PROCEEDINGS OF THE
Cambridge Philosophical Society

VOL. 38 *April* 1942 PART 2

A NOTE ON k-CONNEXES

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Received 16 August 1941

The expression
$$\frac{(n+r)!\,(n+r-1)!}{(n+1)!\,n!\,r!\,(r-1)!}$$
for the number of linearly independent line complexes of degree n in space of r dimensions was derived by Sisam* as a particular case in the determination of the number of linearly independent hyperconnexes of given degrees. The purpose of this note is to generalize Sisam's result by determining the number of linearly independent algebraic forms of a special type to which we give the name k-connex.

1. Let $(x_0^i, \ldots, x_r^i)$, for $i = 0, \ldots, k$ $(k \leqslant r)$, be $k+1$ sets of $r+1$ indeterminates. We denote by $f(x^0, \ldots, x^k)$ a polynomial in these indeterminates which is homogeneous in each of the $k+1$ sets $(x_0^i, \ldots, x_r^i)$ separately. If $f(x^0, \ldots, x^k)$ is of degree l_i in $(x_0^i, \ldots, x_r^i)$, we shall say that $f(x^0, \ldots, x^r)$ is *a form of type* $(l_0, \ldots, l_k)$. If it satisfies the conditions
$$\sum_{\alpha=0}^{r} x_\alpha^i \frac{\partial f}{\partial x_\alpha^j} = 0 \qquad (1_{i,j})$$
identically, for all values of i, j subject to the inequalities $i < j \leqslant k$, we say that it is *a k-connex of type* $(l_0, \ldots, l_k)$. If we regard $(x_0^j, \ldots, x_r^j)$ as current coordinates, and the other indeterminates as parameters, we observe that the conditions that $f(x^0, \ldots, x^k)$ should be a k-connex imply that
$$f(x^0, \ldots, x^k) = 0$$
is the equation of a primal with the property that the polar prime of any point contains $(x_0^0, \ldots, x_r^0)$, ..., $(x_0^{j-1}, \ldots, x_r^{j-1})$, that is, the primal is a cone of kind j whose vertex is the space spanned by these j points. We shall see later that if

* C. H. Sisam, *Atti Torino*, 46 (1911), 481.

https://doi.org/10.1017/S0305004100021824 Published online by Cambridge University Press

$f(x^0, \ldots, x^r)$ is a k-connex, the polynomial can be written in a special form; this will be denoted by

$$F(x^0, |\, x^0x^1\,|, \ldots, |\, x^0 \ldots x^k\,|),$$

and represents a form which is homogeneous in the $r+1$ variables x^0_α, the $\binom{r+1}{2}$ variables $|\, x^0_\alpha x^1_\beta\,|, \ldots,$ the $\binom{r+1}{k+1}$ variables $|\, x^0_{\alpha_0} \ldots x^k_{\alpha_k}\,|$.

In the case $l_0 = l_1 = \ldots = l_k = n$, we see that the k-connex can be written as a homogeneous form of degree n in the determinants $|\, x^0_{\alpha_0} \ldots x^k_{\alpha_k}\,|$. Conversely, it is seen that a form which can be written as a homogeneous expression in these determinants is a k-connex. Thus the determination of the number of k-connexes of given type will lead to the determination of the number of linearly independent k-complexes of given degree. Unfortunately, the present extension of Sisam's method only leads to a formula for the number of k-connexes of type $(l_0, \ldots, l_k)$ when the l's satisfy certain inequalities which exclude the application to complexes. Nevertheless, it is worth while putting the result on record, for it seems plausible that the formula will hold whenever $l_0 \geqslant l_1 \geqslant \ldots \geqslant l_k$, though it is only proved under more severe restrictions. Indeed, it is probable that a simple argument could be found to show that the number of k-connexes of type $(l_0, \ldots, l_k)$ is given by a rational function of $l_0, \ldots, l_k, r, k$ which is valid whenever $l_0 \geqslant l_1 \geqslant \ldots \geqslant l_k$, and if this result can be established the actual function will be given by the result of this paper.

The relations $(1_{i,j})$ impose a number of conditions on the coefficients of the form $f(x^0, \ldots, x^k)$. These are not all independent; indeed, it is easily seen that they are all consequences of the k relations $(1_{0,1})$, $(1_{1,2})$, $\ldots$, $(1_{k-1,k})$. In this note, however, it will be convenient to regard all the relations $(1_{i,j})$ as $\frac{1}{2}k(k-1)$ composite conditions. We shall consider these composite conditions in a given order; $(1_{l,m})$ will precede $(1_{i,j})$ if $m < j$, or if $l < i$, $m = j$, and when these conditions are satisfied we shall write $(l, m) < (i, j)$.

We shall make considerable use of an operator which we shall denote by $\Delta^{\alpha_0 \ldots \alpha_p}_{\alpha_0 \ldots \alpha_p}$. This is defined by the equation

$$\Delta^{\alpha_0 \ldots \alpha_p}_{\alpha_0 \ldots \alpha_p} f = \sum_{\beta_0=0}^{r} \ldots \sum_{\beta_p=0}^{r} \delta^{\beta_0 \ldots \beta_p}_{\alpha_0 \ldots \alpha_p} \frac{\partial^{p+1} f}{\partial x^{\alpha_0}_{\beta_0} \ldots \partial x^{\alpha_p}_{\beta_p}},$$

where $\delta^{\beta_0 \ldots \beta_p}_{\alpha_0 \ldots \alpha_p}$ is zero unless $(\alpha_0, \ldots, \alpha_p)$ and $(\beta_0, \ldots, \beta_p)$ are derangements of the same $p+1$ distinct integers lying between 0 and r (inclusive), and is equal to $+1$ if these derangements are like, and to -1 if they are unlike. The operator $\Delta^{\alpha_0 \ldots \alpha_p}_{\alpha_0 \ldots \alpha_p}$ is antisymmetric in $\alpha_0, \ldots, \alpha_p$, and in $a_0, \ldots, a_p$, and is zero if $p > k$.

In order to avoid an excessive use of the summation symbol Σ we shall make a limited use of the summation convention. We shall agree to sum, from 0 to r, with respect to any repeated *suffix* attached to x^i_α, or $\Delta^{\alpha_0 \ldots \alpha_p}_{\alpha_0 \ldots \alpha_p}$, and we shall always

https://doi.org/10.1017/S0305004100021824 Published online by Cambridge University Press

use Greek suffixes when doing so. If, in applying this convention, we have a combination of factors which are antisymmetric in a number of suffixes the summation is to be applied to each suffix separately, and we shall use the symbol Σ to denote summation over sets of suffixes. Thus,

$$x^{a_0}_{\alpha_0}\dots x^{a_p}_{\alpha_p}\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}f(x^0,\dots,x^k)=\sum_{\alpha_0\dots\alpha_p}\begin{vmatrix}x^{a_0}_{\alpha_0}&\dots&x^{a_0}_{\alpha_p}\\ x^{a_p}_{\alpha_0}&\dots&x^{a_p}_{\alpha_p}\end{vmatrix}\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}f(x^0,\dots,x^k).$$

Equation $(1_{i,j})$ is now written

$$x^i_\alpha\frac{\partial f}{\partial x^j_\alpha}=0.$$

2. We now prove some preliminary formulae.

THEOREM I. *If $f(x^0,\dots,x^k)$ is any form,*

$$\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}\left(x^i_\beta\frac{\partial f}{\partial x^j_\beta}\right)-x^i_\beta\frac{\partial}{\partial x^j_\beta}(\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}f)$$

is zero if i is different from $a_0,\dots,a_p$ and is equal to

$$\Delta^{a_0\dots a_{q-1}ja_{q+1}\dots a_p}_{\alpha_0\dots\qquad\alpha_p}f$$

if $i=a_q$. (It is assumed that $a_0,\dots,a_p$ are distinct.)

