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Differentiating potential functions of SRB measures on hyperbolic attractors

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Abstract. The derivation of Ruelle's derivative formula of the SRB measure depends largely on the calculation of the derivative of the unstable Jacobian. Although Ruelle's derivative formula is correct, the proofs in the original paper and its corrigendum are not complete. In this paper, we re-visit the differentiation process of the unstable Jacobian and provide a complete derivation of its derivative formula. Our approach is to extend the volume form provided by the SRB measure on local unstable manifolds to a system of Hölder continuous local Riemannian metrics on the manifold so that under this system of local metrics, the unstable Jacobian becomes differentiable with respect to the base point and its derivative with respect to the map can be obtained by the chain rule.

1. *Introduction and statements of main result*

In the last 10 years or so, Ruelle's work on differentiability of the SRB measure and its derivative (linear response) for uniformly hyperbolic systems has generated great interest in further developing the theory in many directions. For dynamical systems possessing a certain level of hyperbolicity, it is natural to ask whether quantities or objects associated with hyperbolicity, such as the SRB measure and its entropy, topological entropy, Lyapunov exponents, dimension of the attractor, the rate of mixing, depend on the system in a differentiable way, and, if they do, how to determine their derivative formulas. A far from complete list of recent results include Dolgopyat [**11**] on partially hyperbolic systems, Butterley and Liverani [**8**] on Anosov flows, de la Llave and Jiang [**14, 15**] on coupled hyperbolic attractors, Gouëzel and Liverani [**12**] on Anosov maps, Pollicott [**17**] on stability of mixing rates, Baladi and Smania [**2, 4, 5, 9**] on piecewise expanding unimodal maps, and Ruelle's recent work [**23–25**].

At the core of Ruelle's derivation of the derivative formula of the SRB measure is the calculation of the derivative of the unstable Jacobian whose logarithm is the potential function of the SRB measure. Ruelle used a coordinate-free approach to determine the potential function's derivative after modifying it by a coboundary term associated with a change of volume forms on unstable manifolds. Although the derivative formula is correct, much of the detail of the derivation was left out. Inaccurate and incorrect statements remain

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in the subsequent correction of the original paper [**18, 19**] (see Remark 1 later in this section for details). In this paper, we re-visit the differentiation process of this potential function and provide a complete derivation of the derivative formula. Our approach is to extend the volume form provided by the SRB measure on local unstable manifolds to a system of Hölder continuous local Riemannian metrics on the manifold so that under this system of local metrics, the unstable Jacobian becomes differentiable with respect to the base point and the derivative of the potential function with respect to the map can be obtained using the chain rule.

It should be pointed out that a different approach to obtaining the derivative formula of the SRB measure, avoiding the calculation of the derivative of the unstable Jacobian, was successfully developed by studying the transfer operators on an anisotropic Sobolev space (see Gouëzel and Liverani [**12**] and Baladi and Tsujii [**10**]). The construction of the Sobolev space, which utilizes the hyperbolicity, provides a pure functional analysis approach to the derivation of the formula for both hyperbolic flows and maps (see also [**3, 7, 10, 12**]). What we hope to achieve here is to gain geometric insight into the derivation of the derivative formula of the potential function of SRB measures. In the process of our calculation, we obtain results on the regularity of the unstable Jacobian under the chosen metric (Theorem 3).

Let f be a $C^{1+\alpha}$-diffeomorphism of a smooth (C^∞) compact manifold M possessing a hyperbolic attractor Δ_f, assuming that it is also topologically mixing on Δ_f. The SRB measure of f, ρ_f, is the unique equilibrium state for the potential function $\varphi(x) = -\log J^u f(x)$, where $J^u f(x)$ is the Jacobian of f along the unstable manifold, i.e., $\mu = \rho_f$ is the unique invariant measure satisfying the equation (*variational principle*)

$$H_\mu + \int \varphi(x)\, d\mu = 0,$$

where H_μ is the metric entropy of f with respect to μ.

In [**18**], Ruelle proved that the map $f \to \rho_f$ is differentiable and the derivative, or the *linear response function*, in the direction of $X = \delta f \circ f^{-1}$, is given by

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f\langle \mathrm{grad}(\Phi \circ f^n),\, X\rangle, \tag{1}$$

in a suitable functional setup.

More precisely, assume that f_0 is a C^r-diffeomorphism ($r \geq 3$) of a smooth compact Riemannian manifold possessing a topologically mixing hyperbolic attractor Δ_0. Let $U(f_0)$ denote the C^r-neighborhood of f_0 such that any map $f \in U(f_0)$ possesses a hyperbolic attractor Δ_f and is topologically conjugate to f_0 via a Hölder continuous map $h_f : \Delta_0 \to \Delta_f$:

$$f \circ h_f = h_f \circ f_0.$$

Then, the SRB measure ρ_f pulled back onto Δ_0 by the conjugating map h_f is the unique equilibrium state for the potential function $-\log J^u f(h_f(x))$

$$H_{h_f^*\rho_f} + \int_{\Delta_0} -\log J^u f(h_f(x))\, dh_f^*\rho_f = 0,$$

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where $H_{h_f^*\rho_f}$ is the metric entropy of f with respect to the measure $h_f^*\rho_f$. Given any smooth function Φ on the manifold, the functional

$$f \to \int_{\Delta_f} \Phi \, d\rho_f = \int_{\Delta_0} \Phi \circ h_f \, dh_f^* \rho_f$$

is differentiable in $f \in U(f_0)$ and its derivative formula evaluated at f is given by (1).

The derivation of this formula [**18, 19**] relies mainly on the calculation of the derivative of the potential function $-\log J^u f(h_f(x))$ with respect to f: the SRB measure, as an equilibrium state, is the derivative of the topological pressure at the potential function $-\log J^u f(h_f(x))$ in the Banach space of Hölder continuous functions. Since coboundary potential functions lead to the same equilibrium state, for any Hölder continuous function $\phi_f(x)$ on the hyperbolic attractor Δ_0, a potential function defined by

$$p_f(x) = -\log J^u f(h_f(x)) + \phi_f(f_0(x)) - \phi_f(x)$$

has exactly the same equilibrium state as $-\log J^u f(h_f(x))$ has, provided that the coboundary term $\phi_f(f_0(x)) - \phi_f(x)$ is differentiable with respect to f and, thus, can be used to calculate the derivative of the SRB measure.

Remark 1. The coboundary term $\phi_f(f_0(x)) - \phi_f(x)$ in [**19**] comes from changing volume forms on local unstable manifolds of f at the point $h_f(x)$ by transporting volume forms on local unstable manifolds of f_0 using the holonomy map π defined by the stable foliation of f_0. Since the holonomy map associated with the stable foliation of f_0 is, in general, only Hölder continuous, the coboundary term is not expected to be differentiable in terms of f, contrary to what is indicated in [**19**]. The subsequent calculation of the derivative of the potential function contains an incorrect formula (page 187, line 6) when the volume forms on unstable manifolds are compared. In Ruelle's notation, $\tilde{f}$ and f are for perturbed and unperturbed maps, respectively. The point $jx = h_{\tilde{f}}(x)$ is denoted by $x + \delta x$. In the local coordinate system provided by the stable and unstable subspaces V_x^s and V_x^u, δx is decomposed into $\delta^u x + \delta^s x$. The equation on line 6 page 187,

$$\pi(T_{x+\delta x} f)^{\wedge u} \tilde{F}(x + \delta x) = (T_{x+\delta^u x}) f^{\wedge u} F(x + \delta^u x),$$

does not hold in general since $x + \delta^u x$ is not necessarily the image of $x + \delta x$ under the holonomy map π. (One may try to interpret $x + \delta^u x$ as the image of $x + \delta x$ under π, but then the differentials $T_{x+\delta x} f$ and $T_{x+\delta^u x} f$ are not close in terms of δf since π is not differentiable.)

Nevertheless, the derivative formula is correct, which is also what will be proven in this paper.

MAIN THEOREM. *Given a direction δf, a vector field on M evaluated at $f_0(x)$, let $X = \delta f \circ f_0^{-1}$, there exists a Hölder continuous function $\phi_f(x)$ on Δ_0 such that*

$$\delta(\log J^u f(h_f(x)) + \phi_f(f_0(x)) - \phi_f(x))|_{f=f_0} = \operatorname{div}_\rho X^u(f_0(x)),$$

where $\operatorname{div}_\rho X^u$ is the divergence of the vector field X^u with respect to the volume form induced by the density function of the SRB measure on the local unstable manifold and $X = X^u + X^s$ denotes the projections of X onto unstable and stable subspaces of Df_0.

