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A generic bounded linear cocycle has simple Lyapunov spectrum

NGUYEN DINH CONG

Hanoi Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam
(e-mail: ndcong@math.ac.vn)

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Abstract. We show that the set of cocycles with integral separateness is open and dense in the space of all bounded $Gl(d, \mathbb{R})$-cocycles equipped with uniform topology. As a consequence, a generic bounded linear cocycle has simple Lyapunov spectrum and dominated Oseledets splitting, and a generic bounded $Sl(2, \mathbb{R})$-cocycle is uniformly hyperbolic.

1. *Introduction*

A discrete-time linear deterministic dynamical system is defined by a single matrix A, and its Lyapunov spectrum is nothing but the set of the logarithms of the moduli of the eigenvalues of A. It is well-known that the eigenvalues depend continuously on A (see Kato [**15**]), hence the matrices with simple Lyapunov spectrum form an open (but not dense as the multiplicity 2 of pairs of conjugate complex eigenvalues is stable under small perturbations) set in the space of all non-singular matrices.

Millionshchikov studied linear systems of non-autonomous differential equations and proved that the systems with so-called *integral separateness* form an open and dense set in the space of all linear systems equipped with the uniform topology (see [**17, 18**]). A system with integral separateness has simple Lyapunov spectrum, hence the set of linear systems of differential equations with simple Lyapunov spectrum contains an open and dense set. We note that besides having simple Lyapunov spectrum, systems with integral separateness exhibit an exponentially dichotomous behavior of trajectories which leads to the stronger property that all systems in a neighborhood of it have a similar property. This uniform property is crucial in the investigation of structural stability of dynamical systems.

The objects of interest in this paper are products of random matrices (linear cocycles). Thanks to the multiplicative ergodic theorem of Oseledets [**20**], the Lyapunov spectrum of a cocycle is well defined (under some integrability conditions) and it is a generalization of the Lyapunov spectrum in the deterministic case.

The study of the Lyapunov spectrum of linear cocycles is one of the central tasks of the theory of random dynamical systems. In various situations (see Arnold [**2**]) it is of great

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theoretical and practical importance to know when the Lyapunov spectrum is simple. In the particular case of a product of independent and identically distributed random matrices, the Lyapunov spectrum is fairly well investigated and strong results have been obtained (see Guivarc'h and Raugi [**13**], Goldsheid and Margulis [**12**], Arnold and Cong [**3**]). We mention that in this case, the cocycles with simple Lyapunov spectrum form a residual set [**3**]. This and the result of Millionshchikov mentioned above support the conjecture that cocycles with simple spectrum form a 'big' set. However, the conjecture remained open for a long time. Only an easier problem of density of simple Lyapunov spectrum found a solution for the general stationary case: the cocycles with simple Lyapunov spectrum form a dense set in the space of linear cocycles equipped with the uniform topology (see Knill [**16**] for the two-dimensional case and Arnold and Cong [**4**] for the general d-dimensional case). Johnson [**14**] studied continuous-time cocycles on a compact metric space and proved that in the two-dimensional case the set of cocycles with simple spectrum is C^0-dense. Recently, Arbieto and Bochi [**1**], Bochi [**6**], Bochi and Viana [**7, 8**] and Bonatti and Viana [**9**] have derived some new results on the genericity of hyperbolicity of several classes of dynamical systems including smooth dynamical systems and linear cocycles. Let us mention here a result of Bochi [**6**, Theorem B] stating that a generic (in L^∞ topology) linear bounded $Sl(2, \mathbb{R})$-cocycle is either uniformly hyperbolic or has zero Lyapunov exponent and a result by Bochi and Viana [**7**] stating that the Oseledets splitting of a generic (in C^0 topology) linear cocycle is either dominant (but the Lyapunov spectrum need not be simple) or trivial.

In this paper we give a solution to the problem of genericity of simple Lyapunov spectrum: we prove that the set of cocycles with integral separateness is open and dense in the space of all bounded $Gl(d, \mathbb{R})$-cocycles equipped with uniform topology. As a consequence, a generic bounded linear cocycle has simple Lyapunov spectrum and dominated Oseledets splitting, and a generic bounded $Sl(2, \mathbb{R})$-cocycle is uniformly hyperbolic. In particular, our result improves the above mentioned result of Bochi [**6**]: the zero Lyapunov exponent can only happen in a nowhere dense set in the space of bounded $Sl(2, \mathbb{R})$-cocycles, hence in the theorem of Bochi the second alternative can be dropped. As compared with the above mentioned result of Bochi and Viana [**7**] the difference between L^∞ topology considered in this paper and the C^0 topology considered there is the Lyapunov spectrum of a generic cocycle must be simple in L^∞ topology whereas it must not in C^0 topology. Note that the genericity (in L^∞ topology) of uniform hyperbolicity of $Sl(2, \mathbb{R})$-cocycles derived in this paper cannot be extended to the case of C^0-topology (see Bochi and Viana [**7**] for an example; see also Young [**23**] for an example in C^1-case).

This paper is organized as follows. In §2 we give necessary definitions and the set-up of the problem; we also present the three closely related concepts of exponential dichotomy, integral separateness and dominated splitting, all of which are introduced for the investigation of qualitative behavior of dynamic systems (the definitions we give here are adapted to the case of linear cocycles which is the object of our interest in this paper). Section 3 states and proves the main results of the paper: genericity of integral separateness, and hence the genericity of simple Lyapunov spectrum and dominated Oseledets splitting, and in the $Sl(2, \mathbb{R})$ case we also have genericity of exponential dichotomy, hence uniform hyperbolicity.

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2. *The space of linear cocycles and exponential separation properties*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic Lebesgue probability space, and let θ be an ergodic automorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ preserving the probability measure $\mathbb{P}$. A measurable mapping A from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the topological space $Gl(d, \mathbb{R})$ (for short, $Gl(d)$) of linear non-singular operators of $\mathbb{R}^d$ equipped with its Borel σ-algebra is called a *random linear map*. The mapping A generates a *linear cocycle* over the dynamical system θ via

$$\Phi_A(n, \omega) := \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega) & n > 0, \\ \mathrm{id} & n = 0, \\ A^{-1}(\theta^n \omega) \circ \cdots \circ A^{-1}(\theta^{-1}\omega) & n < 0. \end{cases}$$

Conversely, if we are given a linear cocycle over θ, then its time-one map is a linear random map. Therefore, we usually speak of a linear cocycle A, meaning the cocycle Φ_A generated by A. The above construction applies to any topological group G in place of $Gl(d)$ (in particular, G can be a Lie subgroup of $Gl(d)$, for instance $Sl(d)$), and we shall speak of G-cocycles and random G-maps in that case.

For simplicity of notation we denote by $\|\cdot\|$ both the standard Euclidean norm of $\mathbb{R}^d$ and the operator norm of linear operators of $\mathbb{R}^d$. We shall look at linear cocycles as linear operators of $\mathbb{R}^d$ and identify them with their matrix representations in the standard Euclidean basis of $\mathbb{R}^d$. Thus we also denote by $Gl(d)$ the group of non-singular d-dimensional matrices.

The multiplicative ergodic theorem of Oseledets [**20**] (see also Arnold [**2**]) states that if $A(\cdot)$ satisfies the integrability conditions

$$\log^+ \|A(\cdot)^{\pm 1}\| \in \mathbb{L}^1(\mathbb{P}), \tag{1}$$

then the cocycle Φ_A has Lyapunov exponents $\lambda_1 < \cdots < \lambda_p$ with multiplicities $d_1, \ldots, d_p$, which are independent of ω due to the ergodicity of θ, and the phase space $\mathbb{R}^d$ is decomposed into the direct sum of subspaces $E_i(\omega)$ of dimensions d_i corresponding to the Lyapunov exponents λ_i, $i = 1, \ldots, p$, i.e.

$$\lim_{n \to \pm\infty} n^{-1} \log \|\Phi_A(n, \omega)x\| = \lambda_i \iff x \in E_i(\omega) \backslash \{0\}.$$

The above splitting is called the *Oseledets splitting* of Φ_A, and the subspaces $E_i(\omega)$ are called *Oseledets subspaces* of Φ_A; they are measurable and *invariant with respect to* A, i.e. $A(\omega)E_i(\omega) = E_i(\theta\omega)$. We note that the statements of the multiplicative ergodic theorem hold on an invariant set of full $\mathbb{P}$-measure. The collection $\{(\lambda_i, d_i) : i = 1, \ldots, p\}$ is called the *Lyapunov spectrum* of A. It is said to be *simple* if $p = d$.

Since we deal with discrete-time cocycles we can always neglect sets of null measure, and we shall identify the random mappings which coincide $\mathbb{P}$-almost surely, and when needed we shall assume, without loss of generality, that the assertions of the Oseledets' theorem hold on the whole of Ω.

Denote by $\mathcal{G}(d)$ the set of all $Gl(d)$-valued random maps, and let $\mathcal{G}_{\mathrm{IC}}(d) \subset \mathcal{G}(d)$ denote the subset of those satisfying the integrability conditions (1) of the Oseledets' theorem, and let $\mathcal{G}_\infty(d) \subset \mathcal{G}(d)$ denote the subset of those random maps which are essentially bounded together with their inverses. Clearly $\mathcal{G}_\infty(d) \subset \mathcal{G}_{\mathrm{IC}}(d)$.