The proof is immediate.

THEOREM II. *If $f(x^0,\dots,x^k)$ satisfies $(1_{i,j})$ and i is different from $a_0,\dots,a_p$, then*

$$g(x^0,\dots,x^k)=\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}f$$

satisfies $(1_{i,j})$.

By Theorem I,

$$x^i_\beta\frac{\partial g}{\partial x^j_\beta}=\Delta^{a_0\dots a_p}_{\alpha_0\dots\alpha_p}\left(x^i_\beta\frac{\partial f}{\partial x^j_\beta}\right)=0.$$

THEOREM III. *Let $f(x^0,\dots,x^k)$ be a form of type $(l_0,\dots,l_k)$. If $i<a_1,a_2,\dots,a_p$ and $f(x^0,\dots,x^k)$ satisfies $(1_{i,a_1})\dots(1_{i,a_p})$, then*

$$x^i_\alpha\Delta^{ia_1\dots a_p}_{\alpha\alpha_1\dots\alpha_p}f=(l_i+p)\Delta^{a_1\dots a_p}_{\alpha_1\dots\alpha_p}f.$$

For
$$x^i_\alpha\Delta^{ia_1\dots a_p}_{\alpha\alpha_1\dots\alpha_p}f=x^i_\alpha\left[\frac{\partial}{\partial x^i_\alpha}(\Delta^{a_1\dots a_p}_{\alpha_1\dots\alpha_p}f)-\sum_{j=1}^{p}\frac{\partial}{\partial x^{a_j}_\alpha}(\Delta^{a_1\dots a_{j-1}ia_{j+1}\dots a_p}_{\alpha_1\dots\qquad\alpha_p}f)\right]$$

$$=l_i\Delta^{a_1\dots a_p}_{\alpha_1\dots\alpha_p}f-\sum_{j=1}^{p}\Delta^{a_1\dots a_{j-1}ia_{j+1}\dots a_p}_{\alpha_1\dots\qquad\alpha_p}\left(x^i_\alpha\frac{\partial f}{\partial x^{a_j}_\alpha}\right)$$

$$+\sum_{j=1}^{p}\Delta^{a_1\dots a_{j-1}a_ja_{j+1}\dots a_p}_{a_1\dots\qquad\alpha_p}f\quad\text{(by Theorem I)}$$

$$=(l_i+p)\Delta^{a_1\dots a_p}_{\alpha_1\dots\alpha_p}f;$$

since $f(x^0,\dots,x^k)$ satisfies $(1_{i,a_1}),\dots,(1_{i,a_p})$.

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COROLLARY. *If* $f(x^0, \ldots, x^k)$ *satisfies* $(1_{i,j})$ *for all* $(i, j) \leqslant (k-1, k)$,

$$(l_0+k)(l_1+k-1)\ldots l_k f(x^0, \ldots, x^k) = x^0_{\alpha_0} \ldots x^k_{\alpha_k} \Delta^{0\ldots k}_{\alpha_0\ldots\alpha_k} f$$
$$= \sum_{\alpha_0\ldots\alpha_k} \begin{vmatrix} x^0_{\alpha_0} & \ldots & x^0_{\alpha_k} \\ \vdots & & \\ x^k_{\alpha_0} & \ldots & x^k_{\alpha_k} \end{vmatrix} \Delta^{0\ldots k}_{\alpha_0\ldots\alpha_k} f.$$

THEOREM IV. *If* $f(x^0, \ldots, x^k)$ *is a* k*-connex of type* $(l_0, \ldots, l_k)$, *then*

(i) $f = 0$, *unless* $l_0 \geqslant l_1 \geqslant \ldots \geqslant l_k$, *and*

(ii) *if* $l_0 \geqslant l_1 \geqslant l_2 \geqslant \ldots \geqslant l_k$, *we can write* f *in the form*

$$F(x^0, |x^0x^1|, \ldots, |x^0x^1\ldots x^k|).$$

To prove (i), suppose $l_i < l_j$ for some i, j $(i<j)$. If we regard $(x^a_0, \ldots, x^a_r)$, for all a except $a = i$, $a = j$, as parameters, f is a 1-connex of type (l_i, l_j). It is proved by Sisam that, if $l_i < l_j$, f is zero.

To prove (ii), we first observe that there is nothing to prove if $k = 0$. We proceed by induction, and for this purpose we assume that the theorem is true for $(k-1)$-connexes. If $l_k = 0$, f is a $(k-1)$-connex, and the theorem is true. Hence we may assume the theorem true for $(k-1)$-connexes, and for k-connexes of type $(l'_0, \ldots, l'_k)$, when $l'_k < l_k$.

By Theorem III,

$$(l_0+k)\ldots l_k f = \sum_{\alpha_0\ldots\alpha_k} \begin{vmatrix} x^0_{\alpha_0} & \ldots & x^0_{\alpha_k} \\ \vdots & & \\ x^k_{\alpha_0} & \ldots & x^k_{\alpha_k} \end{vmatrix} \Delta^{0\ldots k}_{\alpha_0\ldots\alpha_k} f.$$

As in Theorem II we show that $\Delta^{0\ldots k}_{\alpha_0\ldots\alpha_k} f$ is a k-connex of type $(l_0-1, \ldots, l_k-1)$, and hence, by the hypothesis of induction, it can be written in the form prescribed in the theorem. It follows that $f(x^0, \ldots, x^k)$ can also be written in the prescribed form.

The converse of this theorem, that a form

$$F(x^0, |x^0x^1|, \ldots, |x^0\ldots x^k|),$$

when regarded as a form $f(x^0, \ldots, x^k)$, is a k-connex can be proved immediately.