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The idea of the proof is similar to what Ruelle used in [**19**], but with a different way of transporting metrics between unstable manifolds. Ruelle expressed [**20**] such that it should be possible to use a metric consistent with the volume forms provided by the SRB measure to calculate the derivative of the potential function. However, such a choice has obvious difficulties: the density function of an SRB measure is defined locally and the density function is expressed in terms of unstable Jacobians. In this paper, we overcome these difficulties by constructing a system of Hölder continuous local Riemannian metrics by extending smooth metrics on unstable manifolds into a neighborhood. Under this system of local metrics, we prove the differentiability of the resulting unstable Jacobian (or, equivalently, the differentiability of the coboundary term) with respect to f and give a complete account of how to obtain this derivative formula. The calculation also reveals regularity of the unstable Jacobian with respect to the base point: under a Hölder continuous local metric, the unstable Jacobian becomes differentiable in the stable direction at the base point, which contrasts its generic non-differentiability in the original smooth metric.

Remark 2. There is no essential difference in evaluating the derivative formula at the point f_0 or any other point $f = h_f \circ f_0 \circ h_f^{-1} \in U(f_0)$. The expression of the formula at any $f \in U(f_0)$ becomes $\mathrm{div}_\rho X^u(h_f(x))$. In the rest of the paper, without loss of generality, we focus on the derivative formula evaluated at the point f_0. The symbol div always denotes the divergence on unstable manifolds.

We now define a system of local metrics in which the unstable Jacobian is calculated. The potential function remains homologous to the original one via a Hölder continuous function.

Definition 1. $\omega(x, \cdot)$ is called a system of local metrics if, for every $x \in M$, $\omega(x, \cdot)$ is a Riemannian metric in an open ϵ-neighborhood of x for some constant $\epsilon > 0$. $\omega(x, \cdot)$ is called a system of Hölder continuous (smooth) local metrics if, for each x, $\omega(x, \cdot)$ is a Hölder continuous (smooth) Riemannian metric in the open ϵ-neighborhood of x. $\omega(x, \cdot)$ is called Hölder continuous in x if $\omega(x, (y, v, w))$ is Hölder continuous in x for each $(y, v, w) \in M \times T_yM \times T_yM$ in any local coordinate chart.

Assume that m is the given smooth Riemannian metric on M. The unstable Jacobian $J^u f(h_f(x))$ is calculated in terms of the induced volume form (denoted by dm) on unstable manifolds at $h_f(x)$ and $f(h_f(x)) = h_f(f_0(x))$. Let $W^u_f(h_f(x))$ be the local unstable manifold of f at the point $h_f(x)$. Assume that f is sufficiently close to f_0 in $U(f_0)$ so that the distance between $h_f(x)$ and x is less than ϵ. The local metric $\omega(x, \cdot)$ induces a volume form on the unstable manifold $W^u_f(h_f(x))$. For simplicity, we denote this volume form by $\omega_x = g_x(z)\, dm$, $z \in W^u_f(h_f(x))$, where $g_x(z)$ denotes the density function and dm denotes the induced volume form from m on the submanifold $W^u_f(h_f(x))$.

With respect to the local volume form $g_x(z)\, dm$, we have the coboundary relation

$$\log J^u f(h_f(x)) = \log J^u_{\omega_x} f(h_f(x)) + \log g_{f_0(x)}(h_f(f_0(x))) - \log g_x(h_f(x)), \qquad (2)$$

where $J^u f$ is calculated with the volume form dm and $J^u_{\omega_x} f$ denotes the unstable Jacobian calculated with respect to the volume form $\omega_x = g_x(z)\, dm$ at corresponding points $h_f(x)$ and $h_f(f_0(x))$.

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Given any system of Hölder continuous local metrics $\omega(x, \cdot)$, if $\omega(x, \cdot)$ is also Hölder continuous in x, then $\log g_x(h_f(x))$ is a Hölder continuous function in x. Thus, we have the following lemma.

LEMMA 1. *For a system of Hölder continuous local metrics $\omega(x, \cdot)$ that are also Hölder continuous in x, the function $\log g_x(h_f(x))$ is Hölder continuous in x. The potential functions $-\log J^u_{\omega_x} f(h_f(x))$, $-\log J^u f(h_f(x))$ have the same equilibrium state on the hyperbolic attractor Δ_0. If $\omega(x, \cdot)$ is a system of smooth local metrics that is Hölder continuous in x, then $\log g_x(h_f(x))$ is differentiable in f whenever the conjugate map h_f is differentiable in f.*

Note that the unstable Jacobian $J^u f(y)$ is, in general, not differentiable in y. This poses the main difficulty in determining the derivative formula of $J^u f(h_f(x))$ with respect to f. In order to obtain the derivative, we construct a special class of systems of Hölder continuous local metrics, denoted by $\omega_\perp = \omega_\perp(x, \cdot)$. Roughly speaking, for each x, the local metric $\omega_\perp(x, \cdot)$ satisfies two conditions: it is smooth when restricted to $W^u_{f_0}(x)$, the local unstable manifold of f_0 at x, and the local unstable and stable subspaces along $W^u_{f_0}(x)$ are orthogonal under the metric.

Definition of the system of local metrics $\omega_\perp = \omega_\perp(x, \cdot)$. Take any smooth metric defined on the local unstable manifold of f_0 at x, for example, the metric inherited from the original Riemannian metric (denoted by $\langle v, w\rangle_z$). We extend it into an open ϵ-neighborhood of x in the following way: identify this open neighborhood of x with a neighborhood of the origin in $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$, B so that $x = (0, 0)$, and the local unstable manifold of f_0 at x, $W^u_{f_0}(x)$, is contained in the subspace $(\mathbb{R}^u, 0)$. This open neighborhood B is foliated by the local stable manifolds of f_0 [**13**]. We denote this stable foliation by $\mathcal{W}^s$. The leaf of the stable foliation passing through the point $x^u_0 = (x^u, 0) \in W^u_{f_0}(x)$ can be expressed as a graph $(F(x^u_0, x^s), x^s)$, $x^s \in (0, \mathbb{R}^s) \cap B$, where $F(x^u_0, x^s)$ is a C^r map from $(0, \mathbb{R}^s) \cap B$ to $\mathbb{R}^u$. Given any point $z = (x^u, x^s) \in B \subset \mathbb{R}^u \oplus \mathbb{R}^s$ near the origin, there exists $x^u_0 \in W^u_{f_0}(x)$ such that $x^u = F(x^u_0, x^s)$. For a fixed x^s, the map $x^u \to x^u_0$ so defined is the holonomy map induced by the stable foliation from $(\mathbb{R}^u, x^s) \cap B$ to $W^u_{f_0}(x)$. Indeed, (x^u_0, x^s) gives another coordinate for each point in this small neighborhood of x. In this coordinate system, the local stable manifolds are simply the coordinate hypersurfaces $x^u_0 = C$. We assume this small neighborhood B of x is in one single coordinate chart. Thus, the tangent space at any point (x^u, x^s) can be identified with $\mathbb{R}^n$. The metric $\omega_\perp(x, \cdot)$ at point $z = (x^u, x^s)$ is defined by the following rules.

(1) *If vectors v, w of the tangent space $T_z M = \mathbb{R}^n$ are in the subspace $\mathbb{R}^u$, then the metric is the same as the metric in $\mathbb{R}^u$ at the point $(x^u_0, 0) : \omega(x, (z, v, w)) = \omega(x, ((x^u_0, 0), v, w))$.*

(2) *If vectors v, w are in the tangent space of the stable leaf, they inherit the original Riemannian metric $\omega(x, (z, v, w)) = \langle v, w\rangle_z$.*

(3) *The metric $\omega_\perp(x, \cdot)$ is defined so that $\mathbb{R}^u$ and the tangent space of the stable leaf are orthogonal.*

At each point along the local stable manifold passing through the point $(x^u_0, 0)$, the metric $\omega_\perp(x, \cdot)$, restricted to the subspace $\mathbb{R}^u$ of the tangent space, is thus dependent only on the coordinate x^u_0, not on x^s. When the metric is restricted to any stable leaf, it is smooth.

