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The asymmetry of $M_0(G)$

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Abstract

Let G be a non-discrete locally compact topological group. A construction is given for a self-adjoint measure μ on G that has independent powers, and is, moreover, in the sub-algebra $M_0(G)$ of $M(G)$. From this, the asymmetry of $M_0(G)$ (and of $M(G)$ itself) is immediate.

1. *Introduction*

It has been known for many years that if G is a non-discrete locally compact abelian group then its measure algebra $M(G)$ is not symmetric (see, e.g. (7), or (6) ch. 5). The proofs so far available all depend to some extent on the structure theory of locally compact abelian groups, and proceed by using special constructions in each of the various standard groups required. They are thus not readily adaptable to the non-abelian situation. It is curious that the case of non-abelian groups, where one might suppose the construction of measures with independent powers would be easier than in the abelian case, should have been apparently more difficult. In the present treatment we are able to avoid any dependence on structure theory, and to give a proof that includes both abelian and non-abelian groups in a single formulation.

The basic strategy, of proving $M(G)$ asymmetric by constructing a suitable measure with independent powers, remains unchanged. It is only the details of the construction that have to be modified. The construction that is given is well-adapted for the inclusion of extra conditions, and as specimens of what can be done we give it in a sufficiently general form to enable us to include proximity to a given positive measure (in an appropriate topology) on the one hand, and membership of $M_0(G)$ on the other.

2. *The basic machinery*

Let G be a locally compact space. We shall be concerned with measures, and sequences of measures, defined on G. All measures considered are in $M(G)$, which may be identified as a Banach space with the dual of $C_0(G)$, the continuous functions on G that vanish at infinity. In $M(G)$ there is of course the usual norm topology. However, we shall be mostly concerned with two other topologies on the measures. One of these

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is defined in terms of representations, in case G is a topological group, and will be described later (in §5). The other is available in any locally compact space G, and is usually described as the $w^{\sim}$-topology. This is the weak*-topology with an extra constraint on the norms, so that basic neighbourhoods of $\mu_0 \in M(G)$ are of the form

$$\{\mu: |\int \phi_r(t)\, d\mu(t) - \int \phi_r(t)\, d\mu_0(t)| < \epsilon, \quad 1 \leqslant r \leqslant n, \quad |\,\|\mu\| - \|\mu_0\|\,| < \epsilon\},$$

where $\phi_1, \ldots, \phi_n$ are elements of $C_0(G)$.

It is convenient to set up this kind of convergence for measures in terms of a sequence of templates, as defined in (1); see also the forthcoming book by Brown and Moran. The technique seems to be due to Körner (3). For convenience, we reproduce here the main ideas, in a context slightly more general than those of the references just given.

A template $(\mathscr{T}, \tau)$ is a pair consisting of a finite collection of sets $T \subset G$, having disjoint compact closures, and a positive real function τ on $\mathscr{T}$ (i.e. $\tau(T)$ is a positive real number for each $T \in \mathscr{T}$). We say that $(\mathscr{S}, \sigma)$ is a refinement of $(\mathscr{T}, \tau)$, and write $(\mathscr{S}, \sigma) < (\mathscr{T}, \tau)$ if each $S \in \mathscr{S}$ is contained in some $T \in \mathscr{T}$, and for each $T \in \mathscr{T}$ we have

$$\sum_{S \subset T} \sigma(S) = \tau(T).$$

Note that the union of the sets in $\mathscr{S}$ may be the same as the union of the sets in $\mathscr{T}$, but $(\mathscr{S}, \sigma)$ and $(\mathscr{T}, \tau)$ may nevertheless be distinct templates. If the sets $T \in \mathscr{T}$ are all open, or all closed, we speak of an open, or closed, template. The support $\operatorname{supp}(\mathscr{T}, \tau)$ of $(\mathscr{T}, \tau)$ is the union of the closures of the sets $T \in \mathscr{T}$. A positive measure μ is subordinate to $(\mathscr{T}, \tau)$, $\mu < (\mathscr{T}, \tau)$, if μ is concentrated on the sets of $\mathscr{T}$ and for each $T \in \mathscr{T}$ we have

$$\mu(T) = \tau(T).$$

In some cases, e.g. if $(\mathscr{T}, \tau)$ is an open template and G is a locally compact group, it is possible, and natural, to identify τ with a canonical measure, supported on the sets of $\mathscr{T}$. In the case just cited, if λ is left Haar measure, then we shall write σ_G for the measure defined by $d\sigma_G = \Delta^{-\frac{1}{2}} d\lambda$, where Δ is the usual modular function, and identify τ with σ_G suitably restricted and normalized: if $T \in \mathscr{T}$ then

$$\tau(T) = \left(\frac{\tau(T)}{\sigma_G(T)}\right) \sigma_G(T)$$

and we may extend τ to all measurable $V \subset T$ by writing

$$\tau(V) = \frac{\tau(T)}{\sigma_G(T)} \sigma_G(V).$$

However, such a canonical measure is not available in general.

Let $\mathscr{U}$ be a finite open cover of $\cup T$; we shall say that $(\mathscr{T}, \tau)$ is subordinate to $\mathscr{U}$ if each $T \in \mathscr{T}$ is contained in some $U \in \mathscr{U}$. Note that here (and in some similar situations) the function τ is irrelevant, and we may sometimes in such contexts simply refer to 'the template $\mathscr{T}$' rather than to 'the template $(\mathscr{T}, \tau)$', as we should. If G is a topological group with identity element e and N is an open neighbourhood of e in G it will be convenient to say that $\mathscr{T}$ is subordinate to N if it is subordinate to some finite open cover consisting of translates of N. We do not in most circumstances require to specify

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left or right translates: but for definiteness let us agree to take always left translates xN as standard.

We shall have occasion in what follows to consider sequences of templates where each is a refinement of the previous one: $(\mathscr{T}_{n+1}, \tau_{n+1}) < (\mathscr{T}_n, \tau_n)$ for $n \geqslant 1$. In this case we shall say that the sequence is ultimately subordinate to the finite open cover $\mathscr{U}$ if there is an n such that $(\mathscr{T}_n, \tau_n)$ (and hence all $(\mathscr{T}_m, \tau_m)$ with $m > n$) is subordinate to $\mathscr{U}$. (Similarly for sequences of templates ultimately subordinate to N.) It will be convenient to write $\tau_{\max}$ for $\max_{T \in \mathscr{T}}(\tau(T))$.

