Combinatorics, Probability and Computing (2006) 15, 355–375. © 2006 Cambridge University Press
doi:10.1017/S0963548305007340 Printed in the United Kingdom

Splitter Theorems for Cubic Graphs

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Received 21 August 2002; revised 21 August 2004

Let $\Gamma_{k,g}$ be the class of k-connected cubic graphs of girth at least g. For several choices of k and g, we determine a set $\mathcal{O}_{k,g}$ of graph operations, for which, if G and H are graphs in $\Gamma_{k,g}$, $G \ncong H$, and G contains H topologically, then some operation in $\mathcal{O}_{k,g}$ can be applied to G to result in a smaller graph G' in $\Gamma_{k,g}$ such that, on one hand, G' is contained in G topologically, and on the other hand, G' contains H topologically.

1. Introduction

Suppose a graph G 'contains' another graph H. Then how can G be built up from H in such a way that certain properties of G and H are preserved during the construction process? Probably the best-known result to answer this kind of question is the one by Seymour [18], for general matroids, and Negami [15], for graphs only, which explains the construction when the containment relation is the minor relation and the property to preserve is the 3-connectedness. In this paper, we investigate results of the same type, which are known as *splitter theorems*, for cubic graphs.

First, let us clarify the graph containment relation used in this paper. A *subdivision* of a graph H is a graph H' obtained from H by replacing each $e \in E(H)$ with a path P_e of positive length (P_e actually is a cycle when e is a loop). We will call each P_e an *edge-path* of H'. For any two given graphs G and H, if a subgraph of G is isomorphic

† Research partially supported by NSA grant H98230-05-1-0081, NSF grant ITR-0326387, and AFOSR grant: F49620-03-1-0238-0239-0241.
‡ Research partially supported by LEQSF(2004-07)-RD-A-22.

https://doi.org/10.1017/S0963548305007340 Published online by Cambridge University Press

to a subdivision of H, then we say that H is *contained topologically* in G and we write $H \preceq G$. We also write $H \prec G$ if $H \preceq G$ and $H \ncong G$. If Γ is a class of graphs, let $\Gamma(H)$ be the class of all graphs G in Γ with $H \preceq G$. Although we do not need this fact in our paper, it is worth mentioning that if H is cubic, then $H \preceq G$ if and only if H is a minor of G, which means that H can be obtained from a subgraph of G by contracting edges.

Next, let us make it clear what we mean by building G from H and also maintaining certain properties. In fact, we will talk about reducing G to H, which is clearly equivalent to building G from H yet it is much more convenient for stating our results. Suppose both G and H belong to a family Γ of graphs. Then we say that G can be *reduced to H within Γ by a set $\mathcal{O}$ of graph operations* if there is a sequence $G_0, G_1, \ldots, G_t$ of graphs in Γ such that $G_0 = G$, $G_t = H$, and each G_i, where $i > 0$, is obtained from G_{i-1} by applying a single operation in $\mathcal{O}$. Moreover, we require that $G_i \preceq G_{i-1}$ holds for all $i > 0$. Clearly, it follows that $H \preceq G_i \preceq G$. Under this terminology, a *splitter theorem* is a result that claims the existence of $\mathcal{O}$ such that every $G \in \Gamma$ can be reduced within Γ by $\mathcal{O}$ to any $H \in \Gamma$ with $H \preceq G$.

In the literature, a splitter theorem also has the following equivalent formulation. There exists a set $\mathcal{O}$ of graph operations for which, if both G and H are in a class Γ of graphs and $H \prec G$, then G can be *reduced within* $\Gamma(H)$ by $\mathcal{O}$, *i.e.*, an operation in $\mathcal{O}$ can be applied to G such that the resulting graph belongs to $\Gamma(H)$. It is clear that this formulation is implied by the first, while the first can also proved by repeatedly using the second (as long as every operation in $\mathcal{O}$ always results in a smaller graph). In this paper, all splitter theorems will be stated using the first formulation but proved using the second.

The graph properties that we try to maintain are connectivity and girth. Let $\Gamma_{k,g}$ be the family of k-connected cubic graphs of girth at least g. Since only cubic graphs are considered, it is natural for us to assume that $k \in \{0, 1, 2, 3\}$. It is also natural to assume $g > 0$ since every cycle has at least one edge. In addition, since the girth of a k-connected cubic graph is at least k, that is, $\Gamma_{k,g} = \Gamma_{k,k}$, for all $g < k$, we also make the assumption that $g \geqslant k$.

In this paper, we follow the notation and terminology of West [24]. In particular, for any graph G, the *degree* of a vertex v, denoted by $d_G(v)$, is the number of non-loop edges incident with v plus twice the number of loops incident with v. The minimum degree of G is denoted by $\delta(G)$. The *length* of a path P, which is denoted by $|P|$, is the number of edges of P. If an edge is contained in a 2-cycle, then it is called a *multiple edge*. If x is an edge or a set of edges, then the deletion and contraction of x in G is denoted by $G\backslash x$ and G/x, respectively. If x is a vertex or a set of vertices, then the deletion of x from G is denoted by $G - x$.

In the next three sections, we will prove splitter theorems for $\Gamma_{k,g}$, for $g = 1, 2$, $g = 3$, and $g = 4$, respectively. In addition, we will also determine the $\preceq$-minimal graphs in each $\Gamma_{k,g}$. Consequently, we have results on how to generate all graphs in each $\Gamma_{k,g}$ from its minimal graphs according to the corresponding splitter theorem. In the last section, we discuss related results concerning cyclically k-connected cubic graphs. A summary of the main results discussed in the paper is given in Table 1.

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Table 1. $\Gamma_{k,g}$ is the class of k-connected cubic graphs of girth at least g; $3K_2$ is the graph obtained from K_2 by adding two multiple edges; K_2^L is the graph obtained from K_2 by adding a loop to each vertex; Q is the cube; $O_0(K)$ is the operation of deleting a component isomorphic to K; other operations listed in the second column are defined in Figures 1–10, respectively.

Graph classes	Graph operations	Minimal graphs	Type of results	Citations and references
$\Gamma_{0,1}$	$\mathscr{R}$	$3K_2$ and K_2^L	splitter theorem generating theorem	Theorem 2.1 Corollary 2.2
$\Gamma_{0,2}$	$\mathscr{R}$	$3K_2$	splitter theorem generating theorem	Theorem 2.3 Corollary 2.4
$\Gamma_{1,1}$	$\mathscr{R}$	$3K_2$ and K_2^L	splitter theorem generating theorem	Theorem 2.5 Corollary 2.6
$\Gamma_{1,2}$	$\mathscr{R}$ and $O_1(3K_2)$	$3K_2$	splitter theorem generating theorem	Theorem 2.7 Corollary 2.8
$\Gamma_{2,2}$	$\mathscr{R}$	$3K_2$	splitter theorem generating theorem	Theorem 2.9 Corollary 2.10
$\Gamma_{0,3}$	$\mathscr{R}$, $O_0(K_4)$ and $O_2(K_4)$	K_4	splitter theorem generating theorem	Theorem 3.1 Corollary 3.5
$\Gamma_{1,3}$	$\mathscr{R}$, $O_1(K_4)$ and $O_2(K_4)$	K_4	splitter theorem generating theorem	Theorem 3.2 [25], Corollary 3.5
$\Gamma_{2,3}$	$\mathscr{R}$ and $O_2(K_4)$	K_4	splitter theorem generating theorem	Theorem 3.3 [25], Corollary 3.5
$\Gamma_{3,3}$	$\mathscr{R}$	K_4	splitter theorem generating theorem	[10], Theorem 3.4 [3, 20, 23], Corollary 3.5
$\Gamma_{0,4}$	$\mathscr{R}$ and $O_i(K)$, for $i = 0, 2, 3$ and $K = K_{3,3}, Q$	$K_{3,3}$ and Q	splitter theorem generating theorem	Theorem 4.1 Corollary 4.5
$\Gamma_{1,4}$	$\mathscr{R}$ and $O_i(K)$, for $i = 1, 2, 3$ and $K = K_{3,3}, Q$	$K_{3,3}$ and Q	splitter theorem generating theorem	Theorem 4.2 Corollary 4.5
$\Gamma_{2,4}$	$\mathscr{R}$ and $O_i(K)$, for $i = 2, 3$ and $K = K_{3,3}, Q$	$K_{3,3}$ and Q	splitter theorem generating theorem	Theorem 4.3 Corollary 4.5
$\Gamma_{3,4}$	$\mathscr{R}$, $O_3(K_{3,3})$, and $O_3(Q)$	$K_{3,3}$ and Q	splitter theorem generating theorem	Theorem 4.4 Corollary 4.5

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2. On non-simple graphs

In this section, we consider the cases when $g = 1, 2$. It is easy to see that there are five classes, $\Gamma_{0,1}$, $\Gamma_{0,2}$, $\Gamma_{1,1}$, $\Gamma_{1,2}$, and $\Gamma_{2,2}$. Most proofs in this section are straightforward. We include them for the purpose of completeness, and also as preparation for the next two sections.