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With this definition, we have a system of Hölder continuous local metrics $\omega_\perp(x, \cdot)$ in an open neighborhood of the attractor Δ_0. The Hölder continuity comes from the fact that the stable foliation is, in general, only Hölder continuous. When the foliation is smooth, we have a smooth local metric. Even though the system of local metrics is not defined on the entire manifold, it is sufficient for calculating the derivative of the potential function. With the choice of the original metric on the unstable manifold, $\omega_\perp(x, \cdot)$ is clearly Hölder continuous in x since the stable and unstable subspaces vary Hölder continuously with respect to the base point. We can also choose other metrics on the unstable manifold in the construction, for example, we can choose the Lyapunov metric [**6**]. In particular, we can choose a smooth metric on local unstable manifolds so that its volume forms coincide with those induced by the SRB measure of f_0, ρ, i.e., the local density function $g_x(z)$ is given by the formula

$$g_x(z) = \frac{p(z)}{p(x)} = \prod_{i=1}^{\infty} \frac{|\det Df_0|_{E^u_{x_i}}|}{|\det Df_0|_{E^u_{z_i}}|} = \prod_{i=0}^{\infty} \frac{|\det Df_0^{-1}|_{E^u_{z_i}}|}{|\det Df_0^{-1}|_{E^u_{x_i}}|}, \tag{3}$$

where $x_i = f^{-i}(x)$, $z_i = f^{-i}(z)$, and z is a given point on the unstable manifold $W^u_{f_0}(x)$. With this choice, the system of local metrics $\omega_\perp(x, \cdot)$ is Hölder continuous in x and it is independent of the original Riemannian metric on the manifold.

For brevity, the potential function calculated using $\omega_\perp(x, \cdot)$ is denoted by $-\log J^u_\perp f(h_f(x))$ for $f \in U(f_0)$.

Let δf be a C^r-vector field evaluated at $f_0(x)$, i.e., for a given $x \in M$, $\delta f(x)$ is a vector in the tangent space $T_{f_0(x)}M$. Thus, $X(x) = \delta f \circ f_0^{-1}(x)$ is a C^r-vector field on M. Let X^u, X^s be the projections of X onto stable and unstable invariant subspaces of Df_0. We have the following theorem.

THEOREM 1. *For each $x \in \Delta_0$, the function $f \to \log J^u_\perp f(h_f(x))$ is differentiable at $f_0 \in U(f_0)$. Its derivative, in a given direction specified by a vector field evaluated at $f_0(x)$, δf, is given by*

$$\delta(\log J^u_\perp f(h_f(x)))|_{f_0} = \operatorname{div} X^u(f_0(x)) + \sum_{k=1}^{\infty} \langle \operatorname{grad}(p_{f_0}),\, D(f_0^{-k})X^u \rangle,$$

where $\operatorname{div} X^u(x)$ *is the divergences of X^u on the local unstable manifold $W^u_{f_0}(x)$ and $p_{f_0} = -\log J^u_\perp f_0$ is the potential function of the SRB measure of f_0.*

If $\omega_\perp$ is so constructed that its induced volume form on local unstable manifolds has the SRB density

$$g_x(z) = \prod_{i=1}^{\infty} \frac{|\det Df_0|_{E^u_{x_i}}|}{|\det Df_0|_{E^u_{z_i}}|}, \tag{4}$$

then

$$\delta(\log J^u_\perp f(h_f(x)))|_{f_0} = \operatorname{div}_\rho X^u(f_0(x)),$$

where the divergence $\operatorname{div}_\rho X^u$ *is calculated with respect to the volume form on unstable manifolds induced by the SRB measure of f_0.*

We note that the differentiability of $\log J^u_\perp f(h_f(x))$ in f does not follow directly from what we know from Ruelle's work in [**18**] since the unstable Jacobian is calculated with

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respect to non-smooth local metrics. The idea of extending the metric on the local unstable manifolds of f_0 into a neighborhood to calculate the unstable Jacobian $J^u f(h_f(x))$ is similar to Ruelle's approach in [**19**]. This system of local metrics is in a sense canonical: the derivative formula is expressed independently of the metric and without a coboundary term.

In the next section, we shall give definitions of some terms used in this theorem and recall some related results, which include the topology of the spaces of the potential functions, the differentiability of the conjugating map h_f, as well as stable and unstable subspaces, in terms of f, and the divergence of a non-smooth vector field.

2. *The preliminaries*

For definitions of commonly used terms such as uniformly hyperbolic maps, attractors, topological mixing, C^r-topology, conjugating map and exponential splitting, we refer to [**16**].

2.1. *Spaces of subbundles of the tangent bundle and Hölder continuous maps.*

Much of the setup is the same as that in [**18**]. We include here a brief description.

2.1.1. *The Banach manifold structure of $U(f_0)$.*

In order to consider perturbations of a diffeomorphism (or, in general, a homeomorphism) f_0 on M, we need to place f_0 in a Banach manifold. The Banach manifold structure of $U(f_0)$ is essentially inherited from M. We only need a description of its tangent space. Let $\mathcal{A}_x$ be the exponential map from the tangent space T_xM to M at the point x. Given a diffeomorphism $f_0 \in \mathrm{Diff}^{C^r}(M)$, for any map g in its small C^r neighborhood, the map $\tilde{g}(x) : x \to \mathcal{A}^{-1}_{f(x)} g(x) \in T_{f_0(x)}M$ defines a C^r vector field on M. Note that this vector field is evaluated at the point $f_0(x)$. This leads us to use the following Banach space to model the tangent space,

$$\mathcal{B} = \{\xi \mid \xi : x \to \xi(x) \in T_{f_0(x)}M,\, C^r\},$$

equipped with the C^r-norm of vector fields on M induced by the Riemannian metric on M.

Thus, in the Banach manifold $U(f_0)$, the exponential map at f_0, denoted by $\mathcal{A}_{f_0}$, is given by

$$\mathcal{A}_{f_0} : \xi \to \mathcal{A}_{f_0}(\xi) = \mathcal{A}_{f_0(x)}(\xi(x)), \quad \xi \in \mathcal{B}.$$

Clearly, the map $\mathcal{A}_{f_0}$ is a C^r diffeomorphism in a small neighborhood of the zero section. Note that the Banach manifold $U(f_0)$ is an open manifold, i.e., a C^r-neighborhood of f_0, and we only need one coordinate chart.

In a similar way, we can define the Banach manifold structure of Hölder continuous maps on M that are close to the identity map. If h is a Hölder continuous map of M close to the identity with a Hölder exponent $0 < \alpha \leq 1$, its C^α neighborhood, as a Banach manifold, is identified with the C^α neighborhood of the zero vector field in the space of all C^α (Hölder continuous with a Hölder exponent $0 < \alpha \leq 1$) vector fields on M. Given a trivialization of the tangent bundle TM by a finite number of coordinate charts, $\{P_1, \ldots, P_l\}$, the C^α norm of a vector field $v(x)$, is defined by

$$\|v(x)\|_{C^\alpha} = \|v(x)\|_{C^0} + \max_{1\leq k\leq l} \sup_{x,y\in P_k, y\neq x} \frac{\|v(x) - v(y)\|}{d(x,\, y)^\alpha},$$

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where $\|v(x) - v(y)\|$ is the norm in the tangent space $T_xM = T_yM$ and $d(x, y)$ is the distance on M. We denote this C^α neighborhood of the identity map by $C^\alpha(M, M)$. If we replace the norm by the absolute value, we have the Banach space of all Hölder continuous functions on M, and we denote it by $C^\alpha(M, \mathbb{R})$.

2.1.2. *The Banach manifold structure of the space of Hölder continuous Grassmannian bundles of the tangent bundle.* Let $G_k(M)$ denote the space of Hölder continuous Grassmannian bundles of order k of the tangent bundle. An element of $G_k(M)$, $\mathcal{J}(x)$, is a k-dimensional linear subspace bundle of the tangent bundle TM. $\mathcal{J}(x)$ is Hölder continuous with respect to x in the Grassmannian metric on the linear subspaces. Such Grassmannian bundles are also defined on any subset of M, in particular, on the hyperbolic attractor of f. With the linear structure of the tangent space equipped with a Riemannian metric, $G_k(M)$ naturally forms a Banach manifold with the Grassmannian distance defined between k-dimensional linear subspaces of the tangent space T_xM induced by this metric.

Let $E^u_f(x)$ and $E^s_f(x)$ denote the unstable and stable invariant subspaces of Df, the differential of f. They are Hölder continuous in x over the hyperbolic attractor Δ_f. Let u and s denote the dimensions of the unstable and stable subspaces of Df. For any given $f \in U(f_0)$, since $h_f(x)$ is close to x, $E^u_f(h_f(x))$, $E^s_f(h_f(x))$ can be identified with subspaces of the tangent space T_xM using local coordinate charts. They can thus be considered as Hölder continuous subbundles of the tangent bundle over the hyperbolic attractor Δ_0. We have that $E^u_f(h_f(x))$ and $E^s_f(h_f(x))$ are elements of $G_u(\Delta_0)$ and $G_s(\Delta_0)$ for each $f \in U(f_0)$.

The following proposition summarizes the preliminary results we need concerning the differentiability of the conjugating map h_f, the exponential splittings $E^u_f(h_f(x))$, $E^s_f(h_f(x))$, and the potential function $-\log J^u f(h_f(x))$.