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We define a metric ρ on $\mathcal{G}(d)$ such that $(\mathcal{G}(d), \rho)$ can be considered as a version of $L^\infty(\mathbb{P})$. For $A, B \in \mathcal{G}(d)$ set

$$\delta(A, B) := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A^{-1}(\omega) - B^{-1}(\omega)\|$$

and

$$\rho(A, B) := \begin{cases} \delta(A, B)(1 + \delta(A, B))^{-1} & \text{if } \delta(A, B) < \infty, \\ 1 & \text{if } \delta(A, B) = \infty. \end{cases}$$

It is known (see Arnold and Cong [**3**]) that $(\mathcal{G}(d), \rho)$ is a complete metric space, and $\mathcal{G}_{\mathrm{IC}}(d)$ and $\mathcal{G}_\infty(d)$ are closed subsets, hence are also complete metric spaces. Moreover, if $A \in \mathcal{G}_{\mathrm{IC}}(d)$ (respectively, $\mathcal{G}_\infty(d)$), $B \in \mathcal{G}(d)$ and $\rho(A, B) < 1$, which is equivalent to $\delta(A, B) < \infty$, then $B \in \mathcal{G}_{\mathrm{IC}}(d)$ (respectively, $\mathcal{G}_\infty(d)$). Being complete metric spaces, $\mathcal{G}(d)$, $\mathcal{G}_{\mathrm{IC}}(d)$ and $\mathcal{G}_\infty(d)$ are Baire spaces. A subset is *residual* if it contains a countable intersection of open and dense sets. A property which holds on a residual set is called *generic*.

For two non-vanishing vectors x, y of $\mathbb{R}^d$ the angle between them is defined by

$$\measuredangle(x, y) := \arccos \frac{\langle x, y\rangle}{\|x\|\|y\|} \in [0, \pi].$$

Clearly, $\operatorname{span}\{x\} = \operatorname{span}\{y\}$ if and only if $\measuredangle(x, y) \in \{0, \pi\}$. The angle between two subspaces $E_1, E_2 \subset \mathbb{R}^d$ is defined by

$$\measuredangle(E_1, E_2) := \inf\{\measuredangle(x, y) : 0 \neq x \in E_1, 0 \neq y \in E_2\}.$$

Obviously, $0 \leq \measuredangle(E_1, E_2) \leq \pi/2$.

Definition 2.1. A linear cocycle $A \in \mathcal{G}(d)$ is said to exhibit an *exponential dichotomy* if there exist positive numbers $K > 0$, $\alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$ depending measurably on $\omega \in \Omega$ such that:

(i) $\|\Phi_A(n, \omega)P_\omega\Phi_A^{-1}(m, \omega)\| \leq K \exp(-\alpha(n - m))$ for all $n \geq m$, $\omega \in \Omega$;

(ii) $\|\Phi_A(n, \omega)(\mathrm{id} - P_\omega)\Phi_A^{-1}(m, \omega)\| \leq K \exp(\alpha(n - m))$ for all $n \leq m$, $\omega \in \Omega$.

Remark 2.2.

(i) If $A \in \mathcal{G}(d)$ has an exponential dichotomy with constants $K > 0$, $\alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$, then $\|P_\omega\| \leq K$ for all $\omega \in \Omega$, but $A(\cdot)$ and $A^{-1}(\cdot)$ might be unbounded.

(ii) The angle between the subspaces $P_\omega\mathbb{R}^d$ and $(\mathrm{id} - P_\omega)\mathbb{R}^d$ is separated from zero by a positive constant independent of $\omega \in \Omega$.

(iii) The random subspaces $E_1(\omega) := P_\omega\mathbb{R}^d$ and $E_2(\omega) := (\mathrm{id} - P_\omega)\mathbb{R}^d$ are invariant with respect to A, i.e. $\Phi_A(n, \omega)E_i(\omega) = E_i(\theta^n\omega)$ for all $n \in \mathbb{Z}$, $\omega \in \Omega$ and $i = 1, 2$.

(iv) The projection P_ω commutes with A in the sense that $\Phi_A(n, \omega) \circ P_\omega = P_{\theta^n\omega} \circ \Phi_A(n, \omega)$ for all $n \in \mathbb{Z}$ and $\omega \in \Omega$.

(v) Exponential dichotomy is also called uniform hyperbolicity by some authors.

Definition 2.3. Let $A \in \mathcal{G}(d)$ and let

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \tag{2}$$

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be an invariant splitting of A, i.e. for almost all $\omega \in \Omega$ we have

$$A(\omega)E_i(\omega) = E_i(\theta\omega), \quad i = 1, \ldots, k.$$

Then the splitting (2) is called *dominated over the orbit of* ω if there exists an integer $m > 0$ such that for any $r \in \mathbb{Z}$ and any $i = 1, \ldots, k-1$, we have

$$\frac{\|\Phi_A(m, \theta^r\omega)x\|}{\|x\|} \leq \frac{1}{2}\frac{\|\Phi_A(m, \theta^r\omega)y\|}{\|y\|}$$

for all non-vanishing vectors $x \in E_1(\theta^r\omega) \oplus \cdots \oplus E_i(\theta^r\omega)$ and $y \in E_{i+1}(\theta^r\omega) \oplus \cdots \oplus E_k(\theta^r\omega)$; the splitting is called simply *dominated* if it is dominated over orbits of almost all $\omega \in \Omega$, in this case m can be chosen to be the same for almost all ω due to ergodicity of θ.

The following definition is a random version of the notion of integral separateness of linear systems of differential equations (see Bylov *et al* [**10**] for the case of ordinary differential equations).

Definition 2.4. A linear cocycle $A \in \mathcal{G}(d)$ is *integrally separated with index* k ($k \in \{1, 2, \ldots, d-1\}$) if there exist positive numbers $K > 0$, $\alpha > 0$ and a projections P_ω of $\mathbb{R}^d$ of rank k depending measurably on $\omega \in \Omega$ such that:

(i) $\Phi_A(n, \omega) \circ P_\omega = P_{\theta^n\omega} \circ \Phi_A(n, \omega)$ for all $n \in \mathbb{Z}$, $\omega \in \Omega$;

(ii) $\|\Phi_A(n, \omega)^{-1}(\mathrm{id} - P_{\theta^n\omega})\|^{-1} \geq K \exp(\alpha n)\|\Phi_A(n, \omega)P_\omega\|$ for all $n \in \mathbb{Z}^+$, $\omega \in \Omega$.

If A is integrally separated with index k for all $k = 1, 2, \ldots, d-1$ then A is called simply *integrally separated*.

The properties of exponential dichotomy, dominance and exponential separateness are similar and they are all robust under small perturbation.

THEOREM 2.5. *If $A \in \mathcal{G}_\infty(d)$ exhibits an exponential dichotomy (respectively, has a dominated splitting, is integrally separated with index k) then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also exhibits an exponential dichotomy (respectively, has a dominated splitting, is integrally separated with index k).*

This is a kind of roughness property. For a proof in the case of exponential dichotomy see Coppel [**11**, Proposition 1, Ch. 4] (see also Sacker and Sell [**22**]). Note that, unlike exponential dichotomy, dominance and integral separateness do not imply that the angle between two invariant subspaces is bounded from below. They do imply boundedness from below of the angle if we assume additionally that the cocycle is bounded, as we did in Theorem 2.5. In the case of exponential dichotomy we may change the condition $A \in \mathcal{G}_\infty(d)$ in Theorem 2.5 into the condition $A \in \mathcal{G}(d)$.

The following theorems reveal a relation between the properties of exponential dichotomy, having dominated splitting and integral separateness, and some of their properties. The proof is straightforward from the definitions of those notions.

THEOREM 2.6.

(i) *If $A \in \mathcal{G}(d)$ is integrally separated with index k for some $1 \leq k < d$ with a family of projections P_ω of $\mathbb{R}^d$ then the random subspaces $P_\omega\mathbb{R}^d$ and $(\mathrm{id} - P_\omega)\mathbb{R}^d$ are*

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invariant with respect to the cocycle A, and if, moreover, $A \in \mathcal{G}_\infty(d)$ then the angle between $P_\omega\mathbb{R}^d$ and $(\mathrm{id} - P_\omega)\mathbb{R}^d$ is separated from zero by a positive constant independent of $\omega \in \Omega$.

(ii) *Let $A \in \mathcal{G}_\infty(d)$ have a dominated splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega)$$

with $\dim E_i(\omega) = d_i$. Then for any $i = 1, \ldots, k$ the angle between $E_i(\omega)$ and

$$E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$$

is separated from zero by a positive constant independent of $\omega \in \Omega$.

Proof. (i) By definition $A(\omega)P_\omega = P_{\theta\omega}A(\omega)$. Therefore,

$$A(\omega)P_\omega\mathbb{R}^d = P_{\theta\omega}A(\omega)\mathbb{R}^d = P_{\theta\omega}\mathbb{R}^d,$$

hence $P_\omega\mathbb{R}^d$ is invariant with respect to the cocycle A. Similarly, $(\mathrm{id} - P_\omega)\mathbb{R}^d$ is invariant with respect to the cocycle A.

If $A \in \mathcal{G}_\infty(d)$ then there exists a positive number $M > 1$ such that $\|A(\omega)\| + \|A(\omega)^{-1}\| < M$ for all $\omega \in \Omega$. Let $x \in P_\omega\mathbb{R}^d$ and $y \in (\mathrm{id} - P_\omega)\mathbb{R}^d$ be arbitrary unit vectors. Let $K > 0$ and $\alpha > 0$ be the positive parameters provided by the definition of integral separateness of A. Choose and fix $n \in \mathbb{N}$ such that $K\exp(\alpha n) > 2$. Then we have

$$2\|\Phi_A(n,\omega)x\| \le \|\Phi_A(n,\omega)y\| \le \|\Phi_A(n,\omega)x\| + \|\Phi_A(n,\omega)(y-x)\|,$$

which implies that

$$\|\Phi_A(n,\omega)x\| \le \|\Phi_A(n,\omega)(y-x)\| \le \|\Phi_A(n,\omega)\|\|y-x\|.$$

Consequently,

$$\|y-x\| \ge \|\Phi_A(n,\omega)\|^{-1}\|\Phi_A(n,\omega)^{-1}\|^{-1} \ge M^{-2n},$$

hence $\measuredangle(x,y) \ge M^{-2n}$ and the angle between $P_\omega\mathbb{R}^d$ and $(\mathrm{id} - P_\omega)\mathbb{R}^d$ is separated from zero by $M^{-2n} > 0$.

(ii) is similar to (i). □

THEOREM 2.7.