Let us first introduce the smallest cubic graphs, the only two on two vertices. It is easy to see that one such graph consists of three multiple edges, and the other consists of a non-loop edge and two non-incident loops. We denote these two graphs by $3K_2$ and K_2^L, respectively. We also introduce a graph operation $\mathscr{R}$ as follows. Let G be a cubic graph on more than two vertices. Let $e = xy$ be a non-loop edge of G and let F_e be the set of edges between $\{x, y\}$ and $V(G) - \{x, y\}$. Since G is cubic, it is clear that $|F_e|$ equals 0, 2, or 4. Then the result of applying $\mathscr{R}$ to e is the *cubic* graph obtained from $G - \{x, y\}$ by adding $|F_e|/2$ edges, where, if $|F_e| \leqslant 2$, the addition is unique and, if $|F_e| = 4$, the only allowed addition is illustrated in Figure 1. In this paper, the resulting graph will be denoted by $(G\backslash e)\urcorner$. It should be mentioned – and it can be easily verified – that $(G\backslash e)\urcorner \preceq G$. When G is a simple graph, operation $\mathscr{R}$ is called *edge reduction* in [13]. We also point out that all other operations defined in this paper can be considered as different combinations of operation $\mathscr{R}$.

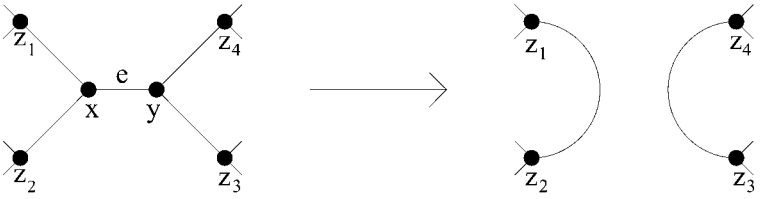


Figure 1. Operations $\mathscr{R}$.

In the following we present the main results in this section. The proofs will be given a little later. We start with our first splitter theorem.

Theorem 2.1. *If G and H are cubic and $H \preceq G$, then G can be reduced to H within $\Gamma_{0,1}$ by $\mathscr{R}$.*

A different way to think of this result is that *a cubic graph H is topologically contained in another cubic graph G if and only if G can be reduced to H by $\mathscr{R}$*. This is an observation that we will use several times.

Since every cubic graph has a non-loop edge, operation $\mathscr{R}$ can be applied to any cubic graph with more than two vertices. Therefore, the following is obvious.

Corollary 2.2. *Every cubic graph can be reduced within $\Gamma_{0,1}$ by $\mathscr{R}$ to either $3K_2$ or K_2^L.*

The situation for $\Gamma_{0,2}$ is similar.

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Theorem 2.3. *If G and H are loopless cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{0,2}$ by $\mathscr{R}$.*

The following corollary of Theorem 2.3 is an analogue of Corollary 2.2 for loopless graphs.

Corollary 2.4. *Every loopless cubic graph can be reduced to $3K_2$ within $\Gamma_{0,2}$ by $\mathscr{R}$.*

Next, we consider connected cubic graphs.

Theorem 2.5. *If G and H are connected cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{1,1}$ by $\mathscr{R}$.*

By combining Corollary 2.2 and Theorem 2.5 we conclude the following immediately.

Corollary 2.6. *Every connected cubic graph can be reduced within $\Gamma_{1,1}$ by $\mathscr{R}$ to $3K_2$ or K_2^L.*

For our next splitter theorem, we need a new graph operation, which is denoted by $O_1(3K_2)$ and is illustrated in Figure 2.

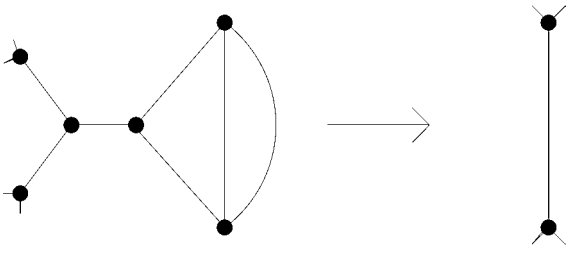


Figure 2. Operation $O_1(3K_2)$.

Theorem 2.7. *If G and H are connected loopless cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{1,2}$ by $\mathscr{R}$ and $O_1(3K_2)$.*

As before, we deduce the following corollary immediately from Corollary 2.4 and Theorem 2.7.

Corollary 2.8. *Every connected loopless cubic graph can be reduced to $3K_2$ within $\Gamma_{1,2}$ by $\mathscr{R}$ and $O_1(3K_2)$.*

Finally, we present our last splitter theorem in this section.

Theorem 2.9. *If G and H are 2-connected cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{2,2}$ by $\mathscr{R}$.*

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Since 2-connected cubic graphs are loopless, the next corollary follows obviously from Corollary 2.4 and Theorem 2.9.

Corollary 2.10. *Every 2-connected cubic graph can be reduced to $3K_2$ within $\Gamma_{2,2}$ by $\mathscr{R}$.*

In the rest of this section, we prove results listed above. We first make an observation on $\mathscr{R}$, which we will use frequently. We omit its proof since the proof is straightforward.

Lemma 2.11. *Let G and H be cubic graphs and let e be a non-loop edge of G. If $H \preceq G\backslash e$, then $H \preceq (G\backslash e)^{\urcorner}$.*

With Lemma 2.11, the proof of Theorem 2.1 is straightforward.

Proof of Theorem 2.1. Let H' be a proper subgraph of G such that it is a subdivision of H. It follows that either G has a vertex x not in H' or H' has a vertex x of degree two. In both cases, it is easy to see that x is incident with a non-loop edge e of $E(G) - E(H')$. By Lemma 2.11, it follows that G can be reduced within $\Gamma_{0,1}(H)$ by $\mathscr{R}$ and thus the theorem is proved. □

To prove Theorem 2.3, we need two lemmas. The proof of the first is again straightforward and is thus omitted.

Lemma 2.12. *Let a subdivision H' of a cubic graph H be a subgraph of a cubic graph G. If the girth of H is at least g, then every cycle of H' must contain at least g degree-three vertices of H'. In particular, if $e = xy \in E(G) - E(H')$ and x is contained in a g-cycle C of $G\backslash e$, then C must have an edge in $E(G) - E(H')$.*

Lemma 2.13. *Let a subdivision H' of $H \in \Gamma_{0,2}$ be a subgraph of $G \in \Gamma_{0,2}$. Suppose an edge e in $F = E(G) - E(H')$ is incident with a multiple edge. Then either F has a multiple edge that is not in a 3-cycle, or F has five distinct edges xy, xy, xz, yz, zv such that no multiple edge is incident with v.*

Proof. We first claim that F has a multiple edge. This is clear if e is a multiple edge. If e is not a multiple edge, then the claim follows obviously from Lemma 2.12. Let $xy \in F$ be a multiple edge. We may assume that xy is contained in a triangle, say xyz, because otherwise we are done. From the fact $\delta(H') > 1$ and Lemma 2.12 we deduce that all the four edges xy, xy, yz, xz are in F. Let zv be the only other edge incident with z. Then zv is also in F, as $d_{H'}(z) \neq 1$. If v is not incident with any multiple edge, then we are done. If v is incident with a multiple edge, then, by Lemma 2.12 again, v is incident with a multiple edge, say f, in F. Since zv is a cut-edge, f is not contained in any triangle and thus the lemma is proved. □

Proof of Theorem 2.3. Let a subdivision H' of H be a proper subgraph of G and let $F = E(G) - E(H')$. By Lemma 2.11, we need only find an edge $f \in F$ such that $(G\backslash f)^{\urcorner}$ is

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loopless. If F has an edge f that is not incident with any multiple edge, then it is easy to see that $(G\backslash f)^{\sim}$ is loopless. By Lemma 2.13, we deduce that F has a multiple edge f, which is not contained in any 3-cycle. But in this case, it is again easy to see that $(G\backslash f)^{\sim}$ is loopless, and thus the theorem is proved. □

Proof of Corollary 2.4. By Theorem 2.3, we need only show that $3K_2 \preceq G$, for all $G \in \Gamma_{0,2}$. Let us apply $\mathscr{R}$ to G repeatedly, whenever it is possible, and as long as no loops are created. Then the resulting graph G' must have a 2-cycle C. In addition, if $e \in E(C)$, then either $G' = 3K_2$ or e is contained in a 3-cycle D. In the second case, it is clear that the union of C and D is a subdivision of $3K_2$. Therefore, in both cases, $3K_2 \preceq G' \preceq G$, as required. □

To prove Theorem 2.5, we need the following lemma, which we will also use in the coming sections.

