A NONLINEAR CONTACT FORCE MODEL FOR REVOLUTE JOINT WITH CLEARANCE

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ABSTRACT

Clearances in the joints of mechanism cause vibrations and noise in mechanical systems. The contact force model is the important factor in dynamic analysis of mechanical systems with clearance. This paper presents a nonlinear contact force model for revolute joint with clearance, which is a hybrid model of the Lankarani-Nikravesh model and the improved elastic foundation model. The framework of the Lankarani-Nikravesh model is used with the nonlinear stiffness coefficient derived using the improved elastic foundation model and the damping applied in introducing the ratio of the nonlinear stiffness coefficient of the improved elastic foundation model and contact stiffness of Lankarani-Nikravesh model. Finally, the nonlinear hybrid contact force model is analyzed and discussed. The hybrid contact force model presented in this paper is a nonlinear continuous contact force model, which extends contact modeling of joints with clearance and has a greater applicable scope.

Keywords: Revolute joint, Clearance, Contact, Hybrid model.

1. INTRODUCTION

Clearances in mechanism are unavoidable due to assemblage, manufacturing errors and wear. Moreover, the clearance occurs in each active joint with the movement of mechanism. The movement of the real mechanisms is deflected from the ideal mechanism and motion accuracy is decreased due to the joint clearances. The existence of clearance in joints also causes the impact dynamic load, affects the transfer of the system load, and may lead to destruction and failure of the mechanism [1-5].

Contact and impact are the typical phenomena of mechanism with joint clearance. The contact force model of revolute joints with clearance is one of the important contents in dynamics analysis of mechanism with clearance. Moreover, reasonable contact model is crucial to the design and analysis of mechanism with clearance. The contact-impact models of mechanism with clearance are mainly focus on the discrete analysis method and continuous contact analysis method [6-8]. The former assumes that the contact-impact is very short and did not change the overall configuration of the object. Then, the contact-impact process is divided into two stages, before and after impact, and relative sliding, viscous stagnation and reverse movement will occur between two objects after the impact. The latter assumes that interaction forces between the impact objects are continuous in the entire contact-impact process. This approach tallies with real contact-impact behavior of objects. The continuous contact force model is widely used for contact-impact analysis of mechanism with clearance and the elastic contact force is widely represented by Hertz contact law. The representative contact force model is Lankarani-Nikravesh model [9-11]. A common Hertz contact force expression is used in this model and considering the damping effects, and it can describe the energy loss during the impact process.

Hertz contact model is available only in solving the contact problem that the geometric shape of contact bodies is non-conformal. The properties of conformal

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contact at the contact surfaces of revolute joints consequentially result in the limitation of Hertz model in solving the contact problem of revolute joints with clearances. And Hertz model is available only in the case that there is a large clearance with a small normal load. However, clearance in actual revolute joint is very small and the contact process of journal and bearing doesn't always satisfy the non-conformal contact condition. Usually, the achieved results are not precise, especially in small clearance for bearing and journal contact. Besides, in most of the past literatures, contact stiffness and damping coefficient were calculated simply according to the impact situation of two sphere bodies or taken as a constant according to experience, which doesn't meet the actual situation [12-16].

Liu *et al.* [12] developed a class of non-conformal contact model caused by joint clearance, which is based on the improved Winkler elastic foundation model and Hertz quadratic pressure distribution assumption. The model is applied to static contact analysis of cylindrical joints with clearance and verified by finite element method. The application of the above model is not only in the contact problem of revolute joint with big clearance but also the contact analysis of revolute joint with small clearance, which developed the clearance contact analysis.

Based on the research mentioned above, nonlinear stiffness coefficient is proposed for contact modeling in this paper. The nonlinear stiffness coefficient is found as the slope of the load-displacement relationship curve by improved elastic foundation model, around the point of instantaneous deformation. And then, a nonlinear continuous contact force model for clearance joint is presented, which is a hybrid model of the Lankarani-Nikravesh model and improved elastic foundation model. The framework of the Lankarani-Nikravesh model is used with the nonlinear stiffness coefficient derived using the improved elastic foundation model and the damping applied in introducing the ratio of the nonlinear stiffness coefficient of the improved elastic foundation model and contact stiffness of Lankarani-Nikravesh model. Finally, the nonlinear hybrid contact force model is analyzed.