THEOREM 2.1. *Suppose that $(\mathscr{T}_n, \tau_n)$ is a sequence of templates with*

$$(\mathscr{T}_{n+1}, \tau_{n+1}) < (\mathscr{T}_n, \tau_n) \quad \text{for} \quad n \geqslant 1.$$

Let μ_n be any (positive) measure with $\mu_n < (\mathscr{T}_n, \tau_n)$. Suppose that for any finite open cover $\mathscr{U}$ of supp $(\mathscr{T}_1, \tau_1)$, *$(\mathscr{T}_n, \tau_n)$ is subordinate to $\mathscr{U}$ for sufficiently large n. Then μ_n is $w^\sim$-convergent to a limit μ which depends only on the sequence of templates. If also $(\tau_n)_{\max} \to 0$ as $n \to \infty$ then μ is continuous.*

Proof. Let ϕ be a continuous function on G with compact support. Given $\epsilon > 0$, for each t there exists $N(t)$ with $|\phi(t') - \phi(t'')| < \epsilon$ for $t', t'' \in N(t)$. The support of $(\mathscr{T}_1, \tau_1)$ is compact, so there is a finite open cover $\mathscr{U}$ of this such that if t', t'' are in the same set of $\mathscr{U}$ then $|\phi(t') - \phi(t'')| < \epsilon$.

Now let n be so large that each $T_n \in \mathscr{T}_n$ is contained in some $U \in \mathscr{U}$. We have, for $m > n$, $\mu_m(T_n) = \mu_n(T_n) = \tau_n(T_n)$ and

$$\left| \int_{T_n} \phi(t)\, d(\mu_m - \mu_n)(t) \right| \leqslant 2\epsilon\, \tau_n(T_n)$$

from which it follows that

$$\left| \int \phi(t)\, d(\mu_m - \mu_n)(t) \right| \leqslant 2\epsilon \sum_{T_n \in \mathscr{T}_n} \tau_n(T_n) = K\epsilon,$$

i.e. μ_n is w^*-Cauchy, hence w^*-convergent to a limit measure μ. Since all the μ_n's are subordinate to $(\mathscr{T}_n, \tau_n)$ the same is true of μ, for each n. In particular, the measure μ is non-zero and indeed is positive with mass equal to $\Sigma_{T_n \in \mathscr{T}_n} \tau_n(T_n)$ (independent of n), i.e. to the mass of μ_n for each n, so that $w^\sim$-convergence is established.

The limit μ depends only on the sequence $(\mathscr{T}_n, \tau_n)$ and not on the choice of μ_n subordinate to $(\mathscr{T}_n, \tau_n)$. For, let μ_n' be any other sequence of measures subordinate to $(\mathscr{T}_n, \tau_n)$. It is immediate that

$$\left| \int \phi(t)\, d(\mu_m' - \mu_n)(t) \right| \leqslant K\epsilon,$$

by exactly the same reasoning as for μ_m and μ_n: and it follows that μ_n, μ_n' tend to the same limit μ.

If $(\tau_n)_{\max} \to 0$ then μ must be continuous. For, let $t \in \text{supp}\, \mu$; t can belong to at most one set cl T for $T \in \mathscr{T}_n$ (since by assumption the closures of the sets $T \in \mathscr{T}_n$ are disjoint). It follows that $\mu(\{t\}) \leqslant \mu_n(T) = \tau_n(T)$ for some $T \in \mathscr{T}_n$, $n = 1, 2, 3, \ldots$; hence

$$\mu(\{t\}) \leqslant (\tau_n)_{\max}$$

for each n, hence $\mu(\{t\}) = 0$, and μ is continuous.

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Let G be a locally compact group with identity element e. Let A be open, B compact in G. Define the B-interior of A, B-int A to be $\{x: xB \subset A\}$. Then

LEMMA 2·2. *The B-interior of A is open.*

Proof. If B-int $A \neq \varnothing$, let $x \in B$-int A. For each $y \in B$ there exists an open neighbourhood N_y of e such that $xN_y^2 y \subset A$. The sets $\{N_y y\}$ cover B so there exists a finite refinement, corresponding to $y_1, \ldots, y_n$ say. Let $N = N_{y_1} \cap \ldots \cap N_{y_n}$ and then

$$xNB \subset xN \,.\, \bigcup_1^n N_{y_r} y_r \subset \bigcup_1^n xN_{y_r}^2 y_r \subset A$$

so x has a neighbourhood xN contained in B-int A and B-int A is open.

LEMMA 2·3. *Let K be a given compact set, $\mathscr{U}$ a given finite open cover of K. Then there is a neighbourhood N of e such that if $\mathscr{T}$ is subordinate to N and* supp $\mathscr{T} \subset K$, *then $\mathscr{T}$ is subordinate to $\mathscr{U}$.*

Proof. For any N with a compact closure $\overline{N}$, and any $U \in \mathscr{U}$, $\overline{N}$-int U is open, by Lemma 2·2. Since $\mathscr{U}$ is an open cover of K, any $x \in K$ is in $\overline{N}$-int U for some N, some U. Since K is compact there exists a finite refinement, say $\overline{N}_1$-int $U_1, \ldots, \overline{N}_m$-int U_m. Let $N^* = \bigcap_{r=1}^m N_r$: then $\overline{N}^*$-int $U_1, \ldots, \overline{N}^*$-int U_m also cover K.

Let $N^2 \subset N^*$, with $N^{-1} = N$ and suppose $\mathscr{T}$ is subordinate to N, supp $\mathscr{T} \subset K$. Then for each $T \in \mathscr{T}$, $T \cap \overline{N}^*$-int $U_r \neq \varnothing$, for some $U_r \in \mathscr{U}$. Suppose $x_1 \in T \cap \overline{N}^*$-int U_r, and $T \subset x_2 N$. Then $x_1 \in x_2 N$, $x_2 \in x_1 N^{-1} = x_1 N$, $T \subset x_1 N^2 \subset x_1 N^*$. But $x_1 \in \overline{N}^*$-int U_r, so $T \subset U_r$, i.e. $\mathscr{T}$ is subordinate to $\mathscr{U}$.