Lemma 2.14. *Let G be a connected graph with $\delta(G) > 1$. If G has a connected proper subgraph A, then $G\backslash e$ is connected for some $e \in E(G) - E(A)$.*

Proof. If A is a spanning subgraph, then every edge in $E(G) - E(A)$ has the required property. If A is not a spanning subgraph, then a spanning tree of A can be extended into a spanning tree, say T, of G. Obviously, T has a pendant vertex v not in A. Since $\delta(G) > 1$, there exists an edge $e \in E(G) - E(T)$ that is incident with v. Now it is easy to see that e has the required property. □

Proof of Theorem 2.5. As before, let a subdivision H' of H be a proper subgraph of G. By Lemma 2.14, $G\backslash e$ is connected for some $e \in E(G) - E(H')$. If e is not a loop, then, by Lemma 2.11, we have $(G\backslash e)^{\sim} \in \Gamma_{1,1}(H)$ and thus we are done. If e is a loop, then the unique edge f that is incident with e cannot be contained in $E(H')$, as $\delta(H') > 1$. Clearly, $(G\backslash f)^{\sim}$ is connected. In addition, by Lemma 2.11 again, we deduce that $(G\backslash f)^{\sim} \in \Gamma_{1,1}(H)$ and thus we are done. □

Proof of Theorem 2.7. Let a subdivision H' of H be a proper subgraph of G and let $F = E(G) - E(H')$. By Lemma 2.14, $G\backslash e$ is connected for some $e \in F$. Without loss of generality, we may assume that e is incident with a multiple edge, because otherwise, by Lemma 2.11, $(G\backslash e)^{\sim} \in \Gamma_{1,2}(H)$ and thus we would be done. As a consequence of this assumption, at least one of the two conclusions of Lemma 2.13 holds. Now it is clear that G can be reduced within $\Gamma_{1,2}(H)$ by $\mathscr{R}$, in the first case, and, by $O_1(3K_2)$, in the second case. □

To prove Theorem 2.9 we need two lemmas. The first is the case $k = 2$ of Theorem 2.2 in [13], and it can also be proved directly without too much difficulty.

Lemma 2.15. *If C is a cycle of a 2-connected cubic graph, then $G\backslash e$ is 2-connected for some $e \in E(C)$.*

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The next is an analogue of Lemma 2.14 for 2-connected cubic graphs.

Lemma 2.16. *If a 2-connected cubic graph G has a 2-connected proper subgraph A, then $G\backslash e$ is 2-connected for some $e \in E(G) - E(A)$.*

Proof. Let $e \in E(G) - E(A)$ and let us assume that $G\backslash e$ is not 2-connected. Then $G\backslash e$ has a cut edge f, as G is cubic. Since A is 2-connected, f cannot be in A. It follows that A is a subgraph of a component, say G', of $G\backslash e\backslash f$. Let G'' be the other component of $G\backslash e\backslash f$. Observe that e and f are not incident, as G is cubic and 2-connected. Thus $\delta(G'') > 1$ and so, G'' has a cycle. Now the result follows obviously from Lemma 2.15. □

Proof of Theorem 2.9. Let a subdivision H' of H be a proper subgraph of G. Since H is 2-connected, H' must also be 2-connected. By Lemma 2.16, $E(G) - E(H')$ has an edge e such that $G\backslash e$ is 2-connected. In this case, it is straightforward to verify that $G\backslash e$ is a subdivision of $(G\backslash e)^{\urcorner}$. Consequently, $(G\backslash e)^{\urcorner}$ is also 2-connected. Now the theorem follows obviously from Lemma 2.11. □

3. On simple graphs

We begin with the definitions of two new operations. If K is a graph, then we denote by $O_0(K)$ the operation that deletes from a graph a component isomorphic to K. Our second operation, $O_2(K_4)$, is illustrated in Figure 3.

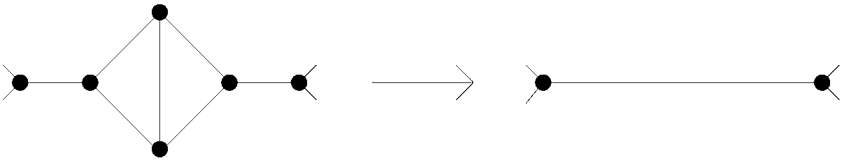


Figure 3. Operation $O_2(K_4)$.

Theorem 3.1. *If G and H are cubic simple graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{0,3}$ by $\mathscr{R}$, $O_0(K_4)$ and $O_2(K_4)$.*

It should be pointed out that Theorem 3.1 slightly improves a result of Statman [19]. Our next splitter theorem needs another operation, $O_1(K_4)$, which is illustrated in Figure 4.

Theorem 3.2. *If G and H are connected cubic simple graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{1,3}$ by $\mathscr{R}$, $O_1(K_4)$ and $O_2(K_4)$.*

Hence we obtain the splitter theorem for 2-connected cubic simple graphs.

Theorem 3.3. *If G and H are 2-connected cubic simple graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{2,3}$ by $\mathscr{R}$ and $O_2(K_4)$.*

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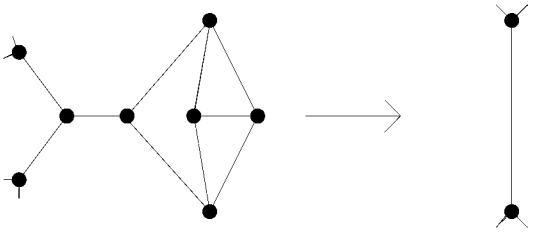


Figure 4. Operation $O_1(K_4)$.

The splitter theorem for $\Gamma_{3,3}$ was proved implicitly in [10]. Here, it is stated without a proof.

Theorem 3.4. *If G and H are 3-connected cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{3,3}$ by $\mathscr{R}$.*

A well-known result of Dirac [5] says that *every simple graph G with $\delta(G) \geqslant 3$ must contain K_4 topologically*. By combining this result with the theorems listed above we deduce the following immediately. For $i = 1, 2, 3, 4$, let $\mathscr{O}_{i-1,3}$ be the set of operations used in Theorem 3.i.

Corollary 3.5. *For $k = 0, 1, 2, 3$, every k-connected cubic simple graph can be reduced to K_4 within $\Gamma_{k,3}$ by operations in $\mathscr{O}_{k,3}$.*

We remark that Corollary 3.5 was partially proved by Wormald [25], for $k = 1$ and 2, by Tutte [23], Barnette and Grünbaum [3], and Titov [21], for $k = 3$, and by Steinitz and Rademacher, in [20] and [8], for planar graphs with $k = 3$. It is also worth mentioning that Johnson [9] gave another construction of $\Gamma_{1,3}$, which also starts with K_4, but uses a different operation, called H-reduction (see [16]). A refinement of Johnson's result is given in [22].

In the rest of this section, we prove theorems listed above. Since there are many overlaps in the proofs, we prove all three theorems together. We proceed by proving a sequence of lemmas.

Lemma 3.6. *Let G be a k-connected graph and let $X \subseteq E(G)$. Suppose G/X is cubic and simple. Then G/X is k-connected.*

Proof. Clearly, G is k-edge-connected. It follows that G/X is also k-edge-connected, since contracting edges does not decrease edge-connectivity. However, it is well known that for cubic simple graphs, being k-edge-connected is the same as being k-connected (see Exercise 4.1.11 in [24], for example). Thus G/X is k-connected. □

We also need the following well-known result, which can be found in [23].