2. MODELING OF REVOLUTE JOINT WITH CLEARANCE

In general, a clearance joint can be included in a mechanical system much like a revolute joint. The classical approach, known as zero-clearance approach, assumes that the connecting points of two bodies linked by a revolute joint are coincident. Then the clearance produced in a joint separates these two points. The difference in radii between the bearing and journal defines the size of the radial clearance. Figure 1 shows the model of revolute joints with clearance. As shown in Fig. 1(a), the radial clearance is defined as follow,

$$c = R_B - R_J \tag{1}$$

where R_B and R_J are the radii of bearing and journal, respectively.

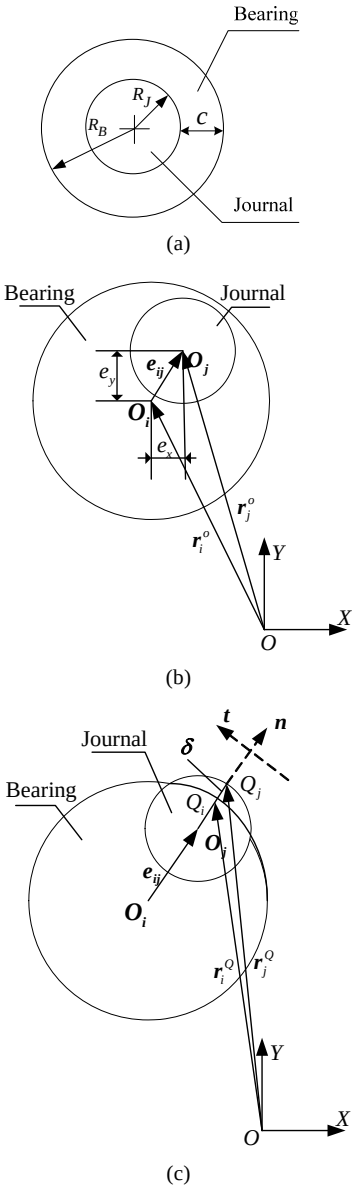


Fig. 1 Modeling of revolute joints with clearance: (a) Revolute joint with clearance; (b) Clearance model in revolute joint of mechanism system without contact; (c) Revolute clearance joint with contact

As shown in Fig. 1(b), Q_i and Q_j define the centers of bearing and journal, respectively, $\boldsymbol{r}_i^o$ and $\boldsymbol{r}_j^o$ are the vectors donating the positions of bearing and journal in the global inertia coordinate, respectively. So in Fig. 1(b), the clearance vector is represented as Eq. (2):

$$\boldsymbol{e}_{ij} = \boldsymbol{r}_j^o - \boldsymbol{r}_i^o \tag{2}$$

where e_{ij} represents the eccentric vector of journal relative to bearing. So the eccentricity of clearance vector can be represented as following:

$$e_{ij} = \sqrt{e_x^2 + e_y^2} \tag{3}$$

The unit normal vector of bearing and journal at the point of their contact is represented as Eq. (4):

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$$\boldsymbol{n} = \boldsymbol{e}_{ij} / e_{ij} \tag{4}$$

That is $\boldsymbol{n} = \dfrac{\boldsymbol{e}_{ij}}{\sqrt{e_x^2 + e_y^2}}$.

Figure 1(c) describes the collision between bearing and journal. $\boldsymbol{r}_i^Q$ and $\boldsymbol{r}_j^Q$ are the positional vectors of the bearing and journal, respectively, which are described in global coordinate frame. The contact deformation caused by collision between bearing and journal can be represented as Eq. (5):

$$\delta = e_{ij} - c \tag{5}$$

where c represents the radial clearance and it is a constant.