3. In order to determine the number of linearly independent k-connexes of given type $(l_0, \ldots, l_k)$ (where we now assume $l_0 \geqslant l_1 \geqslant \ldots \geqslant l_k$), we have to determine the number of conditions imposed on the coefficients of a form $f(x^0, \ldots, x^k)$ of the given type when we impose the composite conditions $(1_{0,1})$, $(1_{0,2})$, ..., successively. Since the ultimate induction is on k, we may suppose that we know the number of conditions imposed by the equations $(1_{i,j})$ for $(i, j) \leqslant (k-2, k-1)$ and consider the effect of imposing the conditions $(1_{0,k})$, $(1_{1,k})$, ... If we denote the number of linearly independent k-connexes of type $(l_0, \ldots, l_k)$ by $\phi_{r,k}(l_0, \ldots, l_k)$, the number of forms of type $(l_0, \ldots, l_k)$ which satisfy the equations $(1_{0,1})$, ..., $(1_{k-2,k-1})$ is

$$\phi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k+r}{r}.$$

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THEOREM V. *Let* $f(x^0, \ldots, x^k)$ *be a form of type* $(l_0, \ldots, l_k)$ *which satisfies the equations* $(1_{i,j})$ *for all* $(i, j) < (a, k)$, *and let* $g(x^0, \ldots, x^k)$ *be the form of type*

$$(l_0, \ldots, l_{a-1}, l_a + 1, l_{a+1}, \ldots, l_{k-1}, l_k - 1)$$

defined by the equation

$$g = x_\alpha^a \frac{\partial f}{\partial x_\alpha^k}.$$

Then $g(x^0, \ldots, x^k)$ *also satisfies the equations* $(1_{i,j})$ *for all* $(i, j) < (a, k)$.

If $(i, j) < (a, k)$,

$$x_\alpha^i \frac{\partial g}{\partial x_\alpha^j} = x_\alpha^i \frac{\partial}{\partial x_\alpha^j}\left(x_\beta^a \frac{\partial f}{\partial x_\beta^k}\right)$$

$$= \delta_j^a x_\alpha^i \frac{\partial f}{\partial x_\alpha^k} + x_\beta^a \frac{\partial}{\partial x_\beta^k}\left(x_\alpha^i \frac{\partial f}{\partial x_\alpha^j}\right) - \delta_k^i x_\beta^a \frac{\partial f}{\partial x_\beta^j}.$$

But, since $(i, j) < (a, k)$, we have $i < j \leqslant k$, and hence $\delta_k^i = 0$; and if $a = j$, $i < a$, and by hypothesis we have

$$x_\alpha^i \frac{\partial f}{\partial x_\alpha^k} = 0, \quad x_\alpha^i \frac{\partial f}{\partial x_\alpha^j} = 0.$$

Therefore

$$x_\alpha^i \frac{\partial g}{\partial x_\alpha^j} = 0.$$

The converse of Theorem V is

THEOREM VI. *If* $g(x^0, \ldots, x^k)$ *is a form of type* $(l_0, \ldots, l_{a-1}, l_a + 1, l_{a+1}, \ldots, l_{k-1}, l_k - 1)$ *which satisfies equations* $(1_{i,j})$ *for all* $(i, j) < (a, k)$, *there exists a form* $f(x^0, \ldots, x^k)$ *of type* $(l_0, \ldots, l_k)$ *which satisfies the equations* $(1_{i,j})$ *for all* $(i, j) < (a, k)$, *and also the equation*

$$x_\alpha^a \frac{\partial f}{\partial x_\alpha^k} = g.$$

This theorem is not true for all values of $l_0, \ldots, l_k$ which are subject only to the conditions $l_0 \geqslant \ldots \geqslant l_k$, and we shall prove it only when the l's satisfy the more restrictive conditions

$$l_i \geqslant l_{i+1} + \ldots + l_k + \rho_i^{a,k} \quad (i = 0, \ldots, k-1),$$

where $\rho_i^{a,k}$ is a number depending only on a, k, i. If the numbers $l_0, \ldots, l_k$ satisfy the equations

$$l_i \geqslant l_{i+1} + \ldots + l_j + \rho_i^{a,j} \quad (i = 0, \ldots, j-1),$$

where the numbers $\rho_i^{a,j}$ are defined inductively by the relations

$$\rho_i^{a,j} = \rho_i^{\overline{a,j}} + a + 1, \quad \rho_i^{i,i+1} = 0,$$

$(\overline{a, j})$ being the immediate predecessor of the (a, j),* we shall say that they satisfy conditions $A_{a,j}$. We observe that if $l_0, \ldots, l_k$ satisfy conditions $A_{i,j}$, they satisfy

* We shall frequently use the symbol $(\overline{a, j})$ to denote the immediate predecessor of (a, j).

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$A_{l,m}$ when $(l,m)<(i,j)$. We therefore propose to prove Theorem VI when the l's satisfy conditions $A_{a,k}$.

Theorem VI is proved by Sisam for the case $k=1$, when the only possible value of a is $a=0$. We now assume that it is true for all forms $g(x^0,\ldots,x^{k'})$, where $k'<k$, and also that it is true for forms in $(k+1)$ sets of indeterminates when (a,k) is replaced by its immediate predecessor $(\overline{a,k})$, subject of course to the restriction that conditions $A_{\overline{a,k}}$ are satisfied. Before going on to the case (a,k) we derive a consequence of the assumption of the truth of the theorem in the case $(\overline{a,k})$.

4. LEMMA. *If ϕ_{ij} is a form of type $(l_0,\ldots,l_{i-1},l_i+1,l_{i+1},\ldots,l_{j-1},l_j-1,l_{j+1},\ldots,l_k)$, and the l's satisfy the conditions $A_{\overline{a,k}}$, the necessary and sufficient conditions that there should exist a form $f(x^0,\ldots,x^k)$ which satisfies the equations*

$$x_\alpha^i\frac{\partial f}{\partial x_\alpha^j}=\phi_{ij} \tag{2}$$

for all $(i,j)\leqslant(\overline{a,k})$, are

$$x_\alpha^l\frac{\partial\phi_{ij}}{\partial x_\alpha^m}-x_\alpha^i\frac{\partial\phi_{lm}}{\partial x_\alpha^j}=\delta_m^i\phi_{lj} \tag{3}$$

for all $(l,m)<(i,j)\leqslant(\overline{a,k})$.

Suppose, first, that a form of $f(x^0,\ldots,x^k)$ satisfying (2) exists. Then

$$\begin{aligned}x_\alpha^l\frac{\partial\phi_{ij}}{\partial x_\alpha^m}-x_\alpha^i\frac{\partial\phi_{lm}}{\partial x_\alpha^j}&=x_\alpha^l\frac{\partial}{\partial x_\alpha^m}\left(x_\beta^i\frac{\partial f}{\partial x_\beta^j}\right)-x_\alpha^i\frac{\partial}{\partial x_\alpha^j}\left(x_\beta^l\frac{\partial f}{\partial x_\beta^m}\right)\\&=\delta_m^i x_\alpha^l\frac{\partial f}{\partial x_\alpha^j}+x_\alpha^l x_\beta^i\frac{\partial^2 f}{\partial x_\alpha^m\partial x_\beta^j}-\delta_j^l x_\alpha^i\frac{\partial f}{\partial x_\alpha^m}-x_\alpha^i x_\beta^l\frac{\partial^2 f}{\partial x_\alpha^j\partial x_\beta^m}\\&=\delta_m^i x_\alpha^l\frac{\partial f}{\partial x_\alpha^j}-\delta_j^l x_\alpha^i\frac{\partial f}{\partial x_\alpha^m}.\end{aligned}$$

Since $l<m$, it follows that $(l,m)>(i,j)$ if $l=j$; hence, by the conditions of the lemma, $\delta_j^l=0$. If $i=m$, $l<i$, and hence $(l,j)<(i,j)<(a,k)$. Hence we have

$$x_\alpha^l\frac{\partial\phi_{ij}}{\partial x_\alpha^m}-x_\alpha^i\frac{\partial\phi_{lm}}{\partial x_\alpha^j}=\delta_m^i\phi_{lj},$$

and the conditions are therefore necessary.