(i) *Let $A \in \mathcal{G}_\infty(d)$ and let*

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \tag{3}$$

be an invariant splitting of A with $\dim E_i(\omega) = d_i$. Then the splitting (3) is dominated if and only if A is integrally separated with index d_1 and with projections P_ω being projections onto $E_1(\omega)$ along $E_2(\omega)$.

(ii) *Suppose that the linear cocycle $A \in \mathcal{G}(d)$ exhibits an exponential dichotomy with constants $K > 0$, $\alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$. Then the invariant splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega)$$

with $E_1(\omega) := P_\omega\mathbb{R}^d$ and $E_2(\omega) := (\mathrm{id} - P_\omega)\mathbb{R}^d$ is dominated.

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Remark 2.8. Exponential dichotomy is strictly stronger than integral separateness: one can construct a one-dimensional cocycle which is non-uniformly hyperbolic (say with negative Lyapunov exponent) and join to it another one-dimensional cocycle which is separated uniformly from the former one and has positive Lyapunov exponent (see Nguyen Dinh Cong [**19**, Proposition 3.2]) to have a two-dimensional cocycle which is integrally separated but exhibits no exponential dichotomy. In the case of ordinary differential equations the concepts of exponential dichotomy and integral separateness are equivalent modulo in the use of shifted equations (see Palmer [**21**]).

Remark 2.9.

(i) There exists an open set $M \subset \mathcal{G}_{\mathrm{IC}}(2)$ such that any $A \in M$ is hyperbolic (one Lyapunov exponent is positive and the other is negative) but exhibits no exponential dichotomy (see Nguyen Dinh Cong [**19**], and we remark here also that any $A \in M$ is not a $Sl(2)$-cocycle).

(ii) A linear cocycle $A \in \mathcal{G}_\infty(d)$ is a point of continuity of all repeated Lyapunov exponents of A if and only if the Oseledets splitting of A is dominated or trivial (see Bochi [**6**] and Bochi and Viana [**7, 8**]).

Now we turn to the notion of cohomology of linear cocycles which is the notion of random basis change for the linear cocycles.

Definition 2.10. Let $A, B \in \mathcal{G}(d)$.

(i) A and B are called *cohomologous* if there exists a measurable map $C : \Omega \to Gl(d)$ such that for almost all $\omega \in \Omega$,

$$A(\omega) = C(\theta\omega)^{-1} \circ B(\omega) \circ C(\omega).$$

(ii) A and B are called *Lyapunov cohomologous* if they are cohomologous by a measurable map C which satisfies

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|C^{\pm 1}(\theta^n \omega)\| = 0 \quad \mathbb{P}\text{-almost surely.}$$

In this case C is called a *Lyapunov cohomology*.

The notion of cohomology of G-cocycles for an arbitrary topological group G is defined similarly. It is well known that if $A, B \in \mathcal{G}_{\mathrm{IC}}(d)$ are Lyapunov cohomologous, then they have the same Lyapunov spectrum.

In the following lemmas we study the relation between cohomology and (dominated) invariant splitting of linear cocycles.

LEMMA 2.11. *Suppose that $A \in \mathcal{G}(d)$ has an invariant splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_r(\omega), \quad \dim E_i(\omega) = d_i,\ i = 1, \ldots, r.$$

If the angles

$$\measuredangle(E_i(\omega), E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_r(\omega)), \quad i = 1, 2, \ldots, r,$$

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are separated from zero by a positive constant independent of $\omega \in \Omega$ *then* A *is cohomologous to a block-diagonal cocycle*

$$B(\omega) = \begin{pmatrix} B_1(\omega) & 0 & \cdots & 0 \\ 0 & B_2(\omega) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_r(\omega) \end{pmatrix}, \quad B_i \in \mathcal{G}(d_i),\ i = 1, \ldots, r,$$

by a Lyapunov cohomology $L(\cdot) : \Omega \to Gl(d)$ *which is essentially bounded together with its inverse* $L(\cdot)^{-1}$.

Proof. Choose orthonormal bases in the random subspaces $E_i(\omega)$, $i = 1, \ldots, r$, and compose from them a random basis $\{f_1(\omega), \ldots, f_d(\omega)\}$ of $\mathbb{R}^d$. Define a random linear map $L(\cdot) : \Omega \to Gl(d)$ by the formula

$$L(\omega) f_i(\omega) = e_i, \quad i = 1, 2, \ldots, d,$$

where $\{e_1, \ldots, e_d\}$ is the standard Euclidean basis of R^d. Put

$$B(\omega) := L(\theta\omega) A(\omega) L(\omega)^{-1}.$$

Clearly B has block-diagonal form as stated in the lemma. Moreover, by assumption there exists a constant $\alpha > 0$ such that

$$\alpha < \measuredangle(E_i(\omega), E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_r(\omega)) \leq \frac{\pi}{2}$$

for all $\omega \in \Omega$ and $i = 1, 2, \ldots, r$. Since the vectors $f_i(\omega)$ are unit vectors and inside each space $E_j(\omega)$ they are orthogonal to each other, from this it follows that the matrix $L(\cdot)$ and its inverse $L(\cdot)^{-1}$ are essentially bounded. □

The proofs of the following three lemmas are immediate.

LEMMA 2.12. *Let* $A \in \mathcal{G}_\infty(d)$ *and let*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad \dim E_i(\omega) = d_i,$$

be a dominated invariant splitting of A. *Then* A *is cohomologous by a cohomology which is bounded together with its inverse to a block-diagonal cocycle*

$$B(\omega) = \begin{pmatrix} B_1(\omega) & 0 & \cdots & 0 \\ 0 & B_2(\omega) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_k(\omega) \end{pmatrix}, \quad B_i \in \mathcal{G}_\infty(d_i),\ i = 1, \ldots, k.$$

LEMMA 2.13. *Suppose that* $A \in \mathcal{G}(d)$ *has a dominated invariant splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad \dim E_i(\omega) = d_i,\ i = 1, \ldots, k,$$

and A *is cohomologous to* $B \in \mathcal{G}(d)$ *by a bounded Lyapunov cohomology* L *which has bounded inverse, i.e.* $A(\omega) = L(\theta\omega)^{-1} B(\omega) L(\omega)$ *for all* $\omega \in \Omega$. *Then*

$$\mathbb{R}^d = L(\omega)E_1(\omega) \oplus \cdots \oplus L(\omega)E_k(\omega) \tag{4}$$

is a dominated splitting of B.

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LEMMA 2.14. *Let $A \in \mathcal{G}_\infty(d)$ be integrally separated with index k and a family of projections P_ω. If $L(\cdot) : \Omega \to Gl(d)$ is a Lyapunov cohomology which is bounded together with its inverse and*

$$B(\omega) = L(\theta\omega)A(\omega)L(\omega)^{-1},$$

then $B \in \mathcal{G}_\infty(d)$ is integrally separated with index k and with the family of projections $L(\omega)P_\omega$.

3. *Cocycles with integral separateness are generic*

In this section we formulate and prove the main result of the paper: an L^∞-generic cocycle is integrally separated. The technique we use is the turning solution method of Millionshchikov [**17**] which, in our case, consists of a construction in an arbitrary open set of a cocycle which is cohomologous to a triangular cocycle with integrally separated diagonal. For this construction we also need a technique of Bochi [**5**] to prove that if a splitting is non-dominated then we can perturb a cocycle on a segment of orbits to get a cocycle which will send any given direction at the beginning of the segment to any given direction at the end of the segment of the orbits.

First we need to study integral separateness of triangular cocycles. This is done in the following two lemmas.

LEMMA 3.1. *Suppose that $A \in \mathcal{G}_\infty(d)$ has the block-triangular form*

$$A(\omega) = \begin{pmatrix} A_{11}(\omega) & 0 \\ A_{21}(\omega) & A_{22}(\omega) \end{pmatrix},$$

where $A_{11} \in \mathcal{G}_\infty(d_1)$, $A_{22} \in \mathcal{G}_\infty(d_2)$, $d_1 + d_2 = d$. If the cocycle

$$\tilde{A}(\omega) = \begin{pmatrix} A_{11}(\omega) & 0 \\ 0 & A_{22}(\omega) \end{pmatrix},$$

which is the block-diagonal part of A, is integrally separated with index d_1 with a family of projections $\tilde{P}_\omega$ being the deterministic orthogonal projection onto the subspace spanned by $\{e_1, \ldots, e_{d_1}\}$, where $\{e_1, \ldots, e_d\}$ is the standard Euclidean basis of $\mathbb{R}^d$, then the cocycle A is integrally separated with index d_1 and with a family of projections being random projections along the subspace spanned by $\{e_{d_1+1}, \ldots, e_d\}$.

Proof. Put

$$L_\beta := \begin{pmatrix} I_{d_1} & 0 \\ 0 & \beta I_{d_2} \end{pmatrix},$$

where I_r is the unit matrix of rank r and $0 \neq \beta \in \mathbb{R}$. Then L_β is a deterministic bounded Lyapunov transformation which has the bounded inverse

$$L_\beta^{-1} := \begin{pmatrix} I_{d_1} & 0 \\ 0 & \beta^{-1} I_{d_2} \end{pmatrix}.$$

Put

$$B_\beta(\omega) := L_\beta A(\omega) L_\beta^{-1} = \begin{pmatrix} A_{11}(\omega) & 0 \\ \beta A_{21}(\omega) & A_{22}(\omega) \end{pmatrix}.$$

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Then we can choose β so that B_β is arbitrarily close to $\tilde{A}$. Therefore, by Theorem 2.5 there exists $\beta > 0$ such that the cocycle B_β is integrally separated with index d_1. By Lemma 2.14, A is integrally separated with index d_1. The characterization of projections P_ω of A comes from the form of L and $\tilde{P}_\omega$ and the fact that, for B_β close to $\tilde{A}$, the projections of B_β are close to those of $\tilde{A}$. □

LEMMA 3.2. *Let $A \in \mathcal{G}_{\mathrm{IC}}(d)$ have the form*

$$A(\omega) = \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1\,d-1}(\omega) & 0 \\ a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2\,d-1}(\omega) & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_{d-1\,1}(\omega) & a_{d-1\,2}(\omega) & \cdots & a_{d-1\,d-1}(\omega) & 0 \\ a_{d1}(\omega) & a_{d2}(\omega) & \cdots & a_{d\,d-1}(\omega) & a_{dd}(\omega) \end{pmatrix}.$$

Assume that there exists $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for any $\omega \in \Omega$ the following inequality holds:

$$\prod_{i=0}^{n-1} |a_{dd}(\theta^i \omega)| \geq \|\Phi_A(n, \omega)\|(1+\varepsilon)^{-n}.$$

Then for any $\varepsilon_1 > 0$ the cocycle

$$\hat{A}(\omega) := \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1\,d-1}(\omega) & 0 \\ a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2\,d-1}(\omega) & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_{d-1\,1}(\omega) & a_{d-1\,2}(\omega) & \cdots & a_{d-1\,d-1}(\omega) & 0 \\ a_{d1}(\omega) & a_{d2}(\omega) & \cdots & a_{d\,d-1}(\omega) & a_{dd}(\omega)(1+\varepsilon+\varepsilon_1) \end{pmatrix}$$

is integrally separated with index $(d-1)$ with a family $(d-1)$-dimensional projection P_ω that are projections along the basis vector e_d of the standard Euclidean basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$.

