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Actions of Lie groups on closed manifolds

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Abstract. We associate to each non-singular action A of a connected Lie group G on a closed manifold M and each A-invariant probability measure μ on M, a linear mapping P_A from the cohomology $H^*(\mathcal{F})$ of the foliation $\mathcal{F}$ given by the orbits of A onto the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. This extends to actions the asymptotic cycles of S. Schwartzman and D. Sullivan and gives a strong connection between the ergodic properties of A and the characteristic mapping χ of $\mathcal{F}$ as defined by dos Santos (*Contemp. Math.* **161** (1994), 41–57). The ergodic notion of E-action introduced by Arraut and dos Santos (*Topology* **31** (1992), 545–555) extends naturally to actions of Lie groups and we prove the main result of this article which shows that a non-singular E-action A of a nilpotent Lie group G completely determines the characteristic mapping χ of the foliation $\mathcal{F}$ given by A. This shows the strength of the above connection and gives as a corollary the vanishing theorem for foliated bundles proved by dos Santos.

0. *Introduction*

We study locally free (l.f.) smooth actions A of a Lie group G of dimension p on a closed, connected and orientable smooth manifold M of dimension m. The orbits of A give a foliation $\mathcal{F}$ of dimension p of M. The cohomology $H^*(\mathcal{F})$ of $\mathcal{F}$ plays an important role in this study. If $I(\mathcal{F})$ is the annihilating ideal of $\mathcal{F}$ and $q = m - p$ is the codimension of $\mathcal{F}$ we denote by $H^*(M, \mathcal{F})$ the cohomology of the differential complex $(I(\mathcal{F})^q, d)$ and by $H^*(M)$ the de Rham cohomology of M. In [**11**] we considered the mapping $\chi : H^*(\mathcal{F}) \to \mathrm{Hom}(H^*(M, \mathcal{F}), H^*(M))$ which was called the *characteristic mapping of the foliated manifold* $(M, \mathcal{F})$. This is a natural generalization of Duminy's Godbillon operator [**5**] and the Heitsch–Hurder Weil operator [**8**]. The action A gives a representation i_A of the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G into $H^*(\mathcal{F})$ and we have the characteristic mapping $\chi_A = \chi \circ i_A$ of A giving non-trivial invariants for the study of l.f. actions of Lie groups. If G has polynomial growth then all the residual secondary classes of $\mathcal{F}$ vanish [**9**], whereas χ_A will in general not vanish. All the above facts are sketched in §2. In §1, to each l.f. action A of G on M and each A-invariant probability measure μ on

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M we associate a linear mapping P_A from the cohomology $H^*(\mathcal{F})$ onto the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. This extends to l.f. actions the asymptotic cycles of Schwartzman [**12**] and Sullivan [**13**] and gives a strong connection between the ergodic properties of A and the characteristic mapping χ of the foliated manifold $(M, \mathcal{F})$. The aim of this article is to prove a factorization theorem that shows the strength of the above connection.

In §3, the ergodic notion of E-actions introduced in [**4**] is naturally extended to l.f. actions of Lie groups and we prove Theorem 3.6 which says that if A is an E-action of a nilpotent Lie group G on a closed, connected and orientable manifold M then the characteristic mapping χ of the foliation $\mathcal{F}$ given by A factors through the characteristic mapping χ_A of A, i.e. $\chi = \chi_A \circ P_A$. *Thus a l.f. E-action of a nilpotent Lie group completely determines the characteristic mapping of the foliation given by it.* As a corollary we obtain the vanishing Theorem 2.11 proved in [**11**] for foliated bundles which is stated at the end of §3 of this article.

The cohomology $H^*(M, \mathcal{F})$ has the distinguished subspace $D^*(M, \mathcal{F})$ generated by $I(\mathcal{F})^q \cap B(M)$, where $B(M)$ is the subspace of the exact differential forms on M and we introduced in [**11**] the derived characteristic mapping

$$\delta : H^*(\mathcal{F}) \to \mathrm{Hom}(D^*(M, \mathcal{F}), H^*(M)).$$

In Example 2.1 we show that the Arnold invariant $\mathcal{A}(Y)$ [**1**] of a non-divergent vector field Y on a 3-manifold M^3 is one of the invariants given by the characteristic mapping δ of the one-dimensional foliation $\mathcal{F}$ given by Y. We show that δ gives more information on Y than does Arnold's invariant. In Example 2.2 we relate Arnold's invariant to the symmetric bilinear form $\alpha_A : R^2 \times R^2$ associated to a l.f. action of R^2 on M^3 in [**2**].

