{12\ldots i}^2$ takes the form

$$\hat{P}_{12\ldots i}^2 = \langle \tilde{q}|P_{23\ldots i}|\tilde{q}\rangle z^{-2} + \langle q|P_{23\ldots i}|q\rangle z^2 - (P_{23\ldots i} \cdot P_{i+1\ldots n-1}) = 0 , \tag{11.70}$$

where $(p_i \cdot p_j) = p_i^\mu p_{j\mu}$ and $\langle i|P|j\rangle \equiv \lambda_i^a P_a{}^b \lambda_{jb}$. One can explicitly write down the values of z that correspond to the propagator going on-shell

$$\left\{ (z_{1,i}^*)^2 , (z_{2,i}^*)^2 \right\} = \frac{(P_{2\ldots i} \cdot P_{i+1\ldots n-1}) \pm \sqrt{(P_{2\ldots i})^2 (P_{i+1\ldots n-1})^2}}{2\langle q|P_{2\ldots i}|q\rangle} . \tag{11.71}$$

[8] One should also make sure that there are no poles at $z = 0$. Exchanging $1/z \leftrightarrow z$ in (11.65) can be compensated by extra sign factors in the kinematics of the shifted legs, so if $\mathcal{A}_n(z)$ vanishes as $z \to \infty$ for generic kinematics, then it also vanishes at $z = 0$.

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▶ **Exercise 11.6**

Prove the following useful identity

$$\left((z^*_{1,i})^2 - 1\right)\left((z^*_{2,i})^2 - 1\right) = \frac{P^2_{12\ldots i}}{\langle q|P_{2\ldots i}|q\rangle}\,. \tag{11.72}$$

As the propagator goes on-shell, the amplitude factorizes into two lower-point amplitudes. This allows us to write the sum of residues at $z \neq 1$ as a sum over distinct single propagator diagrams with legs 1 and n on opposite sides of the propagator. For Chern–Simons matter theories, we also require that only even multiplicity subamplitudes appear on each side of the propagator. For each propagator, one needs to sum over the four solutions, $(z^*_{1,i},\ -z^*_{1,i},\ z^*_{2,i},\ -z^*_{2,i})$ to the on-shell constraint (11.70). The final result is then [162]

$$\mathcal{A}_n = \sum_i \int d^3\eta_I \left(\hat{\mathcal{A}}_L\left(z^*_{1,f};\,\eta_I\right)\frac{H(z^*_{1,f}, z^*_{2,f})}{P^2_{12\ldots i}}\hat{\mathcal{A}}_R\left(z^*_{1,f};\,i\eta_I\right) + \left(z^*_{1,f} \leftrightarrow z^*_{2,f}\right)\right), \tag{11.73}$$

where the function $H(a, b)$ is

$$H(a,b) \equiv \frac{a(b^2-1)}{a^2-b^2} \tag{11.74}$$

and the Grassmann integral takes care of the intermediate state sum. Did you notice the i in $\hat{\mathcal{A}}_R$? That comes from the analytic continuation of the incoming $\to$ outgoing internal line. In 3d massless kinematics, we only have one type of spinor, namely $|p\rangle$, so with $p^{ab} = -|p\rangle^a|p\rangle^b$ we must have

$$|-p\rangle = i\,|p\rangle\,. \tag{11.75}$$

Hence we must also have $\eta_{-p} = i\,\eta_p$, since – as you can check – this ensures that the arguments of the L and R Grassmann delta functions add up to the overall supermomentum $\widetilde{Q}$.

▶ **Exercise 11.7**

Show that the contour deformation of (11.67) gives the representation (11.73).
[Hints: The identity in (11.72) will be useful. Furthermore, since one of the shifted legs, 1 or n, necessarily corresponds to the fermionic multiplet, we have $\mathcal{A}_L(-z)\mathcal{A}_R(-z) = -\mathcal{A}_L(z)\mathcal{A}_R(z)$.]

The validity of (11.73) relies on whether or not the super-shifted superamplitude vanishes as $z \to \infty$. It was shown in [162] that this criterion is satisfied for the ABJM and BLG theories.

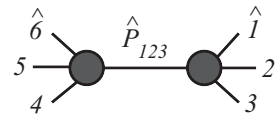
▶ **Exercise 11.8**

Recall that in Section 3.3 we discussed when a BCFW recursion is valid: we showed that the presence of contact terms, for example ϕ^4, in the action tends to spoil the

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recursion since such terms go to a constant as $z \to \infty$. This issue can be avoided in supersymmetric theories since amplitudes where such terms are present are related via supersymmetry to those from which it is absent. This is accomplished via the super-BCFW shifts. One can illustrate the idea by carefully choosing the external states such that contact terms do not contribute to a particular component amplitude; then (loosely speaking) supersymmetry ensures that the superamplitude, which contains this well-behaved component amplitude, also goes to zero for $z \to \infty$. Let us test whether such a component amplitude can be found for ABJM theory at 6-point. Consider the 6-point contact term in (11.56). Show that if we choose all R-symmetry indices to be the same, say 1, then the sextic interaction terms vanish. Thus the 6-point scalar amplitude with all scalars having the same $SU(4)$ indices has good large-z behavior.