PROPOSITION 2·4. *If $(\mathscr{T}_n, \tau_n)$ is a sequence of templates with*

$$\sigma_n < (\mathscr{T}_n, \tau_n), \quad (\mathscr{T}_{n+1}, \tau_{n+1}) < (\mathscr{T}_n, \tau_n) \quad \text{for} \quad n \geqslant 1,$$

as in Theorem 2·1, and if for each N there exists an n_N such that $(\mathscr{T}_n, \tau_n)$ is subordinate to N for $n > n_N$, then for any finite open cover $\mathscr{U}$ of $(\mathscr{T}_1, \tau_1)$, $(\mathscr{T}_n, \tau_n)$ is subordinate to $\mathscr{U}$ for n sufficiently large.

Proof. This is immediate.

COROLLARY. *Under the circumstances specified in the Proposition, μ_n is $w^\sim$-convergent to a positive limit μ (and if $(\tau_n)_{\max} \to 0$ then μ is continuous).*

Proof. See Theorem 2·1.

If G is a group, a template $(\mathscr{T}, \tau)$ on G may satisfy the extra condition of symmetry, that is, $T \in \mathscr{T} \Rightarrow T^{-1} \in \mathscr{T}$, (so that the sets $T \in \mathscr{T}$ either coincide with their inverses or are disjoint from them), and moreover $\tau(T) = \tau(T^{-1})$ for all $T \in \mathscr{T}$. We observe that if $(\mathscr{T}, \tau)$ is a symmetric open template then the restriction and re-normalization of σ_G by:

$$\sigma(V) = \frac{\sigma_G(V)}{\sigma_G(T)} \tau(T) \quad \text{for} \quad V \subset T \in \mathscr{T},$$

produces a self-adjoint measure σ.

If in the above Proposition 2·4 and its Corollary all templates are symmetric then the measures μ_n may or may not be self-adjoint. However, the limit μ is necessarily

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self-adjoint, as the limit is independent of the particular sequence μ_n chosen, and it is always possible to choose each μ_n to be self-adjoint.

At this stage we include the justification of the remark made in the Introduction that it is possible to subject a template construction to the requirement that the measures be arbitrarily close (in the $w^\sim$-sense) to a given positive measure. This will follow from

PROPOSITION 2·5. (i) *Let μ be a given positive measure in $M(G)$ and N a given $w^\sim$-neighbourhood of μ. Then there is an open template $(\mathscr{T}, \tau)$ such that any measure ν subordinate to $(\mathscr{T}, \tau)$ is in N.*

(ii) *If G is a group and μ is self-adjoint then $(\mathscr{T}, \tau)$ may be taken to be symmetric.*

Proof. (i) We note first that we may suppose without loss of generality that μ has compact support K since if the support of μ is not compact we can find in N a measure μ_1 that has compact support, and a neighbourhood N_1 of μ_1 that is contained in N, and then start with μ_1 and N_1 instead of μ and N. Let then N be:

$$\{\nu: |\int \phi_r d(\mu-\nu)| < \epsilon, \quad 1 \leqslant r \leqslant n; \quad |\,\|\mu\| - \|\nu\|\,| < \epsilon\},$$

where each ϕ_r is continuous and of compact support. Given ϵ, let $\epsilon' = \epsilon/2\|\mu\|$ and let $\mathscr{U}$ be a finite open cover of K such that if $t', t'' \in U \in \mathscr{U}$ then

$$|\phi_r(t') - \phi_r(t'')| < \epsilon', \quad 1 \leqslant r \leqslant n.$$

Without loss of generality we may assume that $\mathscr{U}$ is minimal, i.e. no proper subset of $\mathscr{U}$ is a cover of K. Now if $U_1, \ldots, U_m$ are the sets of $\mathscr{U}$, define $V_1 = U_1$, $V_2 = U_2 \backslash U_1$, $V_r = U_r \backslash (U_1 \cup \ldots \cup U_{r-1})$. By assumption, no V_r $(1 \leqslant r \leqslant m)$ is empty. Choose arbitrarily open sets $T_1, \ldots, T_m$ with disjoint closures such that $T_r \subset U_r$ and define τ by $\tau(T_r) = \mu(V_r)$. Let ν be subordinate to $(\mathscr{T}, \tau)$. Then we have

$$\left|\int \phi_r d(\mu-\nu)\right| \leqslant \sum_s \left|\int_{V_s} \phi_r d\mu - \int_{T_s} \phi_r d\nu\right|.$$

Since $V_s \subset U_s$, $T_s \subset U_s$ we have

$$\int_{V_s} \phi_r d\mu - \int_{T_s} \phi_r d\nu = \int_{V_s} (\phi_r(t) - \phi_r(t_0))\, d\mu(t) - \int_{T_s} (\phi_r(t) - \phi_r(t_0))\, d\nu(t) + \int_{V_s} \phi_r(t_0)\, d\mu(t) - \int_{T_s} \phi_r(t_0)\, d\nu(t),$$

for any $t_0 \in U_s$. The last two integrals cancel, by the choice of the template function τ. It follows that

$$\left|\int_{V_s} \phi_r d\mu - \int_{T_s} \phi_r d\nu\right| \leqslant \int_{V_s} |\phi_r(t) - \phi_r(t_0)|\, d\mu(t) + \int_{T_s} |\phi_r(t) - \phi_r(t_0)|\, d\nu(t) < 2\epsilon' \tau(T_s)$$

and hence, summing for $1 \leqslant s \leqslant m$, we have, for $1 \leqslant r \leqslant n$,

$$|\int \phi_r d(\mu-\nu)| < 2\epsilon' \Sigma \tau(T_s) = 2\epsilon' \|\mu\| = \epsilon.$$

It is immediate, of course, that $\|\nu\| = \|\mu\|$, and so ν is in the given $w^\sim$-neighbourhood N.

