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Meromorphic non-integrability of a steady Stokes flow inside a sphere

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Abstract. The non-existence of a real meromorphic first integral for a spherically confined steady Stokes flow of Bajer and Moffatt is proved on the basis of Ziglin's theory and the differential Galois theory. In the proof, the differential Galois group of a second-order Fuchsian-type differential equation associated with normal variations along a particular streamline is shown to be a special linear group according to Kovacic's algorithm. A set of special values of a parameter contained in the Fuchsian-type equation is studied by using the theory of elliptic curves. For this set, a computer algebra system is used in part of Kovacic's algorithm.

1. *Introduction*

The velocity

$$\mathbf{u} : \mathbb{R}^3 \ni (x, y, z) \mapsto (\alpha z - 8xy,\ 11x^2 + 3y^2 + z^2 + \beta xz - 3,\ -\alpha x + 2yz - \beta xy)$$

(with real constants α and β) is a solution to the steady Stokes equations $\Delta\mathbf{u} = \nabla\Pi$ and $\nabla \cdot \mathbf{u} = 0$ with pressure $\Pi = 30y + \text{const}$. It was introduced by Bajer and Moffatt [**1**] as a steady flow which is confined inside the unit sphere S^2 (indeed, $(x, y, z) \cdot \mathbf{u} = 0$ if $x^2 + y^2 + z^2 = 1$) and has stretch–twist–fold structure. This structure causes the non-integrability (i.e. the non-existence of a first integral) of the streamline system of equations for $\mathbf{u}$:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha z - 8xy \\ 11x^2 + 3y^2 + z^2 + \beta xz - 3 \\ -\alpha x + 2yz - \beta xy \end{pmatrix}, \tag{1}$$

as numerically shown for various values of α and β in [**1**]. In particular, the nearly integrable case $0 < \alpha \ll 1$ with $\beta = 0$ was investigated in detail in [**10**].

The aim of this paper is to prove the non-existence of a real meromorphic first integral for (1) inside S^2 in the two cases $\alpha > 0$ with $\beta = 1$ and $\beta > 0$ with $\alpha = 1$. The proof

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is based on combination of Ziglin's theory [**11**] and the differential Galois theory. This combination was originally applied to Hamiltonian systems [**7**]. It was Maciejewski and Przybylska [**6**] who first applied it to a non-Hamiltonian hydrodynamical system (the streamline system of equations for the ABC flow). Unlike the result of [**10**], our result does not require that α or β is small, although it says nothing about the non-existence of a non-meromorphic first integral.

In §2, three main theorems are stated and a second-order Fuchsian-type differential equation associated with normal variations along a particular solution to (1) is derived. In each proof of the theorems, presented in §§3–5, the differential Galois group of that Fuchsian-type equation is shown to be SL(2, $\mathbb{C}$) according to Kovacic's algorithm [**5**]. A set of special values of α is studied by using the theory of elliptic curves in §4. For this set, we use the computer algebra system *Mathematica* to compute recurrence formulas in Kovacic's algorithm.

2. *Main theorems and a preliminary equation*

First, we define the three sets

$$\Lambda_1 = \left\{ \frac{24}{\sqrt{65}}, 4\sqrt{\frac{6}{5}}, 4\sqrt{\frac{21}{17}} \right\} \approx \{2.977, 4.382, 4.446\},$$

$$\Lambda_2 = \{2\sqrt{23}, 8\sqrt{5}, 16\sqrt{2}\} \approx \{9.592, 17.889, 22.627\},$$

$$\Xi = \{\alpha > 0 | \sqrt{25 - \alpha^2} + \alpha \in \mathbb{Q} \text{ and } \sqrt{25 - \alpha^2} - \alpha \in \mathbb{Q}\}.$$

The set Ξ has an infinite number of elements, as shown in §4. Needless to say, the relationship $\Xi \subset \mathbb{Q}$ follows from $25 - \alpha^2 \pm \alpha \in \mathbb{Q}$ for $\alpha \in \Xi$.

We can rigorously prove the following theorem.

THEOREM 1. *If $\alpha > 0$, $\alpha \notin \Lambda_1 \cup \Xi$ and $\beta = 1$, then (1) does not have a real meromorphic first integral inside S^2.*

Furthermore, with the aid of *Mathematica*, we obtain the following theorem.

THEOREM 2. *Theorem 1 is also valid for $\alpha \in \Xi$.*

Analogously to Theorem 1, we can rigorously prove the following theorem.

THEOREM 3. *If $\beta > 0$, $\beta \notin \Lambda_2$ and $\alpha = 1$, then (1) does not have a real meromorphic first integral inside S^2.*

Remark. Although it is unnatural to consider (1) outside S^2, Theorems 1–3 are proved valid outside S^2, too, in almost the same way.