One thing is deriving the recursion relations, another thing is using them! So let us now apply the 3d recursion relations to compute the 6-point amplitude $A_6(\bar{X}^4 X_4 \bar{X}^4 X_4 \bar{\psi}_4 \psi^4)$ in ABJM theory. For simplicity, we drop the $SU(4)$ indices on the component-fields, i.e. $\bar{X}^4 \to \bar{X}$. Choosing lines 1 and 6 for the shift, the only factorization channel is (123|456), so there is only one diagram, namely



(11.76)

The recursion relation (11.73) then reads

$$\mathcal{A}_6\big(\bar{\Psi}\Phi\bar{\Psi}\Phi\bar{\Psi}\Phi\big) = \int d^3\eta \left[\hat{\mathcal{A}}_4\big(\bar{\Psi}_{\hat{1}}\Phi_2\bar{\Psi}_3\Phi_{\hat{P}_{123}}\big)\Big|_{z=z_1^*} \frac{H(z_1^*, z_2^*)}{P_{123}^2}\, \mathcal{A}_4\big(\bar{\Psi}_{-\hat{P}_{123}}\Phi_4\bar{\Psi}_5\Phi_{\hat{6}}\big)\Big|_{z=z_1^*} + (z_1^* \leftrightarrow z_2^*) \right]. \tag{11.77}$$

To project out the amplitude $A_6(\bar{X}X\bar{X}X\bar{\psi}\psi)$ from the superamplitude (11.77) we need the coefficient of the $(\eta_1)^3(\eta_3)^3(\eta_6)^3$ monomial, where $(\eta_i)^3 = \eta_{i1}\eta_{i2}\eta_{i3}$. This follows from (11.51). After manipulation of the Grassmann delta functions and using $\hat{\eta}_1(z)\hat{\eta}_6(z) = \eta_1\eta_6$, we find

$$A_6\big(\bar{X}X\bar{X}X\bar{\psi}\psi\big) = \hat{A}_4\big(\bar{X}_{\hat{1}}X_2\bar{X}_3X_{\hat{P}_{123}}\big)\Big|_{z=z_1^*} \frac{H(z_1^*, z_2^*)}{P_{123}^2}\, \hat{A}_4\big(\bar{X}_{\hat{P}_{123}}X_4\,\bar{\psi}_5\,\psi_{\hat{6}}\big)\Big|_{z=z_1^*} + (z_1^* \leftrightarrow z_2^*), \tag{11.78}$$

where the 4-point amplitudes, obtained from the superamplitude (11.52), are

$$\hat{A}_4\big(\bar{X}_{\hat{1}}X_2\bar{X}_3X_{\hat{P}_{123}}\big) = -\frac{\langle\hat{1}3\rangle^3}{\langle\hat{1}\hat{P}_{123}\rangle\langle\hat{P}_{123}3\rangle} \quad \text{and} \quad \hat{A}_4\big(\bar{X}_{\hat{P}_{123}}X_4\,\bar{\psi}_5\,\psi_{\hat{6}}\big) = \frac{\langle\hat{P}_{123}\hat{6}\rangle^2}{\langle 65\rangle}. \tag{11.79}$$

By (11.71), the poles in the z-plane are located at

$$z_1^{*2} = \frac{\langle 16\rangle^2 - \big(\langle 23\rangle - \langle 45\rangle\big)^2}{\big(\langle 1| + i\langle 6|\big)P_{45}\big(|1\rangle + i|6\rangle\big)}, \qquad z_2^{*2} = \frac{\langle 16\rangle^2 - \big(\langle 23\rangle + \langle 45\rangle\big)^2}{\big(\langle 1| + i\langle 6|\big)P_{45}\big(|1\rangle + i|6\rangle\big)}. \tag{11.80}$$

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After repeated use of momentum conservation and the Schouten identity, we find that the 6-point amplitude is

$$A_6(\bar{X}X\bar{X}X\bar{\psi}\psi) = -\frac{1}{2P_{123}^2}\left[\frac{\big(\langle 2|P_{123}|6\rangle + i\langle 31\rangle\langle 45\rangle\big)^3}{\big(\langle 1|P_{123}|4\rangle + i\langle 23\rangle\langle 56\rangle\big)\big(\langle 3|P_{123}|6\rangle + i\langle 12\rangle\langle 45\rangle\big)} - \frac{\big(\langle 2|P_{123}|6\rangle - i\langle 31\rangle\langle 45\rangle\big)^3}{\big(\langle 1|P_{123}|4\rangle - i\langle 23\rangle\langle 56\rangle\big)\big(\langle 3|P_{123}|6\rangle - i\langle 12\rangle\langle 45\rangle\big)}\right]. \tag{11.81}$$

Here the first term is the result of evaluating the first term in (11.77) while the second term above is the $(z_1^* \leftrightarrow z_2^*)$ contribution.

▶ Exercise 11.9

Let us derive (11.81) from (11.78). First prove

$$\big((z_1^*)^2 + 1\big)\big((z_2^*)^2 - 1\big) = \frac{-i\langle 1|P_{23}|6\rangle + \langle 23\rangle\langle 45\rangle}{\langle q|P_{2\ldots i}|q\rangle}. \tag{11.82}$$

Next, use (11.82) to show that

$$\langle \hat{1}3\rangle\, z_1^*\big((z_2^*)^2 - 1\big) = \frac{i\langle 23\rangle\big(\langle 2|P_{123}|6\rangle + i\langle 31\rangle\langle 45\rangle\big)}{2\langle q|P_{23}|q\rangle}. \tag{11.83}$$

Now continue to manipulate the tree amplitudes (11.79) to derive the first line in (11.81).

You may worry about the apparently spurious poles in the expression (11.81), since each only appears in one term and not the other and thus cannot cancel. But have no fear, these are really local poles in disguise! To see this, we rewrite them as (see Exercise 11.10)

$$\frac{1}{\langle 1|P_{123}|4\rangle - i\langle 23\rangle\langle 56\rangle} = \frac{\langle 1|P_{123}|4\rangle + i\langle 23\rangle\langle 56\rangle}{\langle 1|P_{123}|4\rangle^2 + \langle 23\rangle^2\langle 56\rangle^2} = \frac{\langle 1|P_{123}|4\rangle + i\langle 23\rangle\langle 56\rangle}{P_{123}^2 P_{234}^2}. \tag{11.84}$$

Thus each spurious-looking pole in (11.81) is really a product of local poles. Note that this tells us that the two terms in the BCFW result (11.81) are individually local and free of spurious poles! The reason behind this will be discussed further in Sections 11.3.8 and 11.3.10.

▶ Exercise 11.10

The final manipulation in (11.84) made use of the identity

$$\langle i|p_j + p_k|l\rangle^2 - (p_i + p_j + p_k + p_l)^2\langle jk\rangle^2 = (p_i + p_j + p_k)^2(p_j + p_k + p_l)^2,$$

which holds for any four massless vectors p_i, p_j, p_k, p_l in 3d. Prove it.