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(ii) If G is a group and μ is symmetric then we may proceed much as in (i), but paying due regard to symmetry where appropriate. Let now $\mathscr{U}$ be a minimal symmetric finite open cover of K, say $\mathscr{U} = \{U_1, U_1^{-1}, U_2, \ldots, U_m^{-1}\}$. (One or more of the sets U_r may of course be equal to its inverse.) Write $Y_r = U_r \cap U_r^{-1}$, $Z_r = U_r \cup U_r^{-1}$. If $Y_1 \neq \varnothing$, write $V_1 = Y_1$: if $Y_1 = \varnothing$ write $W_1 = U_1$, then $V_2 = Y_2 \backslash Z_1$ (if $\neq \varnothing$), or $W_2 = U_2 \backslash Z_1$ otherwise: in general $V_r = Y_r \backslash (Z_1 \cup \ldots \cup Z_{r-1})$ if $\neq \varnothing$, or $W_r = U_r \backslash (Z_1 \cup \ldots \cup Z_{r-1})$ otherwise. In any case we can now choose disjoint open sets $T_r \subset U_r$ with $T_r = T_r^{-1}$ if $V_r \neq \varnothing$, $T_r \cap T_r^{-1} = \varnothing$ if $V_r = \varnothing$, and a function τ defined on these sets by:

$$\begin{aligned} \tau(T_r) &= \mu(Z_r \backslash (Z_1 \cup \ldots \cup Z_{r-1})) \quad \text{if} \quad V_r \neq \varnothing \\ &= \mu(U_r \backslash (Z_1 \cup \ldots \cup Z_{r-1})) \quad \text{if} \quad V_r = \varnothing . \end{aligned}$$

The resulting template $(\mathscr{T}, \tau)$ whose sets are all T_r's and such T_r^{-1}'s as are distinct (hence disjoint) from T_r, with τ as above, and $\tau(T_r^{-1}) = \tau(T_r)$ is symmetric, and if ν is any measure subordinate to it the estimates for the difference $\mu - \nu$ follow as before in case (i).

In the application of Proposition 2·5 we shall take μ to be a given positive (probability) measure, and then construct a sequence of templates starting with $(\mathscr{T}, \tau)$ as provided by the Proposition, finally obtaining a continuous limit measure as in Theorem 2·1, that has independent powers and any other properties that may be incorporated in the construction of the sequence of templates: by Proposition 2·5 this limit measure is within the given neighbourhood of μ.

3. *The independence condition*

As in the abelian case, in order to obtain an independent power measure we impose suitable independence properties on its support. These properties are relatively weak, but are sufficient to give us what we require. As usual, the presence (or possible presence) of elements of order 2 in the group presents minor technical problems. We have a choice between separating into two cases, with somewhat differing treatments, according to the abundance of such elements, or of adopting a uniform approach, with a slight increase in complexity. We have adopted the latter course here. In this section G may be any non-discrete topological group, not necessarily locally compact.

Informally, we consider finite subsets $\{x_1, \ldots, x_l\}$ of G, and words using letters from $\{x_i, x_i^{-1}\}$, $1 \leqslant i \leqslant l$. The set $\{x_1, \ldots, x_l\}$ has 'Property X' if no word using at most one entry from each $\{x_i, x_i^{-1}\}$ is equal in G to a shorter word. A similar definition applies when the x_i are replaced by sets X_i. We aim to prove that sets having Property X can be found arbitrarily close to any finite subset of G, and that every finite set with Property X has a neighbourhood that also has this property. We proceed to make these statements precise.

It will be convenient to say that a (not necessarily finite) set of elements of G, $\{x_i : i \in I\}$ is irredundant if and only if

$$\{x_i, x_i^{-1}\} \cap \{x_j, x_j^{-1}\} = \varnothing \quad \text{for} \quad i, j \in I, \quad i \neq j.$$

By analogy, a set $\{X_i : i \in I\}$ of subsets of G will be called irredundant if and only if

$$(X_i \cup X_i^{-1}) \cap (X_j \cup X_j^{-1}) = \varnothing \quad \text{for} \quad i, j \in I, \quad i \neq j$$

(although here the term has a less clear intuitive content than for sets of elements).

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Now fix (for the present) the positive integer l, and distinct symbols $S_1, \ldots, S_l$; $T_1, \ldots, T_l$. Let $\mathscr{W}_l$ denote the set of all words in these symbols (including the empty word). We define $W \in \mathscr{W}_l$ to be restricted iff it has at most one entry from each set $\{S_i, T_i\}$ $(1 \leqslant i \leqslant l)$. Observe that each $W \in \mathscr{W}_l$ gives rise to a map $W^*: G^l \to G$, defining $W^*(x_1, \ldots, x_l)$ to be the element of G obtained by substituting, in W, x_i for S_i and x_i^{-1} for T_i $(1 \leqslant i \leqslant l)$. A finite irredundant set $\{x_1, \ldots, x_l\}$ is then defined to have Property X iff whenever $W_1, W_2 \in \mathscr{W}_l$, W_1 is restricted and W_2 is shorter than W_1, we have

$$W_1^*(x_1, \ldots, x_l) \neq W_2^*(x_1, \ldots, x_l).$$

An arbitrary subset H of G is defined to have Property X iff each finite irredundant subset of H has Property X (where now l is no longer fixed, but varies with the subset, of course).

Returning to a fixed l and $\mathscr{W}_l$, as specified above, note that each $W \in \mathscr{W}_l$ gives rise to a map $W^{**}: (2^G)^l \to 2^G$, defining $W^{**}(X_1, \ldots, X_l)$ to be the subset of G obtained by substituting, in W, X_i for S_i and X_i^{-1} for T_i $(1 \leqslant i \leqslant l)$. A finite irredundant set $\{X_1, \ldots, X_l\}$ is defined to have Property X iff whenever $W_1, W_2 \in \mathscr{W}_l$, W_1 is restricted, and W_2 is shorter than W_1, we have

$$W_1^{**}(X_1, \ldots, X_l) \cap W_2^{**}(X_1, \ldots, X_l) = \varnothing.$$

An arbitrary collection of subsets of G is defined to have Property X iff each finite irredundant sub-collection has Property X.