As a preliminary to proving the above theorems, let us derive a second-order Fuchsian-type differential equation associated with normal variations along a particular solution to (1). Clearly, (1) has an invariant set $\ell = \{(0, y, 0) | -1 \le y \le 1\}$. It is easy to check that a non-equilibrium solution to (1) on ℓ is expressed as $(x, y, z) = (0, w, 0)$ with

$$w = \frac{1 - \tau}{1 + \tau}, \quad \tau = e^{6t}.$$

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In the following sections, we extend τ to $\mathbb{C}-\{-1, 0\}$. Then the non-equilibrium solution $(0, w, 0)$ is also extended to $\gamma = \ell_{\mathbb{C}} - \{(0, \pm 1, 0)\}$, where $\ell_{\mathbb{C}} = \{(0, y, 0) | y \in \mathbb{C}\}$. Setting $(x, y, z) = (\xi, w+\eta, \zeta)$ in (1) and neglecting quadratic terms of ξ, η and ζ, we obtain a variational system of equations

$$\frac{d}{d\tau}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = \frac{1}{6\tau}\begin{pmatrix}-8w & 0 & \alpha\\ 0 & 6w & 0\\ -\alpha-\beta w & 0 & 2w\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix},$$

in particular a normal variational system

$$\frac{d}{d\tau}\begin{pmatrix}\xi\\ \zeta\end{pmatrix} = \frac{1}{6\tau}\begin{pmatrix}-8w & \alpha\\ -\alpha-\beta w & 2w\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}. \tag{2}$$

Eliminating ζ leads to

$$\frac{d^2\xi}{d\tau^2} = \left(-\frac{2}{\tau}+\frac{2}{1+\tau}\right)\frac{d\xi}{d\tau} + \left[\frac{16-\alpha(\alpha+\beta)}{36\tau^2}+\frac{16+\alpha\beta}{18\tau}-\frac{8}{9(1+\tau)^2}-\frac{16+\alpha\beta}{18(1+\tau)}\right]\xi. \tag{3}$$

For $\chi = \tau(1+\tau)^{-1}\xi$, we obtain the desired equation

$$\frac{d^2\chi}{d\tau^2} = \left[\frac{16-\alpha(\alpha+\beta)}{36\tau^2}+\frac{\alpha\beta-20}{18\tau}+\frac{10}{9(1+\tau)^2}+\frac{20-\alpha\beta}{18(1+\tau)}\right]\chi. \tag{4}$$

3. *Proof of Theorem 1*

As pointed out in [**6**, Lemma 5] (see also [**12**]), the proposition of Ziglin [**11**, p. 183] about a Hamiltonian system is also applicable to a non-Hamiltonian dynamical system. In our case, we obtain the following proposition.

PROPOSITION 1. *Assume that there does not exist a non-constant rational function which is invariant under the action of the monodromy group of (2). Then (1) does not have a meromorphic first integral in an arbitrary domain U $(\subset\mathbb{C}^3)$ such that the fundamental group of γ is generated by loops lying in U.*

Moreover, from the fact that the monodromy groups of (2) and (3) are isomorphic [**8**, Proposition 2] and the relationships between the monodromy groups of (3) and (4) and the differential Galois group of (4) [**3**, Propositions 2.2 and 2.3 and Theorem 3.5], we obtain the following proposition.

PROPOSITION 2. *If the differential Galois group of (4) is* $\mathrm{SL}(2, \mathbb{C})$*, then the assumption of Proposition 1 is satisfied.*

Taking Propositions 1 and 2 into account, we can proceed as follows.

Proof of Theorem 1. According to [**5**, §§1.3 and 1.4], the differential Galois group G $(\subseteq \mathrm{SL}(2, \mathbb{C}))$ of (4) with $\beta = 1$, i.e.

$$\frac{d^2\chi}{d\tau^2} = R(\tau)\chi, \quad R(\tau) = \frac{16-\alpha^2-\alpha}{36\tau^2}+\frac{\alpha-20}{18\tau}+\frac{10}{9(1+\tau)^2}+\frac{20-\alpha}{18(1+\tau)}, \tag{5}$$

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is of one of the following types:

(I) it is conjugate to a subgroup of the group

$$\left\{ \begin{pmatrix} c & 0 \\ b & c^{-1} \end{pmatrix} \middle| \, b \in \mathbb{C}, \ c \in \mathbb{C} - \{0\} \right\};$$

(II) it is not of type (I) but conjugate to a subgroup of the group

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| \, c \in \mathbb{C} - \{0\} \right\} \cup \left\{ \begin{pmatrix} 0 & -c \\ c^{-1} & 0 \end{pmatrix} \middle| \, c \in \mathbb{C} - \{0\} \right\};$$

(III) it is neither of type (I) nor of type (II) but a finite group;

(IV) it is $\mathrm{SL}(2, \mathbb{C})$.

Whether G is of type (I) or not is examined by the first part of Kovacic's algorithm [**5**, § 3.1]. Using his notation, we define $\alpha_0^{\pm}$ and $\alpha_{-1}^{\pm}$ by

$$\alpha_0^{\pm} = \frac{1}{2} \pm \frac{\sqrt{25 - \alpha^2 - \alpha}}{6}, \quad \alpha_{-1}^{\pm} = \frac{1}{2} \pm \frac{7}{6}.$$

In addition, since $R(\tau)$ is expanded about ∞ as

$$R(\tau) = \frac{16 - \alpha^2 + \alpha}{36\tau^2} - \frac{\alpha + 20}{18\tau^3} + O(\tau^{-4}),$$

we set

$$\alpha_{\infty}^{\pm} = \frac{1}{2} \pm \frac{\sqrt{25 - \alpha^2 + \alpha}}{6}.$$