Thus, δ can be used to decide whether the bearing and journal have contacted. The kinematic contact condition between the bearing and journal can be given by Eq. (6):

$$\delta = \sqrt{e_x^2 + e_y^2} - c \geq 0 \tag{6}$$

The deformation vector between the potential contact point Q_i and Q_j (see Fig. 1(c)) in the global coordinate frame can be evaluated by

$$\boldsymbol{\delta} = \delta \boldsymbol{n} \tag{7}$$

In order to determine the concrete contact situation, the relative velocity of contact points should be calculated by

$$\dot{\boldsymbol{\delta}} = \delta \dot{\boldsymbol{n}} \tag{8}$$

where the dot denotes the derivative with respect to time. According to the components of the relative velocity of contact points in the normal and tangential plane of collision between the bearing and journal, the concrete contact situation can be determined.

3. HYBRID CONTACT FORCE MODEL

The contact force model during the contact process of revolute joint with clearance is one of the important contents of mechanical system dynamics with joint clearance. This paper presents a nonlinear contact force model for revolute joint with clearance, which is a hybrid model of the Lankarani-Nikravesh model and the improved Winkler elastic foundation model.

A common expression of Hertz contact force is adopted in Lankarani-Nikravesh model, which considers the effect of damping and describe the energy losing in the contact process. The expression of Lankarani-Nikravesh model is shown in Eq. (9) [10]:

$$F_n = K\delta^n + D\dot{\delta} \tag{9}$$

where elastic deformation force is represented by the first item of the right side of Eq.(9) and the energy losing is represented by the second item. δ is the deformation, $\dot{\delta}$ is the relative deformation velocity. K is the contact stiffness coefficient of the impact body, which is obtained from impact experiment of two spheres. K is obtained from the following:

$$K = \frac{4}{3\pi(\sigma_i + \sigma_j)}\left[\frac{R_2 R_1}{R_2 - R_1}\right]^{\frac{1}{2}} \tag{10}$$

$$\sigma_i = \frac{1 - \nu_i^2}{\pi E_i}$$
$$\sigma_j = \frac{1 - \nu_j^2}{\pi E_j} \tag{11}$$

where ν and E are Poisson ratio and Young modulus, respectively. R_1 and R_2 are radii of the two spheres.

Coefficient, D, in Eq. (9) is damping coefficient and $\dot{\delta}$ is relative impact velocity in impact process. The expression of D is shown in Eq. (12) [10]:

$$D = \frac{3K(1 - c_e^2)\,\delta^n}{4\dot{\delta}^{(-)}} \tag{12}$$

where c_e is coefficient of restitution, which is defined as the ratio of relative departure velocity to the relative approach velocity of the contact bodies in the direction of impact. $\dot{\delta}^{(-)}$ is initial relative velocity of the impact point.

Due to the coefficient of restitution, c_e, is closed to unity in Eq. (12) [10], the expression obtained is only with high coefficient of restitution, which is improved by Qin and Lu [17]. The improved relationship between coefficient of restitution and damping coefficient is represented as follows:

$$D_I = \frac{3K(1 - c_e^2)\, e^{2(1-c_e)}\delta^n}{4\dot{\delta}^{(-)}} \tag{13}$$

where D_I is not restricted by coefficient of restitution, it is applicable for both large and small coefficient of restitution.

Hertz theory is available only in solving the contact problem that the geometric shape of contact bodies is non-conformal [12,13]. The properties of conformal contact at the contact surfaces of revolute joints consequentially result in the limitation of Hertz model in solving the contact problem of revolute joints with clearances [12]. Moreover, Hertz theory is not always available, which are strongly depending on the size of the clearance. Hertz model is effective only in the case that there is a large clearance with a small normal load. This can be attributed to the assumptions adopting in the Hertz model. Hertz theory is established based on the plane assumption of the contact region, so the contact between two bodies in the revolute joint can satisfy the assumption of Hertz theory only in the case of that the clearance is larger enough and the load is very small.

