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The constrained quantum mechanical harmonic oscillator

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Abstract. The effect of restricting the displacements of a quantum mechanical harmonic oscillator to semi-infinite and finite regions is studied. It is shown how the eigenvalue problem may be solved, both graphically, and from certain series expansions.

1. *Introduction.* In this paper we consider quantum mechanical systems closely related to that of the one-dimensional harmonic oscillator. The point of difference is that here we consider potential wells of semi-infinite and finite extent, rather than of the infinite range usually associated with the quantum mechanical harmonic oscillator.

The motivation for this work arose, originally, from the study of certain anharmonic effects in solids. The motion of atoms in many crystals can, to a fair approximation, be described in terms of a system of independent (and unconstrained) harmonic oscillators. However, more accurate descriptions must include the effects of coupling between the oscillators and of deviations from the exact quadratic potential. The problem of the constrained harmonic oscillator originated in one particular empirical approach to the general problem associated with the inclusion of these anharmonic and coupling effects.

The present work leads to series expansions for the zeros of Weber parabolic cylinder functions; these series appear to be of some mathematical interest.

We consider two separate systems. In system I the oscillator is confined to the semi-infinite region $q_0 < q < \infty$ (q being the distance variable) by the presence of an infinitely high potential barrier at $q = q_0$. The potential function for this system is, therefore,

$$V(q) = \tfrac{1}{2}m\omega^2 q^2 \quad (q_0 < q < \infty).$$

The parameter ω corresponds to the frequency of a classical oscillator of mass m in an unconstrained system. In system II the oscillator is confined to a region symmetric about the origin and bounded by infinitely high potential barriers at $q = \pm q_0$. The potential function, here, is

$$V(q) = \tfrac{1}{2}m\omega^2 q^2 \quad (-q_0 < q < q_0; \quad q_0 > 0).$$

We wish to find, in each case the energy eigenvalues E_n and the corresponding eigenfunctions ψ_n, of the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dq^2} + V(q)\psi = E\psi$$

($\hbar = h/2\pi$, h being Planck's constant).

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In the case of the unconstrained oscillator, the eigenvalues are known to be

$$E_n = (n+\tfrac{1}{2})\hbar\omega \quad (n=0,1,2,\ldots) \tag{1}$$

with corresponding normalized eigenfunctions

$$u_n = (\sqrt{\pi}n!)^{-\frac{1}{2}} \exp\left(-\frac{m\omega}{2\hbar}q\right) h_n\left(\left\{\frac{2m\omega}{\hbar}\right\}^{\frac{1}{2}} q\right), \tag{2}$$

where

$$h_n(z) = (-)^n e^{\frac{1}{2}z^2} \frac{d^n}{dz^n} e^{-\frac{1}{2}z^2} \tag{3}$$

is the Hermite polynomial of order n.

These results, particularly (1), are well known and are of great importance in several branches of theoretical physics. We shall later refer back to them as special cases of the constrained system.

A useful simplification of notation is achieved by transforming to a new independent variable

$$x = \left(\frac{2m\omega}{\hbar}\right)^{\frac{1}{2}} q.$$

The Schrödinger equation then becomes

$$\frac{d^2\psi}{dx^2} - (\tfrac{1}{4}x^2 + a)\psi = 0, \tag{4}$$

with

$$a = -\frac{E}{\hbar\omega},$$

and applies to the ranges

$$x_0 = \left(\frac{2m\omega}{\hbar}\right)^{\frac{1}{2}} q_0 < x < +\infty \quad \text{(for system I)}$$

and

$$-x_0 < x < +x_0 \quad \text{(for system II)}.$$

Equation (4) is a standard form of a well-known differential equation ((3)), the general solutions of which are the Weber parabolic cylinder functions. Even and odd solutions are given by

$$y_1 = e^{-\frac{1}{4}x^2}\left\{1 + (a+\tfrac{1}{2})\frac{x^2}{2!} + (a+\tfrac{1}{2})(a+\tfrac{5}{2})\frac{x^4}{4!} + \ldots\right\} \tag{5}$$