Proof. First we prove the lemma for the case $n = 1$. Note that $\hat{A}$ has block-triangular form. Clearly the block-diagonal part of $\hat{A}$ is integrally separated with index $(d-1)$. The assertion of the lemma follows from Lemma 3.1. The general case $n \in \mathbb{N}$ is similar. □

The following lemma is a standard technical tool from ergodic theory.

LEMMA 3.3. *Assume that θ is an ergodic invertible transformation of the non-atomic Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any $U \in \mathcal{F}$ with $\mathbb{P}(U) > 0$ and any $n > 0$ there exists a measurable set $V \subset U$ such that $\mathbb{P}(V) > 0$ and the sets $\theta^{-n}V, \ldots, \theta^{-1}V, V, \theta V, \ldots, \theta^n V$ are disjoint.*

Proof. It follows from the Halmos–Rokhlin lemma. □

Now we turn to our main problem of genericity of integral separateness of linear cocycles. First we deal with the two-dimension case in which the main idea of the proof is explained in detail. Denote by

$$R_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \alpha \in [-\pi, \pi],$$

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the rotation of $\mathbb{R}^2$ by the angle α from the direction e_1 into the direction e_2, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. The rotation R_α has the following properties:

$$\|R_\alpha - \mathrm{id}\| \le |\alpha| \quad \text{and} \quad \|R_\alpha^{-1} - \mathrm{id}\| \le |\alpha| \quad \text{for } -\pi \le |\alpha| \le \pi, \tag{5}$$

where id is the identity transformation of $\mathbb{R}^2$.

Using a technique of Millionshchikov [**17**] and Bochi [**5**] we derive the following lemma which will be a key for the proof of the main results of this paper. The idea is if $A \in \mathcal{G}_\infty(2)$ lacks dominance then by small perturbations of A along (long) segments of orbits we can send any given direction (at the beginning of the segments) to any given (even less expanding) direction (at the end of the segments). Note a contrast to the dominance case in which one cannot send the unstable direction into the stable direction by (arbitrarily) small perturbations.

LEMMA 3.4. *Let $A \in \mathcal{G}_\infty(2)$ have simple Lyapunov spectrum. Assume that A is not integrally separated (or, equivalently, the Oseledets splitting of A is non-dominated). Then for any $\varepsilon > 0$ there exists an integer $n \in \mathbb{N}$ and a set $C \in \mathcal{F}$ of positive $\mathbb{P}$-measure such that the sets $C, \theta C, \dots, \theta^n C$ are disjoint and for any random non-vanishing vector $h : \Omega \to \mathbb{R}^2$ there exist a linear cocycle $B \in \mathcal{G}_\infty(2)$ and a random variable $m(\cdot) : \Omega \to \mathbb{R}$ such that*

$$\|B(\omega) - A(\omega)\| + \|B(\omega)^{-1} - A(\omega)^{-1}\| \le \varepsilon \quad \textit{for all } \omega \in \bigcup_{i=0}^{n-1} \theta^i C,$$

$$B(\omega) = A(\omega) \quad \textit{for all } \omega \notin \bigcup_{i=0}^{n-1} \theta^i C,$$

$$\Phi_B(n, \omega)h(\omega) = m(\theta^n \omega)h(\theta^n \omega) \quad \textit{for all } \omega \in C.$$

Proof. Let $\varepsilon > 0$ be an arbitrary given positive number. We can assume that $\varepsilon < 1/10$. Let

$$\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega)$$

be the Oseledets splitting of A corresponding to the Lyapunov exponents $\lambda_1 < \lambda_2$ of the cocycle A. Denote

$$1 < a := \operatorname*{ess\,sup}_{\omega \in \Omega} \|A(\omega)\| + \operatorname*{ess\,sup}_{\omega \in \Omega} \|A(\omega)^{-1}\| < \infty.$$

Choose and fix random unit basis vectors $f_1 \in E_1(\omega)$ and $f_2(\omega) \in E_2(\omega)$ such that

$$\measuredangle(f_1(\omega), f_2(\omega)) = \measuredangle(E_1(\omega), E_2(\omega)) \in \left[0, \frac{\pi}{2}\right] \quad \text{for all } \omega \in \Omega.$$

There are two possible cases: either the angle $\measuredangle(f_1(\omega), f_2(\omega))$ is separated from zero or not.

Case 1. The angle $\measuredangle(f_1(\omega), f_2(\omega))$ is separated from zero. By Lemma 2.11 we may assume that A is diagonal and $f_1(\omega) = e_1$, $f_2(\omega) = e_2$ (because A is cohomologous to a diagonal cocycle by a *bounded* cohomology). Hence A has the form

$$A(\omega) = \begin{pmatrix} a_1(\omega) & 0 \\ 0 & a_2(\omega) \end{pmatrix}$$

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with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log|a_i(\theta^j\omega)| = \lambda_i, \quad i = 1, 2, \ \forall\omega\in\Omega.$$

Since A is not integrally separated there are integers k arbitrarily large such that the set

$$F := \left\{\omega\in\Omega : \left|\prod_{j=0}^{k-1} a_2(\theta^j\omega)\right| \le (1+\varepsilon)^{k/2}\left|\prod_{j=0}^{k-1} a_1(\theta^j\omega)\right|\right\} \tag{6}$$

has positive measure. Fix some $k > 2$ such that $(1+\varepsilon)^k\varepsilon^4 > 1$ and $\mathbb{P}(F) > 0$. Choose and fix a positive number $\varepsilon_1 > 0$ such that $e^{-2k\varepsilon_1}(1+\varepsilon)^k\varepsilon > 1$ and $\varepsilon_1 < (\lambda_2-\lambda_1)/4$.

Since $(1/n)\sum_{j=0}^{n-1}\log|a_2(\theta^j\omega)|$ converges almost surely to λ_2 as $n\to\infty$ the convergence in measure also holds. Therefore if $n > 2k$ is large enough then,

$$\mathbb{P}\left(\left\{\omega\in\Omega : \frac{1}{n}\sum_{j=k}^{n-1}\log|a_2(\theta^j\omega)| > \lambda_2-\varepsilon_1\right\}\right) > 1-\frac{1}{3}\mathbb{P}(F).$$

Similarly, since $(1/n)\sum_{j=0}^{n-1}\log a_1(\theta^j\omega)$ converges almost surely to λ_1 as $n\to\infty$ the convergence in measure also holds. Therefore if $n > 2k$ is large enough then,

$$\mathbb{P}\left(\left\{\omega\in\Omega : \frac{1}{n}\sum_{j=k}^{n-1}\log|a_1(\theta^j\omega)| < \lambda_1+\varepsilon_1\right\}\right) > 1-\frac{1}{3}\mathbb{P}(F).$$

Hence the set

$$\begin{aligned}F' := F &\cap \left\{\omega\in\Omega : \frac{1}{n}\sum_{j=0}^{n-1}\log|a_2(\theta^j\omega)| > \lambda_2-\varepsilon_1\right\}\\ &\cap \left\{\omega\in\Omega : \frac{1}{n}\sum_{j=k}^{n-1}\log|a_1(\theta^j\omega)| < \lambda_1+\varepsilon_1\right\}\\ &= \left\{\omega\in\Omega : \left|\prod_{j=0}^{k-1} a_2(\theta^j\omega)\right| \le (1+\varepsilon)^{k/2}\left|\prod_{j=0}^{k-1} a_1(\theta^j\omega)\right|\right.\\ &\quad \text{and} \quad \left|\prod_{j=k}^{n-1} a_1(\theta^j\omega)\right| < e^{n(\lambda_1+\varepsilon_1)} \quad \text{and} \quad \left.\left|\prod_{j=k}^{n-1} a_2(\theta^j\omega)\right| > e^{n(\lambda_2-\varepsilon_1)}\right\}\end{aligned}$$

has positive measure. Fix one such big number $n\in\mathbb{N}$ with an additional property that

$$\frac{e^{(n-k)(\lambda_1+\varepsilon_1)}}{e^{(n-k)(\lambda_2-\varepsilon_1)}} < \varepsilon^2. \tag{7}$$

By Lemma 3.3 there is a set $C\subset F'$ with positive measure such that the sets $C, \theta C, \ldots, \theta^n C$ are disjoint.

Thus we have found n and C. Let $h : \Omega\to\mathbb{R}^2$ be an arbitrary given random non-vanishing vector. Now we have to construct a cocycle B. Clearly we can assume that $\|h(\omega)\| = 1$ for all $\omega\in\Omega$.