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▶ **Exercise 11.11**

Although the two terms in (11.81) are individually local, they actually need to come in the combination in (11.81): show that the relative minus sign is necessary for the amplitude to have the correct little group properties.

11.3.8 ABJM and dual conformal symmetry

Let us dive straight into the deep end and define 3d dual variables y_i^{ab} and θ_{iA}^a such that $y_i^{ab} - y_{i+1}^{ab} = p_i^{ab}$ and $\theta_{iA}^a - \theta_{i+1,A}^a = \tilde{q}_{iA}^a$. Momentum and supermomentum delta functions for a 4-point superamplitude are then

$$\delta^3\big(P\big)\delta^{(\mathcal{N})}\big(\tilde{Q}\big) \to \delta^3\big(y_1 - y_5\big)\delta^{(\mathcal{N})}\big(\theta_1^a - \theta_5^a\big)\,. \tag{11.85}$$

We define ***dual conformal inversion*** on the variables y_i and θ_i the same way in any spacetime dimension, namely as in (5.37). It then follows from (11.85) that the inversion weights of the momentum and supermomentum delta function exactly cancel for $\mathcal{N} = 6$ supersymmetry. Using

$$y_{i,i+2}^2 = s_{i,i+1} = \langle i, i+1\rangle^2\,, \tag{11.86}$$

we deduce (as in Exercise 5.6) the dual inversion rule for a 3d angle bracket

$$I[\langle i, i+1\rangle] = \frac{\langle i, i+1\rangle}{\sqrt{y_i^2 y_{i+2}^2}}\,. \tag{11.87}$$

For the 4-point superamplitude (11.52) of $\mathcal{N} = 6$ ABJM theory, this then implies

$$I\left[\mathcal{A}_4(\bar{\Psi}_1\Phi_2\bar{\Psi}_3\Phi_4)\right] = \sqrt{y_1^2 y_2^2 y_3^2 y_4^2}\;\mathcal{A}_4(\bar{\Psi}_1\Phi_2\bar{\Psi}_3\Phi_4)\,. \tag{11.88}$$

It can be shown [162], using the $\mathcal{N} = 6$ super-BCFW recursion relations, that dual inversion on the n-point tree-level superamplitude gives

$$I\left[\mathcal{A}_n\right] = \left(\prod_{i=1}^{n}\sqrt{y_i^2}\right)\mathcal{A}_n\,. \tag{11.89}$$

Thus the 3d ABJM tree-level superamplitudes are dual conformal *covariant* with uniform inversion weight $\frac{1}{2}$ on each leg.

Under dual conformal inversion, the superamplitudes of 4d $\mathcal{N} = 4$ SYM transform covariantly with uniform inversion weight 1 on each leg. In Section 5.3 we argued that, as a result, the dual conformal boosts $\mathcal{K}^\mu$ annihilate the superamplitudes only after the non-trivial weights have been compensated by a shift of $\mathcal{K}^\mu$, as below (5.42). This shift is crucial for defining the dual superconformal symmetry and extending it together with the ordinary superconformal symmetry to the $SU(2,2|4)$ Yangian of the 4d planar $\mathcal{N} = 4$ SYM superamplitudes.

In 3d, the dual conformal symmetry can be enlarged into the dual superconformal symmetry group $OSp(6|4)$ [163]. The symmetry group acts on the dual space that consists

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of coordinates $(y_i^{ab}, \theta_{iA}^a, r_{iAB})$, where the extra R-symmetry coordinate r_{iAB} is defined by:

$$r_{iAB} - r_{i+1,AB} = \eta_{iA}\eta_{iB} \,. \tag{11.90}$$

The group $OSp(6|4)$ is also the supergroup for the ordinary superconformal symmetry of the ABJM Lagrangian in (11.53). The combination of the dual and ordinary superconformal symmetries forms an infinite dimensional $OSp(6|4)$ Yangian algebra [157], very similar in nature to the $SU(2,2|4)$ Yangian symmetry of 4d planar $\mathcal{N} = 4$ SYM.

The super-BCFW construction (11.77) actually gives the tree-level 6-point superamplitude in terms of two Yangian invariants,

$$\mathcal{A}_6^{\text{tree}} \equiv \mathcal{A}_6^{\text{tree}}(\bar{\Psi}_1\Phi_2\bar{\Psi}_3\Phi_4\bar{\Psi}_5\Phi_6) = Y_1 + Y_2 \,. \tag{11.91}$$

The two Yangian invariants Y_1 and Y_2 arise precisely from the two BCFW-terms in (11.77). We will not need their explicit form; they can be found in [162].

It will be relevant for us to also consider the tree superamplitude with shifted sites,

$$\mathcal{A}_{6,\text{shifted}}^{\text{tree}} \equiv \mathcal{A}_6^{\text{tree}}(\Phi_1\bar{\Psi}_2\Phi_3\bar{\Psi}_4\Phi_5\bar{\Psi}_6) = \mathcal{A}_6^{\text{tree}}(\bar{\Psi}_2\Phi_3\bar{\Psi}_4\Phi_5\bar{\Psi}_6\Phi_1) \,. \tag{11.92}$$

It has a super-BCFW representation that can be written as

$$\mathcal{A}_{6,\text{shifted}}^{\text{tree}} = \mathcal{A}_6^{\text{tree}}(\Phi_1\bar{\Psi}_2\Phi_3\bar{\Psi}_4\Phi_5\bar{\Psi}_6) = Y_1 - Y_2 \,. \tag{11.93}$$

Now the important point is that two physical objects, $\mathcal{A}_6^{\text{tree}}$ and $\mathcal{A}_{6,\text{shifted}}^{\text{tree}}$, are written as distinct linear combinations of the same two Yangian invariants: this is only possible if each of the two Yangian invariants is local, i.e. free of spurious poles. We have already noted the locality for the particular component amplitude (11.81). Now you see why it was needed. Note that this distinguishes 3d ABJM theory from 4d planar $\mathcal{N} = 4$ SYM in which the dual superconformal invariant 5-brackets had spurious poles.