To prove sufficiency, we suppose that (3) is satisfied. By Sisam's argument (or by that given below in § 6), we can find a form $f_{01}(x^0,\ldots,x^k)$ which satisfies

$$x_\alpha^0\frac{\partial f_{01}}{\partial x_\alpha^1}=\phi_{01}.$$

To solve the equations $\quad x_\alpha^0\dfrac{\partial f_{02}}{\partial x_\alpha^1}=\phi_{01},\quad x_\alpha^0\dfrac{\partial f_{02}}{\partial x_\alpha^2}=\phi_{02},$

we write $\qquad f_{02}=f_{01}+\Psi_{02},$

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and find Ψ_{02} to satisfy

$$x_\alpha^0 \frac{\partial \Psi_{02}}{\partial x_\alpha^1} = 0, \quad x_\alpha^0 \frac{\partial \Psi_{02}}{\partial x_\alpha^2} = \phi_{02} - x_\alpha^0 \frac{\partial f_{01}}{\partial x_\alpha^2}.$$

By the hypothesis of induction, Theorem VI is true when (a, k) is replaced by $(0, 2)$ and, since the conditions $A_{\overline{a,k}}$ imply condition $A_{0,2}$, the conditions for the existence of Ψ_{02} are

$$x_\beta^0 \frac{\partial}{\partial x_\beta^1}\left(\phi_{02} - x_\alpha^0 \frac{\partial f_{01}}{\partial x_\alpha^2}\right) = 0,$$

that is

$$x_\alpha^0 \frac{\partial}{\partial x_\alpha^1}\phi_{02} - x_\alpha^0 \frac{\partial}{\partial x_\alpha^2}\phi_{01} = 0,$$

and this follows from (3). We proceed in this way, seeking to find a form f_{ij} which satisfies the equations

$$x_\alpha^l \frac{\partial f_{ij}}{\partial x_\alpha^m} = \phi_{lm},$$

for $(l, m) \leqslant (i, j)$. Let $(\overline{i, j})$ be the immediate predecessor of (i, j) and suppose we have constructed $f_{\overline{ij}}$. We write

$$f_{ij} = f_{\overline{ij}} + \Psi_{ij},$$

and we have to find Ψ_{ij} to satisfy

$$x_\alpha^l \frac{\partial \Psi_{ij}}{\partial x_\alpha^m} = 0 \quad ((l, m) < (i, j)),$$

$$x_\alpha^i \frac{\partial \Psi_{ij}}{\partial x_\alpha^j} = \phi_{ij} - x_\alpha^i \frac{\partial f_{\overline{ij}}}{\partial x_\alpha^j}.$$

Now, conditions $A_{\overline{a,k}}$ imply conditions $A_{i,j}$, $(i, j) \leqslant (\overline{a, k})$, and therefore, using Theorem VI, with (a, k) replaced by (i, j), we can find Ψ_{ij}, provided that

$$x_\alpha^l \frac{\partial}{\partial x_\alpha^m}\left(\phi_{ij} - x_\beta^i \frac{\partial f_{\overline{ij}}}{\partial x_\beta^j}\right) = 0,$$

for all $(l, m) < (i, j)$. These conditions reduce to

$$x_\alpha^l \frac{\partial \phi_{ij}}{\partial x_\alpha^m} - \delta_m^i x_\alpha^l \frac{\partial f_{\overline{ij}}}{\partial x_\alpha^j} - x_\beta^i \frac{\partial}{\partial x_\beta^j}\left(x_\alpha^l \frac{\partial f_{\overline{ij}}}{\partial x_\alpha^m}\right) = 0.$$

Using the result, proved above, that, since $(l, m) < (i, j)$, we cannot have $l = j$, and remembering that, if $i = m$, $(l, j) < (i, j)$, this equation is just

$$x_\alpha^l \frac{\partial \phi_{ij}}{\partial x_\alpha^m} - x_\alpha^i \frac{\partial \phi_{lm}}{\partial x_\alpha^j} = \delta_m^i \phi_{lj}.$$

The conditions for the existence of Ψ_{ij} are therefore satisfied on account of (3), and we can thus find f_{ij}. The form

$$f(x^0, \ldots, x^k) = f_{\overline{ak}}$$

satisfies the requirements of the lemma.

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5. In order to prove Theorem VI we use this lemma to prove the existence of certain forms $\phi_{p_0 \dots p_a}$, antisymmetric in the suffixes, which can take all values from 0 to $k-1$. The forms $\phi_{p_0 \dots p_a}$ satisfy the following conditions:

(1) $\phi_{p_0 \dots p_a}$ is homogeneous of degree ρ_i in $(x_0^i, \dots, x_r^i)$ where, if $i < a$, ρ_i is equal to $l_i - 1$ or l_i, according as i is not, or is, amongst the numbers $p_0, \dots, p_a$; if $a \leqslant i < k$, $\rho_i = l_i$ or $l_i + 1$, according as i is not, or is, amongst those numbers; and $\rho_k = l_k - 1$;

(2) $\phi_{0 \dots a} = g(x^0, \dots, x^k)$;

$$x_\alpha^i \frac{\partial \phi_{p_0 \dots p_a}}{\partial x_\alpha^j} = \sum_{s=0}^{a} \delta_j^{p_s} \phi_{p_0 \dots p_{s-1} i p_{s+1} \dots p_a}, \tag{3}$$

$$\tag{4}$$

for all $(i, j) < (a, k)$.

It is assumed that the l's satisfy conditions $A_{a,k}$. It follows that if $\phi_{p_0 \dots p_a}$ is of type $(\rho_0, \dots, \rho_k)$ the ρ's satisfy $A_{\overline{a,k}}$.