PROPOSITION 1. **[18]** *If f_0 is C^r, $r \geq 3$, then:*

(1) *the map $f \in U(f_0) \to h_f \in C^\alpha(M, M)$ is C^{r-1} for some $0 < \alpha < 1$. The derivative in the direction of δf at the point f_0 is given by*

$$\delta h_f(x) = \sum_{n=0}^{\infty} Df_0^n X^s - \sum_{n=1}^{\infty} Df_0^{-n} X^u,$$

where $X = \delta f \circ f_0^{-1}$ and $X = X^s + X^u$;

(2) *the maps $f \to E^u_f(h_f(x))$, $E^s_f(h_f(x))$ are C^{r-2} from $U(f_0)$ to $G_u(\Delta_0)$, $G_s(\Delta_0)$, respectively; and*

(3) *the map $f \to -\log J^u f(h_f(x))$ is C^{r-2} from $U(f_0)$ to $C^\alpha(M, \mathbb{R})$.*

2.2. *The divergence of X^u.* Let $\delta f \in TU(f_0)$ be an element in the tangent space of $U(f_0)$. It is a C^r vector field evaluated at the point $f_0(x)$. Thus, the composition $X(x) = \delta f \circ f_0^{-1}(x)$ is a C^r vector field on M. For brevity, we denote the stable and unstable invariant subspaces of Df_0 by $E^s_0(x)$ and $E^u_0(x)$. Since $T_x(M) = E^u_0(x) \oplus E^s_0(x)$ for $x \in \Delta_0$, the vector field X can be projected onto these invariant subspaces of the differential $Df_0 : X(x) = X^u(x) + X^s(x)$. Since Δ_0 is an attractor, the global and local unstable manifolds are smooth submanifolds equipped with an induced Riemannian metric from M. Thus, X^u is a well-defined vector field on either global or local unstable manifolds.

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Since $E_0^s(x)$ is only, in general, Hölder continuous along the unstable manifold $W_{f_0}^u(x)$, the vector field $X^u(x)$ is only Hölder continuous in x. However, using a weak definition of divergence (a distribution), given any smooth volume form on the unstable manifold, the divergence of X^u with respect to the given volume form exists as a result of the absolute continuity of the holonomy map defined by the stable foliation.

Definition 2. [**15**] Let X be a continuous vector field defined in an open set U on a Riemannian manifold with a volume form dw. Let $\mathcal{G}$ denote the family of C^∞ functions in U with compact support. Assume that there is an integrable function $h(x)$ such that

$$\int \langle \text{grad } g,\, X\rangle \, dw = -\int g(x)h(x)\, dw,$$

for all $g \in \mathcal{G}$. Then, we call $h(x)$ the divergence of X and denote it by div X.

The next proposition gives a sufficient condition for the existence of the divergence in our context. Since the definition of the divergence concerns only the vector field in a small open neighborhood, we state the proposition in the case when the unstable manifold is a linear subspace $\mathbb{R}^u$ in an n-dimensional linear space $\mathbb{R}^n$ equipped with a Riemannian metric.

PROPOSITION 2. [**15**] *Let X be a C^1-vector field on $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ equipped with a smooth Riemannian metric with the property that the metric is independent of the coordinate in $\mathbb{R}^s$. Let $\mathcal{W}$ be an absolutely continuous foliation with smooth leaves transversal to $\mathbb{R}^u$. Let $E^s(x)$ be the subspace tangent to $\mathcal{W}$ at $x \in \mathbb{R}^u$ and $X = X^u + X^s$ denote the projections of X onto $\mathbb{R}^u$ and $E^s(x)$, respectively. Then,* div X^u*, the divergence of X^u on the submanifold $\mathbb{R}^u$ with respect to the volume form restricted to the manifold, is well defined. Moreover, when the Jacobian $J\xi(x^u, t)$ of the holonomy map $\xi(x^u, t)$ from $\mathbb{R}^u$ to $\mathbb{R}^u + tX$, defined by the foliation $\mathcal{W}$, is differentiable in t and continuous in x^u, we have*

$$\text{div } X^u = \text{div } X_c^u - \frac{d}{dt} J\xi(x^u, t)\bigg|_{t=0}, \quad x^u \in \mathbb{R}^u, \tag{5}$$

where X_c^u is the projection of the vector field X onto the coordinate subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$: $X = X_c^u + X_c^s$.

Remark 3. The existence of the divergence does not need the special property that the smooth metric is independent of the coordinate in $\mathbb{R}^s$, but the derivation of formula (5) does.

For a uniformly hyperbolic attractor, the stable foliation is absolutely continuous. The Jacobian of the holonomy map $\xi(x^u, t)$ defined by this foliation from the unstable manifold $\mathbb{R}^u$ to $\mathbb{R}^u + tX$ is given by [**6**, p. 255]

$$J\xi(x^u, t) = \prod_{n=0}^{\infty} \frac{|\det(D_{p_n} f|_{E_n})|}{|\det(D_{p_n'} f|_{E_n'})|},$$

where $p_n = f^n(x)$, $p_n' = f^n(\xi(x, t))$, $E_n = D_{p_n} f^n E^u(x)$, and $E_n' = D_{p_n'} f^n E_t^u$, where $E_t^u = E^u(x) + tX(x)$. It is Hölder continuous in x^u and differentiable in t. Thus, div X^u is a well-defined Hölder continuous function on the unstable manifold. In the stable direction,

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since the unstable subspace is Hölder continuous, div X^u is Hölder continuous as well. Thus, div X^u is Hölder continuous on the hyperbolic attractor.

The divergence, by its definition, can be taken with respect to any smooth volume form on the manifold. We denote by $\mathrm{div}_\omega X^u$ the divergence taken with respect to a volume form $d\omega$. For two smooth volume forms $d\omega_1$ and $d\omega_2$, let $p(x) = (d\omega_1/d\omega_2)$ be the relative density function. We have the relation

$$\mathrm{div}_{\omega_1} X^u = \mathrm{div}_{\omega_2} X^u + \langle \mathrm{grad} \log\, p(x),\, X^u \rangle. \tag{6}$$

When volume forms differ only by a constant factor, the divergence is clearly the same. Thus, the divergence can be taken with respect to the volume form given by the density function of the SRB measure ρ on the unstable manifold. We denote this divergence by $\mathrm{div}_\rho X^u$. We use div X^u to denote the divergence taken with respect to the volume form induced by the original Riemannian metric on M.

Note that the volume form given by the SRB measure is independent of the choice of the original Riemannian metric as long as the volume forms induced by the metrics are equivalent. Using the density function $p(z)$ of the SRB measure given by

$$\frac{p(z)}{p(x)} = \prod_{i=1}^{\infty} \frac{|\det Df_{x_i}|_{E^u_{x_i}}|}{|\det Df_{z_i}|_{E^u_{z_i}}|}, \tag{7}$$

where $z_i = f^{-i}(z)$, $x_i = f^{-i}(x)$, and z is a given point on the unstable manifold $W^u_f(x)$, we have

$$\begin{aligned} \mathrm{div}_\rho X^u &= \mathrm{div}\ X^u + \left\langle \mathrm{grad} \log \frac{p(z)}{p(x)},\, X^u \right\rangle \\ &= \mathrm{div}\ X^u + \sum_{n=1}^{\infty} \langle \mathrm{grad}\ p_f,\, D(f^{-n}) X^u \rangle \\ &= \mathrm{div}\ X^u + \sum_{n=1}^{\infty} \langle \mathrm{grad}\ p_f \circ f^{-n},\, X^u \rangle, \end{aligned}$$

where p_f is the potential function $-\log |\det Df|_{E^u} = -\log J^u f(x)$.

2.3. *Systems of smooth local metrics.* Because of the lack of differentiability of local metrics $\omega_\perp(x, \cdot)$, the differentiability of the function $\log J^u_\perp f(h_f(x))$ becomes a problem in the computation of the derivative with respect to f. We need a sequence of systems of smooth local metrics $\omega_n(x, \cdot)$, $n = 1, 2, \ldots$ that approximate the local metrics $\omega_\perp(x, \cdot)$. This sequence of systems of smooth local metrics is obtained by approximating the stable foliation by smooth foliations.