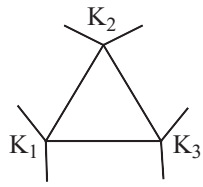
11.3.9 Loops and on-shell diagrams in ABJM

The loop-level superamplitudes can be explored using unitarity methods (Chapter 6). Using the dual inversion property of the tree-level superamplitudes, it can be shown that the planar loop superamplitudes of ABJM, prior to integration, are dual conformal covariant, i.e. they satisfy (11.89). Thus perturbatively, planar ABJM has a structure very similar to planar $\mathcal{N} = 4$ SYM, they are almost baby brothers/sisters. This is rather surprising given that the two theories have very distinct Lagrangians and live in different spacetime dimensions. Moreover, in quantum field theory textbooks, one learns that $D < 4$ theories generically have more severe IR divergences compared to $D = 4$. Thus one might expect that although planar ABJM is very similar to $\mathcal{N} = 4$ SYM at the pre-integrated level, the similarity would be completely scrambled by the potentially more severe IR-divergence in $D = 3$.

To see if this is the case, let us take a look at the planar loop amplitudes in detail. The 1-loop amplitudes in ABJM are purely rational functions [164–166]. This can be understood

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as a consequence of dual conformal symmetry, since the only dual conformal covariant scalar integral is the massive triangle integral, and it integrates to



$$I_3(K_1, K_2, K_3) = -\frac{i\pi}{2} \frac{1}{\sqrt{-K_1^2}\sqrt{-K_2^2}\sqrt{-K_3^2}}, \tag{11.94}$$

where K_1, K_2, K_3 are the sums of the external momenta going out of each of the three corners and $K_i^2 \neq 0$. There are no triangle diagrams with massless corners $K_i^2 = 0$; this follows from generalized unitarity methods using the fact that 3-point loop amplitudes vanish. The integrated result (11.94) has transcendentality 1 thanks to the factor of π.

Since 6-point is the lowest multiplicity at which the triangle integral (11.94) contributes after integration, we conclude that the ***4-point 1-loop amplitude vanishes*** up to $O(\epsilon)$ in dimensional regularization $D = 3 - 2\epsilon$.[9]

The ***6-point 1-loop superamplitude*** is [164, 165, 168][10]

$$\mathcal{A}_6^{\text{1-loop}} = -i \left(\frac{N}{k}\right) \mathcal{A}_{6,\text{shifted}}^{\text{tree}} \times \Big(\langle 12\rangle\langle 34\rangle\langle 56\rangle\, I_3(P_{12}, P_{34}, P_{56}) + \langle 23\rangle\langle 45\rangle\langle 61\rangle\, I_3(P_{23}, P_{45}, P_{61})\Big), \tag{11.95}$$

where N comes from the gauge group $U(N) \times U(N)$, and k is the Chern–Simons level in (11.53). The tree superamplitude $\mathcal{A}_{6,\text{shifted}}^{\text{tree}}$ was defined in (11.92). Using the integrated result (11.94) for the scalar triangle integrals I_3, we find that the 1-loop 6-point superamplitude is

$$\mathcal{A}_6^{\text{1-loop}} = -\frac{\pi}{2} \left(\frac{N}{k}\right) \mathcal{A}_{6,\text{shifted}}^{\text{tree}} \times \Big(\text{sgn}(\langle 12\rangle)\,\text{sgn}(\langle 34\rangle)\,\text{sgn}(\langle 56\rangle) + \text{sgn}(\langle 23\rangle)\,\text{sgn}(\langle 45\rangle)\,\text{sgn}(\langle 61\rangle)\Big). \tag{11.96}$$

Here we have introduced

$$\text{sgn}(\langle ij\rangle) \equiv \frac{\langle ij\rangle}{\sqrt{-K_{ij}^2}} = \frac{\langle ij\rangle}{|\langle ij\rangle|}, \tag{11.97}$$

which equals ± 1 depending on the kinematics. Thus, remarkably, the 1-loop 6-point superamplitude can be either zero or non-vanishing depending on the kinematics! This peculiar behavior has to do with an interesting topological feature of light-like momenta in 3-dimensions. In 3d Minkowski space, a light-like vector can be written as $p_i^\mu = E_i(1, \cos\theta_i, \sin\theta_i)$. This means that light-like vectors can be projected to points on

[9] The 4-point 1-loop integrand is non-trivial. It is given by a loop-momentum dependent integrand that integrates to zero up to $O(\epsilon)$ [167].

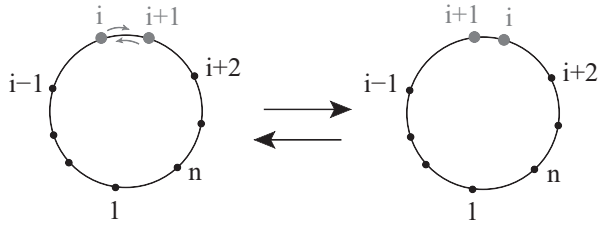
[10] This result is only valid up to $O(\epsilon)$. There are additional integrands, whose coefficients are proportional to the tree amplitude, that integrate to zero up to $O(\epsilon)$.

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a circle S^1. From

$$\langle ij\rangle = \sqrt{-2p_i \cdot p_j} = i\sqrt{E_i\, E_j}\, \sin\left(\frac{\theta_i - \theta_j}{2}\right) \tag{11.98}$$

we see that the sign of $\langle ij\rangle$ changes whenever the two points that represent p_i and p_j cross each other on the S^1. Thus the 1-loop amplitude encounters a sudden jump, from zero to non-vanishing or vice versa, whenever two points on the S^1 cross each other:



(11.99)

The two configurations (11.99) are topologically inequivalent.[11] Thus this sudden jolt is the amplitude's way of telling us that we are changing the topology of momentum space!

▶ **Exercise 11.12**

From 4d, we are familiar with loop amplitudes being proportional to the tree amplitudes, so it may seem odd that here in 3d the 1-loop 6-point amplitude (11.96) is proportional to the shifted tree amplitude. Verify that the LHS and RHS of (11.96) have the same little group scaling thanks to the sign-functions.