and

$$y_2 = e^{-\frac{1}{4}x^2}\left\{x + (a+\tfrac{3}{2})\frac{x^3}{3!} + (a+\tfrac{3}{2})(a+\tfrac{7}{2})\frac{x^5}{5!} + \ldots\right\}, \tag{6}$$

respectively, these series being convergent for all x. In reference (1) the fundamental solutions of the equation are defined as

$$U(a,x) = Y_1 \cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a) - Y_2 \sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a) \tag{7}$$

and

$$V(a,x) = \frac{1}{\Gamma(\frac{1}{2}-a)}\{Y_1 \sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a) + Y_2 \cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a)\}, \tag{8}$$

where

$$Y_1 = \frac{\pi^{\frac{1}{2}} \sec\pi(\frac{1}{4}+\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)} y_1, \tag{9}$$

and

$$Y_2 = \frac{\pi^{\frac{1}{2}} \operatorname{cosec}\pi(\frac{1}{4}+\frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{4}}\Gamma(\frac{1}{4}+\frac{1}{2}a)} y_2. \tag{10}$$

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The fundamental solutions $U(a,x)$ and $V(a,x)$ are linearly independent for all values of x, and for large positive values of x have the asymptotic expressions

$$\left.\begin{aligned} U(a,x) &\sim e^{-\frac{1}{4}x^2}x^{-a-\frac{1}{2}}, \\ V(a,x) &\sim 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}e^{\frac{1}{4}x^2}x^{a-\frac{1}{2}}. \end{aligned}\right\} \tag{11}$$

The asymptotic expressions for large negative x can be obtained from (11) and the relations

$$\left.\begin{aligned} \Gamma(\tfrac{1}{2}+a)\{U(a,-x)+\sin\pi a\, U(a,x)\} &= \pi V(a,x), \\ \pi\sec^2\pi a\{V(a,-x)-\sin\pi a\, V(a,x)\} &= \Gamma(\tfrac{1}{2}+a)\, U(a,x). \end{aligned}\right\} \tag{12}$$

In applying these general results to the systems I and II we shall look for non-degenerate solutions, ψ, which are linear combinations of the standard solutions $U(a,x)$ and $V(a,x)$ in the given ranges and are such that the boundary conditions are satisfied, i.e. $\int\psi^2\,dx < \infty$ (cf. (2)). This integrability condition implies that the function ψ must always vanish at the edge of an infinitely high potential barrier. It is clear, also, that the solutions for system I must contain no component of $V(a,x)$—in view of the limiting behaviour of this function at large x.

2. *Solutions for system I.* We have noted that the eigenfunctions must: (i) contain no component of $V(a,x)$, and (ii) vanish at $x = x_0$. Condition (i) limits the possible solutions to the set $\{U(a,x)\}$, and condition (ii) further restricts the solutions to members of this set which satisfy

$$U(a,x_0) = 0. \tag{13}$$

We shall later expand a as a power series in x_0 which satisfies (13). First, though, we discuss qualitatively the eigenvalue spectrum of system I.

In figure 1 we have plotted the zeros of $U(a,x)$ for $a = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, -\frac{9}{2}, -\frac{11}{2}, -\frac{13}{2}, -\frac{15}{2}$. At these values of a, $U(a,x)$ is simply related to a Hermite polynomial; thus

$$U(-n-\tfrac{1}{2},x) = \text{constant}\times e^{-\frac{1}{4}x^2}h_n(x). \tag{14}$$

Figure 1 is therefore just a plot of the zeros of Hermite polynomials up to order 7.

Let us now consider the particular value $x_0 = 0$. It is clear that the functions

$$U(-n-\tfrac{1}{2}) \quad (n = 1, 3, 5, \ldots, 2j+1, \ldots) \tag{15}$$

constitute valid solutions for system I in this case, for each has a zero at $x = 0$ (the edge of the infinitely high potential barrier) and is square integrable (cf. equation (11)) over the range $(0, +\infty)$. Moreover, the solution for $n = 2j+1$ has exactly j nodes throughout this range, not including the node at x_0. It follows from the well-known node interlacing property of successive eigenfunctions of Sturm–Liouville equations (see (2), section 12) that the functions (15) constitute the complete set of solutions for system I in the particular case $x_0 = 0$.