We write the random vector $h(\omega)$ in the basis $\{e_1, e_2\}$ as

$$h(\omega) = h_1(\omega)e_1 + h_2(\omega)e_2, \quad \omega\in\Omega,\ h_1(\omega), h_2(\omega)\in\mathbb{R}.$$

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Our aim is to construct a cocycle B which differs slightly from A only in $\bigcup_{i=0}^{n-1}\theta^i C$ and such that $\Phi_B(n,\omega)h(\omega)$ is collinear to $h(\theta^n\omega)$ for all $\omega\in C$.

For $\omega\in C$ set

$$B(\omega)=A(\omega)\begin{pmatrix}(1+\varepsilon') & 0\\ 0 & (1+\varepsilon')^{-1}\end{pmatrix}R_{r_1(\omega)},$$

where $r_1(\cdot)$ is a random variable and ε' is a small positive constant which will be specified later. For $\omega=\theta^i\omega_0$ with $\omega_0\in C$ and $1\le i\le k-1$ we set

$$B(\omega)=A(\omega)\begin{pmatrix}(1+\varepsilon') & 0\\ 0 & (1+\varepsilon')^{-1}\end{pmatrix}=\begin{pmatrix}a_1(\omega)(1+\varepsilon') & 0\\ 0 & a_2(\omega)(1+\varepsilon')^{-1}\end{pmatrix}.$$

Notice that if $|r_1(\omega)|<\varepsilon'$ for all $\omega\in C$, then for any $\omega\in\bigcup_{i=0}^{k-1}\theta^i C$ we have

$$\begin{aligned}\|B(\omega)-A(\omega)\|+\|B(\omega)^{-1}-A(\omega)^{-1}\| &\le \sup_{\omega\in\bigcup_{i=0}^{k-1}\theta^i C} 6(\|A(\omega)\|+\|A(\omega)^{-1}\|)\varepsilon'\\ &\le 6\varepsilon'\sup_{\omega\in\Omega}(\|A(\omega)\|+\|A(\omega)^{-1}\|)\\ &\le 6a\varepsilon'. \qquad (8)\end{aligned}$$

For all $\omega\in C$ we denote by $\hat h(\omega)$ the vector $R_{r_1(\omega)}h(\omega)$ and by $g(\theta^k\omega)$ the vector $\Phi_B(k,\omega)h(\omega)$ and write them in the standard basis $\{e_1,e_2\}$ of $\mathbb{R}^2$ as

$$\begin{aligned}\hat h(\omega)&=\begin{pmatrix}\hat h_1(\omega)\\ \hat h_2(\omega)\end{pmatrix}=R_{r_1(\omega)}\begin{pmatrix}h_1(\omega)\\ h_2(\omega)\end{pmatrix},\\ g(\theta^k\omega)&=\begin{pmatrix}g_1(\theta^k\omega)\\ g_2(\theta^k\omega)\end{pmatrix}=\Phi_B(k,\omega)h(\omega)\\ &=\begin{pmatrix}\prod_{i=0}^{k-1}a_1(\theta^i\omega)(1+\varepsilon') & 0\\ 0 & \prod_{i=0}^{k-1}a_2(\theta^i\omega)(1+\varepsilon')^{-1}\end{pmatrix}\begin{pmatrix}\hat h_1(\omega)\\ \hat h_2(\omega)\end{pmatrix}.\end{aligned}$$

Then for all $\omega\in C$ we have

$$\frac{g_2(\theta^k\omega)}{g_1(\theta^k\omega)}=\frac{\big(\prod_{i=0}^{k-1}a_2(\theta^i\omega)(1+\varepsilon')^{-1}\big)\hat h_2(\omega)}{\big(\prod_{i=0}^{k-1}a_1(\theta^i\omega)(1+\varepsilon')\big)\hat h_1(\omega)}.$$

We want to make the vector $\Phi_B(n,\omega)h(\omega)$ collinear to the vector $h(\theta^n\omega)$. For this we aim to make the vector $g(\theta^k\omega)=\Phi_B(k,\omega)h(\omega)$ collinear to the vector $\Phi_B(-(n-k),\theta^n\omega)h(\theta^n\omega)$. Choosing a suitable rotation $R_{r_1(\omega)}$ with $r_1(\omega)\in[-2\varepsilon,2\varepsilon]$ we can make the coordinates of the vector $\hat h(\omega)=R_{r_1(\omega)}h(\omega)$ satisfy the inequalities

$$\varepsilon<|\hat h_1(\omega)|<1,\quad \varepsilon<|\hat h_2(\omega)|<1.$$

Choose $\varepsilon'=\varepsilon$ and taking into account the choice of k and $C\subset F'$, then we have

$$\frac{|g_2(\theta^k\omega)|}{|g_1(\theta^k\omega)|}\le(1+\varepsilon)^{k/2}(1+\varepsilon')^{-k}(1+\varepsilon)^{-k}\frac{1}{\varepsilon}<\varepsilon.$$

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This means that the angle between the subspace generated by the vector $g(\theta^k\omega)$ and the subspace generated by the basis vector e_1 is less than $\frac{3}{2}\varepsilon$.

Now, choose a rotation $R_{r_3(\theta^{n-1}\omega)}$ with $|r_3(\theta^n\omega)| \le 2\varepsilon$ such that the vector

$$h'(\theta^n\omega) = \begin{pmatrix} h'_1(\theta^n\omega) \\ h'_2(\theta^n\omega) \end{pmatrix} := R^{-1}_{r_3(\theta^{n-1}\omega)} h(\theta^n\omega)$$

has coordinates satisfying the inequalities

$$\varepsilon \le |h'_1(\theta^n\omega)| < 1, \quad \varepsilon < |h'_2(\theta^n\omega)| \le 1.$$

Then the vector

$$h''(\theta^k\omega) = \begin{pmatrix} h''_1(\theta^k\omega) \\ h''_2(\theta^k\omega) \end{pmatrix} := \Phi_A(-(n-k), \theta^n\omega) \begin{pmatrix} h'_1(\theta^n\omega) \\ h'_2(\theta^n\omega) \end{pmatrix}$$

has the coordinates

$$h''_1(\theta^k\omega) = \left(\prod_{i=k}^{n-1} a_1(\theta^i\omega)\right)^{-1} h'_1(\theta^n\omega),$$

$$h''_2(\theta^k\omega) = \left(\prod_{i=k}^{n-1} a_2(\theta^i\omega)\right)^{-1} h'_2(\theta^n\omega),$$

which, by virtue of (7) and the choice of F', satisfy the relation

$$\frac{|h''_2(\theta^k\omega)|}{|h''_1(\theta^k\omega)|} \le \frac{\left|\prod_{i=k}^{n-1} a_1(\theta^i\omega)\right|}{\left|\prod_{i=k}^{n-1} a_2(\theta^i\omega)\right|} \frac{1}{\varepsilon} \le \frac{e^{(n-k)(\lambda_1+\varepsilon_1)}}{e^{(n-k)(\lambda_2-\varepsilon_1)}} \frac{1}{\varepsilon} \le \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

This means that the angle between the subspace generated by vector $h''(\theta^k\omega)$ and the subspace generated by the basis vector e_1 is less than $\frac{3}{2}\varepsilon$. Therefore, we can find a rotation $R_{r_2(\theta^k\omega)}$ with $r_2(\theta^k\omega) \in [-4\varepsilon, 4\varepsilon]$, $\omega \in C$, such that the vector $R_{r_2(\theta^k\omega)} g(\theta^k\omega)$ is collinear to the vector $h''(\theta^k\omega)$.

Now for any $\omega \in C$ we put

$$B(\theta^k\omega) = A(\theta^k\omega) R_{r_2(\theta^k\omega)},$$

$$B(\theta^{n-1}\omega) = R_{r_3(\theta^{n-1}\omega)} A(\theta^{n-1}\omega),$$

and for $i = k+1, k+2, \ldots, n-2$ we set

$$B(\theta^i\omega) = A(\theta^i\omega).$$

Thus we have constructed $B(\omega)$ for all $\omega \in \bigcup_{i=0}^{n-1} \theta^i C$. For $\omega \in \Omega \setminus \bigcup_{i=0}^{n-1} \theta^i C$ we set $B(\omega) = A(\omega)$. By the construction of B, the vector $\Phi_B(n, \omega)h(\omega)$ is collinear to the vector $h(\theta^n\omega)$ for any $\omega \in C$ as

$$\begin{aligned} \Phi_B(n, \omega)h(\omega) &= \Phi_B(n-k, \theta^k\omega)\Phi_B(k, \omega)h(\omega) \\ &= R_{r_3(\theta^{n-1}\omega)} \Phi_A(n-k, \theta^k\omega) R_{r_2(\theta^k\omega)} g(\theta^k\omega). \end{aligned}$$

From equations (5) and (8) and the construction of B, it follows that for any $\omega \in \bigcup_{i=0}^{n-1} \theta^i C$ we have

$$\begin{aligned} \|B(\omega) - A(\omega)\| + \|B(\omega)^{-1} - A(\omega)^{-1}\| &\le \sup_{\omega \in \bigcup_{i=0}^{n-1} \theta^i C} (\|A(\omega)\| + \|A(\omega)^{-1}\|) 12\varepsilon \\ &\le 12a\varepsilon. \end{aligned}$$

Since ε is arbitrary the lemma is proved in this case.

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Case 2. The angle $\measuredangle(f_1(\omega), f_2(\omega))$ is not separated from zero. Denote by $\varphi(\omega)$ the angle $\measuredangle(f_1(\omega), f_2(\omega)) \in [0, \pi/2]$. We can assume that the set $\hat{F} := \{\omega \in \Omega : \varphi(\omega) < \varepsilon\}$ has positive measure.