The first non-trivial loop contribution to the ***4-point amplitude*** enters at ***2-loop order*** and it is given by [167, 169]

$$\mathcal{A}_4^{\text{2-loop}} = \left(\frac{N}{k}\right)^2 \mathcal{A}_4^{\text{tree}} \Bigg[-\frac{\left(-\mu^{-2}y_{13}^2\right)^{-\epsilon} + \left(-\mu^{-2}y_{24}^2\right)^{-\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\ln^2\left(\frac{y_{13}^2}{y_{24}^2}\right) + 4\zeta_2 - 3\ln^2 2 + \mathcal{O}(\epsilon)\Bigg]. \tag{11.100}$$

Notice that the IR-divergent part is equivalent to that of the 1-loop 4-point superamplitude (6.42) of planar $\mathcal{N} = 4$ SYM, with $\epsilon \to 2\epsilon$ because this is 2-loops.[12] Not only is the IR-structure of this amplitude identical to that of planar $\mathcal{N} = 4$ SYM, but so is the finite $\ln^2\left(y_{13}^2/y_{24}^2\right)$ piece!

[11] This can be made more precise. By judiciously adding 2π to the angles θ_i, one can arrange the angles such that a given kinematics configuration has all angles strictly increasing according to their color ordering, i.e. $0 < \theta_{i+1} - \theta_i < 2\pi$. This gives a well defined "winding number" $w = (\theta_n - \theta_1)/(2\pi)$. Now as two points cross each other, the winding number changes by one, indicating a distinct topological sector.

[12] The overall scaling is not fixed because we have not identified the 4d $\mathcal{N} = 4$ SYM gauge coupling g with a power of N/k in ABJM.

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Moving on to the ***6-point 2-loop amplitude***, one finds [170]

$$\mathcal{A}_6^{\text{2-loop}} = \left(\frac{N}{k}\right)^2 \left\{ \frac{\mathcal{A}_6^{\text{tree}}}{2} \left[\text{BDS}_6 + R_6 \right] + \frac{\mathcal{A}_{6,\text{shifted}}^{\text{tree}}}{4i} \left[\ln \frac{u_2}{u_3} \ln \chi_1 + \text{cyclic} \times 2 \right] \right\}. \tag{11.101}$$

Here BDS_6 is the 1-loop MHV amplitude (6.57) of $\mathcal{N} = 4$ SYM, again with proper rescaling of the regulator $\epsilon \to 2\epsilon$ to account for being at 2-loops. As the remaining pieces are finite, the BDS Ansatz captures the IR-divergent as well as the resulting non-dual-conformal part of the amplitude. So once again, we observe that the IR structure of planar ABJM theory is identical to that of planar $\mathcal{N} = 4$ SYM! The "remainder" function R_6 in (11.101) is

$$R_6 = -2\pi^2 + \sum_{i=1}^{3} \left[\text{Li}_2(1 - u_i) + \frac{1}{2} \ln u_i \ln u_{i+1} + (\arccos \sqrt{u_i})^2 \right], \tag{11.102}$$

where the u_is are the dual conformal cross-ratios defined in (6.53). The shifted tree $\mathcal{A}_{6,\text{shifted}}^{\text{tree}}$ was encountered in (11.95). Finally, the function χ_1 in (11.101) is

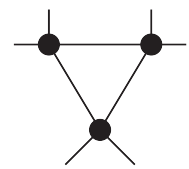
$$\chi_1 = \frac{\langle 12\rangle\langle 45\rangle + i\langle 3|P_{123}|6\rangle}{\langle 12\rangle\langle 45\rangle - i\langle 3|P_{123}|6\rangle}, \tag{11.103}$$

and "cyclic$\times$2" means we sum over all cyclic rotations by two sites, $i \to i + 2$.

▶ **Exercise 11.13**

Seeing both $\mathcal{A}_6^{\text{tree}}$ and $\mathcal{A}_{6,\text{shifted}}^{\text{tree}}$ in the same amplitude, you should check that the other factors in (11.101) indeed compensate for the little group weight difference.

Now that we have seen explicit examples of planar loop amplitudes in ABJM theory, let us turn to the subject of ***Leading Singularities*** and ***on-shell diagrams***. We studied these for planar 4d $\mathcal{N} = 4$ SYM in Chapter 8. Because of the dual superconformal symmetry of the loop-integrands, multi-loop amplitudes of ABJM theory can be calculated with Leading Singularity methods. In 3d, a maximal cut takes three propagators on-shell for each loop-momentum. At 1-loop order, the only dual conformal scalar integral is the massive triangle (11.94), so this plays the equivalent role of the box-diagram in 4d. In 4d, we build the Leading Singularity on-shell diagrams from vertices that are the fundamental 3-point MHV and anti-MHV superamplitudes. These vanish in 3d, so here we use the 4-point superamplitudes instead. In ABJM theory, the first non-trivial 1-loop Leading Singularity is the 6-point diagram

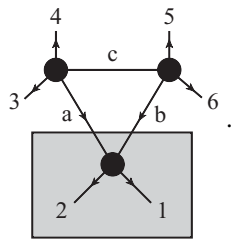


. (11.104)

As noted above, each vertex represents an on-shell 4-point superamplitude of ABJM theory. There is no distinction of "black" and "white" vertices because there is only one type of 4-point superamplitude in ABJM.