By an extension of this argument we notice that whenever x_0 coincides with one of the zeros 1, 3, 6, 10, ... (Figure 1) the corresponding zeroth-order solution to equation (4) is $U(a,x)$ where a is here the ordinate (on Figure 1) of the particular zero defining x_0. Clearly, these functions are satisfactory solutions, for they satisfy the integrability condition and each possesses a zero at x_0; they are referred to as zeroth-order solutions as they contain no node between x_0 and $+\infty$.

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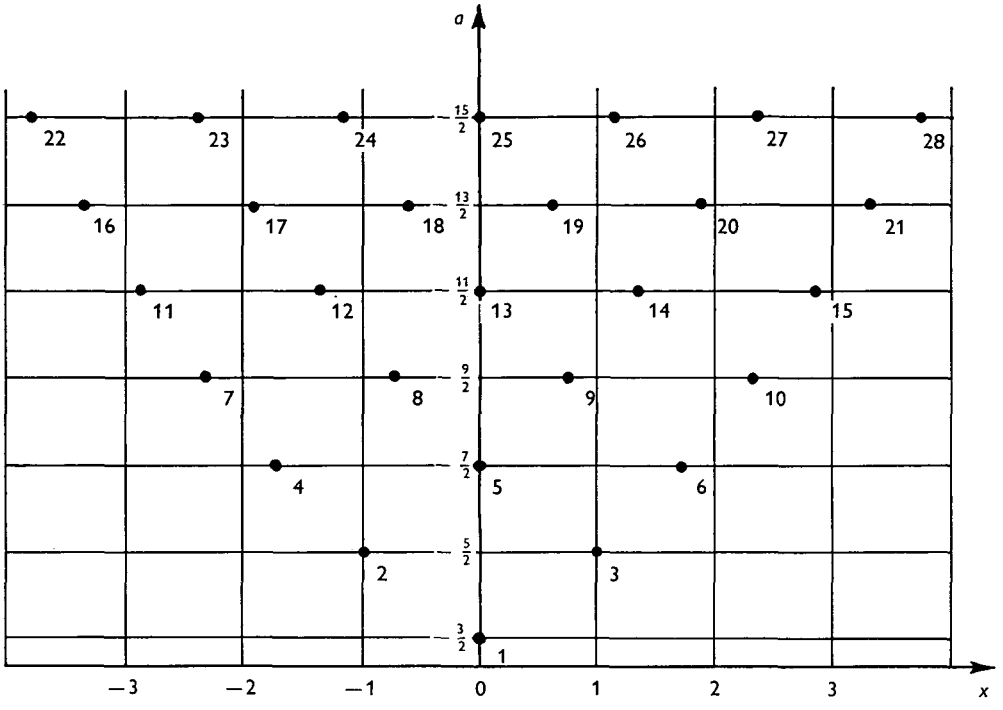


Fig. 1. Zeros of $U(-n-\frac{1}{2}, x)$, or of $h_n(x)$, up to $n = 7$. ($a = -n-\frac{1}{2}$, here.)