By Oseledets theorem, for almost all $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \frac{-1}{n} \log \sin \varphi(\theta^n \omega) = 0,$$

therefore the convergence in measure also holds. Similarly for the limit

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \frac{\|\Phi_A(m,\omega) f_2(\omega)\|}{\|\Phi_A(m,\omega) f_1(\omega)\|} = \lambda_2 - \lambda_1.$$

Hence the measure of the set

$$\begin{aligned} F_m := \Big\{ \omega \in \Omega : \frac{-1}{m} \log \sin \varphi(\theta^m \omega) < \varepsilon \\ \text{and } \frac{1}{m} \log \frac{\|\Phi_A(m+1,\omega) f_2(\omega)\|}{\|\Phi_A(m+1,\omega) f_1(\omega)\|} > \lambda_2 - \lambda_1 - \varepsilon \\ \text{and } \frac{1}{-m} \log \frac{\|\Phi_A(-m,\omega) f_2(\omega)\|}{\|\Phi_A(-m,\omega) f_1(\omega)\|} > \lambda_2 - \lambda_1 - \varepsilon \Big\} \end{aligned}$$

tends to 1 when $m \to \infty$. Clearly, we may assume that $0 < \varepsilon < (\lambda_2 - \lambda_1)/2$. Fix some large $m \in \mathbb{N}$ satisfying

$$e^{(\lambda_2-\lambda_1-\varepsilon)m} > \frac{2}{\varepsilon^2} \quad \text{and} \quad \mathbb{P}(F_m) > 1 - \mathbb{P}(\hat{F}).$$

Therefore, the set $\hat{F}' := \hat{F} \cap F_m$ has positive measure. Choose $n = 2m + 1$. By Lemma 3.3 there is a set $C' \subset \hat{F}'$ with positive measure such that the sets $\theta^{-m}C', \theta^{-m+1}C', \dots, \theta^{-1}C', C', \theta C', \dots, \theta^m C'$ are disjoint. Put $C := \theta^{-m}C'$.

Thus we have found n and C. Let $h : \Omega \to \mathbb{R}^2$ be an arbitrary given random non-vanishing vector. Now we have to construct a cocycle B. Clearly, we can assume that $\|h(\omega)\| = 1$ for all $\omega \in \Omega$.

We use an idea similar to that of the proof of Case 1 above. For the random vector $h(\omega)$ we write it in the basis $\{f_1(\omega), f_2(\omega)\}$ as

$$h(\omega) = h_1(\omega) f_1(\omega) + h_2(\omega) f_2(\omega), \quad \omega \in \Omega,\ h_1(\omega), h_2(\omega) \in \mathbb{R}.$$

For any $\omega \in C$ we set

$$\begin{aligned} B(\omega) &= A(\omega) R_{r_4(\omega)}, \\ B(\theta^m \omega) &= A(\theta^m \omega) R_{r_5(\theta^m \omega)}, \\ B(\theta^{n-1} \omega) &= R_{r_6(\theta^{n-1}\omega)} A(\theta^{n-1} \omega), \end{aligned}$$

and

$$B(\theta^i \omega) = A(\theta^i \omega) \quad \text{for all } 1 \le i \le m-1 \text{ and } m+1 \le i \le n-2,$$

where $r_i(\cdot)$, $i = 4, 5, 6$, are random variables which will be specified later. For $\omega \in \Omega \setminus \bigcup_{i=0}^{n-1} C$ we set

$$B(\omega) = A(\omega).$$

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Notice that if $|r_i(\omega)| < \varepsilon'$ where they are defined ($i = 4, 5, 6$), then we have

$$\|B(\omega) - A(\omega)\| + \|B(\omega)^{-1} - A(\omega)^{-1}\| \le \sup_{\omega \in \bigcup_{i=0}^{n-1} \theta^i C} 2(\|A(\omega)\| + \|A(\omega)^{-1}\|)\varepsilon' \le 2a\varepsilon'. \tag{9}$$

Clearly, we can choose $r_4(\omega) \in [-2\varepsilon, 2\varepsilon]$ such that for all $\omega \in C$ the vector

$$h'(\omega) := R_{r_4(\omega)} h(\omega) = h'_1(\omega) f_1(\omega) + h'_2(\omega) f_2(\omega)$$

has coordinates $h'_1(\omega), h'_2(\omega) \in \mathbb{R}$ satisfying the inequality

$$\frac{|h'_2(\omega)|}{|h'_1(\omega)|} = \frac{|\sin(\measuredangle(f_1(\omega), h'(\omega)))|}{|\sin(\measuredangle(f_2(\omega), h'(\omega)))|} > \varepsilon.$$

For $\omega \in C$ we introduce the notation

$$\Phi_A(m, \omega) h'(\omega) =: \hat{h}(\theta^m \omega) = \hat{h}_1(\theta^m \omega) f_1(\theta^m \omega) + \hat{h}_2(\theta^m \omega) f_2(\theta^m \omega),$$

where $\hat{h}_1(\theta^m \omega), \hat{h}_2(\theta^m \omega) \in \mathbb{R}$ are the coordinates of the vector $\hat{h}(\theta^m \omega)$ in the basis $\{f_1(\theta^m \omega), f_2(\theta^m \omega)\}$. By the choice of m we have (recall that $\theta^m \omega \in C'$)

$$\frac{|\hat{h}_2(\theta^m \omega)|}{|\hat{h}_1(\theta^m \omega)|} = \frac{|h'_2(\omega)|}{|h'_1(\omega)|} \frac{\|\Phi_A(m, \omega) f_2(\omega)\|}{\|\Phi_A(m, \omega) f_1(\omega)\|} \ge \varepsilon e^{m(\lambda_2 - \lambda_1 - \varepsilon)} \ge \frac{2}{\varepsilon}.$$

Hence the angle between the subspace generated by the vector $\hat{h}(\theta^m \omega)$ and the subspace generated by the vector $f_2(\theta^m \omega)$ is less than ε.

Now, for all $\omega \in C$ we choose $r_6(\theta^n \omega) \in [-2\varepsilon, 2\varepsilon]$ such that the vector

$$R^{-1}_{r_6(\theta^n \omega)} h(\theta^n \omega) =: h''(\theta^n \omega) = h''_1(\theta^n \omega) f_1(\theta^n \omega) + h''_2(\theta^n \omega) f_2(\theta^n \omega)$$

has coordinates $h''_1(\theta^n \omega), h''_2(\theta^n \omega) \in \mathbb{R}$ satisfying the inequality

$$\frac{|h''_1(\theta^n \omega)|}{|h''_2(\theta^n \omega)|} > \varepsilon.$$

For all $\omega \in C$ we introduce the notation

$$\Phi_A(-(n-m), \theta^n \omega) h''(\theta^n \omega) =: \tilde{h}(\theta^m \omega) = \tilde{h}_1(\theta^m \omega) f_1(\theta^m \omega) + \tilde{h}_2(\theta^m \omega) f_2(\theta^m \omega),$$

where $\tilde{h}_1(\theta^m \omega), \tilde{h}_2(\theta^m \omega) \in \mathbb{R}$ are the coordinates of the vector $\tilde{h}(\theta^m \omega)$ in the basis $\{f_1(\theta^m \omega), f_2(\theta^m \omega)\}$. By the choice of m we have

$$\frac{|\tilde{h}_1(\theta^m \omega)|}{|\tilde{h}_2(\theta^m \omega)|} = \frac{|h''_1(\theta^n \omega)|}{|h''_2(\theta^n \omega)|} \frac{\|\Phi_A(-(n-m), \theta^n \omega) f_1(\theta^n \omega)\|}{\|\Phi_A(-(n-m), \theta^n \omega) f_2(\theta^n \omega)\|} \ge \varepsilon e^{m(\lambda_2 - \lambda_1 - \varepsilon)} \ge \frac{2}{\varepsilon}.$$

Hence the angle between the subspace generated by the vector $\tilde{h}(\theta^m \omega)$ and the subspace generated by the vector $f_1(\theta^m \omega)$ is less than ε.

Since $\theta^m C = C' \subset \hat{F}$, the angle between the subspaces generated by the vectors $f_1(\theta^m \omega)$ and $f_2(\theta^m \omega)$, $\omega \in C$, is less than ε. Hence the angle between the subspaces generated by the vector $\hat{h}(\theta^m \omega)$ and the vector $\tilde{h}(\theta^m \omega)$ is less than 3ε. Therefore, there exists $r_5(\theta^n \omega) \in [-4\varepsilon, 4\varepsilon]$ such that the vector $R_{r_5(\theta^m \omega)} \hat{h}(\theta^m \omega)$ is collinear

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to the vector $\tilde{h}(\theta^m\omega)$. Thus we have chosen the rotation $R_{r_5(\theta^m\omega)}$, and hence constructed the cocycle B. From the construction of B it follows that the vector $\Phi_B(\omega)h(\omega)$ is collinear to the vector $h(\theta^n\omega)$ for all $\omega \in \Omega$, and for any $\omega \in \bigcup_{i=0}^{n-1} \theta^i C$ we have

$$\begin{aligned}\|B(\omega) - A(\omega)\| + \|B(\omega)^{-1} - A(\omega)^{-1}\| &\le \sup_{\omega\in\bigcup_{i=0}^{n-1}\theta^i C} (\|A(\omega)\| + \|A(\omega)^{-1}\|)8\varepsilon \\ &= 8a\varepsilon.\end{aligned}$$

Since ε is arbitrary the lemma is proved. □

Now we turn to the general d-dimensional case. First we construct a rotation in $\mathbb{R}^d$ which is to be used in this case. Let a and b be two vectors of $\mathbb{R}^d$ such that $\|a\| = \|b\| = 1$ and $|\langle a, b\rangle| < 1$. We define the rotation $R_{a,b,\delta}$ of the vector a by the angle $\delta \in [-\pi, \pi]$ into the direction of the vector b in $\mathbb{R}^d$ as follows.