Definition of systems of smooth local metrics $\omega_n(x, \cdot)$. We recall that for each x, the ϵ open neighborhood of x is identified with an open neighborhood of the origin B in $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ and the unstable manifold of x is contained in the subspace $(\mathbb{R}^u, 0)$. For each integer $n \geq 1$, we define maps $F_n(x^u_0, x^s) : \mathbb{R}^u \oplus \mathbb{R}^s \cap B \to \mathbb{R}^u$ so that it is C^r $(r > 2)$ in both x^u_0 and x^s, and both the map $F_n(x^u_0, x^s)$ and its derivative in the $\mathbb{R}^s$-direction, $D_s F_n(x^u_0, x^s)$, converge to $F(x^u_0, x^s)$ and $D_s F(x^u_0, x^s)$ uniformly in B. These maps surely exist. Indeed, they can be constructed explicitly using Anosov's method

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(see [**1**, pp. 137–138]): since the open neighborhood B is foliated by stable manifolds, the set $f_0^n(B)$ is open and foliated by stable manifolds as well. We choose a smooth foliation of $f_0^n(B)$ so that the leaves are uniformly transversal to the unstable manifold $W^u_{f_0}(f_0^n(x)) \subset f_0^n(B)$, for example, we can project a local smooth foliation in the tangent space onto the manifold using exponential maps. Pull back this local foliation by the diffeomorphism f_0^{-n}. We get a local smooth foliation of B and maps

$$F_n(x_0^u, x^s) : B \to \mathbb{R}^n$$

with desired properties. This is, indeed, how stable manifolds are approximated.

The system of smooth local metrics $\omega_n(x, \cdot)$ is then defined in the same way as $\omega_\perp(x, \cdot)$ by replacing $F(x_0^u, x^s)$ with $F_n(x_0^u, x^s)$ in the definition. For each x and $n \geq 1$, the local metric $\omega_n(x, \cdot)$ is smooth and is Hölder continuous in x. When $\omega_n(x, \cdot)$ is restricted to the subspace $(\mathbb{R}^u, 0) \cap B$, it is the same as $\omega_\perp(x, \cdot)$.

The local smooth foliations can be so constructed that they satisfy the local invariance under f_0: the leaves of the foliations satisfy the relation

$$\begin{aligned} &f_0(\{(F_n(x_0^u, x^s), x^s) : x^s \in (0, \mathbb{R}^s) \cap B\}) \\ &\quad \subset \{(F_{n-1}(f_0(x_0^u, 0), x^s)) : x^s \in (0, \mathbb{R}^s) \cap f_0(B)\}. \end{aligned}$$

The local smooth foliation determined by $F_n(x_0^u, x^s)$ is denoted by $\mathcal{W}_n^s$. For each fixed n, this local smooth foliation gives a local smooth coordinate system (x_0^u, x^s) for B.

3. *Computation of the derivative of the map* $f \to \log J^u_\perp f(h_f(x))$

The computation of the derivative is divided into three parts: reduction of the number of variables from three to two and calculation of the two resulting partial derivatives. Since the metric $\omega_\perp(x, \cdot)$ is not differentiable, we calculate the derivative with respect to the metric $\omega_n(x, \cdot)$. Following a standard real analysis argument (see [**14**, Lemma 6.6]), we obtain both the differentiability and the derivative formula of the map $f \to \log J^u_\perp f(h_f(x))$. We use $J^u f$ to denote the unstable Jacobian under a generic smooth metric.

3.1. *Reduction of the number of variables.*

Even though the function $\log J^u f(y)$ is differentiable in y along unstable manifolds of f, it is, in general, not differentiable over the entire hyperbolic attractor of f, Δ_f [**26**]. We cannot use the chain rule to obtain the derivative of the composition $p_f \circ h_f = -\log J^u f(h_f(x))$. In the potential function $p_f \circ h_f = -\log J^u f(h_f(x))$, there are three terms that vary with f: the unstable Jacobian $J^u f$ in terms of f, the unstable direction E^u_f in which $J^u f$ is calculated, and the base point $h_f(x)$. We describe the dependence of the potential function on f in a neighborhood of f_0 with the help of local coordinate systems on the manifold. Since the negative sign in the potential function does not play any role in differentiability, we omit it in our calculation.

We first extend the domain of the map $f \to \log J^u f(h_f(x))$ to a cross-product of two infinite-dimensional manifolds: $U(f_0)$ and the space of the Grassmannian bundle over M.

We consider the space of the Grassmannian bundle over M of order k, where k is now the dimension of the unstable subspace. The space consisting of all continuous Grassmannian bundles forms a smooth Banach manifold $\mathcal{G}$ with the usual supremum metric provided by the Grassmannian distance. For any diffeomorphism $f \in U(f_0)$,

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its differential Df induces a natural map on $\mathcal{G}$ for every element $g \in \mathcal{G}$, $Df(g)(x) = Df(g(x))$. Note that for a given point x, the differential of f at this point, Df_x, is a linear map on any subspace $g(x)$ of the tangent space T_xM. We denote this linear map by $Df_{g(x)}$. On the product space $\mathcal{G} \times U(f_0)$, we define a map in the form of

$$(g, f) \in \mathcal{G} \times U(f_0) \to Df_{g(x)}(g(x)) = Df(g(x)) \in \mathcal{G}.$$

When $g(x) = E^u_f(x)$, $x \in \Delta_f$, the invariant unstable subspace, we have

$$Df_{E^u_f(x)}(E^u_f(x)) = E^u_f(f(x)).$$

Once the metric is chosen, we have

$$J^u f(h_f(x)) = |\det Df_{E^u_f(h_f(x))}|. \tag{8}$$

Since $h_f(x)$ is close to x when f is close to f_0, $h_f(x)$ and x are necessarily located in one coordinate chart. We identify the tangent spaces at these two points. Thus, $E^u_f(h_f(x))$, the unstable subspace of f at the point $h_f(x)$, can be identified with a subspace in the tangent space T_xM close to the unstable subspace of Df_0, $E^u_0(x)$, and vice versa. We now compare the determinants of two linear maps $Df_{E^u_f(h_f(x))}$ and $Df_{E^u_0(x)}$, where $E^u_0(x)$ is considered as a subspace of the tangent space at the point $h_f(x)$. Note that the map

$$f \to Jf_{E^u_0(x)}(h_f(x)) =: |\det Df_{E^u_0(x)}(h_f(x))| \tag{9}$$

is differentiable in terms of f when the subspace $E^u_0(x)$ is held independent of f. The determinant depends on f and h_f.

What we shall prove next is that these two determinants (8) and (9), when calculated with respect to the local metrics $\omega_\perp(x, \cdot)$ and $\omega_\perp(f_0(x), \cdot)$, differ by a higher-order term in terms of the C^r-distance between f and f_0.

LEMMA 2. *Let $\epsilon = \text{dist}^{C^3}(f, f_0)$ be the C^3-distance between f and f_0. Then, we have*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} |\log J^u_\perp f(h_f(x)) - \log J_\perp f_{E^u_0(x)}(h_f(x))| = 0.$$

The lemma implies that the functionals

$$f \to \log J^u_\perp f(h_f(x)) \quad \text{and} \quad f \to \log J_\perp f_{E^u_0(x)}(h_f(x))$$

will have the same differentiability with respect to f and the same derivative formula at the point f_0 when differentiable.

Lemma 2 is a direct consequence of the following linear algebra result.

PROPOSITION 3. *Assume that $E^k_i \oplus E^s_i$, $i = 1, 2$ are two almost orthogonal decompositions of two n-dimensional inner product spaces $(\mathbb{R}^n, \langle\cdot\rangle_i)$, $i = 1, 2$, respectively, i.e., there exists a small number $\delta > 0$ such that for any unit vectors $e \in E^k_i$ and $w \in E^s_i$, $|\langle e, w\rangle_i| < \delta$, $i = 1, 2$. Let T be a linear transformation between $(\mathbb{R}^n, \langle\cdot\rangle_i)$, $i = 1, 2$ satisfying $TE^k_1 = E^k_2$ and $TE^s_1 = E^s_2$. Assume that E'^k is another linear subspace which is ϵ-close to E^k_1, i.e., E'^k is a linear span of vectors $(e_1 + w_1, e_2 + w_2, \ldots, e_k + w_k)$, where $\{e_1, e_2, \ldots, e_k\}$ is any orthonormal basis of E^k_1 and $w_i \in \mathbb{R}^n$ and $\|w_i\|_1 < \epsilon$, $i = 1, 2, \ldots, k$. Then, there exists a constant C, depending on T continuously, such that*

$$|\det T|_{E^k_1} - \det T|_{E'^k}| < C(\delta\epsilon + \epsilon^2).$$

A proof of this proposition is included as an appendix.