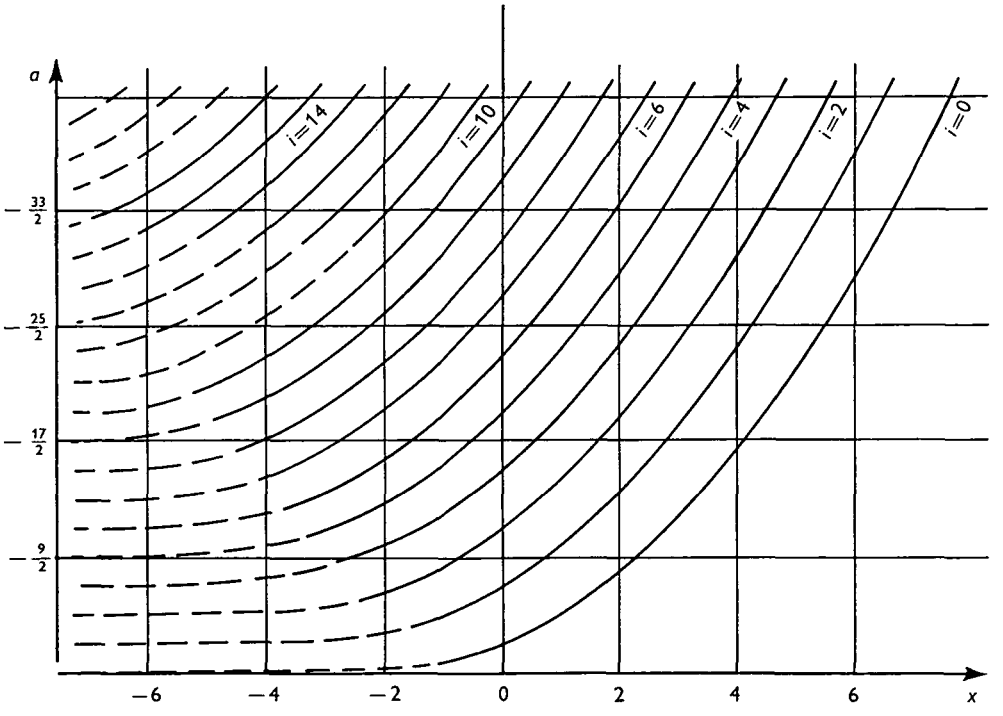


Fig. 2. The curves $a = f_i(x)$. The continuous region of each curve was obtained by interpolation between known points; the broken portion (which is a less accurate representation) by extrapolation.

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By a similar argument it can be seen that if x_0 coincides with any one of the zeros $2, 5, 9, 14, \ldots$ the corresponding *first*-order solution (possessing one node between x_0 and $+\infty$) is $U(a, x)$ where, again, a is the ordinate of the particular zero defining x_0. It is clear that this idea can be extended indefinitely. Thus, $U(a, x)$—with the appropriate values of a inserted— is the second-order solution corresponding to an x_0 at any of the set of points $4, 8, 13, 19, \ldots$; and so on.

In Figure 2 we have connected the various sets of points $1, 3, 6, 10, \ldots$; $2, 5, 9, 14, \ldots$; $4, 8, 13, 19, \ldots$, etc., by smooth curves $a = f_i(x)$ $(i = 0, 1, 2, \ldots)$ which are such that, at all points on these curves, $U(f_i(x), x) = 0$. Following on the arguments of the previous paragraphs it is easy to see that the eigenvalues, a, corresponding to any given x_0 for system I can be found, from the graph, by reading off the ordinates of the intersections of the vertical line through x_0 with the curves. The corresponding eigenfunctions are then, simply, $U(a, x)$.

It is interesting to note that the curves $a = f_i(x)$ $(i = 0, 1, 2, \ldots)$ tend asymptotically to the values $-i - \frac{1}{2}$ as $x \to -\infty$. One can see that this must be so by considering the case $x_0 = -\infty$ in which no constraint is imposed on the oscillator and the eigenvalues must therefore be the negative half-integer values. In this case the eigenfunctions are $U(-i-\frac{1}{2}, x)$ $(i = 0, 1, 2, \ldots)$ and these are, in fact, of the general form (2) as may be seen by inserting any negative half-integer value of a in the expression (7).

Expansions of the functions $f_i(x)$ $(i = 0, 1, 2, \ldots)$, as power series in x, are obtained in the following way. From (7), (8) and (9) we have

$$U(a, x) = \pi^{\frac{1}{2}} 2^{-\frac{1}{2}a - \frac{1}{4}} u(a, x)\, e^{-\frac{1}{4}x^2}, \tag{16}$$

where

$$u(a, x) = \frac{y_1\, e^{\frac{1}{4}x^2}}{\Gamma(\frac{3}{4} + \frac{1}{2}a)} - \frac{2^{\frac{1}{2}} y_2\, e^{\frac{1}{4}x^2}}{\Gamma(\frac{1}{4} + \frac{1}{2}a)}. \tag{17}$$

We write

$$f_i(x) = -\tfrac{1}{2} + a_0(i) + a_1(i)\, x + a_2(i)\, x^2 + \ldots \tag{18}$$

and, using the expansion

$$\frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} c_k z^k \quad (|z| < \infty) \tag{19}$$

(cf. (3), chapter 6, where the first 26 c_k are listed) we equate to zero successive powers of x in the expansion of $u(f_i(x), x)$ as given by (17), (5) and (6).