Several variants of Property X are possible and some of these (not discussed here) are more appropriate in other applications. Here we content ourselves with mentioning one minor variation. We shall say that for a given positive integer k the finite irredundant set $\{x_1, \ldots, x_l\}$ has Property (X, k) iff whenever $W_1, W_2 \in \mathscr{W}_l$, W_1 is restricted, of length not exceeding k, and W_2 is shorter than W_1, we have

$$W_1^*(x_1, \ldots, x_l) \neq W_2^*(x_1, \ldots, x_l).$$

Similar modifications can be made to our other uses of the phrase 'Property X' to convert them into terms of 'Property (X, k)'. It is clear that Property X implies Property (X, k) for all k. Conversely, for $\{x_1, \ldots, x_l\}$, Property (X, k) is the same for all $k \geqslant l$, and then implies (hence is equivalent to) Property X.

For a neighbourhood N of e in G, and $x \in G$, write $N(x) = xN \cap N^{-1}x$; this is a neighbourhood of x such that $(N(x))^{-1} = N(x^{-1})$.

PROPOSITION 3·1. *Let $x_1, x_2, \ldots, x_l \in G$ and let $\mathscr{W} \subset \mathscr{W}_l \times \mathscr{W}_l$ be a finite collection of pairs of words such that for $(W_1, W_2) \in \mathscr{W}$*

$$W_1^*(x_1, \ldots, x_l) \neq W_2^*(x_1, \ldots, x_l).$$

Then there exists a neighbourhood N of e such that

$$W_1^{**}(N(x_1), \ldots, N(x_l)) \cap W_2^{**}(N(x_1), \ldots, N(x_l)) = \varnothing.$$

Proof. Let (W_1, W_2) be fixed in $\mathscr{W}$ and let $y_i = W_i^*(x_1, \ldots, x_l)$ $(i = 1, 2)$. There exists a neighbourhood N' of e such that $y_1 N' \cap y_2 N' = \varnothing$, and because the maps W_1^*, W_2^* are continuous there exists a neighbourhood N'' of e such that

$$W_1^{**}(N''(x_1), \ldots, N''(x_l)) \subset y_i N' \quad (i = 1, 2).$$

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Let N be the intersection of the (finite) collection of neighbourhoods N'' as (W_1, W_2) runs through $\mathscr{W}$, and it is clear that the required condition holds.

COROLLARY. *Under the conditions of the Proposition, if* $y_i \in N(x_i)$ $(1 \leqslant i \leqslant l)$ *then*

$$W_1^*(y_1, \ldots, y_l) \neq W_2^*(y_1, \ldots, y_l).$$

LEMMA 3·2. *Let* i *be fixed,* $1 \leqslant i \leqslant l$, *and let* $W_1, W_2 \in \mathscr{W}_l$ *be such that* W_1 *has precisely one entry from* $\{S_i, T_i\}$ *while* W_2 *has no entry from* $\{S_i, T_i\}$. *Let* $x_1, \ldots, x_l \in G$ *be such that*

$$W_1^*(x_1, \ldots, x_i, \ldots, x_l) = W_2^*(x_1, \ldots, x_i, \ldots, x_l)$$

and let $y_i \neq x_i$. *Then*

$$W_1^*(x_1, \ldots, y_i, \ldots, x_l) \neq W_2^*(x_1, \ldots, y_i, \ldots, x_l).$$

Proof. We may write $W_1 = URV$ where U, V have no entry from $\{S_i, T_i\}$ and R is either S_i or T_i. It follows that

$$\begin{aligned} U^*(x_1, \ldots, y_i, \ldots, x_l) &= U^*(x_1, \ldots, x_i, \ldots, x_l), \\ V^*(x_1, \ldots, y_i, \ldots, x_l) &= V^*(x_1, \ldots, x_i, \ldots, x_l), \\ R^*(x_1, \ldots, y_i, \ldots, x_l) &\neq R^*(x_1, \ldots, x_i, \ldots, x_l). \end{aligned}$$

Then

$$\begin{aligned} W_1^*(x_1, \ldots, y_i, \ldots, x_l) &= U^*(x_1, \ldots, y_i, \ldots, x_l)\, R^*(x_1, \ldots, y_i, \ldots, x_l)\, V^*(x_1, \ldots, y_i, \ldots, x_l) \\ &\neq U^*(x_1, \ldots, x_i, \ldots, x_l)\, R^*(x_1, \ldots, x_i, \ldots, x_l)\, V^*(x_1, \ldots, x_i, \ldots, x_l) \end{aligned}$$

(by the group cancellation law in G)

$$\begin{aligned} &= W_1^*(x_1, \ldots, x_i, \ldots, x_l) \\ &= W_2^*(x_1, \ldots, x_i, \ldots, x_l) \\ &= W_2^*(x_1, \ldots, y_i, \ldots, x_l). \end{aligned}$$

PROPOSITION 3·3. *Given* $x_1, \ldots, x_l \in G$ *and a neighbourhood* N *of* e, *there exist points* $y_i \in N(x_i)$ $(1 \leqslant i \leqslant l)$ *such that* $\{y_i, \ldots, y_l\}$ *is an irredundant set with Property* X.

Proof. We may without loss of generality assume that N is open.

Let $\mathscr{W} \subset \mathscr{W}_l \times \mathscr{W}_l$ be the collection of all pairs (W_1, W_2) such that W_1 is restricted and W_2 is shorter than W_1. Let $\mathscr{W}^{(0)}$ be the subset of consisting of all pairs $(W_1, W_2) \in \mathscr{W}$ such that

$$W_1^*(x_1, \ldots, x_l) \neq W_2^*(x_1, \ldots, x_l).$$

($\mathscr{W}^{(0)}$ might of course be empty, for example if $x_1 = \ldots = x_l = e$.) For each

$$(W_1, W_2) \in \mathscr{W} \setminus \mathscr{W}^{(0)}$$

there is a least index, i, $1 \leqslant i \leqslant l$, such that W_1 contains one of S_i, T_i while W_2 contains neither S_i nor T_i. Let $\mathscr{V}^{(i)}$ be the subset of $\mathscr{W} \setminus \mathscr{W}^{(0)}$ consisting of all pairs (W_1, W_2) which have least index i. The sets $\mathscr{V}^{(i)}$ are disjoint for $1 \leqslant i \leqslant l$ and

$$\mathscr{W} = \mathscr{W}^{(0)} \cup \mathscr{V}^{(1)} \cup \ldots \cup \mathscr{V}^{(l)}.$$

Write $\mathscr{W}^{(i)} = \mathscr{W}^{(0)} \cup \mathscr{V}^{(1)} \cup \ldots \cup \mathscr{V}^{(i)}$, so that we have $\mathscr{W}^{(l)} = \mathscr{W}$.