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Proof of Lemma 2. Let f be sufficiently C^3-close to f_0. The unstable and stable subspaces $E^{u,s}_f(h_f(x))$ and $E^{u,s}_f(h_f(f_0(x)))$ are invariant under $Df : Df E^{u,s}_f(h_f(x)) = E^{u,s}_f(h_f(f_0(x)))$ and ϵ-almost orthogonal since they are within the ϵ-distance of $E^{u,s}_0(x)$ and $E^{u,s}_0(f_0(x))$, respectively. Thus, by Proposition 3, we have

$$\log J^u_\perp f(h_f(x)) - \log J_\perp f_{E^u_0}(h_f(x)) = O(\epsilon^2),$$

where the determinants are calculated with the metrics $\omega_\perp(x, \cdot)$ and $\omega_\perp(f_0(x), \cdot)$ at $h_f(x)$ and $h_f(f_0(x))$, respectively. □

We now consider the differentiability and the derivative formula of the function $\log J_\perp f_{E^u_0}(h_f(x))$ with respect to f. It suffices to show that, for y in a neighborhood of x, the functional $\log J_\perp f_{E^u_0}(y)$, calculated with respect to the metrics $\omega_\perp(x, \cdot)$ and $\omega_\perp(f_0(x), \cdot)$, is differentiable in f and its derivative is continuous in y and the functional $\log J_\perp f_0|_{E^u_0}(y)$ is differentiable in y at $y = x$. In both cases, we use the approximate smooth local metric $\omega_n(x, \cdot)$ to take advantage of the existence of simple formulas when we use a local smooth coordinate system to compute these derivatives. We then obtain the derivative formulas as we take limits when $n \to \infty$.

We denote by $\log J_n f_{E^u_0}(y)$ the same Jacobian, but calculated with the system of local smooth metrics $\omega_n(x, \cdot)$, $n = 1, 2, 3, \ldots$, for any y close to x. Since both the subspace E^u_0 and the metric ω_n are fixed, for each $n \geq 1$, $\log J_n f_{E^u_0}(y)$ is a continuously differentiable function in terms of $f \in U(f_0)$ and it is also continuously differentiable in y when y is close to x.

3.2. *Differentiability and derivative formula of* $\log J_\perp f_{E^u_0}(y)$ *with respect to* f. To determine the differentiability and the derivative formula of $\log J_\perp f_{E^u_0}(y)$ with respect to f at f_0, we take a fixed perturbation δf of f_0. Since we are working in small open neighborhoods of x and $f_0(x)$, we can identify them with open neighborhoods of the origin of $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ and E^u_0 with the subspace $\mathbb{R}^u$. For convenience, we denote these two neighborhoods by $O(x)$ and $O(f_0(x))$, assuming that $f_0(O(x)) \subset O(f_0(x))$. We consider the following sequence of functions of ϵ,

$$g_n(\epsilon) = \log J_n f^\epsilon_{E^u_0}(y),$$

where $f^\epsilon = f_0 + \epsilon\delta f$ for $\epsilon \in (-r, r)$, with $r > 0$ sufficiently small. The unstable Jacobian is computed with respect to the local smooth metrics $\omega_n(x, \cdot)$ and $\omega_n(f_0(x), \cdot)$. For each fixed $y \in O(x)$, since $\lim_{n\to\infty} \omega_n(x, \cdot) = \omega_\perp(x, \cdot)$, we have

$$\lim_{n\to\infty} g_n(\epsilon)(y) = \log J_\perp f^\epsilon_{E^u_0}(y).$$

We show that this sequence of functions is differentiable in $\epsilon \in (-r, r)$, their derivatives with respect to ϵ are bounded uniformly in n, and the sequence of derivatives converges to a function continuous in ϵ. Thus, we conclude by the dominated convergence theorem that the limit function $g_0(\epsilon) = \log J^u_\perp f^\epsilon_{E^u_0}(y)$ is differentiable in ϵ and its derivative is given by the limit of the derivatives of $g_n(\epsilon)$ with respect to ϵ. Since all estimations are uniform in terms of the direction δf, we have the differentiability in f.

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For each $n \geq 1$, the local smooth foliation $\mathcal{W}_n^s$ gives us a coordinate system. Denote the tangent space of leaves of $\mathcal{W}_n^s$ by E_n^s. Under the local smooth metric $\omega_n(x, \cdot)$, the subspaces $\mathbb{R}^u$ and E_n^s are orthogonal by definition of the metric.

Let X be the vector field defined by the composition $\delta f \circ (f^{\epsilon_0})^{-1}$. Let X_n^u and X_n^s denote the projections of X onto $\mathbb{R}^u = E_0^u$ and E_n^s, respectively. We have the following theorem.

THEOREM 2. *For each $y \in O(x)$,*

$$\frac{\partial}{\partial \epsilon} \log J_n f^{\epsilon}_{E_0^u}(y)\bigg|_{\epsilon_0} = \operatorname{div}_{\omega_n} X_n^u(f^{\epsilon_0}(y)).$$

Proof. We first carry out the calculation in the case when $\epsilon_0 = 0$. Note that $f^{\epsilon_0} = f_0$. Since our calculation is restricted to $O(x)$, all tangent spaces of the manifold in $O(x)$ are identified with $\mathbb{R}^n$ and the perturbation $\epsilon\delta f$ can be considered as a map from $O(x)$ to $\mathbb{R}^n$ with the norm $\epsilon\|\delta f\|_{C^r}$ small. For each $y \in O(x)$, we have $Df^{\epsilon}(y) = Df_0(y) + \epsilon D\delta f(y)$. Furthermore,

$$\det(Df_0(y) + \epsilon D\delta f(y))|_{E_0^u} = \det(I_{f_0(y)} + \epsilon D\delta f(y) \cdot Df_0^{-1}(f_0(y)))|_{E_0^u} \det Df_0|_{E_0^u}(y),$$

where $I_{f_0(y)}$ denotes the identity map of the tangent space $T_{f_0(y)}$. Let $X(y)$ denote the vector field $\delta f \circ f_0^{-1}(y)$. We have

$$D\delta f(y) \cdot Df_0^{-1}(f_0(y)) = DX(f_0(y)),$$

the differential of the vector field X evaluated at the point $f_0(y)$.

Since we are using the local metrics $\omega_n(x, \cdot)$, by Lemma 3, we shall next prove that

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(y))\bigg|_{E_0^u} = \operatorname{div}_{\omega_n} X^u(y),$$

where X^u is the projection of X onto $\mathbb{R}^u$.

Thus, we have

$$\frac{\partial}{\partial \epsilon} \log J_n f^{\epsilon}_{E_0^u}(y)\bigg|_{\epsilon=0} = \operatorname{div}_{\omega_n} X_n^u(f_0(y)).$$

In general, at any given point $\epsilon_0 \in (-r, r)$,

$$\begin{aligned} Df^{\epsilon}(y) &= Df_0(y) + \epsilon D\delta f(y) \\ &= Df_0(y) + \epsilon_0 D\delta f(y) + (\epsilon - \epsilon_0) D\delta f(y) \\ &= Df^{\epsilon_0} + (\epsilon - \epsilon_0) D\delta f(y). \end{aligned}$$

Thus, we have

$$\frac{\partial}{\partial \epsilon} \log J_n f^{\epsilon}_{E_0^u}(y)\bigg|_{\epsilon_0} = \operatorname{div}_{\omega_n} X_n^u(f^{\epsilon_0}(y)),$$

where X^u is projection of the vector field $X = \delta f \circ (f^{\epsilon_0})^{-1}$ onto $\mathbb{R}^u$. □

We now state and prove the lemma we need.

LEMMA 3. *Let X be a smooth vector field defined in an open neighborhood U of x on a Riemannian manifold. Assume that the smooth metric w on U satisfies the following conditions.*

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(1) *There exists a split of tangent space* $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ *such that the volume form* $dw = \rho(x)\, dx_1 \wedge \cdots \wedge dx_n$ *induced by the metric at the point* $x = (x^u, x^s) \in U$ *has the property that* $\rho(x^u, x^s) = \rho^u(x^u)$ *for some smooth function* ρ^u *defined on* $\mathbb{R}^u$.

(2) *The subspaces* $\mathbb{R}^u$ *and* $\mathbb{R}^s$ *are orthogonal under the metric* w.

Then,

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(x))\Big|_{\mathbb{R}^u} = \operatorname{div} X^u(x),$$

where the derivative is evaluated at $\epsilon = 0$, $X^u + X^s = X$ *is the projection of the vector field* X *onto* $\mathbb{R}^u$ *and* $\mathbb{R}^s$, *and the divergence is taken with respect to the induced volume form* $dw = \rho^u(x^u)\, dx_1 \wedge \cdots \wedge dx_u$ *on* $\mathbb{R}^u$.