Set

$$\tilde{a} := \frac{a - \langle a, b\rangle b}{\|a - \langle a, b\rangle b\|}, \quad U := \operatorname{span}\{a, b\} = \operatorname{span}\{\tilde{a}, b\}.$$

The rotation $R_{a,b,\delta}$ is defined to be the linear orthogonal operator of $\mathbb{R}^d$ which acts as the identity on $U^\perp$, and acts on the orthogonal basis vectors $b, \tilde{a}$ of U as

$$R_{a,b,\delta}b = b\cos\delta - \tilde{a}\sin\delta, \quad R_{a,b,\delta}\tilde{a} = b\sin\delta + \tilde{a}\cos\delta.$$

The operator $R_{a,b,\delta}$ has the properties

$$\measuredangle(R_{a,b,\delta}a, b) = \measuredangle(a, b) - \delta \quad \text{for } 0 \le |\delta| \le \measuredangle(a, b), \tag{10}$$

$$\|R_{a,b,\delta} - \mathrm{id}\| \le |\delta| \quad \text{and} \quad \|R_{a,b,\delta}^{-1} - \mathrm{id}\| \le |\delta| \quad \text{for } -\pi \le |\delta| \le \pi, \tag{11}$$

where id is the identity transformation of $\mathbb{R}^d$.

The proof of Lemma 3.4 allows us to strengthen it to the following lemma for the general d-dimensional case.

LEMMA 3.5. *Let $d \ge 2$ be an arbitrary integer and let $A \in \mathcal{G}_\infty(d)$ have simple Lyapunov spectrum. Assume that A has no dominated splitting. Then for any $\varepsilon > 0$ there exists an integer $n \in \mathbb{N}$ and a set $C \in \mathcal{F}$ of positive $\mathbb{P}$-measure such that the sets $C, \theta C, \dots, \theta^n C$ are disjoint and for any random non-vanishing vector $h : \Omega \to \mathbb{R}^d$ and any measurable set $Z \subset C$ of positive measure there exist a linear cocycle $B \in \mathcal{G}_\infty(d)$ and a random variable $m(\cdot) : \Omega \to \mathbb{R}$ such that*

$$\|B(\omega) - A(\omega)\| + \|B(\omega)^{-1} - A(\omega)^{-1}\| \le \varepsilon \quad \textit{for all } \omega \in \bigcup_{i=0}^{n-1} \theta^i Z, \tag{12}$$

$$B(\omega) = A(\omega) \quad \textit{for all } \omega \notin \bigcup_{i=0}^{n-1} \theta^i Z, \tag{13}$$

$$\Phi_B(n, \omega)h(\omega) = m(\theta^n\omega)h(\theta^n\omega) \quad \textit{for all } \omega \in Z. \tag{14}$$

Proof. We indicate the necessary modifications to be made in the proof of Lemma 3.4 to make it applicable to our d-dimensional case.

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(1) To change from the two-dimensional case to the d-dimensional case we have to use the rotation $R_{a,b,\delta}$ of the space $\mathbb{R}^d$ instead of the rotation R_α of $\mathbb{R}^2$ considered in Lemma 3.4. Moreover, since instead of two basis directions $\{f_1(\omega), f_2(\omega)\}$ we have d basis directions $\{f_1(\omega), \ldots, f_d(\omega)\}$, the number k in the proof of Lemma 3.4 has to be increased and we have to apply more rotations to move the vector $h(\omega)$ to the direction $h(\theta^n\omega)$ (we move $h(\omega)$ into the direction $f_{d-1}(\theta^{k_1}\omega)$, then move it to the direction $f_{d-2}(\theta^{k_2}\omega)$ and so on to the direction $f_1(\theta^{k_{d-1}}\omega)$ and then rotate it by a small rotation such that we will arrive at the direction $h(\theta^n\omega)$).

(2) It is straightforward to make a change from the set C in Lemma 3.4 to a measurable subset $Z \subset C$ in this lemma. □

The following theorem is the main result of this paper. In this theorem we use the technique of Millionshchikov to perturb an arbitrary cocycle $A \in \mathcal{G}_\infty(d)$ to get a cocycle which is cohomologous to an integrally separated cocycle.

THEOREM 3.6. *Let $A \in \mathcal{G}_\infty(d)$. In any neighborhood of A there is a cocycle which is integrally separated.*

Proof. The idea of the proof of this theorem is to follow the technique of Millionshchikov and use the result of Lemma 3.5 to perturb a non-integrally separated cocycle to get a cocycle with one invariant direction with good growth rate; then perturb it further to get a cocycle with one integrally separated direction. By restriction to the $(d-1)$-dimensional (integrally separated) invariant subspace we reduce the dimension by one and the induction does the remaining work.

By a theorem of Arnold and Cong [**4**] on the density of simple Lyapunov spectrum we may assume that A has simple Lyapunov spectrum. If A is integrally separated then we are done. Suppose that A is not integrally separated with any index. Let $\varepsilon > 0$ be an arbitrary given positive number. Take a number n and a set C according to Lemma 3.5. By Lemma 3.3 we may choose a measurable $Z \subset C$ of positive measure such that the sets $Z, \theta Z, \ldots, \theta^{nn_1} Z$ are disjoint, where $n_1 \in \mathbb{N}$ is a given integer whose value will be specified later. Clearly, we can assume that $n \in \mathbb{N}$ is so big that $e^{-\varepsilon n} \le \varepsilon/2$. Set

$$1 < a := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)^{-1}\| < \infty.$$

Now we construct a cocycle $\hat{A}$ near to A and an invariant one-dimensional subspace of $\hat{A}$ which is generated by a random vector $g(\omega)$ in several steps as follows.

Step (i). For all $\omega \in \theta^n Z$ we set $\hat{A}(\omega) = A(\omega)$, and denote by $g(\omega)$ a unit vector of $\mathbb{R}^d$ such that

$$\|\Phi_A(n,\omega)g(\omega)\| = \|\Phi_A(n,\omega)\|. \tag{15}$$

To make the choice of $g(\omega)$ more precise and make it measurably dependent on ω, in case there are more than one such unit vector satisfying equality (15), we choose the unit vector $g(\omega)$ which has the smallest coordinates in the standard basis of $\mathbb{R}^d$ (the vectors are written in the standard Euclidean basis of $\mathbb{R}^d$ and they are compared by coordinates in the lexicographical way). Thus, in this step, $\hat{A}(\omega)$ is defined for $\omega \in \theta^n Z$ and $g(\omega)$ is defined for $\omega \in \theta^n Z$.

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Step (ii). For $\omega = \theta^i\omega_0$ with $\omega_0 \in \theta^n Z$ and $i = 1, 2, \ldots, n-1$ we set $\hat{A}(\omega) = A(\omega)$ and

$$g(\omega) = \Phi_{\hat{A}}(i, \omega_0)g(\omega_0).$$

For $\omega = \theta^n\omega_0$ with $\omega_0 \in \theta^n Z$ we set

$$g(\omega) = \Phi_{\hat{A}}(n, \omega_0)g(\omega_0).$$

Thus, in this step, $\hat{A}(\omega)$ is defined for $\omega \in \theta^i Z$ with $i = n+1, n+2, \ldots, 2n-1$ and $g(\omega)$ is defined for $\omega \in \theta^i Z$ with $i = n+1, n+2, \ldots, 2n$.

Step (iii). For $\omega = \theta^n\omega_0$ with $\omega_0 \in \theta^n Z$ we add a small rotation to A to obtain $\hat{A}$ as follows. Denote by $\tilde{g}(\omega)$ a unit vector of $\mathbb{R}^d$ such that

$$\|\Phi_A(n, \omega)\tilde{g}(\omega)\| = \|\Phi_A(n, \omega)\|. \tag{16}$$

To make the choice of $\tilde{g}(\omega)$ more precise and make it measurably dependent on ω, in case there are more than one such unit vector satisfying equality (16), we choose the unit vector $\tilde{g}(\omega)$ which has the smallest coordinates in the standard basis of $\mathbb{R}^d$ (the vectors are written in the standard Euclidean basis of $\mathbb{R}^d$ and they are compared by coordinates in the lexicographical way). If $\pi/2 - \varepsilon < \measuredangle(g(\omega), \tilde{g}(\omega)) \leq \pi/2$ then we put

$$\hat{A}(\omega) = A(\omega)R_{g(\omega),\tilde{g}(\omega),\varepsilon};$$

if $\pi/2 < \measuredangle(g(\omega), \tilde{g}(\omega)) < \pi/2 + \varepsilon$ then we put

$$\hat{A}(\omega) = A(\omega)R_{g(\omega),\tilde{g}(\omega),-\varepsilon};$$

if $\measuredangle(g(\omega), \tilde{g}(\omega)) \leq \pi/2 - \varepsilon$ or $\measuredangle(g(\omega), \tilde{g}(\omega)) \geq \pi/2 + \varepsilon$ we put $\hat{A}(\omega) = A(\omega)$.

For $\omega = \theta^i\omega_0$ with $\omega_0 \in \theta^n Z$ and $i = n+1, n+2, \ldots, 2n-1$ we set $\hat{A}(\omega) = A(\omega)$ and

$$g(\omega) = \Phi_{\hat{A}}(i, \omega_0)g(\omega_0).$$

For $\omega = \theta^{2n}\omega_0$ with $\omega_0 \in \theta^n Z$ we set

$$g(\omega) = \Phi_{\hat{A}}(2n, \omega_0)g(\omega_0).$$

Thus, in this step, $\hat{A}(\omega)$ is defined for $\omega \in \theta^i Z$ with $i = 2n, 2n+1, \ldots, 3n-1$ and $g(\omega)$ is defined for $\omega \in \theta^i Z$ with $i = 2n+1, 2n+2, \ldots, 3n$.

Step (iv). Continuing as in Step (iii) we define $\hat{A}(\omega)$ for $\omega \in \theta^i Z$ with $i = 3n, 3n+1, \ldots, n_1 n - 1$ and define $g(\omega)$ for $\omega \in \theta^i Z$ with $i = 3n+1, 3n+2, \ldots, n_1 n$.

Step (v). Note that we can decompose the set $\theta^n Z$ into the disjoint union

$$\theta^n Z = \bigcup_{i=n(n_1-1)}^{\infty} V_i \tag{17}$$

with the sets V_i defined by

$$V_i := \{\omega \in \theta^n Z \mid \theta^i\omega \in Z \text{ and } \theta^j\omega \notin Z \text{ for all } j = 0, 1, \ldots, i-1\}.$$

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Some of the sets V_i might be empty (or have zero measure). Consequently, the space Ω is decomposed into the following disjoint union of measurable sets:

$$\Omega = \Big(\bigcup_{j=0}^{n} \theta^j Z\Big) \bigcup \Big(\bigcup_{i=n(n_1-1)}^{\infty} \bigcup_{k=1}^{i-1} \theta^k V_i\Big).$$

Step (vi). Let V_m be an arbitrary set from (17), $m \geq n(n_1 - 1)$. Let $m = kn + r$ with $0 \leq r < n, k \geq n_1$.