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Let k be an integer with $1 \leqslant k \leqslant l$, and let P_k be the statement: we can choose $y_1, \ldots, y_k \in G$ to satisfy

(i_k) $y_i \in N(x_i)$, $(1 \leqslant i \leqslant k)$,

(ii_k) the set $\{y_1, \ldots, y_k\}$ is irredundant,

(iii_k) for $(W_1, W_2) \in \mathscr{W}^{(k)}$ we have

$$W_1^*(y_1, \ldots, y_k, x_{k+1}, \ldots, x_l) \neq W_2^*(y_1, \ldots, y_k, x_{k+1}, \ldots, x_l).$$

By Proposition 3·1 there is a neighbourhood $N^{(0)}$ of e such that $N^{(0)} \subset N$, hence $N^{(0)}(x_i) \subset N(x_i)$ $(1 \leqslant i \leqslant l)$ and, for $(W_1, W_2) \in \mathscr{W}^{(0)}$,

$$W_1^{**}(N^{(0)}(x_1), \ldots, N^{(0)}(x_l)) \cap W_2^{**}(N^{(0)}(x_1), \ldots, N^{(0)}(x_l)) = \varnothing.$$

Choose arbitrarily $y_1 \in N^{(0)}(x_1)$, $y_1 \neq x_1$ and (i_1) is immediate, (ii_1) is vacuously true, and (iii_1) follows from Lemma 3·2 with $i = 1$. That is, P_1 holds.

Now assume that P_k holds. By Proposition 3·1 there is a neighbourhood $N^{(k)}$ of e such that for $(W_1, W_2) \in \mathscr{W}^{(k)}$ we have

$$W_1^{**}(N^{(k)}(y_1), \ldots, N^{(k)}(y_k), N^{(k)}(x_{k+1}), \ldots, N^{(k)}(x_l)) \cap W_2^{**}(N^{(k)}(y_1), \ldots, N^{(k)}(y_k), N^{(k)}(x_{k+1}), \ldots, N^{(k)}(x_l)) = \varnothing.$$

Without loss of generality, we may assume that also $N^{(k)}$ is small enough to satisfy the conditions $N^{(k)} \subset N$ and $N^{(k)}(y_i) \subset N(x_i)$ $(1 \leqslant i \leqslant k)$. Now make any choice of y_{k+1} to satisfy the conditions $y_{k+1} \in N^{(k)}(x_{k+1})$, $y_{k+1} \neq x_{k+1}$, and such that the set $\{y_1, \ldots, y_k, y_{k+1}\}$ is irredundant. Infinitely many such choices are possible, since G is not discrete. We then have immediately (i_{k+1}) and (ii_{k+1}). Moreover if $(W_1, W_2) \in \mathscr{W}^{(k)}$ then

$$W_1^*(y_1, \ldots, y_k, y_{k+1}, x_{k+2}, \ldots, x_l) \neq W_2^*(y_1, \ldots, y_k, y_{k+1}, x_{k+2}, \ldots, x_l),$$

by Proposition 3·1, Corollary. If $(W_1, W_2) \in \mathscr{V}^{(k+1)}$ the same inequality holds by Lemma 3·2, with $i = k+1$. Hence the inequality holds for $(W_1, W_2) \in \mathscr{W}^{(k+1)}$, i.e. (iii_{k+1}) holds. That is, P_k implies P_{k+1}.

Since P_1 is true, it follows that P_k is true for all k, $1 \leqslant k \leqslant l$. In particular, P_l is true, which is the assertion of the Proposition.

We obtain the main result of this section by combining Propositions 3·1 and 3·3.

PROPOSITION 3·4. *Let $x_1, \ldots, x_l \in G$ and let N be a neighbourhood of e. Then there are elements $y_1, \ldots, y_l$ and a neighbourhood N' of e such that the set $\{N'(y_1), \ldots, N'(y_l)\}$ has Property X, and $N'(y_i) \subset N(x_i)$ $(1 \leqslant i \leqslant l)$.*

Proof. Without loss of generality we may assume that N is open. By Proposition 3·3 we can find y_i $(1 \leqslant i \leqslant l)$ such that $y_i \in N(x_i)$ and $\{y_1, \ldots, y_l\}$ is an irredundant set with Property X. By Proposition 3·1 we can find a neighbourhood N'' of e such that $\{N''(y_1), \ldots, N''(y_l)\}$ is irredundant and has Property X. Let N' be any neighbourhood of e satisfying $N' \subset N''$ and $N'(y_i) \subset N(x_i)$ $(1 \leqslant i \leqslant l)$: then the conditions of the Proposition clearly hold.

Because we are working with templates in the construction of measures with the required properties, we are often concerned with Property X for collections of sets. This follows from the corresponding point property in the following way.

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LEMMA 3·5. *Suppose that $H \subset G$ has Property X. Then the collection of all subsets of H has Property X.*

Proof. Suppose that H has Property X and that $\{H_1, \ldots, H_l\}$ is a finite irredundant collection of subsets of H. We have to show that if $W_1, W_2, \in \mathscr{W}_l$ with W_1 restricted and W_2 shorter than W_1 then

$$W_1^{**}(H_1, \ldots, H_l) \cap W_2^{**}(H_1, \ldots, H_l) = \varnothing;$$

that is, if h_i, k_i are arbitrary in H_i $(1 \leqslant i \leqslant l)$ then

$$W_1^*(h_1, \ldots, h_l) \neq W_2^*(k_1, \ldots, k_l).$$

Let $J = \{i\colon 1 \leqslant i \leqslant l, k_i \notin \{h_i, h_i^{-1}\}\} = \{i_1', \ldots, i_m'\}$, say and let $n = l+m$. The result will be proved by considering words in $\mathscr{W}_n$ and applying Property X to the set $\{k_i\} \cup \{k_j\}$ $(1 \leqslant i \leqslant l, j \in J)$. Note first that there is a natural injection ψ of $\mathscr{W}_l$ into $\mathscr{W}_n$, where $\psi(W)$ is the word in $\mathscr{W}_n$ with the same formal expression in terms of $S_1, \ldots, S_l$; $T_1, \ldots, T_l$ as $W \in \mathscr{W}_l$. Now define $W_1' = \psi(W_1)$. Define next a word $W_2' \in \mathscr{W}_n$ as follows. Take W_2: if $k_i = h_i$ leave S_i and T_i unaltered, if $k_i = h_i^{-1}$ replace S_i by T_i and T_i by S_i. If $i \in J$, that is, $k_i \neq h_i$ and $k_i \neq h_i^{-1}$ then $i = i_r'$ for some r, $1 \leqslant r \leqslant m$; in this case replace S_i by S_{l+r} and T_i by T_{l+r} at each occurrence in W_2. We then have a word $W_2' \in \mathscr{W}_n$.