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Lemma 3.7. *If e is an edge of a 2-connected graph G, then either $G\backslash e$ or G/e is 2-connected.*

In the next few lemmas, let G and H be graphs in $\Gamma_{k,3}$. Let a proper subgraph H' of G be isomorphic to a subdivision of H. Let $F = E(G) - E(H')$. We will make a few observations on when G can be reduced within $\Gamma_{k,3}(H)$. A *tripod* T is a subgraph of G with distinct vertices $t_1, t_2, \ldots, t_6$ and edges t_1t_2, t_2t_3, t_3t_1, t_1t_4, t_2t_5, t_3t_6.

Lemma 3.8. *If T has an edge in F, then $(G\backslash e\urcorner \in \Gamma_{k,3}(H)$ for some $e \in F$.*

Proof. By Lemma 2.12, we may assume that the 3-cycle C of T has an edge e in F. Clearly, $(G\backslash e\urcorner$ is simple. Therefore, by Lemma 2.11, $(G\backslash e\urcorner \in \Gamma_{0,3}(H)$. Notice that $(G\backslash e\urcorner$ is actually isomorphic to $G/E(C)$, thus the result follows from Lemma 3.6. □

A *necklace* N is a subgraph of G with vertices $n_1, n_2, \ldots, n_6$ and distinct edges n_1n_3, n_2n_4, n_3n_5, n_3n_6, n_4n_5, n_4n_6, n_5n_6. Vertices n_1 and n_2, which could be identical, are the *ends* of N.

Lemma 3.9. *Suppose an edge f of F is not contained in K_4 or any necklace. If $G\backslash f$ is k-connected, then $(G\backslash e\urcorner \in \Gamma_{k,3}(H)$ for some $e \in F$.*

Proof. If $(G\backslash f\urcorner$ is simple, then, since this graph can be considered as obtained from $G\backslash f$ by contracting edges, it follows from Lemmas 2.11 and 3.6 that f can be chosen as e. Therefore, we may assume that $(G\backslash f\urcorner$ is not simple. Next, let us verify that an end of f is contained in a 3-cycle C of $G\backslash f$. Since G is simple, $(G\backslash f\urcorner$ has no loops, and so must have 2-cycles. Since f is not in K_4 or any necklace, the two new edges of $(G\backslash f\urcorner$ do not form a 2-cycle. It follows that each 2-cycle of $(G\backslash f\urcorner$ consists of an old edge and a new edge. Clearly, such a 2-cycle corresponds to a 3-cycle C as claimed above. Now it is easy to see that the six edges incident with the three vertices of C form a tripod and thus the result follows from Lemma 3.8. □

A necklace N is *short* if its two ends are identical; it is *closed* if its two ends are adjacent; and it is *open* if its two ends are neither identical nor adjacent. Next we consider these three situations under the assumption that $E(N) \cap F \neq \emptyset$. First, we have a simple observation, which follows directly from the fact that H is simple.

Lemma 3.10. *$E(N) \cap E(H')$ is a (possibly empty) subpath of an edge-path P_e of H'.*

Lemma 3.11. *Suppose N is short. If e is the only edge between $V(N)$ and $V(G) - V(N)$, then $E(N) \cup \{e\} \subseteq F$. In addition, $(G\backslash e\urcorner$ is simple, unless e is contained in tripod or an open necklace.*

Proof. Since H is simple, it follows from Lemma 3.10 that $E(N) \cap E(H') = \emptyset$, and thus, $e \in F$. Let x be the end of e that is not in N. If x is not in a 3-cycle, then $(G\backslash e\urcorner$ is simple.

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If x is in a 3-cycle, then this 3-cycle can be extended into a tripod or an open necklace that contains e. □

Lemma 3.12. *Suppose N is closed. If e is the edge between the ends of N, then $(G\backslash e)^\curlywedge \in \Gamma_{k,3}(H)$.*

Proof. Let P be a path of N between its two ends. Since H is simple, it follows from Lemma 3.10 that, if $e \in E(H')$, then $E(N) \cap E(H') = \emptyset$. Therefore, by replacing e with P, if necessary, we may assume that $e \in F$. Clearly, $(G\backslash e)^\curlywedge$ is simple, and thus, by Lemma 2.11, we have $(G\backslash e)^\curlywedge \in \Gamma_{0,3}(H)$. Now, by Lemma 3.6, it remains to show that $G\backslash e$ is k-connected. This is trivial if $k = 0$. It is also obvious if $k = 1$, since adding e to P is a cycle containing e. If $k = 2$, we deduce from Lemma 3.7 that $G\backslash e$ is 2-connected, as G/e is not 2-connected. Finally, notice that the two ends of N form a cut of G, which means that G is not 3-connected, so the proof of the lemma is complete. □

Lemma 3.13. *If N is open, then applying $O_2(K_4)$ to N results in a graph in $\Gamma_{k,3}(H)$.*

Proof. Let G' be the resulting graph. From Lemma 3.6 and the definition of $O_2(K_4)$, it is straightforward to verify that $G' \in \Gamma_{k,3}$. Then we deduce from Lemma 3.10 that $H \preceq G'$, and thus the lemma is proved. □

Proof of Theorems 3.1–3.3. As before, let a proper subgraph H' of G be a subdivision of H and let $F = E(G) - E(H')$. By Lemmas 2.14 and 2.16, F has an edge f such that $G\backslash f$ is k-connected. Then, by Lemma 3.9, we may assume that f is contained in a K_4 or a necklace. If f is contained in a subgraph $A \cong K_4$, we deduce from $\delta(H') > 1$ and Lemma 2.12 that $E(A) \subseteq F$. Since $E(H') \neq \emptyset$, we must have $k = 0$, and thus applying $O_0(K_4)$ to A results in a graph in $\Gamma_{k,3}(H)$.

Next, we consider that case when f is contained in a necklace N. By Lemmas 3.12 and 3.13, we may assume that N is short, which implies that $k \leqslant 1$. In addition, by Lemmas 3.8 and 3.13, we may also assume that no edge of F is contained in a tripod or an open necklace. Consequently, by Lemma 3.11, $E(N) \cup \{e\} \subseteq F$ and $(G\backslash e)^\curlywedge$ is simple, where e is the only edge between $V(N)$ and $V(G) - V(N)$. Now it is clear that G is reduced within $\Gamma_{k,3}(H)$ by applying $\mathscr{R}$ to e, if $k = 0$, and by applying $O_1(K_4)$ to N, if $k = 1$. □

4. On graphs of girth at least 4

As before, we state our main results in the first half and prove them in the second half. We will see that, for each $\Gamma_{k,4}$, the only two $\preceq$-minimal graphs are $K_{3,3}$ and the Cube, which will be denoted by Q in this paper. We first define in Figures 5–8 four operations.

In our first splitter theorem, we allow H to be a *null graph*, a graph with neither vertices nor edges.

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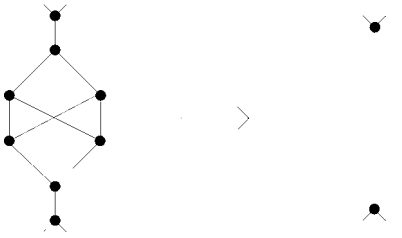


Figure 5. Operation $O_2(K_{3,3})$.

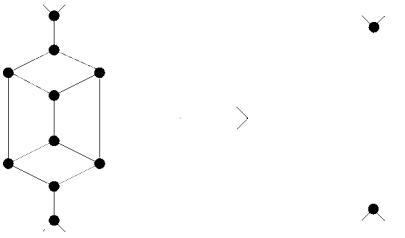


Figure 6. Operation $O_2(Q)$.

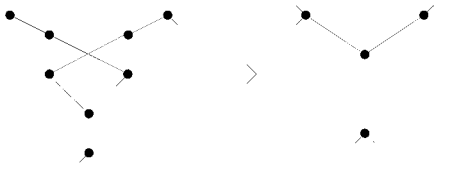


Figure 7. Operation $O_3(K_{3,3})$.

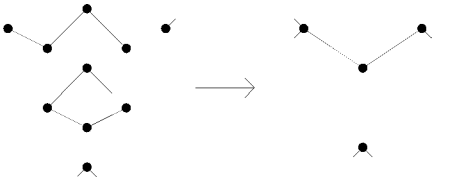


Figure 8. Operation $O_3(Q)$.