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In 4d $\mathcal{N} = 4$ SYM, we found that the 4-point Leading Singularity box diagram represents the 4-point tree amplitudes (see Section 8.3). Similarly, in ABJM, it turns out that the 6-point Leading Singularity triangle diagram (11.104) reproduces the tree-level 6-point superamplitude. To see this, we isolate the 3rd vertex in (11.104) and label the legs as



(11.105)

The internal momenta a and b are on-shell, so they each have two degrees of freedom. Of the combined $4 = 2 + 2$ degrees of freedom in a and b, three are fixed in terms of momenta 1 and 2 by the momentum conservation delta function at the bottom vertex in (11.105). Thus the spinor variables of a and b can be parameterized in terms of $|1\rangle$ and $|2\rangle$ using a single free variable. With a little thought – or, even better, a little calculation – one finds that the following parameterization solves the momentum conservation constraints

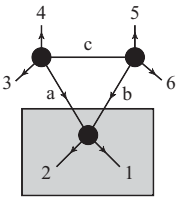
$$|a\rangle = \cos\theta\,|1\rangle - \sin\theta\,|2\rangle\,, \qquad |b\rangle = \sin\theta\,|1\rangle + \cos\theta\,|2\rangle\,. \tag{11.106}$$

This is exactly the BCFW deformation (11.65) of legs 1 and 2. Indeed, the final on-shell condition $p_c^2 = 0$ becomes the factorization condition that the parameter θ (i.e. z) must satisfy.

▶ **Exercise 11.14**

Verify that the supermomentum delta function on the bottom vertex enforces the following identification $\eta_a = \cos\theta\,\eta_1 - \sin\theta\,\eta_2$ and $\eta_b = \sin\theta\,\eta_1 + \cos\theta\,\eta_2$.

Now it is very tempting to conclude that the on-shell diagram (11.105) can be also understood as a BCFW diagram for the 6-point tree superamplitude in ABJM. This is true, but we have to make sure that we produce the BCFW recursion formula (11.73), including the weight-factor $H(a, b)$ defined in (11.74) and the propagator of the factorization channel. Taking into account the Jacobian factors associated with the triple cut and the bottom vertex, it has been shown [168] that $H(a, b)$ and the factorization propagator are indeed produced. Thus we have found the 3d analogue of the "BCFW bridge," namely



$$= \int d^3\eta \left(\mathcal{A}_4\big(\Phi\bar{\Psi}\Phi\bar{\Psi}\big)\frac{H(z_1^*, z_2^*)}{(P_{234})^2}\mathcal{A}_4\big(\bar{\Psi}\Phi\bar{\Psi}\Phi\big) + (z_1^* \leftrightarrow z_2^*)\right) = \mathcal{A}_6^{\text{tree}}\,. \tag{11.107}$$

Recall that in 4d, the Leading Singularity is closely related to the integral coefficients in expressions like (6.26). Previously we have seen that the 1-loop 6-point amplitude is

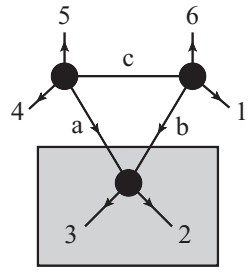
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proportional to $\mathcal{A}^{\text{tree}}_{6,\text{shifted}}$, so it is perhaps puzzling that the 6-point Leading Singularity (11.107) is just $\mathcal{A}^{\text{tree}}_{6}$. This has to do with a subtlety of the Jacobian factors. Recall that the integral coefficients can be determined by unitarity cuts. When we apply unitarity cuts, we are substituting the propagators with delta functions, as discussed in Section 6.2. As we solve the delta function constraints, we generate a Jacobian factor with an absolute value. On the other hand, when we are computing the Leading Singularity, we treat the delta functions as contour integrals, thus while localizing on a pole, the Jacobian factor does not come with an absolute value. In the 1-loop cases that we encountered in 4d, the Jacobian factors for the two loop-momentum solutions are identical, so the presence of an absolute value did not make a difference. However, in 3d, the Jacobian factors for the two loop-momentum solutions differ by a sign, so whether or not there is an absolute value on the Jacobian makes a difference [170]. The result of this is that the 1-loop 6-point amplitude is proportional to $\mathcal{A}^{\text{tree}}_{6,\text{shifted}}$ while the 6-point Leading Singularity is just $\mathcal{A}^{\text{tree}}_{6}$.

► **Exercise 11.15**

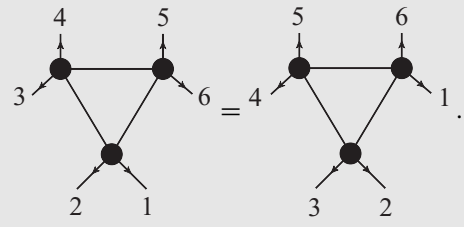
The above discussion indicates that if we had a relative plus sign for the two BCFW terms on the RHS of (11.81), the result would be $\mathcal{A}^{\text{tree}}_{6,\text{shifted}}$ instead of $\mathcal{A}^{\text{tree}}_{6}$. Verify that with a relative plus sign, (11.81) has the correct little group property of $\mathcal{A}^{\text{tree}}_{6,\text{shifted}}$. [Hint: you need to take into account that the coefficient for the η-polynomial corresponds to a different component amplitude in the shifted amplitude.]

Instead of (11.105), we could have computed the on-shell diagram



(11.108)

and the result would have been exactly the same, namely $\mathcal{A}^{\text{tree}}_{6}$. This gives us the ABJM equivalent of the "square move" (8.48) in 4d $\mathcal{N} = 4$ SYM. The ABJM "triangle move" is

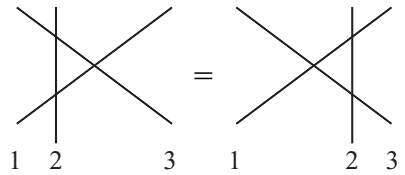


(11.109)

This is dubbed the ***Yang–Baxter move***, because it is precisely the graphical representation of the Yang–Baxter equation that plays an important role in integrable theories. It is

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usually represented as



(11.110)

Just as in $\mathcal{N} = 4$ SYM, the on-shell diagrams at higher multiplicity have an interesting structure. A description of this can be found in [120].