We think to clarify the meaning of the characteristic mapping δ of the foliation given by a non-singular vector field is important.

This article also surveys a theory of characteristic classes for l.f. actions of Lie groups which has been developed in [**2, 4, 11**] and the present paper.

1. *Cohomology and actions of Lie groups*

Let $\mathcal{F}$ be a p-dimensional foliation of a closed, orientable, connected m-manifold M and let $\Lambda(M)$ be the graded algebra of all C^∞ differential forms on M. If $I(\mathcal{F})$ is the annihilating ideal of $\mathcal{F}$, then $I(\mathcal{F})^{q+1} = 0$ where $q = m - p$ is the codimension of $\mathcal{F}$ and $\Lambda(\mathcal{F}) = \Lambda(M)/I(\mathcal{F})$ is the graded algebra of *differential forms tangent* to $\mathcal{F}$. Since by Frobenius' Theorem $dI(\mathcal{F}) \subset I(\mathcal{F})$, the differential $d : \Lambda(M) \to \Lambda(M)$ induces the *tangent differential* $d_f : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F})$. We have a short exact sequence of differential complexes

$$0 \to I(\mathcal{F}) \to \Lambda(M) \xrightarrow{\pi} \Lambda(\mathcal{F}) \to 0 \tag{1.1}$$

which gives a long exact cohomology sequence

$$\cdots \to H^j(I(\mathcal{F})) \to H^j(M) \to H^j(\mathcal{F}) \xrightarrow{\delta^*} H^{j+1}(I(\mathcal{F})) \to \cdots$$

where $\pi(\xi) = \xi^f$ is the projection and $\delta^*[\xi^f] = [d\xi]$.

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Let $A : G \times M \to M$ be a (left) l.f. action of the Lie group G on M and let $\mathcal{F}$ be the foliation given by the orbits of A. Associated to A we have a non-singular homomorphism $\mathcal{G} \to \chi(\mathcal{F})$ of the Lie algebra $\mathcal{G}$ of G into the Lie algebra $\chi(\mathcal{F})$ of all smooth vector fields tangent to $\mathcal{F}$. This homomorphism is given by $\overline{E}(x) = DA_x(e) \cdot E$, $x \in M$, where $A_x : G \to M$, $A_x(g) = A(g, x)$. *Non-singular* means: $\overline{E}(x) = 0$ for some $x \in M$ implies $E = 0$. Dually, we have a natural homomorphism i_A of the exterior algebra $\Lambda_R(\mathcal{G}^*)$ of the left invariant forms on G into the exterior algebra $\Lambda(\mathcal{F})$ of smooth forms tangent to $\mathcal{F}$. This homomorphism commutes with the differentials, i.e. $i_A(dw) = d_f i_A(w)$ and it is non-singular [**4**].

We now introduce the dual description of l.f. actions in terms of differential forms, leading to the cohomology.

PROPOSITION 1.1. *Let M be a closed, orientable, connected m-manifold and $\mathcal{F}$ be a p-dimensional foliation of M. Let G be a Lie group of dimension p. The following data are equivalent:*

(i) *a locally free action A of G* tangent *to $\mathcal{F}$, i.e. the orbits of A are the leaves of $\mathcal{F}$;*

(ii) *a non-singular homomorphism $\mathcal{G} \to \chi(\mathcal{F})$ of the Lie algebra $\mathcal{G}$ of G into the Lie algebra $\chi(\mathcal{F})$ of the smooth vector fields tangent to $\mathcal{F}$;*