It is then straightforward to check that

$$W_1'^*(h_1, \ldots, h_l, k_{i_1'}, \ldots, k_{i_m'}) = W_1^*(h_1, \ldots, h_l)$$

and

$$W_2'^*(h_1, \ldots, h_l, k_{i_1'}, \ldots, k_{i_m'}) = W_2^*(k_1, \ldots, k_l)$$

so that Property X for the set $\{h_1, \ldots, h_l, k_{i_1'}, \ldots, k_{i_m'}\}$ gives the required conclusion.

4. *The asymmetry of $M(G)$ for metrisable G*

We are now in a position to establish the asymmetry of $M(G)$ where G is a metrisable non-discrete locally compact group. To begin with, however, we do not require to assume that G is metrisable. We recall that a Banach algebra with an involution $x \to x^\sim$ is symmetric if the following condition holds:

(i) $\mathrm{sp}(x^\sim x) \subset \mathbb{R}^+$ for all x,

which implies at once

(ii) $\mathrm{sp}(x) \subset \mathbb{R}$ for all self-adjoint x $(x = x^\sim)$.

((i) and (ii) are in fact equivalent, by the Shirali–Ford lemma.)

We shall therefore be able to establish asymmetry as soon as we have a self-adjoint element in the algebra with an unreal point in its spectrum. We recall also that an element x has independent powers if

$$\|p(x)\| = \Sigma|c_r| \, \|x\|^r \quad \text{for all polynomials} \quad p(x) = \Sigma c_r x^r.$$

In the next result we assume that the algebra is unital, i.e. there is a unit element 1 with $\|1\| = 1$.

LEMMA 4·1. *If x has independent powers then* $\mathrm{sp}(x)$ *contains the circle* $\{\lambda\colon |\lambda| = \|x\|\}$.

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Proof. Suppose that $(\lambda 1 - x)^{-1}$ exists for such a λ: then, for arbitrary n,

$$\lambda^n 1 + \lambda^{n-1}x + \ldots + x^n = (\lambda 1 - x)^{-1}(\lambda^{n+1}1 - x^{n+1}),$$

and so
$$\|\lambda^n 1 + \lambda^{n-1}x + \ldots + x^n\| \leqslant \|(\lambda 1 - x)^{-1}\|\,(|\lambda|^{n+1} + \|x\|^{n+1}).$$

But the norm of the left side is

$$|\lambda|^n + \ldots + \|x\|^n = (n+1)\,\|x\|^n,$$

whereas the norm of the right side does not exceed $K\|x\|^{n+1}$, so we have a contradiction.

If the algebra has no unit, then it is symmetric if and only if the algebra with a unit adjoined is symmetric. This is the situation for certain sub-algebras of $M(G)$ such as $M_0(G)$. If the sub-algebra is not complete we shall say that it is symmetric or not according as the condition (i) holds for the spectrum of the element $x^{\sim}x$ in the completion (= closure in $M(G)$, here) of the sub-algebra. We thus have the following criterion:

PROPOSITION 4·2. *If there is in the sub-algebra A of $M(G)$ a self-adjoint measure μ with independent powers then the algebra A is asymmetric.*

We now proceed to examine the implications for the limit measure μ of the imposition of suitable independence conditions on the templates of a sequence. We begin with two elementary lemmas of a technical nature that may perhaps also be of relevance in contexts other than the present one; cf. (1).

LEMMA 4·3. *Let $\mu = \mu_1 + \ldots + \mu_N$ where $\mu_r \geqslant 0$, $1 \leqslant r \leqslant N$, $\|\mu\| = 1$, $\|\mu_r\| \leqslant h$ $(1 \leqslant r \leqslant N)$. Then if m is a given positive integer there is a constant κ_m, depending only on m, such that the sum of the norms of the terms in the multinomial expansion of*

$$(\mu_1 + \ldots + \mu_N)^m$$

that contain two or more occurrences of the same factor is $\leqslant \kappa_m h$. (We could take, e.g. *$\kappa_m = \frac{1}{2}(m!)$.)*

Proof. Let $\nu_1 \ldots \nu_{m-1}$ be a typical product of $m-1$ factors that contains a repeated factor. Then if $m \geqslant 3$ the products

$$\mu_1\nu_1 \ldots \nu_{m-1},\ \mu_2\nu_1 \ldots \nu_{m-1},\ \ldots\ldots,\ \mu_N\nu_1 \ldots \nu_{m-1},$$
$$\nu_1\mu_1 \ldots \nu_{m-1},\ \nu_1\mu_2 \ldots \nu_{m-1},\ \ldots\ldots,\ \nu_1\mu_N \ldots \nu_{m-1},$$
$$\nu_1 \ldots \nu_{m-1}\mu_1,\ \nu_1 \ldots \nu_{m-1}\mu_2,\ \ldots\ldots,\ \nu_1 \ldots \nu_{m-1}\mu_N,$$

as $\nu = \nu_1 \ldots \nu_{m-1}$ runs through all words of length $m-1$ with a repeated factor, will include all words of length m with a repeated factor. For fixed ν, the sum of the norms of the elements in each row is $\|\nu\|$, and there are m rows, so the total, summing over all ν, is $\leqslant m\Sigma\|\nu\|$. If $\Sigma\|\nu\| \leqslant \kappa_{m-1}h$, then the sum of the terms of degree m with repeated factors is $\leqslant \kappa_m h$ with $\kappa_m = m\kappa_{m-1}$. Now if $m = 2$ the terms with repeated factors are $\mu_1^2, \ldots, \mu_N^2$ and the sum of the norms of these is $\leqslant h = \frac{1}{2}\cdot 2!h$ so the result follows by induction.