11.3.10 The orthogonal Grassmannian

Let us see if we can reason our way to a proper Grassmannian formula for the 3d superamplitudes of ABJM theory. We begin with

$$\prod_{\mathsf{a}=1}^{k} \delta^{2|3}\Big(\sum_i C_{\mathsf{a}i}\,\Lambda_i\Big) \equiv \prod_{\mathsf{a}=1}^{k} \delta^{2}\Big(\sum_i C_{\mathsf{a}i}\,|i\rangle\Big)\delta^{(3)}\Big(\sum_i C_{\mathsf{a}i}\,\eta_i\Big)\,, \tag{11.111}$$

where $\Lambda_i = (|i\rangle, \eta_i)$. As in the 4d case, treated in Chapter 9, the function (11.111) is invariant under generators in (11.39) which are linear in derivatives. If we consider the generators that are quadratic in derivatives, for example the conformal boost generator, we find

$$\Big(\sum_{i=1}^{n} \frac{\partial}{\partial|i\rangle^a}\frac{\partial}{\partial|i\rangle^b}\Big)\prod_{\mathsf{a}=1}^{k} \delta^{2}\Big(\sum_j C_{\mathsf{a}j}\,|j\rangle\Big) = \Big(\sum_{i=1}^{n} C_{\mathsf{a}'i}C_{\mathsf{b}'i}\Big) f_{ab} \prod_{\mathsf{a}\neq\mathsf{a}',\mathsf{b}'} \delta^{2}\Big(\sum_j C_{\mathsf{a}j}\,|i\rangle\Big)\,. \tag{11.112}$$

Here f_{ab} is a function that includes either a single derivative or double derivatives of the delta functions, depending on whether $a = b$ or not. The important piece in (11.112) is the prefactor: it tells us that to ensure invariance under conformal boosts, we need to dress (11.111) with an extra term that enforces $CC^T = 0$, i.e.

$$\delta^{\frac{k(k+1)}{2}}\Big(CC^T\Big)\prod_{\mathsf{a}=1}^{k} \delta^{2|3}\Big(\sum_i C_{\mathsf{a}i}\,\Lambda_i\Big)\,. \tag{11.113}$$

The product CC^T is a symmetric $k \times k$ matrix, so setting it to zero takes $k(k+1)/2$ constraints, as indicated in the delta function.

► Exercise 11.16

Show that the combination in (11.113) is also invariant under the multiplicative generators such as P^{ab}.

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Now we can readily write down an Ansatz for an $\mathcal{N} = 6$ superconformal invariant integral over a Grassmannian $\mathrm{Gr}(k, n)$ subject to the orthogonal constraint $CC^T = 0$:

$$\int d^{n\times k}C \; f(M)\, \delta^{\frac{k(k+1)}{2}}\Big(CC^T\Big) \prod_{\mathsf{a}=1}^{k} \delta^{2|3}\Big(\sum_i C_{\mathsf{a}i}\, \Lambda_i\Big)\,, \tag{11.114}$$

where $f(M)$ is a function that only depends on the minors of the Grassmannian, so that it preserves $SL(k)$ invariance. In order to interpret (11.114) as an integral over a Grassmannian manifold, it has to be $GL(k)$ invariant. All terms in (11.114) are $SL(k)$ invariant, and the $GL(1)$ weight count of the delta functions plus the measure gives $nk - k(k+1) - 2k + 3k = k(n-k)$. This tells us that the function $f(M)$ needs to have $GL(1)$ weight $-k(n-k)$.

We need more input to fix $f(M)$ and the extra information comes from little group analysis. Under the $\mathbb{Z}_2$ little group, we have $|i\rangle \to -|i\rangle$ and $\eta_i \to -\eta_i$, so invariance of the delta functions in (11.114) requires $C_{\mathsf{a}i} \to -C_{\mathsf{a}i}$. For an amplitude with a $\bar{\Psi}$-supermultiplet on the odd-sites, the superamplitude should pick up a minus sign whenever we perform a $\mathbb{Z}_2$ transformation on the odd-numbered legs, while it should be inert for even legs with their Φ-supermultiplet. Take $n = 2k$, and $k =$ even: then the product of k consecutive minors,

$$f(M) = \prod_{i=1}^{k} \frac{1}{M_i}\,, \tag{11.115}$$

indeed satisfies the little group criteria. (Exercise 11.17 helps you see this.) Furthermore, since $n = 2k$, the function (11.115) has $GL(1)$ weight $-k^2$, precisely as needed for overall $GL(1)$ invariance.

▶ **Exercise 11.17**

For $k = 3$ (and hence $n = 6$), verify that (11.115) indeed picks up a minus sign under little group scaling for odd legs and is invariant for even legs. Show that for $k =$ odd, the function $f(M) = \prod_{i=2}^{k+1} \frac{1}{M_i}$ does the right job.

We conclude that the 3d Grassmannian formula for ABJM theory is given by an orthogonal Grassmannian integral [171] which for $k =$ even is

$$\mathcal{L}^{\mathrm{O}}_{2k,k} = \int \frac{d^{2k^2}C}{GL(k)} \left(\prod_{i=1}^{k} \frac{1}{M_i}\right) \delta^{\frac{k(k+1)}{2}}\Big(CC^T\Big) \prod_{\mathsf{a}=1}^{k} \delta^{2|3}\Big(\sum_i C_{\mathsf{a}i}\, \Lambda_i\Big)\,. \tag{11.116}$$

The superscript "O" indicates it is an ***orthogonal Grassmannian***. When $k =$ odd, the product of minors is replaced by $\prod_{i=2}^{k+1} \frac{1}{M_i}$, as shown in Exercise 11.17. Some comments are in order:

- *Momentum conservation* is enforced in (11.116) in a slightly differently manner than in the 4d version (9.17) of the Grassmannian integral because we only have the $|i\rangle$-spinors in 3d. Here is how it goes. The orthogonality condition forces the Grassmannian to be a collection of null vectors in an n-dimensional space. The bosonic delta function $\delta^2(C \cdot |i\rangle)$ requires the two n-dimensional vectors $\{|i\rangle\}$ to be orthogonal to C. This means that $\{|i\rangle\}$ lies in the complement of C, which is nothing but C^T, and thus $\{|i\rangle\}$ must also be null: $\sum_i |i\rangle^a |i\rangle^b = 0$.

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- *Two-site cyclicity?* The integral $\mathcal{L}^{\rm O}_{2k,k}$ does not appear to have the correct cyclic invariance by two sites as discussed (11.58). However, thanks to the orthogonality condition it can be shown that

$$M_i M_{i+1} = (-1)^k M_{i+k} M_{i+1+k} \,. \tag{11.117}$$

Therefore the formula (11.116) is indeed invariant under cyclic rotation by two sites up to a factor of $(-1)^{k-1}$, as required.