(iii) *a non-singular homomorphism $\Lambda_R(\mathcal{G}^*) \to \Lambda(\mathcal{F})$, $w \to w'$ which commutes with the differentials.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are proved in [**4**, Example 1.5]. (ii) $\Rightarrow$ (i) can be seen in [**7**]. To prove (iii) $\Rightarrow$ (ii), choose a basis $\{E_1, \ldots, E_p\}$ of $\mathcal{G}$ and let $\{w_1, \ldots, w_p\}$ be the dual basis in $\Lambda^1_R(\mathcal{G}^*)$. Thus $\{w'_1, \ldots, w'_p\}$ is a coframe of $T\mathcal{F}$. Let $X = \{X_1, \ldots, X_p\}$ be the frame of $\mathcal{F}$ given by $w'_i(X_j) = \delta_{ij}$. Since the homomorphism $w \mapsto w'$ commutes with the differentials, then the mapping $E_j \mapsto X_j$ extends to a non-singular homomorphism of Lie algebras $\mathcal{G} \to \chi(\mathcal{F})$. □

Remark 1.2. Since the injective homomorphism $\Lambda_R(\mathcal{G}^*) \to \Lambda(\mathcal{F})$ associated with a l.f. action A of G tangent to $\mathcal{F}$ commutes with the differentials, then it induces a representation

$$i_A : H^*(\mathcal{G}) \to H^*(\mathcal{F}) \tag{1.2}$$

of the cohomology of the Lie algebra $\mathcal{G}$ of G into the cohomology of the foliated manifold $(M, \mathcal{F})$.

We now discuss the homology directions associated with l.f. actions, extending the notion of asymptotic cycles of Schwartzman [**12**] and Sullivan [**13**]. This gives a strong connection between the topological properties and the ergodic properties of l.f. actions of Lie groups.

Let $A : G \times M \to M$ be a l.f. action, as before, and let μ be a probability measure on M which is invariant under A. Such a measure always exists in the case when G has polynomial growth [**10**]. One has a surjective linear mapping

$$\widetilde{P}_A : \Lambda^k(M) \to \Lambda^k_R(\mathcal{G}^*), \quad 0 \leq k \leq p \tag{1.3}$$

given by

$$\widetilde{P}_A w(E_1, \ldots, E_k) = \int_M w(\overline{E}_1, \ldots, \overline{E}_k)\, d\mu$$

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where $E \mapsto \overline{E}$ is the homomorphism $G \to \chi(\mathcal{F})$ given by Proposition 1.1. We observe that $\widetilde{P}_A$ does not preserve the exterior product.

PROPOSITION 1.3. *Let $A : G \times M \to M$ be a (left) l.f. action of a Lie group G into a closed, orientable, connected m-manifold and let μ be an A-invariant probability measure on M. Let $\mathcal{F}$ be the foliation given by A. Then:*

(i) *the operator $\widetilde{P}_A : \Lambda(M) \to \Lambda_R(\mathcal{G}^*)$ commutes with the differentials;*

(ii) $I(\mathcal{F}) \subset \ker \widetilde{P}_A$.

Proof. (i) Recall the standard formula

$$dw(\overline{E}_0, \ldots, \overline{E}_k) = \sum_{i=0}^{k} (-1)^i \overline{E}_i w(\overline{E}_0, \ldots, \widehat{E}_i, \ldots, \overline{E}_k) + \sum_{i<j} (-1)^{i+j} w([\overline{E}_i, \overline{E}_j].\overline{E}_0, \ldots, \widehat{E}_j, \ldots, \overline{E}_k) \tag{1.4}$$

where $w \in \Lambda^k(M)$ and $E_i \in \mathcal{G}$, $1 \le i \le p$. We claim that

$$\mu(\overline{E}h) = \int_M \overline{E}h \, d\mu = 0 \tag{1.5}$$

for every smooth real function h on M and every E in $\mathcal{G}$. This holds since

$$\overline{E}h = \lim_{t \to 0} \frac{h \circ \varphi_t - h}{t} \tag{1.6}$$

where $\varphi_t(x) = A(\exp tE, x)$ is the flow of $\overline{E}$. By Lebesgue's dominated convergence theorem and the A-invariance of μ, one gets $\mu(\overline{E}h) = 0$, proving our claim. Now since $E \to \overline{E}$ is a homomorphism of the Lie algebra $\mathcal{G}$ into the Lie algebra $\chi(\mathcal{F})$, from (1.4) and (1.5) one sees that $\widetilde{P}_A$ commutes with the differentials. One sees immediately from the definition of $I(\mathcal{F})$ that $I(\mathcal{F}) \subset \ker \widetilde{P}_A$, proving (ii). □