A necessary condition for G to be of type (I) is the existence of a non-negative integer d such that $d = \alpha_{\infty}^{\sigma(\infty)} - \alpha_0^{\sigma(0)} - \alpha_{-1}^{\sigma(-1)}$ holds, where $\sigma(\cdot)$ denotes $+$ or $-$. Clearly, such d does not exist when $25 - \alpha^2 - \alpha \leq 0$, i.e. $\alpha \geq \alpha_M$ with

$$\alpha_M = \frac{\sqrt{101} - 1}{2} \approx 4.525.$$

Let

$$f^{\pm}(\alpha) = \sqrt{25 - \alpha^2 + \alpha} \pm \sqrt{25 - \alpha^2 - \alpha}.$$

Then it is not hard to check that both $f^+(\alpha)$ and $f^-(\alpha)$ are monotone in $\alpha \in (0, \alpha_M)$. Therefore,

$$0 < f^-(\alpha) < \sqrt{2\alpha_M} < f^+(\alpha) < 10 \tag{6}$$

for $\alpha \in (0, \alpha_M)$. Taking this into account, we notice that, if d exists, then it is one of:

- $d = 2$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (+, -, -)$ and $f^+(\alpha) = 8$;
- $d = 1$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (+, +, -)$ and $f^-(\alpha) = 2$;
- $d = 0$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (-, +, -)$ and $f^+(\alpha) = 4$.

The solutions to $f^+(\alpha) = 8$, $f^-(\alpha) = 2$ and $f^+(\alpha) = 4$ in $(0, \alpha_M)$ are $24/\sqrt{65}$, $4\sqrt{6/5}$ and $4\sqrt{21/17}$, respectively, which are elements of Λ_1. This means that, for α of the theorem, d does not exist and G is not of type (I).

Whether G is of type (II) or not is examined by the second part of Kovacic's algorithm [**5**, § 4.1]. Still using his notation, we define

$$e_0 \in \left\{ 2, \, 2 \pm \frac{2\sqrt{25 - \alpha^2 - \alpha}}{3} \right\} \cap \mathbb{Z}, \quad e_{\infty} \in \left\{ 2, \, 2 \pm \frac{2\sqrt{25 - \alpha^2 + \alpha}}{3} \right\} \cap \mathbb{Z} \tag{7}$$

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and $e_{-1} = 2$. A necessary condition for G to be of type (II) is the existence of e_0 and e_∞ such that at least one of them is odd and $(e_\infty - e_0 - e_{-1})/2 = -1 + (e_\infty - e_0)/2$ equals a non-negative integer. Clearly, it is impossible that both e_0 and e_∞ are odd for $\alpha \notin \Xi$, and thus G is not of type (II).

Since a necessary condition for G to be of type (III) is that α belongs to Ξ [**5**, § 2.1, Theorem, Case 3], G is not of type (III) either. Therefore, $G = \mathrm{SL}(2, \mathbb{C})$ follows and the non-existence of a (complex) meromorphic first integral for (1) near γ in $\mathbb{C}^3$ is deduced from Propositions 1 and 2. The non-existence of a real meromorphic first integral for (1) is obtained in the same way as in [**6**, Proof of Theorem 4] and [**12**], namely by choosing an arbitrarily thin vicinity of ℓ in $\ell_{\mathbb{C}}$ as U. □

For $\alpha \in \Lambda_1$, G is of type (I) and Proposition 2 cannot be applied. As a result of Kovacic's algorithm, we obtain solutions to (5):

$$\chi = \left(\tau^2 + \frac{121 - 3\sqrt{65}}{28}\,\tau + \frac{7613 - 363\sqrt{65}}{7028}\right)\tau^{(-1+3/\sqrt{65})/6}(\tau+1)^{-2/3}$$

$$\text{for } \alpha = \frac{24}{\sqrt{65}},$$

$$\chi = (\tau + 11 + 2\sqrt{30})\tau^{(1+\sqrt{6/5})/3}(\tau+1)^{-2/3} \quad \text{for } \alpha = 4\sqrt{\frac{6}{5}},$$

$$\chi = \tau^{(5-\sqrt{21/17})/6}(\tau+1)^{-2/3} \quad \text{for } \alpha = 4\sqrt{\frac{21}{17}}.$$

4. *The set Ξ and proof of Theorem 2*

The first goal of this section is to prove that Ξ is an infinite set by using the theory of elliptic curves. The proof of Theorem 2, in which we make use of *Mathematica*, is given at the end of the section.

For $\alpha \in \Xi$, we set

$$\sqrt{25 - \alpha^2 + \alpha} + \sqrt{25 - \alpha^2 - \alpha} = \frac{q}{p}, \quad \sqrt{25 - \alpha^2 + \alpha} - \sqrt{25 - \alpha^2 - \alpha} = \frac{s}{r} \tag{8}$$

with positive integers p, q, r and s such that $\mathrm{GCD}(p, q) = 1$ and $\mathrm{GCD}(r, s) = 1$. Then

$$\alpha = \frac{qs}{2pr} \tag{9}$$

holds. From (6), we have

$$0 < \frac{s}{r} < \sqrt{2\alpha_M} < \frac{q}{p} < 10. \tag{10}$$

Intuitively, it seems very special for $\alpha \in (0, \alpha_M)$ to belong to Ξ. Indeed, as far as is known, even the simplest element of Ξ is $23256/6305$, which corresponds to $(p, q, r, s) = (97, 646, 65, 72)$ obtained from (21) and (24) below. In general, we can prove the following lemma.