Equating the coefficient of x^0 to zero leads to the condition $1/\Gamma\{\frac{1}{2}(a_0 + 1)\} = 0$, and consequently $a_0(i) = -2i - 1$ $(i = 0, 1, 2, \ldots)$. In general, the coefficient of x^s in $u(f_i(x), n)$ is

$$\frac{1}{n!} \left(\frac{d^s u(f_i(x), x)}{dx^s} \right)_{x=0}.$$

Application of this formula, and subsequently equating each coefficient to zero, leads to the following results for the first few coefficients.

$$a_0(n) = -(2n + 1) \quad (n = 0, 1, 2, \ldots), \tag{20}$$

$$\left.\begin{aligned} a_1(n+1) &= \left(\frac{2n+3}{2n+2}\right) a_1(n), \\ a_1(0) &= -2^{\frac{1}{2}} \pi^{-\frac{1}{2}}. \end{aligned}\right\} \tag{21}$$

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$$\left.\begin{aligned} a_2(n+1) &= \left(\frac{2n+3}{2n+2}\right)^2 a_2(n) + \frac{(2n+3)!\,(2n+2)!}{16\pi[(n+1)!]^4\,(n+1)\,2^{4n}}, \\ a_2(0) &= -2\pi^{-1}(1-\ln 2), \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} a_3(n) = {} & \frac{\sqrt{2}}{4}\frac{(-)^n}{n!}a_1(n)^2\left[\frac{1}{\Gamma(-n-\frac{1}{2})}\right]^{(2)} + 2^{\frac{1}{2}}\frac{(-)^n}{n!}a_2(n)\left[\frac{1}{\Gamma(-n-\frac{1}{2})}\right]^{(1)} \\ & + \frac{\sqrt{2}}{3}\frac{(2n+1)!}{2^{2n}(n-1)!\,\pi^{\frac{1}{2}}n!} \\ & + \frac{(-)^{n+1}}{n!}\frac{a_1(n)^3}{24}\left[\frac{1}{\Gamma(-n)}\right]^{(3)} + \frac{(-)^{n+1}a_1(n)\,a_2(n)}{2n!}\left[\frac{1}{\Gamma(-n)}\right]^{(2)} \\ & + a_1(n)\,(n+\tfrac{1}{2}), \end{aligned}\right\} \tag{23}$$

where

$$\left[\frac{1}{\Gamma(p)}\right]^{(s)} = \left(\frac{d^s}{dx^s}\frac{1}{\Gamma(x)}\right)_{x=p}. \tag{24}$$

Table 1 gives the numerical values, accurate to seven decimal places, of a_0, a_1, a_2 and a_3 for $n \leqslant 10$. The table indicates that in each series the coefficients of successive powers of x decrease rapidly in magnitude. For any given x the accuracy of the truncated series increases with increasing n. For example, at points in the neighbourhood of $x = +1$, the absolute error in values of a as given by the lowest-order series ($n = 0$) is of the order of 0·001; the corresponding error for the series $n = 8$ is of the order of 0·00001.