It follows from the above proposition that we have a short exact sequence of differential complexes

$$0 \to \widetilde{K} \to \Lambda(M) \xrightarrow{\widetilde{P}_A} \Lambda_R(\mathcal{G}^*) \to 0$$

which gives a long exact cohomology sequence

$$\cdots H^j(\widetilde{K}) \to H^j(M) \to H^j(\mathcal{G}) \xrightarrow{\delta^*} H^{j+1}(\widetilde{K}) \to \cdots . \tag{1.7}$$

Since $I(\mathcal{F}) \subset \ker \widetilde{P}_A$ then we also have a split short exact sequence of differential complexes

$$0 \to K \to \Lambda(\mathcal{F}) \underset{i_A}{\overset{P_A}{\rightleftarrows}} \Lambda_R(\mathcal{G}^*) \to 0 \tag{1.8}$$

which gives the split short exact cohomology sequences

$$0 \to H^j(K) \to H^j(\mathcal{F}) \to H^j(\mathcal{G}) \to 0, \quad 0 \le j \le p.$$

We have proved the following.

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PROPOSITION 1.4. *With the above assumptions we have*

$$H^j(\mathcal{F}) = H^j(\mathcal{G}) \oplus H^j(K), \quad 0 \leq j \leq p.$$

We notice that if A has an invariant probability measure μ then the linear mapping $i_A : H^*(\mathcal{G}) \to H^*(\mathcal{F})$ is injective since $P_A \circ i_A$ is the identity mapping.

2. *Characteristic classes of actions of Lie groups*

Let $\mathcal{F}$ be a p-dimensional foliation of a closed, orientable, connected m-manifold and $\Lambda(M, \mathcal{F}) = I(\mathcal{F})^q$, where, as before, $I(\mathcal{F})$ denotes the annihilating ideal of $\mathcal{F}$ and $q = m - p$ is the codimension of $\mathcal{F}$. The cohomology of the differential complex $(\Lambda(M, \mathcal{F}), d)$ is denoted by $H^*(M, \mathcal{F})$. Associated to the foliated manifold $(M, \mathcal{F})$, in [**11**] we introduced the characteristic mappings

$$\chi_{jk} : H^j(\mathcal{F}) \to \mathrm{Hom}(H^k(M, \mathcal{F}), H^{j+k}(M)) \tag{2.1}$$

for $0 \leq j \leq p$ and $q \leq k \leq m - j$. To each *tangent cohomology* class $\overline{\xi}$ in $H^j(\mathcal{F})$ and each *normal cohomology* class $\langle\theta\rangle$ in $H^k(M, \mathcal{F})$ there corresponds the de Rham cohomology class $[\xi \wedge \theta]$. These mappings are natural generalization of Duminy's Godbillon operator [**5**] and the Heitsch–Hurder Weil operator [**9**]. They also give non-trivial invariants for non-singular actions of Lie groups.

For each integer k, $q \leq k \leq m$, we have the subspaces

$$D^k(M, \mathcal{F}) = [\langle\theta\rangle; \theta \in \Lambda(M, \mathcal{F}) \cap B^k(M)] \tag{2.2}$$

of $H^k(M, \mathcal{F})$ where, as usual, $B^k(M)$ is the space of exact k-forms on M. We also introduced in [**11**] the *derived characteristic mappings*

$$\delta_{jk} : H^j(\mathcal{F}) \to \mathrm{Hom}(D^k(M, \mathcal{F}), H^{j+k}(M)) \tag{2.3}$$

for $0 \leq j \leq p$ and $q \leq k \leq m - j$.

We now relate some known invariants to the characteristic classes that arise from the tangent cohomology.

Example 2.1. (Arnold's invariant) Let v be a smooth volume form on a compact, orientable, connected 3-manifold M. Let X be a smooth divergence-free vector field on M. X is *homologous to zero* if the closed 2-form $i_X v$ is exact, i.e. $i_X v = d\alpha$ for some smooth 1-form α on M. Arnold introduces in [**1**] the invariant

$$\mathcal{A}(X) = \int_M \alpha \wedge d\alpha. \tag{2.4}$$

We notice that if X is non-singular then it generates a l.f. action $A : R \times M \to M$. Let $\mathcal{F}$ be the one-dimensional foliation given by X. We claim that $i_X v$ belongs to $I(\mathcal{F})^2$. In fact, extend X to a frame $\{X, Y, Z\}$ of M and let $\{\xi, \eta, \zeta\}$ be the dual coframe. We may assume that $v = \xi \wedge \eta \wedge \zeta$. Thus $i_X v = \eta \wedge \zeta$ and $\eta, \zeta \in I(\mathcal{F})$, proving our claim. Therefore $i_X v = d\alpha$ gives a *normal* cohomology class in $D^2(M, \mathcal{F})$.