LEMMA 1. *The following facts are true for p, q, r and s from (8):*

(i) $p \neq 1$ *and* $r \neq 1$*;*

(ii) $\mathrm{GCD}(p, r) = 1$ *and* $\mathrm{GCD}(qr \pm ps,\ pr) = 1$*;*

(iii) *p and r are odd, while q and s are even.*

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Proof. From (9) and $q^2/p^2 + s^2/r^2 = 100 - 4\alpha^2$, we obtain

$$\frac{(p^2+q^2)(r^2+s^2)}{p^2r^2} = 1 + \frac{q^2}{p^2} + \frac{s^2}{r^2} + \frac{q^2s^2}{p^2r^2} = 101,$$

that is

$$(p^2+q^2)(r^2+s^2) = 101\,p^2r^2. \tag{11}$$

The case $p = r = 1$ is impossible. Indeed, if $p = r = 1$, then $1 + q^2 = 1$ (with $1 + s^2 = 101$) or $1 + s^2 = 1$ (with $1 + q^2 = 101$) follows from (11). When $p \neq 1$ and $r = 1$, (11) with $\mathrm{GCD}(p, q) = 1$ yields $p^2 + q^2 = 101$ and $1 + s^2 = p^2$. The only solution to the former equality is $(p, q) = (10, 1)$, which conflicts with the latter equality as $s^2 = 99$. For the same reason, the case $p = 1$ and $r \neq 1$ is impossible, too. Thus (i) is proved.

Because of $\mathrm{GCD}(p, q) = \mathrm{GCD}(r, s) = 1$, one of the following two systems is true:

$$p^2 + q^2 = 101r^2, \quad r^2 + s^2 = p^2 \tag{12}$$

or

$$p^2 + q^2 = r^2, \quad r^2 + s^2 = 101\,p^2. \tag{13}$$

Clearly, any of these four equalities proves $\mathrm{GCD}(p, r) = 1$. In consequence, $\mathrm{GCD}(qr \pm ps,\ p) = \mathrm{GCD}(qr \pm ps,\ r) = 1$ holds, and thus (ii) is proved.

The systems (12) and (13) are rewritten as

$$q^2 + 101s^2 = 100p^2, \quad q^2 + s^2 = 100r^2 \tag{14}$$

and

$$101q^2 + s^2 = 100r^2, \quad q^2 + s^2 = 100p^2, \tag{15}$$

respectively. Clearly, q and s have the same parity. If q and s are assumed odd, then it follows that $q^2 + s^2 \equiv 2 \pmod 4$, which contradicts $100p^2 \equiv 100r^2 \equiv 0 \pmod 4$. Hence (iii) is proved. □

In connection with (14) and (15), it is useful to note the following lemma.

LEMMA 2. *The set*

$$\{(x_0, x_1, x_2, x_3) \in \mathbb{N}^4 \mid x_0^2 + 101x_1^2 = x_2^2,\ x_0^2 + x_1^2 = x_3^2,\ \mathrm{GCD}(x_0, x_1, x_2, x_3) = 1\} \tag{16}$$

is infinite.

Proof. Let $P^n(\mathbb{Q})$ be the n-dimensional projective space over $\mathbb{Q}$, whose elements are written as $[x_0, x_1, \ldots, x_n]$ $(=[ax_0, ax_1, \ldots, ax_n]$ for any $a \in \mathbb{Q} - \{0\})$. Then, according to [**9**, Theorem 2.7], the two sets

$$\{[x_0, x_1, x_2, x_3] \in P^3(\mathbb{Q}) \mid x_0^2 + 101x_1^2 = x_2^2,\ x_0^2 + x_1^2 = x_3^2\} \tag{17}$$

and

$$\{[\widetilde{X}, \widetilde{Y}, \widetilde{Z}] \in P^2(\mathbb{Q}) \mid 101\widetilde{X}^2\widetilde{Y} - \widetilde{X}\widetilde{Y}^2 + (\widetilde{X} - \widetilde{Y})\widetilde{Z}^2 = 0\} \tag{18}$$

are biregularly equivalent and their elements are mapped to each other as

$$[\widetilde{X}, \widetilde{Y}, \widetilde{Z}] = [(x_2 - x_0)/101,\ x_3 - x_0,\ x_1],$$
$$[x_0, x_1, x_2, x_3] = [\widetilde{Y}(\widetilde{Z}^2 - 101\widetilde{X}^2),\ 2\widetilde{X}\widetilde{Y}\widetilde{Z},\ \widetilde{Y}(\widetilde{Z}^2 + 101\widetilde{X}^2),\ \widetilde{X}(\widetilde{Z}^2 + \widetilde{Y}^2)]. \tag{19}$$

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Furthermore, setting $\widetilde{X} = (X - 33Z)/100$, $\widetilde{Y} = (X + 67Z)/100$ and $\widetilde{Z} = Y/100$, we see that (18) is biregularly equivalent to

$$\left\{ [X, Y, Z] \in P^2(\mathbb{Q}) \;\middle|\; \begin{array}{l} Y^2Z = (X - 33Z)(X - 34Z)(X + 67Z) \\ \qquad = X^3 - 3367XZ^2 + 75174Z^3 \end{array} \right\}. \tag{20}$$