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However, clearance in actual revolute joint is very small and the achieved results are not precise, especially in small clearance for bearing and journal contact. Besides, in most of the published literatures, contact stiffness and damping coefficient are calculated simply or taken as a constant according to experience, which doesn't tally with the actual situation. Liu *et al*. [12] presented a new relationship between load and displacement for cylindrical joint with clearance based on the improved Winkler elastic foundation model and Hertz quadratic pressure distribution assumption. Bases on the improved elastic foundation model, the relationship between load, F_N, and deformation, δ, of cylindrical joint clearance is obtained as shown in Eq. (14):

$$F_N = \frac{1}{2}\pi\delta E^* \sqrt{\frac{\delta}{2(R_i - R_j + \delta)}} \tag{14}$$

where E^* is compound elastic modulus in Eq. (14) and the expression is represented as:

$$\frac{1}{E^*} = \frac{1-v_i^2}{E_i} + \frac{1-v_j^2}{E_j} \tag{15}$$

The above model is applied in the contact analysis of revolute joint with both big clearance and small clearance. Compared to the results of finite element numerical calculation, this contact model has a better precision and a greater applicable scope than the Hertz model [12].

Based on the improved elastic foundation model mentioned above Eq. (14), nonlinear stiffness coefficient is proposed in this paper. The nonlinear stiffness coefficient is found as the slope of the load-displacement relationship curve by improved elastic foundation model, around the point of instantaneous deformation. Further, based on the contact theory, nonlinear stiffness coefficient, K_n, is achieved from Eq. (14) as shown in Eq. (16):

$$K_n = \frac{1}{8}\pi E^* \sqrt{\frac{2\delta\left(3(R_i - R_j) + 2\delta\right)^2}{(R_i - R_j + \delta)^3}} \tag{16}$$

The nonlinear stiffness coefficient, K_n, is related to the material property, geometry property, clearance size and deformation of contact bodies, and it varies with δ and is not constant during the contact process.

Contact force model in clearance is improved by adding some form of damping force based on elastic contact force. Furthermore, the stiffness coefficient of Lankarani-Nikravesh model, K, as shown in Eq. (13) is modified by the nonlinear stiffness coefficient, K_n, so the modified damping coefficient is achieved as following:

$$D_{\text{mod}} = \frac{3K_n(1-c_e^2)\, e^{2(1-c_e)}\delta^n}{4\dot{\delta}^{(-)}} \tag{17}$$

where D_{mod} is the modified damping coefficient and it is related to the nonlinear stiffness coefficient, K_n.

Thus, it can be found that the modified damping coefficient, Eq. (17), introduced the ratio of nonlinear stiffness coefficient and stiffness coefficient of Lankarani-Nikravesh model, it is also can be simply expressed as Eq. (18):

$$D_{\text{mod}} = \frac{K_n}{K} D_I \tag{18}$$

Based on the research above, a hybrid model of the Lankarani-Nikravesh model and the improved elastic foundation model is established with the framework of Lankarani-Nikravesh model. Stiffness coefficient of the hybrid model is the nonlinear stiffness coefficient from the improved elastic foundation model. And the damping force applied in introducing the ratio of the nonlinear stiffness coefficient of the improved elastic foundation model and contact stiffness of Lankarani-Nikravesh model, with the modified damping coefficient D_{mod} in the hybrid model.

Hence, the expression of hybrid contact force model of revolute joint with clearance is represented in Eq. (19):

$$F_{n\text{mod}} = K_n\, \delta^n + D_{\text{mod}}\, \dot{\delta} \tag{19}$$

where the elastic deformation force is represented by the first item of the right side of Eq. (19) and the energy losing is represented by the second item. δ is the deformation and $\dot{\delta}$ is the relative deformation velocity. K_n is the nonlinear stiffness coefficient and D_{mod} is the modified damping coefficient.

Therefore, in the contact force model, nonlinear stiffness coefficient is proposed based on contact theory and it is useful for calculating and determining stiffness coefficient. The nonlinear stiffness coefficient is related to material property, geometry property, clearance size and deformation of contact bodies. It varies with δ and is not constant during the contact process. Therefore, it is varies with condition process and it is more realistic. Moreover, the hybrid contact force model of revolute joint is applicable for both big clearance and small clearance.