Since α is d_f-closed, i.e. $d\alpha \in I(\mathcal{F})$, it gives a *tangent* cohomology class $\bar{\alpha} \in H^1(\mathcal{F})$ and we have

$$\mathcal{A}(X) = \delta(\bar{\alpha})\langle d\alpha\rangle. \tag{2.5}$$

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Example 2.2. Let A be a l.f. action of R^2 into a closed, connected, orientable smooth 3-manifold. In [**2**] we associate to A a symmetric bilinear form

$$\alpha_A : R^2 \times R^2 \to H^3(M) \simeq R$$

which is defined as follows. Let $\{X_1, X_2\}$ be the generators of A. Choose a Riemannian metric *adapted* to A, i.e. $\langle X_i, X_j \rangle = \delta_{ij}$, and let $\xi_j = i_{X_j}\langle \cdot, \cdot \rangle$. Now

$$\alpha_A(e_i, e_j) = \int_M \xi_i \wedge d\xi_j \tag{2.6}$$

and α_A does not depend on the choice of the adapted metric and $d\xi_j \in I(\mathcal{F})$. Thus $\alpha_A(e_i, e_j) = \delta_A(\overline{\xi}_i)\langle d\xi_j \rangle$. We proved in [**2**] that there exists a reparametrization $\widetilde{A}$ of A, i.e. $\widetilde{A}(u, x) = A(Lu, x)$, $L \in O_+(2, R)$, so that $\alpha_{\widetilde{A}}(e_1, e_2) = 0$. So, we may assume that, up to a reparametrization, α_A is diagonal. Now let v be the volume form corresponding to the adapted metric chosen. Then

$$\alpha_A(e_j, e_j) = \int_M \xi_j \wedge d\xi_j = \mathcal{A}(\operatorname{curl} X_j), \quad 1 \le j \le 2 \tag{2.7}$$

because $d\xi_j = i_{\operatorname{curl} X_j} v$. Thus the invariants α_A and $\mathcal{A}$ are related by (2.7) and we also have

$$\alpha_A(e_i, e_j) = \int_M \langle X_i, \operatorname{curl} X_j \rangle \, dv \tag{2.8}$$

since $\xi_i \wedge d\xi_j = \langle X_i, \operatorname{curl} X_j \rangle v$.

Remark 2.3. For the horocycle Y given in [**4**, Example 2.7] we have $\mathcal{A}(Y) = 0$, but $\delta_Y \neq 0$. This shows that the characteristic mapping δ_Y of Y gives more information on Y than does $\mathcal{A}(Y)$. It is important to clarify the meaning of the characteristic mapping δ for the foliation given by a non-singular vector field.

3. *Ergodic properties of actions and characteristic classes*

In this section we study the relation between the topological properties of l.f. actions and their ergodic properties.

Let G be a connected and simply connected Lie group. We denote by $|B|$ the (left) Haar measure of a Borel set $B \subset G$. A *summing sequence* S in G is a sequence of compact sets $S_n \subset G$ of positive Haar measure $|S_n|$ so that:

(i) $S_n \subset S_{n+1}$ for all positive integers n;

(ii) $G = \bigcup_{n=1}^{\infty} S_n$;

(iii) S satisfies the *Folner condition*

$$\lim_{n\to\infty} \frac{|gS_n \Delta S_n|}{|S_n|} = 0$$

for all $g \in G$, Δ being the symmetric difference of sets.

S is *regular* [**6**] if

$$|S_n^{-1} S_n| \le K|S_n| \tag{3.1}$$

for some constant $K > 0$ and all positive integers n.