An example of elements of (20) is $[-47, 360, 1]$. According to the theory of elliptic curves [**2**], (20) forms an additive group. It follows from the Nagell–Lutz theorem [**2**, Theorem 7.11] that, if $[X_0, Y_0, 1]$ ($Y_0 \in \mathbb{Z} - \{0\}$) belongs to a finite subgroup of that additive group, then Y_0^2 is a divisor of $D = -4(-3367)^3 - 27(75174)^2 = (2^2 \cdot 5^2 \cdot 101)^2$. Clearly, 360^2 is not a divisor of D, and thus $[-47, 360, 1]$ belongs to an infinite subgroup. In other words, (20) is infinite, and so is (17). Hence the lemma is proved. □

The element $[X, Y, Z] = [-47, 360, 1]$ of (20) corresponds to $[\widetilde{X}, \widetilde{Y}, \widetilde{Z}] = [-4, 1, 18]$ of (18), from which (19) yields $[x_0, x_1, x_2, x_3] = [323, 36, -485, 325]$ in (17), and then an element of (16):

$$(x_0, x_1, x_2, x_3) = (323, 36, 485, 325) = (17 \cdot 19, 2^2 \cdot 3^2, 5 \cdot 97, 5^2 \cdot 13). \tag{21}$$

Since $[323, 36, -485, 325]$ belongs to an infinite subgroup of the additive group which (17) forms, we can calculate as many elements of (16) as we like, by using the formula

$$\begin{aligned} [x_0, x_1, x_2, x_3] + [y_0, y_1, y_2, y_3] = [&x_0^2y_0^2 - 101x_1^2y_1^2, x_0x_1y_2y_3 + x_2x_3y_0y_1, \\ &x_0x_2y_0y_2 + 101x_1x_3y_1y_3, x_0x_3y_0y_3 + x_1x_2y_1y_2] \end{aligned} \tag{22}$$

[**9**, Definition 4.1]. Indeed, from

$$\begin{aligned} 2 \cdot [323, 36, -485, 325] &= [323, 36, -485, 325] + [323, 36, -485, 325] \\ &= [10714899025, -3665727000, 38366679025, 11324602225] \\ &= [428595961, -146629080, 1534667161, 452984089], \end{aligned} \tag{23}$$

the second element of (16) is obtained as

$$\begin{aligned} (x_0, x_1, x_2, x_3) &= (428595961, 146629080, 1534667161, 452984089) \\ &= (23 \cdot 47 \cdot 521 \cdot 761, 2^3 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 97, \\ &\qquad 1621 \cdot 946741, 29 \cdot 1889 \cdot 8269), \end{aligned}$$

and the nth element ($n = 3, 4, 5, \ldots$) is given by $n \cdot [323, 36, -485, 325]$.

Let us note relationships between x_0, x_1, x_2 and x_3.

LEMMA 3. *Every element* (x_0, x_1, x_2, x_3) *of (16) satisfies*

$$\mathrm{GCD}(x_j, x_k) = \begin{cases} 1 & \text{if } (j, k) \neq (2, 3), \\ 1 \text{ or } 5 & \text{if } (j, k) = (2, 3), \end{cases}$$

where $0 \leq j < k \leq 3$. *One of* x_0 *and* x_1 *is even.*

Proof. It is easy to check that a contradiction with $\mathrm{GCD}(x_0, x_1, x_2, x_3) = 1$ is deduced from the equalities of (16) when $\mathrm{GCD}(x_j, x_k) > 1$ is assumed for an arbitrary

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pair $(j, k) \neq (2, 3)$. From this, it follows that $x_0^2 + 101x_1^2 \equiv x_0^2 + x_1^2 \equiv 1$ or $2 \pmod 4$, while $x_2^2 \equiv x_3^2 \equiv 1$ or $0 \pmod 4$. Therefore, one of x_0 and x_1 is even. Since $x_2^2 - x_3^2 = (10x_1)^2$ and $101x_3^2 - x_2^2 = (10x_0)^2$, $\mathrm{GCD}(x_2, x_3)$ is an odd divisor of 10, i.e. 1 or 5. □

We are now ready to prove the following lemma.

LEMMA 4. *The set* Ξ *is infinite.*

Proof. We use (x_0, x_1, x_2, x_3) from (16) to determine (p, q, r, s) in (14) or (15), which gives $\alpha \in \Xi$ as (9). By setting $\widetilde{q} = q/2$ and $\widetilde{s} = s/2$ due to Lemma 1(iii), the systems (14) and (15) lead to

$$\widetilde{q}^{\,2} + 101\widetilde{s}^{\,2} = (5p)^2, \quad \widetilde{q}^{\,2} + \widetilde{s}^{\,2} = (5r)^2$$

and

$$101\widetilde{q}^{\,2} + \widetilde{s}^{\,2} = (5r)^2, \quad \widetilde{q}^{\,2} + \widetilde{s}^{\,2} = (5p)^2,$$

respectively. Comparing these with the equalities of (16) and taking Lemma 3 into account, we can determine (p, q, r, s) as