Table 1

n	a_0	a_1	a_2	a_3
0	−1	−0·7978846	−0·1953486	−0·0078238
1	−3	−1·1968268	−0·2008019	−0·0037306
2	−5	−1·4960336	−0·2018471	−0·0026868
3	−7	−1·7453725	−0·2022048	−0·0022090
4	−9	−1·9635440	−0·2023667	−0·0019246
5	−11	−2·1598984	−0·2024532	−0·0017304
6	−13	−2·3398900	−0·2025046	−0·0015867
7	−15	−2·5070250	−0·2025377	−0·0014745
8	−17	−2·6637140	−0·2025601	−0·0013836
9	−19	−2·8116982	−0·2025760	−0·0013080
10	−21	−2·9522831	−0·2025878	−0·0012438

3. *Solutions for system II.* In system II the potential function is symmetric about the origin and limited to a finite region by infinitely high potential barriers at $\pm x_0$. The eigenfunctions, ψ, are therefore either even or odd functions of x (cf. (2), section 14) and are such that $\psi(+x_0) = \psi(-x_0) = 0$.

The general form of even and odd solutions to the differential equation (4) are given by $y_1(a, x)$ and $y_2(a, x)$ in (5) and (6), respectively. The problem, given any particular x_0, is to find the appropriate eigenvalues a which are such that y_1 (or y_2) is zero at x_0 for these values of a. First (as in the previous case) we discuss a graphical method of solution; later we derive certain expansions for the eigenvalues.

We refer back to Figure 1 and consider, first, the particular case of x_0 coinciding with the point 3. Clearly, the even function $y_1(-\frac{5}{2}, x) = ce^{-\frac{1}{4}x^2}h_2(x)$ (where c is a constant)

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is the lowest-order solution here, as $h_2(x_0) = h_2(-x_0) = 0$, and there is no node between $-x_0$ and $+x_0$. Again, if we take the case of x_0 at the point 9 in Figure 1, the lowest-order solution is $y_1(-\frac{9}{2}, x)$, a multiple of $e^{-\frac{1}{4}x^2}h_4(x)$. Similarly, the lowest-order solution for x_0 at the point 19 is $y_1(-\frac{13}{2}, x)$; and so on.

Similar considerations show that $y_2(-\frac{7}{2}, x)$ is the first-order solution (one node) corresponding to an x_0 at the point 6 in Figure 1; and that $y_2(-\frac{11}{2}, x)$ is the first-order solution corresponding to x_0 at the point 14, etc.

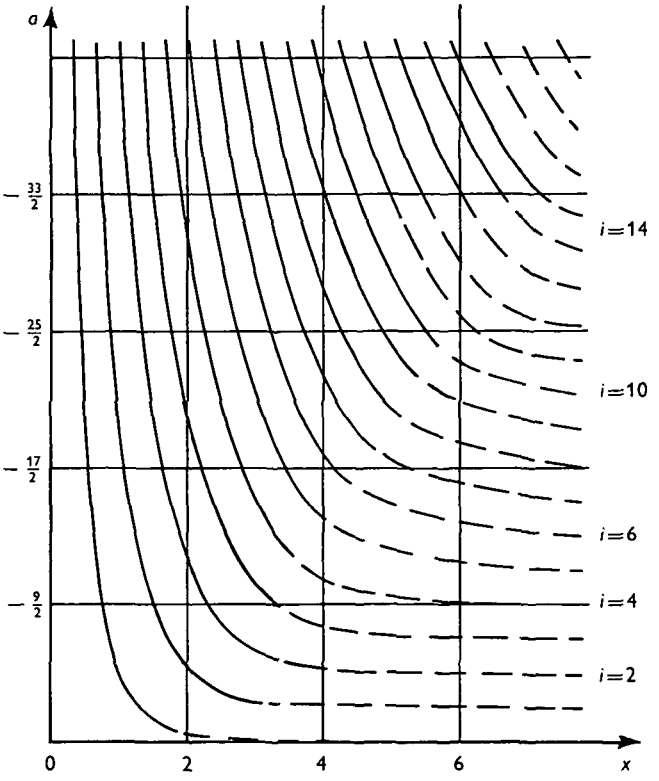


Fig. 3. The curves $a = g_i(x)$. The continuous region of each curve was obtained by interpolation between known points; the broken portion (which is a less accurate representation) by extrapolation.