Theorem 4.1. *If G and H are cubic graphs of girth at least four and $H \preceq G$, where H could be the null graph, then G can be reduced to H within $\Gamma_{0,4}$ by $\mathscr{R}$ and $O_i(K)$, for $K = K_{3,3}, Q$, and $i = 0, 2, 3$.*

For connected graphs, we need two more operations, which are illustrated in Figures 9–10.

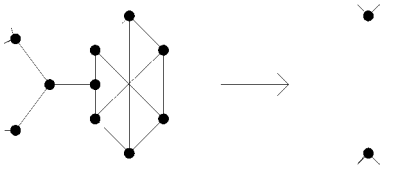


Figure 9. Operation $O_1(K_{3,3})$.

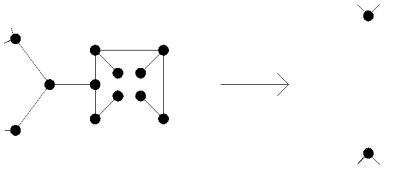


Figure 10. Operation $O_1(Q)$.

Theorem 4.2. *If G and H are connected cubic graphs of girth at least four and $H \preceq G$, then G can be reduced to H within $\Gamma_{1,4}$ by $\mathscr{R}$ and $O_i(K)$, for $K = K_{3,3}, Q$, and $i = 1, 2, 3$.*

The situations for 2- and 3-connected graphs are similar.

Theorem 4.3. *If G and H are 2-connected cubic graphs of girth at least four and $H \preceq G$, then G can be reduced to H within $\Gamma_{2,4}$ by $\mathscr{R}$ and $O_i(K)$, for $K = K_{3,3}, Q$, and $i = 2, 3$.*

Theorem 4.4. *If G and H are 3-connected cubic graphs of girth at least four and $H \preceq G$, then G can be reduced to H within $\Gamma_{3,4}$ by $\mathscr{R}$, $O_3(K_{3,3})$, and $O_3(Q)$.*

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It is not difficult to check, as remarked before, that applying $\mathscr{R}$ or any $O_i(K)$ results in a graph that is topologically contained in the original graph. Therefore, by taking H to be the null graph, we deduce from Theorem 4.1 that *every cubic graph of girth at least four must topologically contain either* $K_{3,3}$ *or* Q. This observation and the last four theorems clearly imply the following corollary. For $i = 1, 2, 3, 4$, let $\mathscr{O}_{i-1}$ be the set of operations used in Theorem 4.i.

Corollary 4.5. *For* $k = 0, 1, 2, 3$, *every graph in* $\Gamma_{k,4}$ *can be reduced within* $\Gamma_{k,4}$ *to* $K_{3,3}$ *or* Q *by operations in* $\mathscr{O}_k$.

In the remainder of this section, we prove the four theorems listed above. As in the last section, we prove all four theorems together. We proceed by proving a sequence of lemmas. Our first lemma is a well-known result, which is a special case of (11.1) in [18].

Lemma 4.6. *If* $G \neq K_4$ *is a* 3*-connected cubic graph, then, for each* $e \in E(G)$, *at least one of* G/e *and* $(G\backslash e)^\sim$ *is* 3*-connected*

As before, let G and H be graphs in $\Gamma_{k,4}$ with $H \prec G$. Let a subgraph H' of G be isomorphic to a subdivision of H. Let $F = E(G) - E(H')$. Without loss of generality, we assume that G is *irreducible*, that is, G has no edge e with $(G\backslash e)^\sim \in \Gamma_{k,4}(H)$.

Lemma 4.7. *If the girth of* $(G\backslash e)^\sim$ *is less than four, for some* $e = xy \in E(G)$, *then at least one of* x *and* y *is contained in a* 4*-cycle of* $G\backslash e$.

Proof. Suppose the other two neighbours of x are x_1, x_2, and the other two neighbours of y are y_1, y_2. Since the girth of G is at least four, vertices x, y, x_1, x_2, y_1, y_2 are all distinct and x_1x_2, y_1y_2 are not edges of G. Consequently, the girth of $(G\backslash e)^\sim$ is at least three, which means, by our assumption, that the girth of $(G\backslash e)^\sim$ is exactly three. Let C be a 3-cycle of $(G\backslash e)^\sim$. Then, at least (exactly) one of the two new edges x_1x_2 and y_1y_2, say x_1x_2, is contained in C. Let x_3 be the other vertex of C. It is clear now that $xx_1x_3x_2x$ is a 4-cycle of $G\backslash e$ that contains x. □

Lemma 4.8. *Let* $Z \subseteq E(G)$ *and let* J *be a component of* $G\backslash Z$ *with* $|V(J)| \leqslant 8$. *Suppose* $E(J) \cap F \neq \emptyset$ *and at least one end of each* $e \in E(J)$ *is on a* 4*-cycle of* $J\backslash e$. *If* $|Z| \leqslant 2$, *then no degree-three vertex of* H' *is in* J. *If* $|Z| = 3$, *then at most one degree-three vertex of* H' *is in* J.

Proof. Let S be the set of degree-three vertices of H' and let $S_J = V(J) \cap S$. We first show that $|V(J) - S_J| \geqslant 4$. Let $e = xy$ be an edge in $E(J) \cap F$. Clearly, $\{x, y\} \subseteq V(J) - S_J$, as $e \in F$. From our assumption, at least one of x and y, say x, is on a 4-cycle C of $J\backslash e$. By Lemma 2.12, C is not a subgraph of H'. Let xu, uv, vw, and wx be the four edges of C. By symmetry, we may assume that either xu or uv is in F. The first case implies $wx \in F$, as $d_{H'}(x) \neq 1$, and thus $\{y, x, u, w\} \subseteq V(J) - S_J$, while the second case implies

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$\{y, x, u, v\} \subseteq V(J) - S_J$. Since G has girth at least four, $y \notin \{u, v, w\}$, and it follows that $|V(J) - S_J| \geqslant 4$ in both cases.

From $|V(J)| \leqslant 8$ we deduce that $|S_J| \leqslant 4$. If $|S_J| = 4$, since H is cubic and has girth at least four, H' must have at least four edge-disjoint paths between S_J and $S - S_J$, which implies $|Z| \geqslant 4$. Similarly, $|S_J| = 3$ implies $|Z| \geqslant 5$, $|S_J| = 2$ implies $|Z| \geqslant 4$, and $|S_J| = 1$ implies $|Z| \geqslant 3$. Then the result follows clearly. □

When an operation $O_i(K)$ is applied to G, we often need to specify the subgraph J of G to which the operation is applied. If $i = 0$, it is clear that J is the component (which is isomorphic to K) to be deleted. For $i = 1, 2, 3$ and $K = K_{3,3}, Q$, notice that, when $O_i(K)$ is applied, G has an edge-cut Z with $|Z| = i$ such that $G \backslash Z$ has a component on $|V(K)| - i + 2$ vertices (see Figures 5–10). This component will be the subgraph referred as J. It is easy to see that: if $i = 1$, then J is obtained from K by subdividing an edge; if $i = 2$, then J is obtained from K by deleting an edge; if $i = 3$, then J is obtained from K by deleting a vertex.

Lemma 4.9. *Let G' be the result of applying $O_2(K)$ or $O_3(K)$ to a subgraph J of G. Suppose G' is simple and J has an edge in F. Then G' is a k-connected cubic graph with $H \preceq G'$.*

Proof. It is easy to see from the definitions of the operations (see Figures 5–8) that G' is cubic. Then it follows from Lemma 3.6 that G' is k-connected. Finally, by Lemma 4.8, it is straightforward to verify that $H \preceq G'$. □

Lemma 4.10. *Suppose an induced subgraph J of G is isomorphic to $K_{3,3} \backslash e$ or $Q \backslash e$. Let x and y be the two degree-two vertices of J and let xx' and yy' be the unique edges of $G \backslash E(J)$ that are incident with x and y, respectively. If $E(J) \cap F \neq \emptyset$, then x' and y' are not adjacent.*