$$(p, q, r, s) = \begin{cases} (x_2/5,\ 2x_0,\ x_3/5,\ 2x_1) & \text{if } 5 \mid x_2,\ 5 \mid x_3 \text{ and } x_2 < \sqrt[4]{101}\, x_3, \\ (x_2,\ 10x_0,\ x_3,\ 10x_1) & \text{if } 5 \nmid x_2,\ 5 \nmid x_3 \text{ and } x_2 < \sqrt[4]{101}\, x_3, \\ (x_3/5,\ 2x_1,\ x_2/5,\ 2x_0) & \text{if } 5 \mid x_2,\ 5 \mid x_3 \text{ and } x_2 > \sqrt[4]{101}\, x_3, \\ (x_3,\ 10x_1,\ x_2,\ 10x_0) & \text{if } 5 \nmid x_2,\ 5 \nmid x_3 \text{ and } x_2 > \sqrt[4]{101}\, x_3, \end{cases} \tag{24}$$

for which $\mathrm{GCD}(p, q) = \mathrm{GCD}(r, s) = 1$ is satisfied. The conditions $x_2 \lessgtr \sqrt[4]{101}\, x_3$ come from (10). Indeed, in the first two of the four cases, we have

$$\frac{q^2}{p^2} = \frac{100x_0^2}{x_2^2} = \frac{101(x_3^2 - x_1^2) - (x_2^2 - 101x_1^2)}{x_2^2} = \frac{101x_3^2}{x_2^2} - 1 > 2\alpha_M,$$
$$\frac{s^2}{r^2} = \frac{100x_1^2}{x_3^2} = \frac{(x_2^2 - x_0^2) - (x_3^2 - x_0^2)}{x_3^2} = \frac{x_2^2}{x_3^2} - 1 < 2\alpha_M.$$

If one of x_2 and x_3 is divisible by 5 and the other is not, then (p, q, r, s) cannot be determined so that $\mathrm{GCD}(p, q) = \mathrm{GCD}(r, s) = 1$. However, such a case does not occur if we restrict (x_0, x_1, x_2, x_3) so that it is obtained from $n \cdot [323, 36, -485, 325]$ of (17) for all $n \in \mathbb{N}$. Indeed, taking (23) as a term on the left-hand side of (22), we inductively verify that each $2n \cdot [323, 36, -485, 325]$ yields (x_0, x_1, x_2, x_3) with only x_1 divisible by 5, while each $(2n + 1) \cdot [323, 36, -485, 325]$ yields that with x_2 and x_3 divisible by 5. Thus the lemma is proved. □

Finally, we prove Theorem 2.

Proof of Theorem 2. By the proof of Theorem 1, the differential Galois group $G = G_\Xi$ from (5) for $\alpha \in \Xi$ $(\subset \mathbb{Q})$ is not of type (I). A necessary condition for G_Ξ to be of type (II) is the existence of odd numbers e_0 and e_∞ of (7). This means that each of $(qr \pm ps)/(2pr) = (\widetilde{q}r \pm p\widetilde{s})/(pr)$ $(= \sqrt{25 - \alpha^2} \pm \alpha)$ is equal to $(3/2) \cdot (\text{odd number})$. However, this is impossible because of Lemma 1, and G_Ξ is not of type (II).

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Whether G_Ξ is of type (III) or not is judged by the third part of Kovacic's algorithm [**5**, § 5.1] with his $n = 12$. We define $\widetilde{e}_0$, $\widetilde{e}_{-1}$ and $\widetilde{e}_\infty$ by

$$\widetilde{e}_0 \in \left\{ 6 \pm \frac{k\sqrt{25-\alpha^2-\alpha}}{3} \;\middle|\; k = 0, 1, \ldots, 6 \right\} \cap \mathbb{Z},$$
$$\widetilde{e}_{-1} \in \left\{ 6 \pm \frac{7k}{3} \;\middle|\; k = 0, 1, \ldots, 6 \right\} \cap \mathbb{Z} = \{-8, -1, 6, 13, 20\},$$
$$\widetilde{e}_\infty \in \left\{ 6 \pm \frac{k\sqrt{25-\alpha^2+\alpha}}{3} \;\middle|\; k = 0, 1, \ldots, 6 \right\} \cap \mathbb{Z}.$$

Because of Lemma 1, neither of $(qr \pm ps)/(2pr) = (\widetilde{q}r \pm p\widetilde{s})/(pr)$ $(= \sqrt{25-\alpha^2} \pm \alpha)$ has the form $(3/k) \cdot (\text{integer})$ with $k \in \{1, 2, \ldots, 6\}$. Therefore, $\widetilde{e}_0 = \widetilde{e}_\infty = 6$ follows. Since $\widetilde{e}_\infty - \widetilde{e}_0 - \widetilde{e}_{-1} = -\widetilde{e}_{-1}$ has to equal a non-negative integer $\widetilde{d}$, we have $\widetilde{e}_{-1} = -8$ or -1. A necessary condition for G_Ξ to be of type (III) is the existence of a monic polynomial $P(\tau)$ of degree $\widetilde{d}$ such that, if the polynomials $P_{12}(\tau)$, $P_{11}(\tau)$, $\ldots$, $P_{-1}(\tau)$ are recursively defined by

$$\begin{aligned} P_{12} &= -P, \\ P_{j-1} &= -\tau(\tau+1)\frac{dP_j}{d\tau} + \left[(12-j)(2\tau+1) - 6(\tau+1) - \widetilde{e}_{-1}\tau\right] P_j \\ &\quad - (12-j)(j+1)\tau^2(\tau+1)^2 R(\tau) P_{j+1} \quad (j = 12, 11, \ldots, 0) \end{aligned} \tag{25}$$