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Let $A : G \times M \to M$ be a smooth l.f. action and let f be a continuous real function on M. We consider the *ergodic averages*

$$A_n f = \frac{1}{|S_n|} \int_{S_n} (f \circ A_g)\, dg. \tag{3.2}$$

We now restrict ourselves to l.f. actions of connected and simply connected *nilpotent* Lie groups. Such groups have regular summing sequences and non-trivial centers [**10**]. We considered an important class of actions in [**11**].

Definition 3.1. A (left) l.f. action A of G into M is an *E-action* if A leaves invariant a probability measure μ on M and there exists a regular summing sequence S of the center C of G so that for every continuous function $f : M \to R$ the ergodic averages $A_n f$ converge pointwise to $\mu(f) = \int_M f d\mu$ in a set of total Lebesgue measure. The measure μ will be referred to as a *good measure* for the action A.

We notice that if the restriction of A to the center C of G is uniquely ergodic then A is an E-action. See [**4**] and [**11**] for more important examples of E-actions.

Emerson proved a theorem in [**6**] that in our case gives the following.

THEOREM 3.2. *Let G be a Lie group which has a regular summing sequence $S = \{S_n; n \in N\}$ of the center C of G and let A be a C^r, $r \geq 0$, (left) action of G on M. Then for each A-invariant probability measure μ on M the ergodic averages $A_n f$ converge in L^2 and pointwise, μ a.e. to an A-invariant function $f^* \in L^2(M, \mu)$.*

It follows from the above theorem that if the measure μ is absolutely continuous with full support and A is ergodic then A is an E-action.

LEMMA 3.3. *Let $S = \{S_n\}$ be a summing sequence of the center C of a Lie group G and let A be a l.f. action of G on a closed connected smooth manifold M. Then the ergodic averages*

$$A_n \xi = \frac{1}{|S_n|} \int_{S_n} A_g^* \xi \, dg$$

are uniformly bounded for every continuous k-form ξ tangent to the foliation $\mathcal{F}$ given by A, i.e.

$$|A_n \xi|_0 \leq |\xi|_0$$

for all positive integers n where $|\cdot|_0$ denotes the C^0 norm.

Proof. Choose a basis $\{E_1, \ldots, E_p\}$ of the Lie algebra $\mathcal{G}$ of G and let $\{\theta_1, \ldots, \theta_p\}$ be its dual basis. Notice that the exterior algebra $\Lambda(\mathcal{F})$ of tangent forms to $\mathcal{F}$ is generated over the ring $C^\infty(M)$ of smooth functions on M by the 1-forms $\{\theta'_1, \ldots, \theta'_p\}$ as in Proposition 1.1(iii). Now

$$A_g^* \theta' = \theta', \quad \text{for all } \theta \in \Lambda_R(\mathcal{G}^*) \tag{3.3}$$

and all g in the center C of G. Thus $\xi = \sum_I h_I \theta'_I$, where $I = (i_1, \ldots, i_k)$, $1 \leq i_1 \leq \cdots \leq i_k \leq p$, $\theta'_I = \theta'_{i_1} \wedge \cdots \wedge \theta'_{i_p}$ and h_I are smooth functions on M, and

$$A_g^* \xi = \sum_I (h_I \circ A_g) \theta'_I. \tag{3.4}$$

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Thus

$$|A_g^*\xi|_0 = |\xi|_0, \quad \text{where } |\xi|_0 = \sum_I |h_I|_0 \text{ and } g \in C$$

and $|A_n\xi|_0 \leq |\xi|_0$ for all integer $n > 0$. □

We consider the *homotopy operators* on the category of foliated manifolds $(M, \mathcal{F})$. They are linear operators $K : \Lambda^j(\mathcal{F} \times R) \to \Lambda^j(\mathcal{F})$, $1 \leq j \leq p$, such that

$$d_f \circ K + K \circ d_f = j_1^* - j_0^* \tag{3.5}$$

where $j_t(x) = (x.t)$, $t = 0, 1$.

Each l.f. action $A : G \times M \to M$ of a connected Lie group G can be lifted to a unique l.f. action $\widetilde{A} : \widetilde{G} \times M \to M$ where $\widetilde{G}$ is the simply connected covering group of G. Thus we may restrict ourselves to actions of simply connected Lie groups.