In Figure 3 points related in this way to zeroth-order solutions (points 3, 9, 19, ...), to first-order solutions (points 6, 14, 26, ...), to second-order solutions (containing two nodes) (points 10, 20, 34, ...), and to higher-order solutions, have been joined by curves $a = g_i(x)$ $(i = 0, 1, 2, \ldots)$ which are such that, at all points on these curves, $y_1(g_i(x), x) = 0$ for even i, and $y_2(g_i(x), x) = 0$ for odd i. One can find the eigenvalues corresponding to system II for any given x_0 simply by noting the ordinates of the points of intersection of these curves with the vertical line at x_0. The eigenfunction corresponding to the jth lowest intersection on the line is y_2 if j is even and y_1 if j is odd.

The curves $a = g_i(x)$ in Figure 3 tend asymptotically to the values

$$-i-\tfrac{1}{2} \quad (i = 0, 1, 2, \ldots),$$

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in the limit as $x \to \infty$. This is clear from the fact that in the limit the system approaches that of the unconstrained harmonic oscillator.

Expansions for the functions $g_i(x)$, for even i, were obtained as follows. We write

$$a = g_i(x) = \frac{c_2}{x^2} + b_0 + b_2 x^2 + b_4 x^4 + \ldots, \tag{25}$$

substitute this series in the right-hand side of

$$y_1 e^{\frac{1}{2}x^2} = 1 + \frac{(a+\frac{1}{2})x^2}{2!} + \frac{(a+\frac{1}{2})(a+\frac{5}{2})x^4}{4!} + \ldots, \tag{26}$$

and equate each power of x to zero.

Thus, the coefficient of x^0 in the right-hand side of (26), using (25), is

$$1 + \frac{c_2}{2!} + \frac{c_2^2}{4!} + \frac{c_2^3}{6!} + \ldots.$$

Equating this to zero gives

$$\cosh c_2^{\frac{1}{2}} = 0$$

or

$$c_2 = -(s+\tfrac{1}{2})^2 \pi^2 \quad (s = 0, 1, 2, \ldots). \tag{27}$$

The algebra connected with the determination of $b_0, b_2, b_4, \ldots$ becomes increasingly cumbersome. However, the general procedure is the same in all cases; it is simply an extension of that used above for c_2 and we illustrate now the method used in deriving the higher coefficients.

The coefficient of x^m in the right-hand side of (26) is

$$\sum_{n=0}^{\infty} [\text{coefficient of } x^{-2n+m-2} \text{ in } E_n],$$

where

$$E_n = \frac{(a+\frac{1}{2})(a+\frac{5}{2})(a+\frac{9}{2})\ldots(a+2n+\frac{1}{2})}{(2n+2)!}. \tag{28}$$

Using (25) we can, for any particular value of m, write the coefficient of $x^{-(2n-m+2)}$ in E_n as the sum of contributions each of which relates to a certain distribution of powers of x among the factors of E_n. We illustrate for $m = 6$. In this case we require the coefficient of x^{-2n+4} in E_n. Now, x^{-2n+4} can be chosen from E_n in just three possible ways:

(1) x^{-2} from n factors (right-hand side of (28)) and x^4 from one factor;

(2) x^{-2} from $n-1$ factors, x^2 from one factor and a constant from one factor;

(3) x^{-2} from $n-2$ factors and constants from 3 factors.

Consequently, we can write the required coefficient in E_n as the sum of terms $t_n^{(6)}(1)$, $t_n^{(6)}(2)$ and $t_n^{(6)}(3)$, where

$$t_n^{(6)}(1) = (n+1)\frac{c_2^n b_4}{(2n+2)!},$$

$$t_n^{(6)}(2) = n[\tfrac{1}{2} + \tfrac{5}{2} + \ldots + (2n+\tfrac{1}{2})]\frac{c_2^{n-1} b_2}{(2n+2)!},$$

$$t_n^{(6)}(3) = [\tfrac{1}{2}\cdot\tfrac{5}{2}\cdot\tfrac{9}{2} + \tfrac{1}{2}\cdot\tfrac{5}{2}\cdot\tfrac{9}{2} + \ldots + (2n-\tfrac{7}{2})(2n-\tfrac{3}{2})(2n+\tfrac{1}{2})]\frac{c_2^{n-2}}{(2n+2)!}.$$