with $R(\tau)$ from (5), then P_{-1} identically vanishes. By substituting $\sum_{k=0}^{\widetilde{d}} c_k \tau^k$ (with $c_{\widetilde{d}} = 1$) for $P(\tau)$ in (25) and setting the coefficients of $P_{-1}(\tau)$, which depend on α and c_k, equal to zero, we can judge whether $P(\tau)$ exists for $\alpha \in \Xi$ or not. However, hundreds of terms appear in $P_{-1}(\tau)$ and it is a formidable task to treat them by hand. For this reason, the author used *Mathematica* to obtain a rational function $\widetilde{P}_{-1}(\tau)$, which should identically vanish as $P_{-1}(\tau)$, by computing

$$\begin{aligned} \widetilde{P}_{12} &= -P, \\ \widetilde{P}_{j-1} &= -\frac{d\widetilde{P}_j}{d\tau} - \left(\frac{6}{\tau} + \frac{\widetilde{e}_{-1}}{\tau+1}\right) \widetilde{P}_j - (12-j)(j+1) R(\tau) \widetilde{P}_{j+1} \\ &\quad (j = 12, 11, \ldots, 0). \end{aligned} \tag{26}$$

The formulas (26) were introduced in [**4**] and are computed faster than (25). We can derive (26) from (25) by setting $\widetilde{P}_j = \tau^{j-12}(\tau+1)^{j-12} P_j$.

In the case $\widetilde{d} = -\widetilde{e}_{-1} = 8$, $\widetilde{P}_{-1}$ has the form

$$\widetilde{P}_{-1}(\tau) = \frac{1}{6561\tau^{12}(\tau+1)^{12}} \sum_{k=0}^{8} \sum_{l=0}^{18} c_k \, \phi_{k,l}(\alpha) \, \tau^l,$$

according to the result of the computation. Here each $\phi_{k,l}(\alpha)$ is zero or a polynomial of degree 12 or 11. The author ran *Mathematica* on the computer cluster at Yamaguchi University to compute $\sum_{l=0}^{18} \phi_{k,l}(\alpha) \, \tau^l$ for $k = 0, 1, \ldots, 8$ separately (and simultaneously) by replacing $P(\tau)$ by τ^k in (26). It took about 20–50 min to get

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the result for each k. In order for $(c_0, c_1, \ldots, c_7)$ to exist, the determinant of any 9×9 submatrix of the 9×19 matrix $(\phi_{k,l}(\alpha))$ should vanish. For example, when the $0, 1, \ldots,$ 8th columns are chosen, the vanishing determinant yields $\sum_{k=1}^{99} a_k \alpha^k = 0$ with $\mathrm{GCD}(a_{99}, a_{98}, \ldots, a_1) = 1$ and

$$a_{99} = 2^{46} \cdot 3^3 \cdot 5^{16} \cdot 2701393, \quad a_{98} = 2^{47} \cdot 3 \cdot 5^{17} \cdot 5297 \cdot 53161.$$

When the $10, 11, \ldots,$ 18th columns are chosen, $\sum_{k=1}^{99} a_k (-\alpha)^k = 0$ is obtained. Therefore, α should satisfy both

$$\sum_{k=0}^{49} a_{2k+1} \alpha^{2k} = 0 \tag{27}$$

and

$$\sum_{k=1}^{49} a_{2k} \alpha^{2k-2} = 0. \tag{28}$$

It is well known that every rational root of a $\mathbb{Z}$-coefficient monic polynomial is an integer. In addition, the substitution of $\widetilde{\alpha}/a_{99}$ and $\widetilde{\alpha}/a_{98}$ for α in (27) and (28), respectively, yields monic polynomials of $\widetilde{\alpha}$ which equal zero. Consequently, every α that satisfies both (27) and (28) is written (if it exists) as

$$\alpha = \frac{(\text{integer})}{a_{99}} = \frac{(\text{integer})}{a_{98}},$$

where the fractions may be reducible. From this, (9) and Lemma 1, we derive that one of p and r is 3 and the other is $5, 5^2, \ldots, 5^{15}$ or 5^{16}. It is easy to check that these p and r satisfy neither (12) nor (13). Hence, for $\alpha \in \Xi$, there does not exist $(c_0, c_1, \ldots, c_7)$, i.e. $P(\tau)$ of degree $\widetilde{d} = 8$.

In the case $\widetilde{d} = -\widetilde{e}_{-1} = 1$, we obtain

$$\widetilde{P}_{-1}(\tau) = \frac{1}{6561 \tau^{12} (\tau + 1)^{12}} \sum_{k=0}^{1} \sum_{l=0}^{11} c_k \, \psi_{k,l}(\alpha) \, \tau^l,$$

where each $\psi_{k,l}(\alpha)$ is a polynomial of degree 12. In order for c_0 to exist, the determinant of any 2×2 submatrix of the 2×12 matrix $(\psi_{k,l}(\alpha))$ should vanish. For example, when the zeroth and the first columns are chosen, $\sum_{k=0}^{22} b_k \alpha^k = 0$ follows, where

$$\mathrm{GCD}(b_{22}, b_{21}, \ldots, b_0) = 1, \quad b_{22} = 2^{14} \cdot 3 \cdot 5^4, \quad b_{21} = 2^{11} \cdot 5^4 \cdot 127.$$

When the 10th and the 11th columns are chosen, $\sum_{k=0}^{22} b_k (-\alpha)^k = 0$ is obtained. In the same way as in the case $\widetilde{d} = 8$, we deduce the non-existence of $P(\tau)$ of degree $\widetilde{d} = 1$ for $\alpha \in \Xi$.