In addition, due to the coefficient of restitution, c_e, is closed to unity in Eq. (12) [10], the expression obtained is only with higher coefficient of restitution, and the calculation error is higher for the lower coefficient of restitution according to Schwab [18]. The hybrid contact force model contains modified damping coefficient, D_{mod}, which is not restricted by coefficient of restitution and has better calculation precision for lower coefficient of restitution. Therefore the nonlinear hybrid contact force model describes the energy loss during contact process accurately for different coefficients of restitution and extends the application range of coefficient of restitution.

Hereby, the hybrid contact force model is a nonlinear continuous contact force model, which combines Lankarani-Nikravesh model and the improved elastic foundation model. It extends contact modeling of clearance in revolute joints and has a greater applicable scope.

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4. ANALYSIS OF THE HYBRID CONTACT FORCE MODEL

To show the variations of the impact responses associated with the different parameters, the hybrid contact force model, as shown in Eq. (19), is applied to journal and bearing contact in planar revolute joint with clearance, as shown in Fig. 2. The radii of journal and bearing are 1cm and 0.95cm, respectively. The elasticity modulus is E = 207GPa and Poisson ratio is υ = 0.3. The mass of journal is 1kg, the initial velocity of journal is 1m/s and the bearing is fixed boundary.

4.1 Contact Analysis with Different Coefficients of Restitution

First, the nonlinear hybrid contact force model for revolute joint with clearance is analyzed for different coefficients of restitution and one initial impact velocity (V_0 = 1m/s). Figure 3 shows the contact force law with one initial impact velocity and different coefficients of restitution, and here the clearance is 0.5mm.

Figure 3 indicates that the contact processes are different with the different coefficients of restitution. The higher coefficient of restitution, the larger maximum deformation and the longer contact time to reaching the maximum deformation. When the coefficient of restitution is larger, the recover time is shorter and the deformation velocity during recover process is larger. The reason is that the damping force during the contact process is smaller with the higher coefficient of restitution. And the energy loss is smaller, so the loss of energy is slower. When the coefficient of restitution is higher, the recovery procedure is shorter, so the contact duration time is shorter.

The relationship of force-deformation shows the hysteresis damping characteristic and reflects the energy loss during contact process. And the smaller coefficient of restitution, the larger energy loss is. These facts are accompanied with the increase of the contact duration and the hysteresis loop, which in turn is associated with the energy dissipated during the contact process. Observing Fig. 3(d), it is important to note how the hysteresis loop increases when the coefficient of restitution decreases. In addition, the calculated results based on the hybrid contact force model are compared to other studies from previous literatures [10,19] and the global data produced here are corroborated by similar analysis available in the thematic literature. It is indicated that the nonlinear hybrid contact model presented in this paper describes the energy loss during contact process accurately for different coefficients of restitution.

4.2 Contact Analysis with Different Initial Contact Velocities

The nonlinear hybrid contact force model for revolute joint with clearance is analyzed for different initial impact velocities and one coefficient of restitution (c_e = 0.9). Figure 4 shows the contact force law with one coefficient of restitution and different initial impact velocities, and here the clearance is 0.5mm.