If we arrange the forms $\phi_{p_0 \dots p_a}$ in any order subject to the condition that $\phi_{i_0 \dots i_a}$ precedes $\phi_{j_0 \dots j_a}$ whenever $\Sigma i_s < \Sigma j_s$, we see that the forms $\phi_{p_0 \dots p_a}$ are defined recurrently, since $i < j$ in (i, j), and in order to see that the forms can be found we have merely to show that, for a given $\phi_{p_0 \dots p_a}$, the right-hand sides of (4) satisfy equations (3). If $(l, m) < (i, j) < (a, k)$,

$$\begin{aligned} x_\alpha^l \frac{\partial}{\partial x_\alpha^m} \Big[\sum_s \delta_j^{p_s} \phi_{p_0 \dots p_{s-1} i p_{s+1} \dots p_a} \Big] - x_\alpha^i \frac{\partial}{\partial x_\alpha^j} \Big[\sum_t \delta_m^{p_t} \phi_{p_0 \dots p_{t-1} l p_{t+1} \dots p_a} \Big] \\ = \delta_m^i \sum_s \delta_j^{p_s} \phi_{p_0 \dots p_{s-1} l p_{s+1} \dots p_a} - \delta_j^l \sum_t \delta_m^{p_t} \phi_{p_0 \dots p_{t-1} i p_{t+1} \dots p_a} \\ + \sum_s \sum_{t \neq s} \delta_j^{p_s} \delta_m^{p_t} \phi_{p_0 \dots p_{s-1} i p_{s+1} \dots p_{t-1} l p_{t+1} \dots p_a} \\ - \sum_t \sum_{s \neq t} \delta_j^{p_s} \delta_m^{p_t} \phi_{p_0 \dots p_{s-1} i p_{s+1} \dots p_{t-1} l p_{t+1} \dots p_a}. \end{aligned}$$

The last two terms cancel. Also we show, as in § 4, that if $(l, m) < (i, j)$ we cannot have $l = j$, and if $i = m$, $(l, j) < (i, j) < (a, k)$. Hence the equations (3) are satisfied and the forms $\phi_{p_0 \dots p_a}$ can be found.

For simplicity in notation we shall write

$$\phi_{p_1 \dots p_s} = \phi_{0 \dots a-s\, p_1 \dots p_s}.$$

We now prove that, if $s < a$,

$$(l_s + k - s) \sum_{p<s} \Delta^{p_{s+1} \dots p_a k}_{\alpha_{s+1} \dots \alpha_a \beta} \phi_{p_{s+1} \dots p_a} = \sum_{p>s-1} x_{\alpha_s}^s \Delta^{p_s \dots p_a k}_{\alpha_s \dots \alpha_a \beta} \phi_{p_s \dots p_a}. \tag{5}$$

Using equation (4), we see that

$$\phi_{p_{s+1} \dots p_a} = \frac{1}{k-a-1} \sum_{q=s+1}^{k-1} x_\alpha^s \frac{\partial}{\partial x_\alpha^q} \phi_{q p_{s+1} \dots p_a}.$$

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Hence
$$F_s = \sum_{p>s} \Delta^{p_{s+1}\dots p_a k}_{\alpha_{s+1}\dots\alpha_a\beta}\,\phi_{p_{s+1}\dots p_a}$$
$$= \frac{1}{k-a-1}\sum_{p>s}\sum_{q>s} x^s_\alpha \Delta^{p_{s+1}\dots p_a k}_{\alpha_{s+1}\dots\alpha_a\beta}\frac{\partial}{\partial x^q_\alpha}\phi_{qp_{s+1}\dots p_a}$$
$$= \frac{1}{k-a-1}\sum_{p>s} x^s_{\alpha_s}\Delta^{p_s\dots p_a k}_{\alpha_s\dots\alpha_a\beta}\phi_{p_s\dots p_a},$$

since
$$x^s_\alpha \frac{\partial \phi_{qp_{s+1}\dots p_a}}{\partial x^k_\alpha} = 0.$$

Hence
$$F_s = \frac{1}{k-a-1}\sum_{p>s-1} x^s_{\alpha_s}\Delta^{p_s\dots p_a k}_{\alpha_s\dots\alpha_a\beta}\phi_{p_s\dots p_a}$$
$$-\frac{1}{k-a-1}\sum_{p>s} x^s_{\alpha_s}\Delta^{sp_{s+1}\dots p_a k}_{\alpha_s\dots\alpha_a\beta}\phi_{sp_{s+1}\dots p_a}$$
$$= \frac{1}{k-a-1}\sum_{p>s-1} x^s_{\alpha_s}\Delta^{p_s\dots p_a k}_{\alpha_s\dots\alpha_a\beta}\phi_{p_s\dots p_a}$$
$$-\frac{l_s+a-s+1}{k-a-1}\sum_{p>s}\Delta^{p_{s+1}\dots p_a k}_{\alpha_{s+1}\dots\alpha_a\beta}\phi_{p_{s+1}\dots p_a},$$

using Theorem III. Hence
$$(k-a-1+l_s+a-s+1)F_s = \sum_{p>s-1} x^s_{\alpha_s}\Delta^{p_s\dots p_a k}_{\alpha_s\dots\alpha_a\beta}\phi_{p_s\dots p_a},$$
and this is the required result.

6. We are now in a position to construct a form $f(x^0, x^1, \dots, x^k)$ to satisfy Theorem VI. We define $\phi_p = \phi_{0\dots a-1\,p}$ as in § 5, and consider the form
$$F = \frac{1}{(l_a+k-a)\,l_k}\sum_{p=a}^{k-1} x^k_\alpha \frac{\partial \phi_p}{\partial x^p_\alpha}.$$
Consider $(i,j) < (a,k)$. Then
$$x^i_\beta \frac{\partial F}{\partial x^j_\beta} = \frac{1}{(l_a+k-a)\,l_k}\sum_{p=a}^{k-1}\left[\delta^j_k x^i_\alpha \frac{\partial\phi_p}{\partial x^p_\alpha} + x^k_\alpha \frac{\partial}{\partial x^p_\alpha}\left(x^i_\beta \frac{\partial\phi_p}{\partial x^j_\beta}\right) - \delta^i_p x^k_\alpha \frac{\partial\phi_p}{\partial x^j_\alpha}\right]$$
$$= \frac{1}{(l_a+k-a)\,l_k}\sum_{p=a}^{k-1}\left[\delta^j_k\phi_i + \delta^j_p x^k_\alpha \frac{\partial\phi_i}{\partial x^p_\alpha} - \delta^i_p x^k_\alpha \frac{\partial\phi_p}{\partial x^j_\alpha}\right].$$

If $j = k$, $i < a$ and therefore $\phi_i = 0$. Hence
$$x^i_\beta \frac{\partial F}{\partial x^j_\beta} = \frac{1}{(l_a+k-a)\,l_k}\left[x^k_\alpha \frac{\partial\phi_i}{\partial x^j_\alpha} - x^k_\alpha \frac{\partial\phi_i}{\partial x^j_\alpha}\right] = 0.$$