As a result, G_Ξ is not of type (III) but $\mathrm{SL}(2, \mathbb{C})$. The rest of the proof is the same as the last part of the proof of Theorem 1. □

5. *Proof of Theorem 3*

It is convenient to note the following lemma.

LEMMA 5. *There does not exist a number β such that $\sqrt{24 + \beta} \in \mathbb{Q}$ and $\sqrt{24 - \beta} \in \mathbb{Q}$.*

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Proof. If $\sqrt{24+\beta} = \nu/\mu$ holds with $\mu \in \mathbb{N}$ and $\nu \in \mathbb{N}$ such that $\mathrm{GCD}(\mu, \nu) = 1$, then we have $\sqrt{24-\beta} = \sqrt{48\mu^2 - \nu^2}/\mu$. In addition, if $\sqrt{48\mu^2 - \nu^2}$ is equal to $\lambda \in \mathbb{N}$, then $\lambda^2 + \nu^2 = 48\mu^2$ $(=2^4 \cdot 3 \cdot \mu^2)$ is deduced. However, there do not exist such λ, μ and ν because a positive integer whose factorization contains 3 to an odd power cannot be expressed by the sum of two squares [**2**, Corollary 3.15]. Hence the lemma is proved. □

Theorem 3 is proved analogously to Theorem 1 as follows.

Proof of Theorem 3. Let us show that the differential Galois group H $(\subseteq \mathrm{SL}(2, \mathbb{C}))$ of (4) with $\alpha = 1$, i.e.

$$\frac{d^2\chi}{d\tau^2} = \left[\frac{15-\beta}{36\tau^2} + \frac{\beta-20}{18\tau} + \frac{10}{9(1+\tau)^2} + \frac{20-\beta}{18(1+\tau)}\right]\chi, \tag{29}$$

is $\mathrm{SL}(2, \mathbb{C})$. Using Kovacic's notation in [**5**, § 3.1], we set

$$\alpha_0^{\pm} = \frac{1}{2} \pm \frac{\sqrt{24-\beta}}{6}, \quad \alpha_{-1}^{\pm} = \frac{1}{2} \pm \frac{7}{6}, \quad \alpha_{\infty}^{\pm} = \frac{1}{2} \pm \frac{\sqrt{24+\beta}}{6}.$$

In order for H to be of type (I), there needs to exist a non-negative integer d which satisfies $d = \alpha_{\infty}^{\sigma(\infty)} - \alpha_0^{\sigma(0)} - \alpha_{-1}^{\sigma(-1)}$ $(\sigma(\cdot) = +$ or $-)$. Such d does not exist except in the following three cases:

- $d = 2$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (+, -, -)$ and $g^+(\beta) = 8$;
- $d = 1$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (+, +, -)$ and $g^-(\beta) = 2$;
- $d = 0$ for $(\sigma(\infty), \sigma(0), \sigma(-1)) = (-, -, -)$ and $g^-(\beta) = 4$;

with $g^{\pm}(\beta) = \sqrt{24+\beta} \pm \sqrt{24-\beta}$, which satisfy $0 < g^-(\beta) < 4\sqrt{3} < g^+(\beta) < 4\sqrt{6}$ for $\beta \in (0, 24)$. Since the solutions to $g^+(\beta) = 8$, $g^-(\beta) = 2$ and $g^-(\beta) = 4$ in $(0, 24)$ are $16\sqrt{2}$, $2\sqrt{23}$ and $8\sqrt{5} \in \Lambda_2$, respectively, H is not of type (I). Next, using Kovacic's notation in [**5**, § 4.1], we define

$$e_0 \in \left\{2, 2 \pm \frac{2\sqrt{24-\beta}}{3}\right\} \cap \mathbb{Z}, \quad e_{-1} = 2, \quad e_{\infty} \in \left\{2, 2 \pm \frac{2\sqrt{24+\beta}}{3}\right\} \cap \mathbb{Z}.$$

In order for H to be of type (II), both e_0 and e_{∞} should be odd. However, this is impossible because of Lemma 5, and thus H is not of type (II). Lemma 5 also implies that H is not of type (III) (see [**5**, § 2.1, Theorem, Case 3]). Therefore, $H = \mathrm{SL}(2, \mathbb{C})$ follows. The rest of the proof is the same as the last part of the proof of Theorem 1. □

For $\beta \in \Lambda_2$, H is of type (I) and Proposition 2 cannot be used. As a result of Kovacic's algorithm, we obtain solutions to (29):

$$\chi = \left(\tau^2 + \frac{4(3+\sqrt{2})}{7}\tau + \frac{11+6\sqrt{2}}{7}\right)\tau^{(-1+2\sqrt{2})/6}(\tau+1)^{-2/3} \quad \text{for } \beta = 16\sqrt{2},$$

$$\chi = \left(\tau + \frac{27+4\sqrt{23}}{19}\right)\tau^{(2+\sqrt{23})/6}(\tau+1)^{-2/3} \quad \text{for } \beta = 2\sqrt{23},$$

$$\chi = \tau^{(5-2\sqrt{5})/6}(\tau+1)^{-2/3} \quad \text{for } \beta = 8\sqrt{5}.$$

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