Proof. Suppose, on the contrary, that $f = x'y'$ is an edge of G. We deduce a contradiction by showing that $(G \backslash f) \in \Gamma_{k,4}(H)$. First, by Lemma 4.7, the girth of $(G \backslash f)$ is at least four, since $G \backslash f$ does not have a 4-cycle that contains either x' or y'. Next, we claim that $(G \backslash f)$ is k-connected. The claim is trivial if $k = 0$. It is also obvious if $k = 1$, since f is clearly contained in a cycle of G, which guarantees that f is not a cut-edge. If $k = 2$, we deduce the claim from Lemma 3.7, as G/f is not 2-connected. Furthermore, notice that $\{x, y\}$ is a cut of G, which means $k \neq 3$, so the claim is proved. Finally, we verify that $H \preceq (G \backslash f)$. This is clear if $f \in F$. If $f \in E(H')$, we claim that $E(J') \subseteq F$, where J' is the subgraph of G induced by $E(J) \cup \{xx', yy'\}$. Suppose otherwise. Since, by Lemma 4.8, J does not contain degree-three vertices of H', it follows that edges in $E(J') \cap E(H')$ form a path between x' and y'. Consequently, this path and f form a cycle of H'. Notice that this cycle has at most two vertices, namely, x' and y', that are of degree three in H'. This is a contradiction, by Lemma 2.12, and so our second claim is proved. Let P be a path of J'

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between x' and y'. Now it is easy to see that, in H', replacing f with P results in another subdivision of H and thus we have $H \preceq G\backslash f$, which clearly implies $H \preceq (G\backslash f)^{\urcorner}$. □

Lemma 4.11. *Suppose a subgraph J of G is isomorphic to $K_{3,3} - v$ or $Q - v$. Let x_1, x_2, x_3 be the three degree-two vertices of J and let y_1, y_2, y_3 be three distinct vertices of $V(G) - V(J)$ such that $x_i y_i \in E(G)$, for $i = 1, 2, 3$. If $E(J) \cap F \neq \emptyset$, then $y_i y_j \notin E(G)$, for all $i \neq j$.*

Proof. Suppose otherwise. By symmetry, let us assume that $e = y_1 y_2 \in E(G)$. We deduce a contradiction by showing that $(G\backslash e)^{\urcorner} \in \Gamma_{k,4}(H)$. First, by Lemma 4.7, the girth of $(G\backslash e)^{\urcorner}$ is at least four, since $G\backslash e$ does not have a 4-cycle that contains either y_1 or y_2. Next, we claim that $(G\backslash e)^{\urcorner}$ is k-connected. The claim is trivial if $k = 0$. It is also obvious if $k = 1$, since e is contained in a cycle of G. If $k = 3$, notice that G/e is not 3-connected, as the new vertex and x_3 form a cut of size two. Thus we deduce from Lemma 4.6 that $(G\backslash e)^{\urcorner}$ is 3-connected. If $k = 2$, we need to observe that the path $P = x_1 y_1 y_2 x_2$ can be extended in G into two cycles, say C_1 and C_2, such that edges in P are the only common edges of these two cycles. Therefore, if $(G\backslash e)^{\urcorner}$ is not 2-connected, it must have a cut-edge, as it is cubic. It follows that $G\backslash e$ has a cut-edge, say f. This implies that $\{e, f\}$ is an edge-cut of G, and so, every cycle that contains e must also contain f. From our choice of C_1 and C_2 we deduce that f must be an edge in P, and thus f and e are incident. Consequently, since e and f form an edge-cut, their common end is a cut vertex, which is impossible in a 2-connected graph. This contradiction completes the proof of our claim.

It remains to verify that $H \preceq (G\backslash e)^{\urcorner}$. Clearly, we only need to prove that $G\backslash e$ has a subgraph H'' that is isomorphic to a subdivision of H. Let S be the set of degree-three vertices of H' and let J_0 be the subgraph of G induced by $V(J) \cup \{y_1, y_2\}$. By considering the edge-cut $\{x_1 y_1, x_2 y_2, x_3 y_3\}$, we deduce from Lemma 4.8 that $|V(J) \cap S| \leqslant 1$, which implies that $S_0 = S \cap V(J_0)$ has at most three members. Since H' is a subdivision of a graph of girth at least four, it is easy to see that, when $|S_0| = 2$ or 3, H' has at least $|S_0| + 2$ edge-disjoint paths between S_0 and $S - S_0$. Notice that J_0 is separated from the rest of the graph by three edges. Thus we conclude that $|S_0| \leqslant 1$. Now it is straightforward to verify that by changing $E(H') \cap E(J_0)$ we can get H'', as required. □

Lemma 4.12. *Let $e = xy$ and $f = xz$ be two edges of G and let $C = xzuwx$ be a 4-cycle of G that contains f but not e. If $e \in F$, then $H \preceq G\backslash\{e, f\}$.*

Proof. The result is clear if $f \in F$, so we assume that $f \in E(H')$. From $e \in F$ we deduce that $xv \in E(H')$, as $d_{H'}(x) \neq 1$. We also deduce from Lemma 2.12 that $F' = F \cap \{zu, uw\}$ is not empty. Let H'' be obtained from H' by deleting x and adding edges in F'. Clearly, H'' contains neither e nor f. We claim that H'' is isomorphic to a subdivision of H, which proves the lemma. If $|F'| = 2$, then $d_{H'}(u) \leqslant 1$ and thus u is not in H'. It follows that H'' is the result of replacing the path vxz of H' by another path vuz, so the claim holds. Consequently, we may assume that $|F'| = 1$. By symmetry, we may assume $F' = \{uw\}$, that is, $uw \in F$ and $zu \in E(H')$. Let h be the edge of G between z and $V(G) - V(C)$.

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If $h \in F$, then H'' is the result of replacing the path $vxzu$ of H' by an edge vu, so the claim also holds. If $h \in E(H')$, then $d_{H'}(z) = 3$ and $d_{H'}(x) = d_{H'}(u) = d_{H'}(v) = 2$. Let P be the edge-path of H' that contains x. Then H'' is the result of 'shifting' one end of P from z to u. Clearly, H'' is also a subdivision of H and thus the claim holds again. □

Lemma 4.13. *If $e \in F$ is contained in a 4-cycle of G, then $(G\backslash e)$ is k-connected.*

Proof. The result is trivial when $k = 0$. For $k = 1$, since e is in a cycle, it follows that $G\backslash e$, and thus $(G\backslash e)$, is connected. Next, let $e = v_1v_2$ be contained in a 4-cycle $C = v_1v_2v_3v_4v_1$ of G. For $k = 2, 3$, we prove that, if $(G\backslash e)$ is not k-connected, then $(G\backslash f) \in \Gamma_{k,4}(H)$, for some f of C, which would contradict the assumption that G is irreducible.

We first consider the case when $k = 2$. Suppose $(G\backslash e)$ is not 2-connected. Since $(G\backslash e)$ is cubic and connected, it must have a cut-edge. Consequently, as a subdivision of $(G\backslash e)$, $G\backslash e$ has a cut-edge h. Since $\{e, h\}$ is an edge-cut of G, h must be contained in every cycle that contains e, including C. However, h is not incident with e, because otherwise, their common end would be a cut vertex of G. It follows that $h = v_3v_4$.

Since H is 2-connected, H' is also 2-connected. Since e is not in H', it is clear that h is not in H' either. Therefore, $G\backslash\{e, h\}$ has a component, say A, that is vertex-disjoint from H'. Without loss of generality, let us assume that $f = v_2v_3 \in E(A)$. Notice that G/f is not 2-connected, we thus conclude from Lemma 3.7 that $G\backslash f$ is 2-connected. Consequently, $(G\backslash f)$ is 2-connected. Furthermore, as $f \in E(A) \subseteq F$, we have $H \preceq (G\backslash f)$. Finally, since $\{e, h\}$ is an edge-cut of G, no 4-cycles of $G\backslash f$ contain either end of f. It follows that the girth of $(G\backslash f)$ is at least four, which implies $(G\backslash f) \in \Gamma_{2,4}(H)$, a contradiction.

The proof for the case $k = 3$ is similar. An edge-cut is *trivial* if it separates only one vertex from the rest of the graph. Suppose $(G\backslash e)$ is not 3-connected. Then, as the graph is cubic, $(G\backslash e)$ has an edge-cut of size at most two. Consequently, $G\backslash e$ has a nontrivial edge-cut of size at most two, which in turn implies that, as G is 3-connected, G has a nontrivial edge-cut Z such that $e \in Z$ and $|Z| = 3$. Since G is cubic, it is easy to see that edges in Z are pairwise non-incident. Therefore, the edge v_3v_4 must be contained in Z, as $|E(C) \cap Z| \neq 1$.