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Again

$$x_\alpha^a \frac{\partial F}{\partial x_\alpha^k} = \frac{1}{(l_a+k-a)\,l_k} \sum_{p=a}^{k-1} \left[x_\alpha^a \frac{\partial \phi_p}{\partial x_\alpha^p} + x_\alpha^a x_\beta^k \frac{\partial^2 \phi_p}{\partial x_\alpha^k \partial x_\beta^p} \right]$$

$$= \frac{1}{(l_a+k-a)\,l_k} \left[(l_a+k-a)\,g + x_\alpha^a x_\beta^k \sum_p \frac{\partial^2 \phi_p}{\partial x_\alpha^p \partial x_\beta^k} - x_\alpha^a x_\beta^k \sum_p \Delta_{\alpha\beta}^{pk} \phi_p \right]$$

$$= g - \frac{1}{(l_a+k-a)\,l_k} x_\alpha^a x_\beta^k \sum_{p>a-1} \Delta_{\alpha\beta}^{pk} \phi_p$$

$$= g - \frac{1}{l_k \prod_{i=0}^{a} (l_i+k-i)} x_{\alpha_0}^0 \dots x_{\alpha_a}^a x_\beta^k \sum_p \Delta_{\alpha_0 \dots \alpha_a \beta}^{p_0 \dots p_a k} \phi_{p_0 \dots p_a}$$

$$= g - \frac{1}{l_k \prod_{i=0}^{a} (l_i+k-i)} \sum_{\alpha_0 \dots \alpha_{a+1}} \begin{vmatrix} x_{\alpha_0}^0 \dots x_{\alpha_{a+1}}^0 \\ \vdots \\ x_{\alpha_0}^a \dots x_{\alpha_{a+1}}^a \\ x_{\alpha_0}^k \dots x_{\alpha_{a+1}}^k \end{vmatrix} \sum_p \Delta_{\alpha_0 \dots \alpha_a \alpha_{a+1}}^{p_0 \dots p_a k} \phi_{p_0 \dots p_a},$$

by repeated application of equation (5).

We now consider the expression

$$\Psi_{\alpha_0 \dots \alpha_{a+1}} = \sum_p \Delta_{\alpha_0 \dots \alpha_{a+1}}^{p_0 \dots p_a k} \phi_{p_0 \dots p_a}.$$

This is a form of type $(l_0-1, \dots, l_{a-1}-1, l_a, \dots, l_{k-1}, l_k-2)$. Moreover, if $(i,j) < (a,k)$,

$$x_\beta^i \frac{\partial \Psi_{\alpha_0 \dots \alpha_{a+1}}}{\partial x_\beta^j} = \sum_p \Delta_{\alpha_0 \dots \alpha_{a+1}}^{p_0 \dots p_a k} \left(\sum_s \delta_j^{p_s} \phi_{p_0 \dots p_{s-1} i p_{s+1} \dots p_a} \right)$$

$$- \sum_p \sum_s \delta_i^{p_s} \Delta_{\alpha_0 \dots}^{p_0 \dots p_{s-1} j p_{s+1} \dots p_a k}{}_{\alpha_{a+1}} \phi_{p_0 \dots p_a}$$

$$= 0.$$

If $l_k = 1$, Theorem VI is now proved, since $\Psi_{\alpha_0 \dots \alpha_{a+1}} = 0$. We now prove the theorem generally by induction on l_k.

We observe that the numbers $l_0-1, \dots, l_a-1, l_{a+1}, \dots, l_{k-1}, l_k-1$ also satisfy conditions $A_{a,k}$. Since the $\Psi_{\alpha_0 \dots \alpha_{a+1}}$ satisfy the conditions of Theorem VI we can, by the hypothesis of induction, find forms $f_{\alpha_0 \dots \alpha_{a+1}}$ which satisfy

$$x_\beta^i \frac{\partial f_{\alpha_0 \dots \alpha_{a+1}}}{\partial x_\beta^j} = 0 \quad (i,j) < (a,k),$$

$$x_\beta^a \frac{\partial f_{\alpha_0 \dots \alpha_{a+1}}}{\partial x_\beta^k} = \Psi_{\alpha_0 \dots \alpha_{a+1}} \Big/ l_k \prod_{i=0}^{a} (l_i+k-i).$$

It may be verified immediately that

$$f(x^0, \dots, x^k) = F + \sum_{\alpha_0 \dots \alpha_{a+1}} \begin{vmatrix} x_{\alpha_0}^0 & \dots & x_{\alpha_{a+1}}^0 \\ \vdots & & \\ x_{\alpha_0}^k & \dots & x_{\alpha_{a+1}}^k \end{vmatrix} f_{\alpha_0 \dots \alpha_{a+1}}$$

is a form of type $(l_0, \dots, l_k)$ which satisfies the requirements of Theorem VI.

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7. We use Theorems V and VI in order to calculate the number of linearly independent k-connexes of type $(l_0, \ldots, l_k)$. We shall assume that $l_0, \ldots, l_k$ satisfy the conditions $A_{k-1,k}$. We denote the required number by $\phi_{r,k}(l_0, \ldots, l_k)$.

The number of forms $f(x^0, \ldots, x^k)$ which satisfy equations $(1_{0,1}), \ldots, (1_{k-2,k-1})$ is

$$\phi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k + r}{r}.$$

We take any such form f, and consider

$$x^0_\alpha \frac{\partial f}{\partial x^k_\alpha} = g_0.$$

g_0 is a form of type $(l_0 + 1, l_1, \ldots, l_{k-1}, l_k - 1)$ which satisfies $(1_{0,1}), \ldots, (1_{k-2,k-1})$, and no further conditions since, given any g_0 satisfying $(1_{0,1}), \ldots, (1_{k-2,k-1})$, there exists an f satisfying these equations and the equation

$$x^0_\alpha \frac{\partial f}{\partial x^k_\alpha} = g_0.$$

The number of extra conditions imposed on the coefficients of $f(x^0, \ldots, x^k)$ by making it satisfy $(1_{0,k})$ is therefore equal to the number of independent forms g_0, that is, since the numbers $l_0 + 1, l_1, \ldots, l_{k-1}, l_k - 1$ satisfy condition $A_{k-1,k}$,

$$\phi_{r,k-1}(l_0 + 1, l_1, \ldots, l_{k-1}) \binom{l_k + r - 1}{r},$$

and so the number of forms $f(x^0, \ldots, x^k)$ satisfying $(1_{i,j})$ for all $(i, j) \leqslant (0, k)$ is

$$\phi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k + r}{r} - \phi_{r,k-1}(l_0 + 1, l_1, \ldots, l_{k-1}) \binom{l_k + r - 1}{r},$$

whenever $l_0, \ldots, l_k$ satisfy condition $A_{k-1,k}$.