Proof. Let $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ denote the matrix representation of the operator DX under the orthonormal bases provided by the subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$ at the corresponding points x and $x + \epsilon X(x)$. We have

$$\begin{aligned} \det(I + \epsilon DX(x))|_{\mathbb{R}^u} &= \sqrt{\det\left[(I + \epsilon A_{11} \;\; \epsilon A_{21})^T \begin{pmatrix} I + \epsilon A_{11} \\ A_{21} \end{pmatrix} \right]} \cdot \frac{\rho(x + \epsilon X(x))}{\rho(x)} \\ &= \sqrt{\det(I + 2\epsilon A_{11} + \epsilon^2 A_{11}^T A_{11} + \epsilon^2 A_{21}^T A_{21})} \cdot \frac{\rho(x + \epsilon X(x))}{\rho(x)}. \end{aligned}$$

This gives

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(x))\Big|_{\mathbb{R}^u} = \operatorname{trace}(A_{11}) + \frac{d}{d\epsilon} \log \rho(x + \epsilon X(x)).$$

We write $x + \epsilon X(x) = (x^u + \epsilon X^u(x), x^s + \epsilon X^s(x))$, where $X(x) = X^u(x) + X^s(x)$ is the decomposition over $\mathbb{R}^u \oplus \mathbb{R}^s$. Since $\rho(x + \epsilon X(x)) = \rho^u(x^u + \epsilon X^u(x))$, we have

$$\frac{d}{d\epsilon} \log \rho(x + \epsilon X(x)) = \langle \operatorname{grad} \log \rho^u(x^u), X^u(x) \rangle.$$

Thus,

$$\det(I + \epsilon DX(x))|_{\mathbb{R}^u} = \operatorname{trace}(A_{11}) + \langle \operatorname{grad} \log \rho^u(x^u), X^u(x) \rangle,$$

i.e., the divergence of the vector field X^u on $\mathbb{R}^u$ with respect to the induced volume form $\rho^u\, dx_1 \wedge \cdots \wedge dx_u$. □

To see that the derivatives $\operatorname{div}_{\omega_n} X_n^u(f^{\epsilon_0}(y))$ are uniformly bounded in n and the sequence's limit exists as $n \to \infty$, we use the fact that the local smooth foliations $\mathcal{W}_n^s$ converge to the stable foliation of f_0 uniformly in $y \in O(x)$. Note that the density function ρ^u on E_0^u from the metric $\omega_n(x, \cdot)$ does not depend on n. Following Lemma 3, we consider

$$\operatorname{div}_{\omega_n} X_n^u(y) = \operatorname{trace}(A_{11}(n)) + \langle \operatorname{grad} \log \rho^u(y_n^u), X_n^u(y) \rangle,$$

where $(A_{11}(n))$ is the first block in the matrix representation of DX under the coordinate system induced by the foliation $\mathcal{W}_n^s$ and y_n^u is the component in the E_0^u-direction in the coordinate system: $y = (y_n^u, y_n^s)$. Both the trace of $A_{11}(n)$ and $\langle \operatorname{grad} \log \rho^u(x_n^u), X_n^u(y) \rangle$ converge as $n \to \infty$ uniformly in y and, thus, are uniformly bounded in $\epsilon_0 \in (-r, r)$. The uniform convergence of foliations $\mathcal{W}_n^s$ to the stable foliations of f_0 implies that $X_n^u(y)$ converges to $X^u(y)$ uniformly in $O(x)$. Using the generalized divergence definition, we have that $\operatorname{div}_{\omega_n} X_n^u(y)$ converges weakly to $\operatorname{div}_{\omega_\perp} X^u(y)$. By the uniqueness of the weak

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limit, we must have

$$\lim_{n\to\infty} \mathrm{div}_{\omega_n} X_n^u(y) = \mathrm{div}_{\omega_\perp} X^u(y),$$

for any $y \in O(x)$. The (Hölder) continuity of $\mathrm{div}_{\omega_\perp} X^u(y)$ in y follows from Proposition 2. Thus, we have the first partial derivative.

COROLLARY 1. *For y in a neighborhood of x, the functional* $\log J_\perp f_{E_0^u}(y)$*, calculated with respect to the metric $\omega_\perp(x,\cdot)$ and $\omega_\perp(f_0(x),\cdot)$, is differentiable in f and the derivative at f_0 in the direction $X = \delta f \circ f_0$ is a Hölder continuous function in y:* $\mathrm{div}_{\omega_\perp} X^u(f_0(y))$*, where $X = X^u + X^s$ is the decomposition of X in the unstable and stable subspaces of Df_0.*

Remark 4. Even though the value of the derivative $(\partial/\partial\epsilon)\log J_n f^\epsilon_{E_0^u}(x)|_{\epsilon=0}$ does not depend on the choice of distributions complementary to E_0^u, its calculation is made to be dependent on the complementary subspaces $E_n^s(x)$ to represent its value in terms of a divergence of a vector field on the unstable submanifold.

3.3. *Calculating the second partial derivative* $(\partial/\partial y)\log J_\perp f_0|_{E_0^u}(y)|_{y=x}$. We again consider the function $\log J_\perp f_0|_{E_0^u}(y)$ in a small open neighborhood of x, $O(x)$. Since the metric in the unstable manifold is smooth, this function is differentiable at x in the unstable direction E_0^u. But its differentiability in other directions needs to be proven. Since both f_0 and E_0^u are now fixed, the unstable Jacobian $J_n f_0|_{E_0^u}(y)$ calculated with the smooth metric $\omega_n(x,\cdot)$ is differentiable in y for each $n \geq 1$. Moreover, the convergence of the local smooth metric $\omega_n(x,\cdot)$ to $\omega_\perp(x,\cdot)$ in $O(x)$ (as well as in $O(f_0(x))$) implies that

$$\lim_{n\to\infty} J_n f_0|_{E_0^u}(y) = J_\perp f_0|_{E_0^u}(y), \quad y \in O(x).$$

For any given point $y \in O(x)$, under the local coordinate system provided by the stable foliation and the local unstable manifold $W^u_{f_0}(x) \subset E_0^u$, we write $y = (y_0^u, y_0^s)$. Similarly, when we use the local smooth foliation $\mathcal{W}_n^s$, y is represented by a unique pair of coordinates (y_n^u, y_n^s), where y_n^u is the projection of y on E_0^u induced by the holonomy map of the foliation $\mathcal{W}_n^s$ and y_n^s is the coordinate of y on the leaf of $\mathcal{W}_n^s$ that passes through y.

LEMMA 4. *For any point $y \in O(x)$ and $\|y - x\| \leq \epsilon$, for sufficiently small ϵ,*

$$J_n f_0|_{E_0^u}(y) = J_n f_0|_{E_0^u}(y_n^u, 0) + O(\epsilon^2),$$

where $O(\epsilon^2)$ is bounded by $C\epsilon^2$ for some constant C uniformly in n.

Proof. To obtain an easy estimation of $J_n f_0|_{E_0^u}(y)$, the foliation $\mathcal{W}_n^s$ of the neighborhood at the point $f_0(x)$ can be taken to be the image of the foliation at x under the map f_0. This does not affect the convergence of these smooth foliations to the stable foliations at these points. The metric $\omega_n(f_0(x),\cdot)$ is also adjusted accordingly. In the coordinate system provided by the unstable subspace E_0^u and the local smooth foliations $\mathcal{W}_n^s$, the differential $Df_0(y)$ is represented by a block matrix

$$Df_0(y) = \begin{pmatrix} A_{uu} & 0 \\ A_{us} & A_{ss} \end{pmatrix},$$

where the submatrix A_{uu} depends only on the coordinate y_n^u since the leaves of $\mathcal{W}_n^s$ are invariant coordinate hypersurfaces. Note that E_0^u is invariant under $Df_0(y)$ when $y \in E_0^u$.

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We have $A_{us} = 0$ when $y \in E_0^u$ and, thus, when $y \in O(x)$ and $\|y - x\| \leq \epsilon$, the norm of A_{us} is on the order of ϵ.

By the orthogonality of the metric $\omega_n(x, \cdot)$ at both points y and $f_0(y)$, we have

$$|\det Df_0(y)| = \sqrt{\det[A_{uu}^T A_{uu} + A_{us}^T A_{us}]}.$$

Since the norm of $A_{us}^T A_{us}$ is on the order of ϵ^2, we have

$$J_n f_0|_{E_0^u}(y) = J_n f_0|_{E_0^u}(y_n^u, 0) + O(\epsilon^2).$$

To see that $O(\epsilon^2)$ is uniform in n, we estimate $A_{us} = A_{us}(y_n^u, y_n^s)$. The submatrix A_{us} represents the projection of images of unit vectors in E_0^u under the linear map Df_0 onto the tangent space to the leaves of the foliation $\mathcal{W}_n^s$. Since the tangent spaces are converging to the stable subspace at y, the limit of the block $A_{us} = A_{us}(y_n^u, y_n^s)$ exists as $n \to \infty$, which implies $O(\epsilon^2) \leq C\epsilon^2$ for some constant C independent of n. □

The lemma implies

$$J_\perp f_0|_{E_0^u}(y) = J_\perp f_0|_{E_0^u}(y_0^u, 0) + O(\epsilon^2)$$

as we take the limit when $n \to \infty$ and, thus,

$$\langle \text{grad}(\log J_\perp f_0|_{E_0^u}(y))|_{y=x}, X^s \rangle$$

exists and is equal to zero, where $X^s \in E_0^s(x)$, the stable subspace of Df_0 at x. We obtain the next theorem as a direct consequence of the lemma.