Let A and B be the two components of $G\backslash Z$ such that B has at least one degree-three vertex of H'. Since $e \in F$ and H is 3-connected, A cannot have any degree-three vertex of H'. Let f be the unique common edge of C and A. Without loss of generality, let $f = v_2v_3$. Then, let vu be the third edge of Z with $v \in V(A)$ and let h be the only edge of $A\backslash f$ that is incident with v_3. We prove that $(G\backslash f) \in \Gamma_{3,4}(H)$, which gives us the desired contradiction.

Observe that G/f is not 3-connected, as v and the new vertex form a vertex cut of size two. Thus, by Lemma 4.6, $(G\backslash f)$ is 3-connected. To prove $(G\backslash f)$ has girth at least four, by Lemma 4.7, we only need to check that no 4-cycles of $G\backslash f$ contain either v_2 or v_3. But this is clear since Z is an edge-cut and all its edges are pairwise non-incident. Finally, we verify that $H \preceq (G\backslash f)$. This is obvious if $f \in F$. If $f \in E(H')$, since $e \in F$ and all degree-three vertices of H' are in B, we deduce that $H'\backslash E(B)$ is a path between v_4

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and u. Notice that both f and v_3v_4 are in this path, therefore, h is not and thus $h \in F$. Now we conclude from Lemma 4.12 that $H \preceq (G\backslash f)^{\urcorner}$ and that completes the proof of the Lemma. □

Lemma 4.14. *If a* 4*-cycle* C *of* G *contains an edge* e *in* F*, then either some* $O_i(K)$ $(i = 0, 2, 3$ *and* $K = K_{3,3}, Q)$ *can be applied to* G *to result in a graph in* $\Gamma_{k,4}(H)$ *or* C *is contained in a subgraph isomorphic to some* U_j *illustrated in Figure 11.*

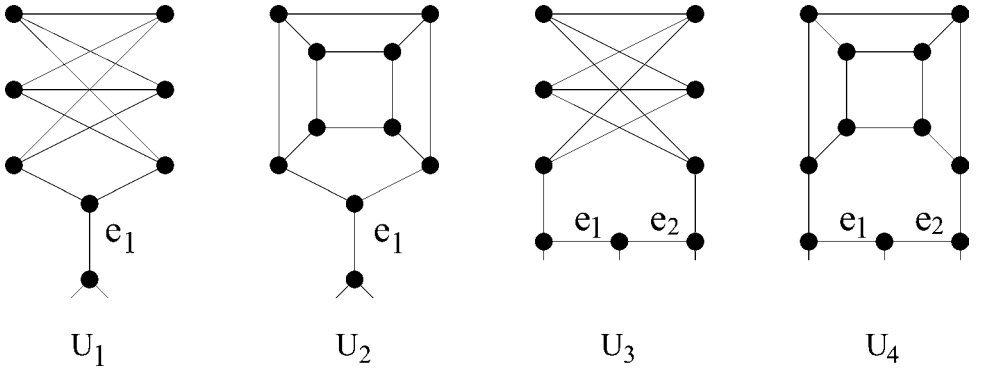


Figure 11. Subgraphs U_1, U_2, U_3, and U_4.

Proof. We first show that C is contained in an induced subgraph J of G such that J is isomorphic to either $K_{3,3} - v$ or $Q - v$. By Lemmas 4.13 and 2.11, $(G\backslash e)^{\urcorner} \in \Gamma_{k,1}(H)$. Since G is irreducible, by Lemma 4.7, at least one end of e is contained in a 4-cycle C' of $G\backslash e$. Notice that $5 \leqslant |V(C) \cup V(C')| \leqslant 6$. If $V(C) \cup V(C')$ has five vertices, then the subgraph it induces is isomorphic to $K_{2,3}$, and that gives us the subgraph J we are looking for. If $|V(C) \cup V(C')| = 6$, then $L = (V(C) \cup V(C'), E(C) \cup E(C'))$ consists of a 6-cycle and a chord f, which is the only common edge of C and C'. Clearly, e is one of the edges that are incident with f. By Lemma 4.12, we can assume that f is also in F. Then, by Lemma 4.13, we deduce that $(G\backslash f)^{\urcorner} \in \Gamma_{k,1}(H)$. Since G is irreducible, at least one end of f is contained in a 4-cycle C'' of $G\backslash f$. Suppose C' was chosen with $V(C) \cup V(C')$ minimal. Then L has to be an induced subgraph of G. Consequently, the subgraph induced by $V(L) \cup V(C'')$ is isomorphic to $Q - v$, which is the subgraph J we are looking for.

Let $K = K_{3,3}$ if $|V(J)| = 5$ and let $K = Q$ if $|V(J)| = 7$. If applying $O_3(K)$ to J results in a graph in $\Gamma_{k,4}(H)$, then we are done. Else, by Lemma 4.9, the shortest path P of $G\backslash E(J)$ that is between two distinct vertices of J must have length either two or three. From Lemma 4.11 we deduce that $|P| = 2$. Let $J' = J \cup P$ and let A be the component of G that contains J'. If $V(A) = V(J')$, then A is isomorphic to K and $k = 0$. It follows that applying $O_0(K)$ to A results in a graph in $\Gamma_{k,4}(H)$ and thus we are done. Next, we consider the case when $|V(A)| > |V(J')|$. In this case, $O_2(K)$ can be applied to J'. If the result is

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not in $\Gamma_{k,4}(H)$, then, by Lemma 4.9 again, $G\backslash E(J')$ has a path P' between two distinct vertices of J' such that $2 \leqslant |P'| \leqslant 4$. We deduce from Lemma 4.10 that $|P'| \neq 3$, and so $|P'| = 2$ or 4. If $|P'| = 4$, $J' \cup P'$ is isomorphic to U_3 or U_4; if $|P'| = 2$, $J' \cup P' \cup \{e_1\}$ is isomorphic to U_1 or U_2, where e_1 is the edge not in P' but is incident with the middle vertex of P'. □

Lemma 4.15. *Suppose no $O_i(K)$ ($i = 0, 2, 3$ and $K = K_{3,3}, Q$) can be applied to G to result in a graph in $\Gamma_{k,4}(H)$. If a subgraph U of G is isomorphic to U_3 or U_4, then all its 4-cycles are contained in H'.*

Proof. Suppose, on the contrary, that some 4-cycle of U contains an edge in F. We prove that $(G\backslash e_i) \curlywedge \in \Gamma_{k,4}(H)$, for some $i \in \{1, 2\}$, where e_1 and e_2 are the two edges of U that are indicated in Figure 11. This would be a contradiction, as G is irreducible. Let J be the union of all the 4-cycles of U. Then J has an edge in F and J is isomorphic to either $K_{3,3}\backslash e$ or $Q\backslash e$. Let the two edges between $V(J)$ and $V(G) - V(J)$ be f_1 and f_2, where f_i ($i = 1, 2$) is the one incident with e_i. For $i = 1, 2$, let the common vertex of f_i and e_i be x_i. Let J^+ be obtained from J by adding vertices x_1, x_2 and edges f_1, f_2. Let the common vertex of e_1 and e_2 be x.

We first prove that, for some $i \in \{1, 2\}$, $G\backslash e_i$ contains a subdivision H'' of H such that $E(J) \nsubseteq E(H'')$. This is clear if at least one e_i is in F, so we assume that both e_1 and e_2 are in $E(H')$. By Lemma 4.8, J has no degree-three vertices of H', which means that either $E(H') \cap E(J^+) = \emptyset$ or edges in $E(H') \cap E(J^+)$ form a nonempty path between x_1 and x_2. However, in the second case, the path and e_1, e_2 would make a cycle of H', which contains at most three degree-three vertices, namely, x_1, x_2, and x, of H'. By Lemma 2.12, this is impossible, so $E(J^+)$ must be a subset of F. Consequently, $d_{H'}(x_1) = d_{H'}(x_2) = 2$. Let P be a path of J^+ between x_1 and x_2. If $d_{H'}(x) = 2$, then replacing the path x_1xx_2 of H' with P results in another subdivision H'' of H, which avoids e_1, e_2 and at least one edge of J. If $d_{H'}(x) = 3$, then $H'' = (H'\backslash e_1) \cup P$ avoids e_1 and at least one edge of J. Furthermore, H'' is another subdivision of H, as it is the result of 'shifting' one end of P', the edge-path of H' that contains x_1, from x to x_2. In conclusion, we can always find H'' and thus we may assume that $e_1 \in F$, which in turn implies that $H \preceq (G\backslash e_1) \curlywedge$.