In the same way, we show that the number of forms of type $(l_0, \ldots, l_k)$ which satisfy the equation $(1_{i,j})$ for all $(i, j) \leqslant (1, k)$ is

$$\begin{aligned}
&\phi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k + r}{r} - \phi_{r,k-1}(l_0 + 1, l_1, \ldots, l_{k-1}) \binom{l_k + r - 1}{r} \\
&- \left[\phi_{r,k-1}(l_0, l_1 + 1, \ldots, l_{k-1}) \binom{l_k + r - 1}{r} - \phi_{r,k-1}(l_0 + 1, l_1 + 1, l_1, \ldots, l_{k-1}) \binom{l_k + r - 2}{r} \right],
\end{aligned}$$

whenever $l_0, \ldots, l_k$ satisfy condition $A_{k-1,k}$; and by a simple induction we prove that the number $\phi_{r,k}(l_0, \ldots, l_k)$ of forms of type $(l_0, \ldots, l_k)$ which satisfy equations $(1_{0,1}), \ldots, (1_{k,k-1})$ is given by

$$\begin{aligned}
\phi_{r,k}(l_0, \ldots, l_k) = {} & \phi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k + r}{r} \\
& - \sum_i \phi_{r,k-1}(l_0, \ldots, l_{i-1}, l_i + 1, l_{i+1}, \ldots, l_{k-1}) \binom{l_k + r - 1}{r} \\
& + \sum_{i,j} \phi_{r,k-1}(l_0, \ldots, l_{i-1}, l_i + 1, l_{i+1}, \ldots, l_{j-1}, l_j + 1; l_{j+1}, \ldots, l_{k-1}) \binom{l_k + r - 2}{r} \\
& - \ldots \\
& + (-)^k \phi_{r,k-1}(l_0 + 1, \ldots, l_{k-1} + 1) \binom{l_k + r - k}{r}. \qquad (6)
\end{aligned}$$

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We also observe that

$$\phi_{r,0}(l) = \binom{l+r}{r}.$$

8. We now prove that the solution of the difference equation

$$\begin{aligned}\Psi_{r,k}(l_0, \ldots, l_k) = {} & \Psi_{r,k-1}(l_0, \ldots, l_{k-1}) \binom{l_k+r}{r} \\ & - \sum_i \Psi_{r,k-1}(l_0, \ldots, l_i+1, \ldots, l_{k-1}) \binom{l_k+r-1}{r} \\ & + \sum_{i,j} \Psi_{r,k-1}(l_0, \ldots, l_i+1, \ldots, l_j+1, \ldots, l_{k-1}) \binom{l_k+r-2}{r} \\ & - \ldots \\ & + (-)^k \Psi_{r,k-1}(l_0+1, \ldots, l_k+1) \binom{l_k+r-k}{r}, \end{aligned} \tag{7}$$

when the initial value is $\Psi_{r,0}(l) = \binom{l+r}{r}$, is

$$\Psi_{r,k}(l_0, \ldots, l_k) = \frac{(l_0+r)! \ldots (l_k+r-k)!}{(l_0+k)! \ldots l_k!} \frac{\prod_{i<j} (l_i - l_j - i + j)}{r! \ldots (r-k)!}.$$

We prove this by showing inductively that the difference equation is satisfied when this expression is substituted in it. We first show that, when this substitution is made, the polynomials in l_0 forming the two sides of (7) have the same coefficients of l_0^r, l_0^{r-1}.

The coefficient of l_0^r in $\Psi_{r,k}(l_0, \ldots, l_k)$ is

$$\frac{(l_1+r-1)! \ldots (l_k+r-k)!}{(l_1+k-1)! \ldots l_k!} \frac{\prod_{0<i<j} (l_i - l_j - i + j)}{(r-k)! \ldots r!} = \frac{1}{r!} \Psi_{r-1,k-1}(l_1, \ldots, l_k),$$

and, similarly the coefficient of l_0^{r-1} is

$$\begin{aligned} & \frac{1}{r!} \Psi_{r-1,k-1}(l_1, \ldots, l_k) \left[\sum_{k+1}^{r} s - \sum_{j=1}^{k} l_j + \sum_{1}^{k} j \right] \\ & \qquad = \frac{1}{r!} \Psi_{r-1,k-1}(l_1, \ldots, l_k) \left[\tfrac{1}{2} r(r+1) - \sum_{j=1}^{k} l_j \right]. \end{aligned}$$

To calculate the coefficients of l_0^r and l_0^{r-1} on the right-hand side of (7) we must separate the terms $\Psi_{r,k-1}(m_0, \ldots, m_{k-1})$ in which m_0 is l_0 from those in which m_0 is l_0+1. We find that the coefficient of l_0^r is

$$\begin{aligned} \frac{1}{r!} \Bigg[& \Psi_{r-1,k-2}(l_1, \ldots, l_{k-1}) \left\{ \binom{l_k+r}{r} - \binom{l_k+r-1}{r} \right\} \\ & - \sum_i \Psi_{r-1,k-2}(l_1, \ldots, l_i+1, \ldots, l_{k-1}) \left\{ \binom{l_k+r-1}{r} - \binom{l_k+r-2}{r} \right\} + \ldots \\ & + (-1)^{k-1} \Psi_{r-1,k-2}(l_1+1, \ldots, l_{k-1}+1) \left\{ \binom{l_k+r-k+1}{r} - \binom{l_k+r-k}{r} \right\} \Bigg] \end{aligned}$$

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$$= \frac{1}{r!}\Big[\Psi_{r-1,k-2}(l_1,\ldots,l_{k-1})\binom{l_k+r-1}{r-1} - \sum_i \Psi_{r-1,k-2}(l_1,\ldots,l_i+1,\ldots,l_{k-1})\binom{l_k+r-2}{r-1}+\ldots + (-1)^{k-1}\Psi_{r-1,k-2}(l_1+1,\ldots,l_{k-1}+1)\binom{l_k+r-k}{r-1}\Big]$$

$$= \frac{1}{r!}\Psi_{r-1,k-1}(l_1,\ldots,l_k),$$

by the hypothesis of induction.

Similarly the coefficient of l_0^{r-1} in the right-hand side of (7) is

$$\frac{1}{r!}\Big[\Psi_{r-1,k-2}(l_1,\ldots,l_{k-1})\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j\Big\}\binom{l_k+r}{r} - \sum_i \Psi_{r-1,k-2}(l_1,\ldots,l_i+1,\ldots,l_{k-1})\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j-1\Big\}\binom{l_k+r-1}{r}+\ldots + (-1)^{k-1}\Psi_{r-1,k-2}(l_1+1,\ldots,l_{k-1}+1)\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j-k+1\Big\}\binom{l_k+r-k+1}{r}\Big]$$

$$-\frac{1}{r!}\Big[\Psi_{r-1,k-2}(l_1,\ldots,l_{k-1})\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j+r\Big\}\binom{l_k+r-1}{r} - \sum_i \Psi_{r-1,k-2}(l_1,\ldots,l_i+1,\ldots,l_{k-1})\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j+r-1\Big\}\binom{l_k+r-2}{r}+\ldots + (-1)^{k-1}\Psi_{r-1,k-2}(l_1+1,\ldots,l_{k-1}+1)\Big\{\tfrac{1}{2}r(r+1)-\sum_1^{k-1}l_j+r-k+1\Big\}\binom{l_k+r-k}{r}\Big]$$

$$= \Big[\tfrac{1}{2}r(r+1)-\sum_1^k l_j\Big]\frac{1}{r!}\Psi_{r-1,k-1}(l_1,\ldots,l_k)$$

$$+\frac{1}{r!}\Big[\Psi_{r-1,k-2}(l_1,\ldots,l_{k-1})\Big\{l_k\binom{l_k+r}{r}-(l_k+r)\binom{l_k+r-1}{r}\Big\} - \sum_i \Psi_{r-1,k-2}(l_1,\ldots,l_i+1,\ldots,l_{k-1})\Big\{(l_k-1)\binom{l_k+r-1}{r} - (l_k+r-1)\binom{l_k+r-2}{r}\Big\}+\ldots + (-1)^{k-1}\Psi_{r-1,k-2}(l_1+1,\ldots,l_{k-1}+1)\Big\{(l_k-k+1)\binom{l_k+r-k+1}{r} - (l_k+r-k+1)\binom{l_k+r-k}{r}\Big\}\Big]$$

$$= \frac{1}{r!}\Psi_{r-1,k-1}(l_1,\ldots,l_k)\Big[\tfrac{1}{2}r(r+1)-\sum_1^k l_j\Big].$$