PROPOSITION 3.4. *Let $A : G \times M \to M$ be a l.f. action of a connected and simply connected nilpotent Lie group on a closed orientable manifold M and let $\mathcal{F}$ be the foliation given by the orbits of A. Let $\{S_n\}$ be a summing sequence of the center C of G, then:*

(i) *the action $A^* : G \to \mathrm{Aut}(H^*(\mathcal{F}))$ induced by A is trivial;*

(ii) *the ergodic averages $A_n : \Lambda(\mathcal{F}) \hookleftarrow$ commute with the tangent differential d_f and induce the identity on the tangent cohomology $H^*(\mathcal{F})$;*

(iii) *the characteristic mapping χ of $\mathcal{F}$ is invariant under A_n, i.e. $\chi \circ A_n = \chi$ for all integer $n > 0$.*

Proof. (i) Since G is a nilpotent simply connected Lie group then $\exp : \mathcal{G} \to G$ is a diffeomorphism of the Lie algebra $\mathcal{G}$ onto G. The mapping $H : G \times M \times R \to M$, $H(g, x, t) = A(t \exp^{-1}(g), x)$ gives a f-homotopy of A_g with the identity mapping for every g in G. Thus $A_g^* = \mathrm{id}$ [**4**, Proposition 2.3].

(ii) Clearly $d_f \circ A_n = A_n \circ d_f$ because d_f commutes with A_g^* for every g in G. Now observe that (3.5) gives $A_g^*\xi - \xi = d_f K H_g^*\xi$ for every d_f-closed tangent form ξ. Thus

$$A_n\xi - \xi = d_f\left(\frac{1}{|S_n|}\int_{S_n} K H_g^*\xi \, dg\right) \tag{3.6}$$

proving (ii).

(iii) It follows from (ii) and the definition of ξ. □

We now prove an ergodic theorem for E-actions.

THEOREM 3.5. *Let $A : G \times M \to M$ be a l.f. action of a connected and simply connected nilpotent Lie group G on a closed orientable manifold M. Suppose A is an E-action with good measure μ and let $\{S_n\}$ be a summing sequence of the center C of G and P_A be the operator associated to μ as in (1.8). Then $A_n(\xi)$ converges to $(i_A \circ P_A)(\xi)$ pointwise on a set of total Lebesgue measure and this convergence is dominated, for each tangent k-form $\xi \in \Lambda^k(\mathcal{F})$.*

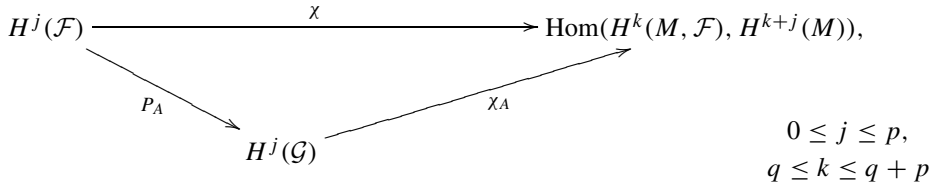
Proof. Since $A_g^*\theta = \theta$ for all $g \in C$ and all θ in the image of $i_A : \Lambda_R(\mathcal{G}^*) \to \Lambda(\mathcal{F})$ and the exterior algebra $\Lambda(\mathcal{F})$ of the differential forms tangent to the foliation $\mathcal{F}$ given by A is generated by $\mathrm{Im}\, i_A$ as module over the ring of the smooth functions, then it suffices

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to consider the case $k = 0$. Now since A is an E-action, the ergodic averages $A_n(h)$, $h \in C^\infty(M)$, converge pointwise to $\mu(h)$ in a set of total Lebesgue measure. Moreover, Lemma 3.3 says $|A_n\xi|_0 \leq |\xi_0|$ for all $\xi \in \Lambda(\mathcal{F})$ and every integer $n > 0$, proving the theorem. □

We now prove the Factorization Theorem which is the main result of this article.

THEOREM 3.6. *Let A be a non-singular smooth action of a connected and simply connected nilpotent Lie group G on a closed, connected and orientable manifold M. If A is an E-action, then we have the commuting diagrams*



where $\mathcal{G}$ is the Lie algebra of G, $\mathcal{F}$ is the foliation given by the orbits of A, P_A is the operator given in (1.8) and $\chi_A = \chi \circ i_A$ is the characteristic mapping of A.