Next, we prove that $(G\backslash e_1) \curlywedge$ is k-connected. This is trivial when $k = 0$, and it is also obvious when $k = 1$, as e_1 is contained in a cycle. Since $\{x_1, x_2\}$ is a cut of the graph, G is not 3-connected, and so we only need to consider the case when $k = 2$. Let $A = G - V(J) - \{x, x_1, x_2\}$. Observe that G has only three edges between $V(A)$ and $V(G) - V(A)$. Since G is 2-connected, A must be connected. It follows that $G - V(J) - x_2$ has a cycle C_1 containing e_1. Let C_2 be a cycle of $G - V(A)$ that contains e_1. Now, if $(G\backslash e_1) \curlywedge$ is not 2-connected, it must have a cut-edge, as it is cubic. This means that $G\backslash e_1$ has a cut-edge, say e, and thus every cycle that contains e_1 has to contain e. Therefore, e should belong to both $C_1\backslash e_1$ and $C_2\backslash e_1$, which is impossible from our choice of these two cycles. This contradiction completes the proof that $(G\backslash e_1) \curlywedge$ is k-connected.

It remains to show that the girth of $(G\backslash e_1) \curlywedge$ is at least four. Suppose otherwise. Then, by Lemma 4.7, $G\backslash e_1$ has a 4-cycle C that contains either x_1 or x. Observe that f_i ($i = 1, 2$) is

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not contained in any 4-cycle of G, so C must contain x, x_2, and another two vertices in $G - V(J) - \{x, x_1, x_2\}$. Since $e_1 \in F$, as proved earlier, we deduce from Lemma 2.12 that C also has an edge in F. Thus, by Lemma 4.14, C is contained in a subgraph U' of G such that U' is isomorphic to some U_j defined in Figure 11. Let J' be the subgraph of U' such that J' is isomorphic to $K_{3,3}\backslash e$ or $Q\backslash e$. Then it is clear that C is contained in J'. In particular, x_2x is an edge of J'. Since x_2 is incident with f_2, an edge does not belong to any 4-cycle, it follows that x_2 must be a degree-two vertex of J'. This fact and $x_2x \in E(J')$ imply that x_2xx_1 is a path of J', which in turns implies that x_1 is a degree-three vertex of J'. But this is a contradiction because, as a degree-three vertex of J', each edge incident with it must be contained in a 4-cycle, while we do know that x_1 is incident with f_1, an edge that does not belong to any 4-cycle. This contradiction proves that C does not exist, which completes our proof of the lemma. □

Lemma 4.16. *Suppose no $O_i(K)$ $(i = 0, 2, 3$ and $K = K_{3,3}, Q)$ can be applied to G to result in a graph in $\Gamma_{k,4}(H)$. If a subgraph U of G is isomorphic to either U_1 or U_2, and $E(U) \cap F \neq \emptyset$, then $(G\backslash e_1\urcorner \in \Gamma_{0,4}(H)$, where e_1 is the edge indicated in Figure 11.*

Proof. Let J be the subgraph of U that is isomorphic to either $K_{3,3}\backslash e$ or $Q\backslash e$. By Lemma 2.12, we may assume that J has an edge in F. Let $e_1 = xy$, where x is the one adjacent to the degree two vertices of J. We first claim that $e_1 \in F$. Suppose otherwise. Then one end of the edge-path that contains e_1 is in $V(J) \cup \{x\}$, which means that $V(J) \cup \{x\}$ contains at least one degree-three vertex of H'. By Lemma 4.8, J has no degree-three vertices of H'. It follows that $d_{H'}(v) = 0$ or 2, for all $v \in V(J)$, and $d_{H'}(x) = 3$. Therefore, $U - y$ contains a cycle of H', but contains only one degree-three vertex of H', contradicting Lemma 2.12, and so our claim is proved.

Suppose $(G\backslash e_1\urcorner \notin \Gamma_{0,4}(H)$. By the above claim, y is contained in 4-cycle C. Then, by Lemma 2.12, C has an edge in F. From Lemma 4.14 we deduce that C is contained in a subgraph U' of G such that U' is isomorphic to some U_j. Let J' be the union of all 4-cycles of U'. Then y is a vertex of J' and J' is isomorphic to either $K_{3,3}\backslash e$ or $Q\backslash e$. From the definition of y it is clear that y is a cut vertex of G. However, it is straightforward to verify (see Figure 11) that, as a vertex of J', which is a subgraph U', y is not a cut vertex of G. This contradiction completes our proof of the lemma. □

Lemma 4.17. *G has an edge f, which is contained in a 4-cycle and such that $H \preceq G\backslash f$.*

Proof. We first claim that G has an edge $e = xy$ such that $(G\backslash e\urcorner \in \Gamma_{k,1}(H)$. The claim is trivial when $k = 0$, as every $e \in F$ has the required property. If $k > 0$, then the claim follows easily from Lemma 2.14, Lemma 2.16, and Theorem 3.4. Since G is irreducible, $(G\backslash e\urcorner$ must have girth less than four. Therefore, by Lemma 4.7, either x or y is contained in a 4-cycle C of $G\backslash e$. Now the lemma follows obviously from Lemma 4.12. □

Proof of Theorems 4.1–4.4. By Lemma 4.17, we may assume that G has a 4-cycle C that contains an edge in F. Then, by Lemmas 4.14 and 4.15, we may further assume that C is

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contained in a subgraph U of G such that U is isomorphic to either U_1 or U_2 defined in Figure 11. Since U has a cut edge of G, we deduce that k is either 0 or 1. By Lemma 4.16, we know that k can only be 1. Let J be obtained from U by deleting its degree-one vertex. Then J is isomorphic to a subdivision of K, where $K = K_{3,3}$ or Q. By Lemma 4.16 again, we deduce that applying $O_1(K)$ to J results in a graph in $\Gamma_{1,4}(H)$. The proof is complete. □

5. Further remarks

A connected cubic graph is *cyclically k-connected* if every edge-cut of size less than k can only separate a tree from the rest of the graph. It is not difficult to prove (see [13], for example) that, for $k \leqslant 3$, being cyclically k-connected is the same as being k-connected, unless the graph is $3K_2$, K_4, or $K_{3,3}$, which are considered as cyclically 2-, 3-, and 4-connected, respectively. Therefore, $\Gamma_{k,g}$ can be considered as the class of cyclically k-connected cubic graphs of girth at least g. By modifying the definition of $\Gamma_{k,g}$, we can consider cases for $k > 3$. Notice that the girth of a cyclically k-connected cubic graph is at least k, so we may still assume $k \leqslant g$, as before.

We have considered in this paper all cases when $g \leqslant 3$. For $g = 4$, the only class not considered in our paper is $\Gamma_{4,4}$. The following is the splitter theorem (see [11]) for this class.

Theorem 5.1. *If G and H are cyclically 4-connected cubic graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{4,4}$ by $\mathscr{R}$.*

Corollary 5.2. *Every cyclically 4-connected cubic graph can be reduced within $\Gamma_{4,4}$ by operation $\mathscr{R}$ to either $K_{3,3}$ or Q.*

This corollary was proved by Wormald [25] and Fontet [7], while its planar version was proved by Kotzig [12] and Faulkner and Younger [6].

For $g \geqslant 5$, as far as we know, the only class that has been studied is $\Gamma_{5,5}$. Let $\Gamma^p_{5,5}$ be the class of planar graphs in $\Gamma_{5,5}$. Barnette [2] and Butler [4] proved that every graph in $\Gamma^p_{5,5}$ can be reduced within $\Gamma^p_{5,5}$ to the Dodecahedron by $\mathscr{R}$ and two other operations, O_a and O_b, which we do not define here. Martinov [14] and McCuaig [13] generalized this result by showing that every graph in $\Gamma_{5,5}$ can be reduced within $\Gamma_{5,5}$ by $\mathscr{R}$ and O_a to either one of three special graphs or a graph in an infinite sequence, which includes the Petersen graph and the Dodecahedron. A weaker version of the last result was proved in [1], which uses three operations, instead of two. Finally, we point out that the splitter theorem for $\Gamma_{5,5}$ is established by Robertson, Seymour and Thomas [17]. To be a little more specific, they proved that, if G and H are graphs in $\Gamma_{5,5}$ with $H \preceq G$, then, modulo well-described exceptions, G can be reduced to H within $\Gamma_{5,5}$ by $\mathscr{R}$ and O_a. A natural remaining problem is to establish splitter theorems for each class $\Gamma_{k,5}$ with $k \leqslant 4$.

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