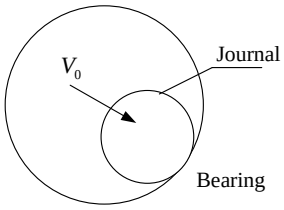


Fig. 2 Contact of bearing and journal in clearance joint

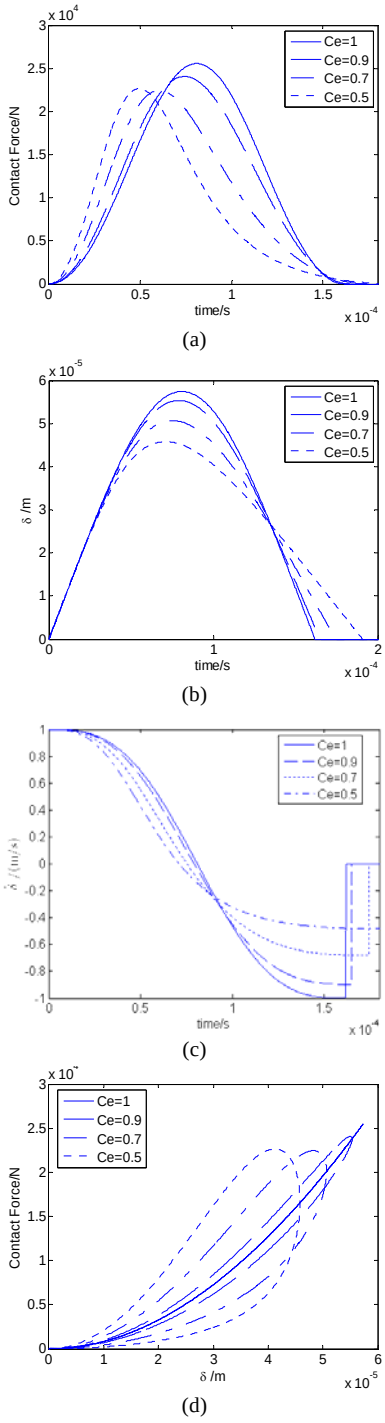


Fig. 3 Contact force model for different coefficients of restitution: (a) variation of contact force with time; (b) variation of deformation with time; (c) variation of deformation velocity; (d) force-deformation relationship. (Here initial impact velocity V_0 = 1m/s and clearance size c = 0.5mm)

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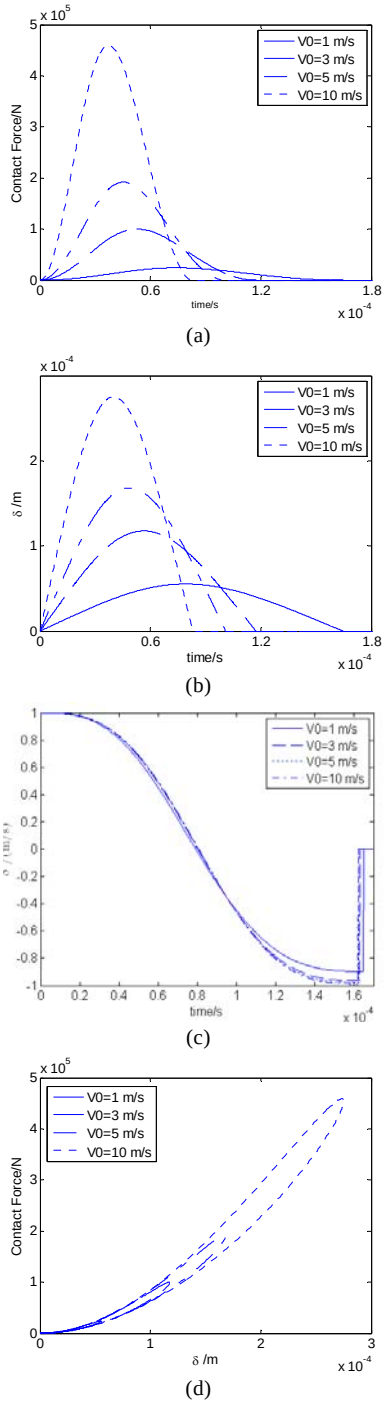


Fig. 4 Contact force model for different initial impact velocities: (a) variation of contact force with time; (b) variation of deformation with time; (c) variation of deformation velocity; (d) force-deformation relationship. (Here coefficient of restitution c_e = 0.9 and clearance size c = 0.5mm)

Figure 4 shows that the contact processes are different with the different initial impact velocities. And the contact force, deformation, and deformation velocity are different. With the increase of the initial impact velocity, the contact force increases rapidly, resulting in sharp increase of the contact deformation, and the contact duration time is shorter, agreeing with the actual situation.

As described by the theoretical formula, when the contact force reaches the maximum, the deformation reaches the maximum at the same time. The contact force-deformation curve shows there is energy loss during the contact process, reflecting the hysteresis damping characteristics.

In addition, the calculated results based on the hybrid contact force model are compared to other studies from previous literatures [10] and in which the similar results are found. So the model proposed in this paper is validated by other data published on the field of contact models.

4.3 Contact Analysis with Different Clearance Sizes

The hybrid contact force model for revolute joint with clearance is analyzed for different clearance sizes and one coefficient of restitution ($c_e = 0.9$). Figure 5 shows the contact force law with one coefficient of restitution and different clearance sizes, and here the initial impact velocity is 1m/s.