▶ Exercise 11.18

Show that (11.117) holds true at 4-points: using $GL(2)$ invariance and the orthogonality condition, we can choose to fix the 2×4 matrices C to take the form

$$C = \begin{pmatrix} 1 & 0 & i\sin\theta & -i\cos\theta \\ 0 & 1 & i\cos\theta & i\sin\theta \end{pmatrix}. \tag{11.118}$$

Verify that $CC^T = 0$ and that (11.117) holds.

- *The dimension of the integral* (11.116) is found by counting how many free variables are left after localization by the delta functions. To start with, there is a total of $2k^2$ integration variables. The bosonic delta functions fix $k(k+1)/2 + 2k - 3$ constraints, with the -3 coming from the removal of the constraints that enforce momentum conservation. Subtracting the k^2 redundancy of $GL(k)$, the dimension of the integral is then $\frac{(k-2)(k-3)}{2}$. Thus for 4- and 6-point amplitudes ($k = 2, 3$), the delta functions completely localize the Grassmannian integral.

Let us now take a closer look at (11.116) for $n = 4$. Gauge fix the $GL(2)$ by taking

$$C = \begin{pmatrix} c_{21} & 1 & c_{23} & 0 \\ c_{41} & 0 & c_{43} & 1 \end{pmatrix}. \tag{11.119}$$

This leaves four parameters that can be fixed by the four delta functions in $\delta\big(C \cdot |i\rangle\big)$. Denote the solutions to $C \cdot |i\rangle = 0$ by $c^*_{\bar{r}s}$, with barred labels indicating even legs and un-barred odd legs. Then the delta functions can be rewritten as

$$\delta^4\big(C \cdot |i\rangle\big) = \frac{1}{\langle 13\rangle^2} \prod_{\bar{r},s} \delta^4(c_{\bar{r}s} - c^*_{\bar{r}s}) \,, \qquad \begin{pmatrix} c^*_{21} & c^*_{23} \\ c^*_{41} & c^*_{43} \end{pmatrix} = -\frac{1}{\langle 13\rangle} \begin{pmatrix} \langle 23\rangle & \langle 12\rangle \\ \langle 43\rangle & \langle 14\rangle \end{pmatrix}. \tag{11.120}$$

▶ Exercise 11.19

Which property of the external momenta does it take for the above solution $c^*_{\bar{r}s}$ to solve the orthogonality constraint? (Show it!)

Localizing the Grassmannian integral on to $c^*_{\bar{r}s}$, we then find

$$\delta^3\big(CC^T\big) = \frac{\langle 13\rangle^6}{\langle 24\rangle^3}\, \delta^3\big(P\big) \,, \qquad \delta^6\big(C \cdot \eta\big) = \frac{\langle 24\rangle^3}{\langle 13\rangle^6}\, \delta^{(6)}\big(\widetilde{Q}\big) \,, \qquad \frac{1}{M_1 M_2} = \frac{\langle 13\rangle^2}{\langle 14\rangle\langle 34\rangle} \,. \tag{11.121}$$

Combining (11.120) and (11.121), we recover the superamplitude (11.52) of ABJM theory.

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For $n = 6$, the integral (11.116) is again completely localized by the bosonic delta functions onto two solutions, each corresponding to one of the BCFW terms in (11.77). Recall that these two terms are individually local (see (11.81)). We can now understand why. At 6-point these are the only possible invariants produced by the Grassmannian integral, so this means that *if the orthogonal Grassmannian integral produces all possible dual conformal invariants*[13] the Leading Singularity of the 6-point superamplitude must be some linear combination of them. However, we already know that there are two distinct local rational functions for $n = 6$, namely $\mathcal{A}_n^{\text{tree}}$ and $\mathcal{A}_n^{\text{1-loop}} \propto \mathcal{A}_{n,\text{shifted}}^{\text{tree}}$. As we noted at the end of Section 11.3.8, since they are distinct, this can only mean one thing, namely that the two terms in (11.77) are individually local and free of spurious poles.

We conclude this section with a comparison of the 3d and 4d Grassmannians. In the 4d Grassmannian, the choice of contour that gives the tree superamplitude forces the Grassmannian $\text{Gr}(k, n)$ to be localized to a $\text{Gr}(2, n)$ submanifold. For $n = 6$ we saw how this is intimately related to locality, since the tree contour selected three residues whose sum was free of spurious poles. In 3d, on the other hand, the orthogonal Grassmannian integral localizes completely for $n = 6$ and gives us two local objects without any need for us to pick any contour. Does this mean that the localization to the $\text{Gr}(2, n)$ submanifold is not really related to locality? The answer turns out to be "no" in an interesting way. It was found in [173] that for $n = 6$, the orthogonality constraint indeed forces the Grassmanian to localize to a $\text{Gr}(2, n)$ submanifold. Thus, the Grassmannian for $\mathcal{N} = 4$ SYM achieves locality for the 6-point NMHV superamplitude by choosing a particular "tree-contour," while for 6-point ABJM superamplitudes, the Grassmannian achieves locality by subjecting itself to the orthogonal constraint. The invariant between the two cases is the $\text{Gr}(2, n)$ submanifold, which was previously [125, 126] linked to Witten's twistor string formulation [54]. So is there perhaps a 3d twistor string theory? A twistor-like string theory with target space $SU(2, 3|5)$ was constructed in [174] and it reproduces the $\text{Gr}(2, n)$ formula of the ABJM amplitudes. It is quite fascinating how the study of scattering amplitudes reveals the existence of a new twistor string theory!

There is also a momentum twistor formulation of the Grassmannian integral for ABJM amplitudes (see Elvang *et al.* [60]). It could be the first step towards an ABJM amplituhedron.

[13] Using on-shell diagrams, one can show that the Leading Singularities obtained from the result of loop-level recursion can always be identified with residues of the orthogonal Grassmannian integral [172].

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