Proof. We show that the closed differential forms $\xi \wedge w$ and $(i_A \circ P_A)(\xi) \wedge w$ are cohomologous for every d_f-closed j-form ξ tangent to $\mathcal{F}$ and every closed k-form w in $I(\mathcal{F})^q$, where q is the codimension of $\mathcal{F}$. By the Poincaré Duality Theorem it is sufficient to show that

$$\int_M \xi \wedge w \wedge \lambda = \int_M (i_A \circ P_A)(\xi) \wedge w \wedge \lambda \tag{3.7}$$

for every closed $n - (j + k)$-differential form on M. Observe that Proposition 3.4 gives

$$\int_M \xi \wedge w \wedge \lambda = \int_M A_n(\xi) \wedge w \wedge \lambda \tag{3.8}$$

for all positive integers n.

Now by Theorem 3.5 and the Lebesgue Dominated Convergence Theorem we see that

$$\int_M A_n(\xi) \wedge w \wedge \lambda \quad \text{converges to} \quad \int_M (i_A \circ P_A)\xi \wedge w \wedge \lambda. \tag{3.9}$$

Thus (3.7)–(3.9) prove the theorem. □

We recall that $P_A \circ i_A$ is the identity on $H^*(\mathcal{G})$ but $i_A \circ P_A$ is not the identity on $H^*(\mathcal{F})$ even if A is an E-action. For instance, if A is a linear Liouville action on T^p then $H^j(K)$ given in Proposition 1.4 is a non-Hausdorff infinite dimension vector space $1 \leq j \leq p$ [**3**].

The factorization theorem relates ergodic properties of non-singular actions of nilpotent Lie groups on closed manifolds to the characteristic mappings of the foliation $\mathcal{F}$ given by the orbits of the actions. *It says that an E-action completely determines the characteristic mapping of $\mathcal{F}$.* We obtain as a corollary the vanishing theorem proved in [**11**, Theorem 2.11].

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THE VANISHING THEOREM. *Let G be a connected simply-connected nilpotent Lie group, let Γ be a discrete cocompact subgroup of G and let F be a closed smooth manifold. If $\varphi : \Gamma \to \mathrm{Diff}(F)$ is an E-action, then the derived characteristic mapping δ of its suspension foliated bundle $F \to (M, \mathcal{F}) \to B$ vanishes.*

This follows from the factorization theorem by observing that $\delta_H = 0$ for the canonical action H of G on M given by right translations on the first factor of $G \times F$ [**11**].

REFERENCES

[1] V. I. Arnold. The asymptotic Hopf invariant and its applications. *Sel. Math. Sov.* **5** (1986), 328–345.
[2] J. L. Arraut and N. M. dos Santos. Actions of R^p on closed manifolds. *Topology Appl.* **29** (1988), 41–54.
[3] J. L. Arraut and N. M. dos Santos. Linear foliations of T^n. *Bol. Soc. Brasil. Mat.* **21** (1991), 189–204.
[4] J. L. Arraut and N. M. dos Santos. The characteristic mapping of a foliated bundle. *Topology* **31** (1992), 545–555.
[5] G. Duminy. L'invariant de Godbillon–Vey d'un feuilletage se localise sur les feuilles ressort. *Preprint*, Lille, 1982.
[6] W. Emerson. The pointwise ergodic theorem for amenable groups. *Amer. J. Math.* **96** (1974), 472–487.
[7] C. Godbillon. *Feuilletages, Études Géométriques I*. Irma, Strasbourg, 1985.
[8] J. Heitsch and S. Hurder. Secondary classes, Weil measures and the geometry of foliations. *J. Diff. Geom.* **23** (1986), 347–365.
[9] S. Hurder and A. Katok. Ergodic theory and Weil measures for foliations. *Ann. Math.* **126** (1987), 221–275.
[10] J. W. Jenkins. Growth of connected locally compact groups. *J. Func. Anal.* **12** (1973), 113–127.
[11] N. M. dos Santos. Foliated cohomology and characteristic classes. *Contemp. Math.* **161** (1994), 41–57.
[12] S. Schwartzman. Asymptotic cycles. *Ann. Math.* **66** (1957), 270–284.
[13] D. Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.* **36** (1976), 225–255.

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