Figure 5 shows that the contact processes are different with different clearance sizes. With the increase of the clearance size, the contact force decrease, however, the contact deformation increase, and the contact duration time is longer, agreeing with the actual situation. The bigger size of clearance, the lower maximum contact force and the longer contact time to reaching the maximum contact force. However, the bigger size of clearance, the larger maximum deformation and the longer contact time to reaching the maximum deformation. As described by the theoretical formula, when the contact force reaches the maximum, the deformation reaches the maximum at the same time. The contact force-deformation curve also shows there is energy loss during the contact process, reflecting the hysteresis damping characteristics.

It is indicated that the hybrid contact force model presented in this paper describes the contact process accurately for different clearance sizes in revolute joints.

5. CONCLUSIONS

This paper presents a nonlinear contact force model in the joint clearance, which combines Lankarani-Nikravesh model and the improved elastic foundation model. The nonlinear contact force model for clearance in revolute joints are analyzed and discussed based on the numerical results.

The relationship of force-deformation shows the hysteresis damping characteristic and reflects the energy loss during contact process. And the smaller coefficient of restitution, the larger energy loss is. With the increase of the initial impact velocity, the contact force increases rapidly, resulting in sharp increase of the contact deformation, and the contact duration time is shorter. With the increase of the clearance, the contact force decrease, however, the contact deformation increase, and the contact duration time is longer, agreeing with the actual situation.

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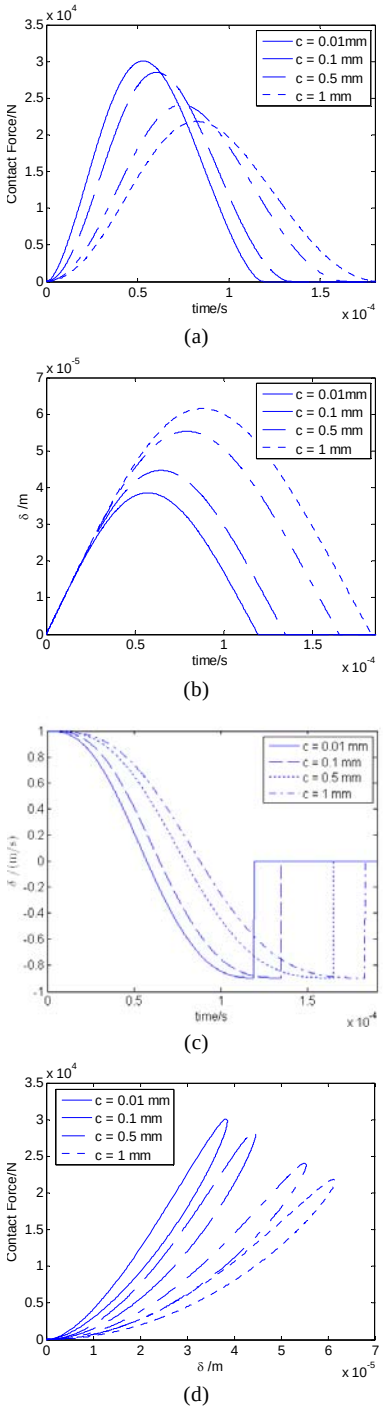


Fig. 5 Contact force model for different clearance sizes: (a) variation of contact force with time; (b) variation of deformation with time; (c) variation of deformation velocity; (d) force-deformation relationship. (Here coefficient of restitution $c_e = 0.9$ and initial impact velocity $V_0 = 1$m/s)

The nonlinear contact force model presented in this paper describes the energy loss during contact process accurately for different coefficients of restitution. Besides, the nonlinear contact force model describes the contact process accurately for different initial impact velocities as well as different clearance sizes in revolute joints. This work extends the contact modeling of clearance in revolute joints and has a greater applicable scope.

However, the work is only a theoretical investigation and analyzed based on the numerical results, and the practicability and applications of the model should be validated with the experiments. That will be of the main concern in the further investigations.

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