Since the two sides of equation (7) have the same coefficients of l_0^r and l_0^{r-1}, they will be identical if we can show that they have $r-1$ common zeros. The zeros of $\Psi_{r,k}(l_0,\ldots,l_k)$ are $l_0 = -r, -r+1, \ldots, -(k+1), l_1-1, l_2-2, \ldots, l_k-k$.

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We show that the first $r-1$ of these are zeros of the right-hand side of (7). Clearly $-r, -r+1, \ldots, -(k+1)$ are zeros of this expression, since they are zeros of each term. To prove that $l_1-1, \ldots, l_{k-1}-k+1$ are also zeros we prove the following lemma:

Let

$$\chi(p,q) = \Psi_{r,k-1}(m_0, \ldots, m_{i-1}, p+1, m_{i+1}, \ldots, m_{j-1}, q, m_{j+1}, \ldots, m_{k-1}) + \Psi_{r,k-1}(m_0, \ldots, m_{i-1}, p, m_{i+1}, \ldots, m_{j-1}, q+1, m_{j+1}, \ldots, m_{k-1}).$$

Then

$$\chi(p, p-i+j) = 0.$$

Let

$$A = \frac{(m_0+r)! \ldots (m_{i-1}+r-i+1)!\,(m_{i+1}+r-i-1)! \ldots (m_{j-1}+r-j+1)!}{(m_0+k-1)! \ldots (m_{i-1}+k-i)!\,(m_{i+1}+k-i-2)! \ldots (m_{j-1}+k-j)!} \times \frac{(m_{j+1}+r-j-1)! \ldots (m_{k-1}+r-k+1)! \prod_{s<t}^{s,t\neq i,j} (m_s-m_t-s+t)}{(m_{j+1}+k-j-2)! \ldots m_{k-1}!\,(r-k+1)! \ldots r!}$$

Then

$$\chi(p,q) = A\Bigg[\frac{(p+1+r-i)!}{(p+1+k-i-1)!}(p+1-i-q+j)\prod_{s<i}(m_s-s-p-1+i) \times \prod_{s>i}^{s\neq j}(p+1-i-m_s+s) + \frac{(q+r-j)!}{(q+k-j-1)!}\prod_{s<j}^{s\neq i}(m_s-s-q+j)\prod_{s>j}(q-j-m_s+s) + \frac{(p+r-i)!}{(p+k-i-1)!}(p-i-q+j-1)\prod_{s<i}(m_s-s-p+i)\prod_{s>i}^{s\neq j}(p-i-m_s+s) \times \frac{(q+1+r-j)!}{(q+1+k-j-1)!}\prod_{s<j}^{s\neq i}(m_s-s-q-1+j)\prod_{s>j}(q+1-j-m_s+s)\Bigg],$$

and in this form it is obvious that

$$\chi(p, p-i+j) = 0.$$

We apply this lemma to the right-hand side of (7), taking $i = 0$. It follows that this expression has a factor (l_0-l_j+j) for $j = 1, \ldots, k-1$. Hence the two sides of (7) have $r-1$ common roots, and it follows that they are identical. Thus we have obtained the solution of the difference equation (7).

9. Now suppose $l_0, \ldots, l_k$ satisfy conditions $A_{k-1,k}$. Then $l_0, \ldots, l_{k-1}$ satisfy conditions $A_{k-2,k-1}$. Let us assume, as hypothesis of induction, that when $l_0, \ldots, l_{k-1}$ satisfy conditions $A_{k-2,k-1}$, $\phi_{r,k-1}(l_0, \ldots, l_{k-1}) = \Psi_{r,k-1}(l_0, \ldots, l_{k-1})$. This is true when $k = 1$. Then, since $l_0, \ldots, l_k$ satisfy $A_{k-1,k}$, it is easily seen that the l's in each $\phi_{r,k-1}$ actually present in (6) satisfy $A_{k-2,k-1}$, and hence each $\phi_{r,k-1}$ in (6) can be replaced by the corresponding $\Psi_{r,k-1}$. It follows from § 7 that

$$\phi_{r,k}(l_0, \ldots, l_k) = \Psi_{r,k}(l_0, \ldots, l_k),$$

and hence we have

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THEOREM VII. *The number of linearly independent k-connexes of type* $(l_0, \ldots, l_k)$, *where the l's satisfy conditions* $A_{k-1,k}$ *is*

$$\frac{(l_0+r)!\ldots(l_k+r-k)!}{(l_0+1)!\ldots l_k!}\frac{\prod_{i<j}(l_i-l_j-i+j)}{r!\,(r-1)!\ldots(r-k)!}.$$

10. Although we cannot apply Theorem VII to the case $l_0 = \ldots = l_k = n$, the expression

$$\phi(n,r,k) = \frac{(n+r)!\ldots(n+r-k)!}{(n+k)!\ldots n!}\frac{1!\ldots k!}{(r-k)!\ldots r!}$$

obtained by putting $l_0 = \ldots = l_k = n$ in $\Psi_{r,k}(l_0, \ldots, l_k)$ does indeed give the number of linearly independent complexes of k-spaces in an r-space which are of degree n in the cases in which it is possible to compute this number otherwise. Thus one should expect that this number will indeed give the number of such complexes for all values of n. If this is so, $\phi(n,r,k)$ is the postulation of the Grassmannian of k-spaces in an r-space. If we write it in the form

$$\phi(n,r,k) = \sum_0^t (-1)\,p_i\binom{n+t-i-1}{t-i},$$

where $t = (r-k)(k+1)$, $p_i - (-1)^i$ is the arithmetic genus of the section of the Grassmannian by i general primes. We should thus have a method of computing these invariants. Again, the coefficient of n^t in $\phi(n,r,k) \times t!$ would be the order of the Grassmannian. Thus if $\phi(n,r,k)$ is indeed the postulation the order should be

$$\frac{1!\,2!\ldots k!}{(r-k)!\ldots r!}t!.$$

This expression for the order agrees with Schubert's formula.

I wish to thank Mr P. M. Grundy for help in the preparation of this note.

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https://doi.org/10.1017/S0305004100021824 Published online by Cambridge University Press