THEOREM 3. *The function* $\log J_\perp f_0|_{E_0^u}(y)$ *is differentiable at* x. *The derivative of* $\log J_\perp f_0|_{E_0^u}(y)$ *in the direction of* X *is equal to that in the direction of* X^u, *where* $X = X^u + X^s$ *is the decomposition of* X *in the unstable and stable directions.*

Remark 5. Theorem 3 contrasts the general non-differentiability of $\log J^u f_0(y)$ in the stable direction, when the unstable Jacobian is calculated along the unstable subspace of f_0 at y.

4. *Conclusion*

We recall that

$$\frac{\partial}{\partial \epsilon} h_{f^\epsilon}\bigg|_{\epsilon=0} = -\sum_{k=1}^{\infty} Df_0^{-k} X^u + \sum_{k=0}^{\infty} Df_0^k X^s,$$

where $X^u + X^s = X$, $X = \delta f \circ f_0^{-1}$, X^u and X^s are projections of $X = \delta f \circ f_0^{-1}$ onto the unstable and stable subspaces of Df_0. Thus, by Theorem 3,

$$\begin{aligned}
\delta(\log J_\perp f_0|_{E_0^u}(h_f(x)))|_{f_0} &= \left\langle \text{grad} \log J_\perp f_0|_{E_0^u}(y), -\sum_{k=1}^{\infty} Df_0^{-k} X^u + \sum_{k=0}^{\infty} Df_0^k X^s \right\rangle\bigg|_{y=x} \\
&= \left\langle \text{grad} \log J_\perp f_0|_{E_0^u}(y), -\sum_{k=1}^{\infty} Df_0^{-k} X^u \right\rangle\bigg|_{y=x} \\
&= \langle \text{grad}(p_{f_0} \circ {f_0}^{-k}), X^u \rangle,
\end{aligned}$$

where $p_{f_0}(y) = -\log J_\perp f_0|_{E_0^u}(y)$ is exactly the potential function of the SRB measure.

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Using Lemma 2 and Corollary 1, we have the complete derivative formula

$$\delta(\log J^u_\perp f(h_f(x)))|_{f_0} = \mathrm{div}_{\omega_\perp} X^u(f_0(x)) + \sum_{k=1}^{\infty} \langle \mathrm{grad}(p_{f_0} \circ {f_0}^{-k}),\, X^u \rangle.$$

Since

$$\mathrm{div}_\rho X^u = \mathrm{div}_{\omega_\perp} X^u + \sum_{k=1}^{\infty} \langle \mathrm{grad}(p_{f_0} \circ {f_0}^{-k}),\, X^u \rangle,$$

we have

$$\delta(\log J^u_\perp f(h_f(x)))|_{f_0} = \mathrm{div}_\rho X^u(x) + \mathrm{div}_{\omega_\perp} X^u(f_0(x)) - \mathrm{div}_{\omega_\perp} X^u(x).$$

If we choose the system of local metrics $\omega_\perp(x, \cdot)$ so that its density function on the unstable manifold $W^u_{f_0}(x)$ coincides with that induced by the SRB measure ρ, we have

$$\delta(\log J^u_\perp f(h_f(x)))|_{f_0} = \mathrm{div}_\rho X^u(f_0(x)),$$

as claimed in Theorem 1.

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A. *Appendix. Proof of Proposition 3*

Proof. Without loss of generality, we may assume $E^k_1 = E^k_2 = E^k$ and $E^s_1 = E^s_2 = E^s$, and $\langle\cdot\rangle_1 = \langle\cdot\rangle_2$. Let $(e_1, e_2, \ldots, e_k)$ be an orthonormal basis of E^k. Let $(v_1, v_2, \ldots, v_s)$ be an orthonormal basis of E^s. Since $\delta > 0$ is a small number and $E^k \oplus E^s = \mathbb{R}^n$, $(e_1, e_2, \ldots, e_k, v_1, v_2, \ldots, v_s)$ is a basis of $\mathbb{R}^n$. Under this basis, the matrix representation of the linear transformation T takes a block diagonal form since E^k and E^s are invariant under T:

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Using the Gram–Schmidt orthogonalization procedure, we can obtain an orthonormal basis of $\mathbb{R}^n$ $(e_1, e_2, \ldots, e_k, v'_1, v'_2, \ldots, v'_s)$, where

$$v'_1 = (1 - \beta_1)(v_1 + \alpha_{11} e_1 + \alpha_{12} e_2 + \cdots + \alpha_{1k} e_k),$$

$\alpha_{1i} = -\langle e_i, v_1\rangle$, $i = 1, 2, \ldots, k$, $\beta_1 = 1 - \sqrt{1 - \alpha^2_{11} - \alpha^2_{12} - \cdots - \alpha^2_{1k}}$. Note that

$$\langle v_2, v'_1\rangle = (1 - \beta_1)(\alpha_{11}\langle v_2, e_1\rangle + \cdots + \alpha_{1k}\langle v_2, e_k\rangle).$$

We have

$$v'_2 = (1 - \beta_2)(v_2 + \eta_1 v'_1 + \alpha_{21} e_1 + \alpha_{22} e_2 + \cdots + \alpha_{2k} e_k),$$

where $\alpha_{2i} = -\langle e_i, v_2\rangle$, $\eta_1 = -\langle v_2, v'_1\rangle$, and

$$\beta_2 = 1 - \sqrt{1 - \eta^2_1 - \alpha^2_{21} - \alpha^2_{22} - \cdots - \alpha^2_{2k}}.$$

Inductively, we conclude

$$(v'_1, v'_2, \ldots, v'_s) = (e_1, e_2, \ldots, e_k, v_1, v_2, \ldots, v_s)\begin{bmatrix} P \\ I + Q \end{bmatrix},$$

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where the absolute value of the entries of P is bounded by δ and the absolute value of the entries of Q is bounded by $C\delta^2$ for some constant C. Under this new orthonormal basis $(e_1, \ldots, e_k, v'_1, \ldots, v'_s)$, the transformation's matrix representation becomes

$$\begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix} = \begin{bmatrix} I & P^* \\ 0 & I+Q^* \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix} = \begin{bmatrix} A & AP+P^*B+P^*BQ \\ 0 & B+Q^*B+BQ+Q^*BQ \end{bmatrix},$$

where $Q^* = -\sum_{i=1}^{\infty} Q^n$ and $P^* = -P - PQ^*$. The matrices Q^* and P^* have entries of the order of δ^2 and δ, respectively. We denote $\tilde{P} = AP + P^*B + P^*BQ$ and $\tilde{Q} = Q^*B + BQ + Q^*BQ$. Clearly, $\det T|_{E^k} = \det(A)$. To calculate $\det T|_{E'^k}$, we take an orthonormal basis $(e'_1, \ldots, e'_k)$ of E'^k. Under the basis of $(e_1, \ldots, e_k, v'_1, \ldots, v'_s)$, the k-column vectors $(Te'_1, \ldots, Te'_k)$ form a matrix H of size $n \times k$. Then, $|\det T|_{E'^k}| = \sqrt{\det(H'H)}$. We now compute H.

Since E'^k is ϵ-close to E^k, we can take a basis of E'^k in the form of

$$(e_1, \ldots, e_k, v'_1, \ldots, v'_s) \begin{bmatrix} I \\ E \end{bmatrix},$$

with entries of E bounded by ϵ. We again apply the Gram–Schmidt orthogonalization procedure to this basis of E'^k to obtain an orthonormal basis $(e'_1, \ldots, e'_k)$. By the same argument, we have

$$(e'_1, \ldots, e'_k) = (e_1, \ldots, e_k, v'_1, \ldots, v'_s) \begin{bmatrix} I \\ E \end{bmatrix} (I+R),$$

where R is a matrix of size $k \times k$ whose entries are on the order of ϵ^2. Then,

$$H = \begin{bmatrix} A & \tilde{P} \\ 0 & B+\tilde{Q} \end{bmatrix} \begin{bmatrix} I \\ E \end{bmatrix} (I+R) = \begin{bmatrix} A + AR + \tilde{P}E(I+R) \\ (B+\tilde{Q})E(I+R) \end{bmatrix}.$$

Finally, we obtain the leading terms of $H'H$ up to the second order of ϵ, δ:

$$H'H = A'A + A'AR + A'PE + (AR)'A + (PE)'A + (BE)'BE.$$

Thus, we have

$$|\det T|_{E'^k}| = \sqrt{\det(H'H)} = \sqrt{\det(A'A)} + C(\delta\epsilon + \epsilon^2),$$

for some constant C. □

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