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The coefficient of x^6 in the right-hand side of (26) is then

$$\sum_{n=0}^{\infty} [t_n^{(6)}(1) + t_n^{(6)}(2) + t_n^{(6)}(3)].$$

The sums occurring here, in the expressions for $t_n^{(6)}(2)$ and $t_n^{(6)}(3)$, typify a more general form of sum occurring in this work—the sum of all possible distinct products of p numbers from the set $\{1, 5, 9, 13, \ldots, 4n+1\}$, $(p = 1, 2, \ldots)$. Such sums are conveniently evaluated using standard formulae and tables of Stirling Numbers ((3), chapter 24). The summations over n are always of a standard kind involving hyperbolic functions, and these present no formal difficulties of evaluation.

Table 2

i	c_2	b_2	b_6
0	−2·4674011	−0·0326727	+0·0000740
1	−9·8696044	−0·0706682	+0·0000697
2	−22·2066099	−0·0777044	−0·0000147
3	−39·4784176	−0·0801670	−0·0000194
4	−61·6850275	−0·0813069	−0·0000159
5	−88·8264396	−0·0819261	−0·0000124

Expansions for $g_i(x)$, for odd i, are obtained in an analogous manner, except that the function

$$y_2 e^{\frac{1}{2}x^2} = x + (a+\tfrac{3}{2})\frac{x^3}{3!} + (a+\tfrac{3}{2})(a+\tfrac{7}{2})\frac{x^5}{5!} + \ldots \tag{29}$$

is used instead of (26). The coefficients up to b_6 have been determined in both cases (i.e. for both even and odd i) and these are as follows:

$$\left.\begin{aligned} c_2 &= -(s+\tfrac{1}{2})^2\pi^2 \quad (s = 0, 1, 2, \ldots \text{ for even } i \text{ in } g_i(x)), \\ c_2 &= -s^2\pi^2 \qquad\quad (s = 1, 2, 3, \ldots \text{ for odd } i \text{ in } g_i(x)), \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} b_0 &= b_4 = 0, \\ b_2 &= -\frac{1}{12} - \frac{1}{8c_2}, \\ b_6 &= \frac{1}{720c_2} + \frac{5}{192c_2^2} + \frac{7}{128c_2^3}, \end{aligned}\right\} \quad \text{for both even and odd } i. \tag{31}$$

It is convenient to condense the pair of equations (30) into the single equation

$$c_2 = -\left(\frac{i+1}{2}\right)^2 \pi^2 \quad (i = 0, 1, 2, 3, \ldots). \tag{32}$$

The index i, here, is identical to the suffix on $g_i(x)$, and (31) and (32) therefore summarize the first few coefficients in the expansion (25) for all values of i.

Table 2 gives the numerical values of the coefficients c_2, b_2 and b_6 for i up to 5. Notice that $|b_6| \ll |b_2| \ll |c_2|$, indicating good convergence of (25) for small values of x. As an example of the usefulness of the series for small x, $g_0(1)$ as computed from the coefficients in the table is $-2{\cdot}4999\,998$, in very good agreement with its true value of $-2\frac{1}{2}$.

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4. *Summary and discussion.* The general features of the eigenvalue dependence on x_0 for systems I and II is summarized in the graphs of Figures 2 and 3, respectively. The associated eigenfunctions are $U(a,x)$ for system I, and the even and odd functions $y_1(a,x)$ and $y_2(a,x)$ (alternately, starting with y_1 for the lowest eigenvalue) for system II.

The expansions (18) (for system I) and (25) (for system II), for the curves of these diagrams, have been carried out up to a_3 and b_6, respectively. Successive coefficients decrease rapidly in both cases (Tables 1 and 2) and convergence, for small x, is particularly good in the case of (25).

The series (18) and (25) may be of some general mathematical interest (apart from their value in the context of the constrained harmonic oscillator) as they provide a means, on reversion, of obtaining the zeros of parabolic cylinder functions. We draw attention to the particularly simple form of the coefficients of (25) (cf. (31) and (32)); it may prove possible to provide a general formula for all the coefficients of this series.

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