In Steps (i)–(iv) we have constructed the random matrix $\hat{A}(\omega)$ for $\omega \in \theta^i Z$ with $i = n, n+1, \ldots, n_1 n - 1$ and $g(\omega)$ for $\omega \in \theta^i Z$ with $i = n, n+2, \ldots, n_1 n$. Consequently, we have constructed the random matrix $\hat{A}(\omega)$ for $\omega \in \theta^i V_m$ with $i = n, n+1, \ldots, n_1 n - 1$ and $g(\omega)$ for $\omega \in \theta^i V_m$ with $i = n, n+2, \ldots, n_1 n$.

Performing a similar procedure further for V_m like that of Steps (iii)–(iv) for Z we define the random matrix $\hat{A}(\omega)$ for $\omega \in \theta^i V_m$ with $i = n, n+1, \ldots, kn - 1$ and $g(\omega)$ for $\omega \in \theta^i V_m$ with $i = n, n+2, \ldots, kn$.

For $\omega \in \theta^i V_m$ with $i = kn, kn+1, \ldots, kn + r - 1$, hence $\omega = \theta^i \omega_0$ with $\omega_0 \in V_m$ we set $\hat{A}(\omega) = A(\omega)$ and $g(\omega) = \Phi_{\hat{A}}(i, \omega_0) g(\omega_0)$.

For $\omega = \theta^m \omega_0$ with $\omega_0 \in V_m$, hence $\omega \in \theta^m V_m \subset Z$, we set $g(\omega) = \Phi_{\hat{A}}(m, \omega_0) g(\omega_0)$.

Step (vii). Perform the procedure in Step (vi) for all V_m, $m \geq n(n_1 - 1)$. Then we have constructed the random matrix $\hat{A}(\omega)$ for $\omega \in \Omega \setminus \bigcup_{i=0}^{n-1} \theta^i Z$ and $g(\omega)$ for $\omega \in \Omega \setminus \bigcup_{i=1}^{n-1} \theta^i Z$.

On the orbit segments $\bigcup_{i=0}^{n-1} \theta^i Z$ we use Lemma 3.5 to perturb the random matrix $A(\omega)$ to get a random matrix $\hat{A}(\omega)$ (and to define $g(\omega)$ on $\bigcup_{i=1}^{n-1} \theta^i Z$) such that $\hat{A}(\omega)$ is ε-close to A on this measurable set and leave the one-dimensional subspace generated by the random vector $g(\omega)$ invariant, i.e.

$$\Phi_{\hat{A}}(i, \omega_0) g(\omega_0) \text{ is collinear to } g(\theta^i \omega_0) \text{ for all } \omega_0 \in Z,\ i = 1, 2, \ldots, n.$$

After performing these steps we have constructed a random matrix $\hat{A}(\omega)$ which is ε-close to the random matrix $A(\omega)$ and a random vector $g(\omega)$ which generates a one-dimensional invariant subspace of $\hat{A}$. Moreover, by the construction of g and $\hat{A}$, for all $\omega \in V_i \subset \theta^n Z$ ($i = n(n_1 - 1), n(n_1 - 1) + 1, n(n_1 - 1) + 2, \ldots$) and all $k = jn \leq i - n$ we have

$$\|\Phi_{\hat{A}}(n, \theta^k \omega) g(\theta^k \omega)\| \geq \frac{\varepsilon}{2} \|\Phi_{\hat{A}}(n, \theta^k \omega)\| \geq e^{-\varepsilon n} \|\Phi_{\hat{A}}(n, \theta^k \omega)\|, \tag{18}$$

i.e. the vector $g(\omega)$ has growth rate near to the growth rate of the norm of the random matrix $\hat{A}(\omega)$. Choose $n_1 > 4$ so large that

$$(1 + a)^{2n} \leq e^{\varepsilon n n_1},$$

then we have for all $\omega \in V_i \subset \theta^n Z$ ($i = n(n_1 - 1), n(n_1 - 1) + 1, n(n_1 - 1) + 2, \ldots$), $k = -n + 1, \ldots, 0, 1, \ldots, i$ the inequality

$$\begin{aligned} \|\Phi_{\hat{A}}(nn_1, \theta^k \omega) g(\theta^k \omega)\| &\geq e^{-\varepsilon n (n_1 - 1)} \|\Phi_{\hat{A}}(nn_1, \theta^k \omega)\| (1 + a)^{-2n} \\ &\geq e^{-2\varepsilon n n_1} \|\Phi_{\hat{A}}(nn_1, \theta^k \omega)\|. \end{aligned}$$

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Consequently, for all $\omega \in \Omega$ we have

$$\|\Phi_{\hat{A}}(nn_1, \omega)g(\omega)\| \geq e^{-2\varepsilon nn_1}\|\Phi_{\hat{A}}(nn_1, \omega)\|. \tag{19}$$

For all $\omega \in \Omega$ set

$$h_d(\omega) := \frac{g(\omega)}{\|g(\omega)\|}$$

and extend it to a random orthonormal basis $\{h_1(\omega), h_2(\omega), \ldots, h_d(\omega)\}$ of $\mathbb{R}^d$. Define a random orthogonal transformation $L(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(\omega)e_i = h_i(\omega), \quad i = 1, 2, \ldots, d.$$

Then L transforms the cocycle $\hat{A}$ into a block-triangular cocycle

$$B(\omega) = \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) & \cdots & 0 \\ b_{21}(\omega) & b_{22}(\omega) & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ b_{d1}(\omega) & b_{d2}(\omega) & \cdots & b_{dd}(\omega) \end{pmatrix},$$

that is

$$B(\omega) = L(\theta\omega)^{-1}\hat{A}(\omega)L(\omega).$$

Since L is orthogonal, $\|B(\omega)\| = \|\hat{A}(\omega)\|$ for all $\omega \in \Omega$. By the construction of L and B, from (19) it follows that for all $\omega \in \Omega$ the following inequality holds:

$$\prod_{i=0}^{n_1 n-1} |b_{dd}(\theta^i\omega)| \geq \|\Phi_B(n_1 n, \omega)\| \exp(-2\varepsilon n_1 n). \tag{20}$$

For all $\omega \in \Omega$ set

$$\hat{B}(\omega) := \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) & \cdots & 0 \\ b_{21}(\omega) & b_{22}(\omega) & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ b_{d1}(\omega) & b_{d2}(\omega) & \cdots & b_{dd}(\omega)(1+4\varepsilon) \end{pmatrix}. \tag{21}$$

Then, due to equation (20), by Lemma 3.2 and Theorem 2.7(i), $\hat{B}$ has a dominated splitting

$$\mathbb{R}^d = E_1(\omega) \oplus E_2,$$

where E_1 is a random $(d-1)$-dimensional subspace and E_2 is the subspace of $\mathbb{R}^d$ generated by the basis vector e_d. Since L is an orthogonal transformation the splitting

$$\mathbb{R}^d = L(\omega)E_1(\omega) \oplus L(\omega)E_2$$

of the cocycle

$$C(\omega) := L(\theta\omega)\hat{B}(\omega)L(\omega)^{-1}$$

is dominated. Furthermore, by construction, the cocycle C is $\tilde{a}\varepsilon$-close to $\hat{A}$ in the space $\mathcal{G}_\infty(d)$, where $\tilde{a}$ is a positive constant depending only on a. Hence C is $(\tilde{a}+1)\varepsilon$-close to A.

Now we restrict the consideration to C and to the invariant $(d-1)$-dimensional subspace $L(\omega)E_1(\omega)$ from the dominated splitting of C and apply again the above procedure.

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After $(d-1)$ steps we get a cocycle D which has an invariant splitting of $\mathbb{R}^d$ into d one-dimensional random subspaces which is dominated and D is $M\varepsilon$-close to A in the space $\mathcal{G}_\infty(d)$. The cocycle D is clearly cohomologous by a bounded cohomology $\hat{L}$ which has bounded inverse to a diagonal cocycle which is integrally separated, hence by Lemma 2.14 D is integrally separated. Since ε is arbitrary the theorem is proved. □

THEOREM 3.7. *The set $\mathcal{H}$ of cocycles with integral separateness is open and dense in the space $\mathcal{G}_\infty(d)$.*

Proof. $\mathcal{H}$ is open by Theorem 2.5 and dense by Theorem 3.6. □

THEOREM 3.8. *The set of cocycles with simple Lyapunov spectrum and dominated Oseledets splitting is generic in $\mathcal{G}_\infty(d)$.*

Proof. It follows from Theorem 3.7 because a cocycle with integral separateness has simple Lyapunov spectrum and dominated Oseledets splitting, hence every cocycle from $\mathcal{H}$ has simple Lyapunov spectrum and dominated Oseledets splitting. Note that the set $\mathcal{H}$ is 'very large': it is open and dense in $(\mathcal{G}_\infty(d), \rho)$. □

COROLLARY 3.9. *The set of cocycles which exhibit exponential dichotomy (hence are uniformly hyperbolic) is generic in the space of bounded $Sl(2, \mathbb{R})$-cocycles.*

Proof. It follows from the fact that any $Sl(2, \mathbb{R})$-cocycle with integral separateness exhibits an exponential dichotomy, and the fact that the rotation and perturbations we performed in the proofs of Lemmas 3.4 and 3.5 and Theorem 3.6 keep the perturbed cocycles in the class of $Sl(2, \mathbb{R})$-cocycles (there is a minor change to be made in the construction of $\hat{B}$ in the formula (21) to keep the perturbed cocycle $\hat{B}$ in the class of $Sl(2, \mathbb{R})$-cocycles). □

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