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LEMMA 4·4. *Let* $\mu = \mu_1 + \ldots + \mu_N$, *where* $\mu_r \geqslant 0$, $\max_{1 \leqslant r \leqslant N} \|\mu_r\| = h$ *and* $\|\mu\| = 1$. *Let* $(\mu^r)'$ *be the sum of the terms in the multinomial expansion of* μ^r *that are products of* r *distinct factors* μ_i $(1 \leqslant i \leqslant N)$. *Let* $p(\mu) = \sum_{r=0}^{k} c_r \mu^r$ *be a given polynomial. Then*

$$\|p(\mu)\| \geqslant \Big\| \sum_{r=0}^{k} c_r (\mu^r)' \Big\| - \sum_{r=0}^{k} |c_r| \kappa_r h$$

where κ_r *is as in Lemma* 4·3.

Proof. $$\|p(\mu) - \Sigma c_r (\mu^r)'\| \leqslant \Sigma |c_r| \, \|\mu^r - (\mu^r)'\|,$$

and, by Lemma 4·3, $\|\mu^r - (\mu^r)'\| \leqslant \kappa_r h$, from which the result follows.

THEOREM 4·5. *Let* $(\mathscr{T}_n, \tau_n)$ *be a sequence of symmetric templates with Property* X, *and* $(\tau_n)_{\max} \to 0$ *as* $n \to \infty$. *Let* μ *be subordinate to* $(\mathscr{T}_n, \tau_n)$ *for all* n. *Then* μ *has independent powers.*

Proof. Let $p(\mu)$ be a given polynomial, say $p(\mu) = \sum_{r=0}^{k} c_r \mu^r$. Let $\mathscr{T}_n$ be generated by $\{T_1, \ldots, T_N\}$ $(N = N(n))$ and let μ_i $(1 \leqslant i \leqslant N)$ be the restriction of μ to $T_i \cup T_i^{-1}$. Since $(\tau_n)_{\max} \to 0$, it follows that $N(n) \to \infty$ as $n \to \infty$; choose n large enough so that $N(n) \geqslant k$. Write h_n for h in Lemma 4·4 and we obtain (with $(\mu^r)'$ as in Lemma 4·4)

$$\begin{aligned} \|p(\mu)\| &\geqslant \Big\| \sum_{r=0}^{k} c_r (\mu^r)' \Big\| - \sum_{r=0}^{k} |c_r| \kappa_r h_n \\ &= \sum_{r=0}^{k} |c_r| \, \|(\mu^r)'\| - \sum_{r=0}^{k} |c_r| \kappa_r h_n, \end{aligned}$$

because the sets on which the measures $(\mu^r)'$ are supported are disjoint for $0 \leqslant r \leqslant k$. Also, by Lemma 4·3,

$$\|(\mu^r)'\| > \|\mu^r\| - \kappa_r h_n$$

and we have $\|\mu^r\| = \|\mu\|^r$ since μ is positive, so

$$\|p(\mu)\| \geqslant \sum_{r=0}^{k} |c_r| \, \|\mu\|^r - 2 \sum_{r=0}^{k} |c_r| \kappa_r h_n.$$

Since $h_n \to 0$ as $n \to \infty$ and $\sum_{r=0}^{k} |c_r| \kappa_r$ is independent of n, we have

$$\|p(\mu)\| \geqslant \sum_{r=0}^{k} |c_r| \, \|\mu\|^r,$$

which is of course equivalent to the required independent powers condition.

PROPOSITION 4·6. *Let* $(\mathscr{T}_0, \tau_0)$ *be a given symmetric open template,* N *a given neighbourhood of* e *in* G *and* ϵ *a given positive number. Then there is an open symmetric refinement* $(\mathscr{T}, \tau)$ *of* $(\mathscr{T}_0, \tau_0)$ *such that*

(i) $(\mathscr{T}, \tau)$ *has property* X

(ii) $(\mathscr{T}, \tau)$ *is subordinate to* N

(iii) $\tau_{\max} < \epsilon$.

Proof. Let m be an integer greater than $(\tau_0)_{\max}/\epsilon$. In each T_0 in $\mathscr{T}_0$ choose m distinct points and label the totality of these by $\{x_1, \ldots, x_l\}$. Choose a neighbourhood N_1 of e

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such that $N_1 \subset N$ and the closure of $N_1(x_1)$ is contained in the corresponding member of $\mathscr{T}_0$ for each $i = 1, \ldots, l$. Using Proposition 3·4 we find a neighbourhood N_2 of e contained in N_1 and an irredundant set $\{y_1, \ldots, y_l\}$ with property X such that

$$N_2(y_i) \subset N_1(x_i)$$

for each i. Since the sets $\{y_i, y_i^{-1}\}$ are pairwise disjoint, we may also suppose that the sets $N_2(y_i) \cup N_2(y_i^{-1})$ are pairwise disjoint (by passing to a still smaller neighbourhood if necessary). In fact we can further specify that

$$N_2(y_i) = N_2(y_i^{-1}) \quad \text{if} \quad y_i^2 = e, \quad N_2(y_i) \cap N_2(y_i^{-1}) = \varnothing \text{ otherwise.}$$

(Note that $N(x^{-1}) = N(x)^{-1}$). Accordingly we may take $\mathscr{T} = \{N_2(y_i), N_2(y_i^{-1}), i = 1, \ldots, l\}$ to define the sets of an open symmetric template. For each T_0 in $\mathscr{T}_0$, we count the number $q = q(T_0)$ of members T of $\mathscr{T}$ which are subsets of T_0. We define $\tau(T)$ for any such set to be the number $q^{-1}\tau(T_0)$. In this way we have defined a template $(\mathscr{T}, \tau)$ which obviously satisfies (i) and (ii).

The only property which remains to be checked is that $\tau_{\max} < \epsilon$, and this follows because $q \geqslant m$. In fact

$$\tau_{\max} \leqslant (\tau_0)_{\max}/m < (\tau_0)_{\max}/((\tau_0)_{\max}/\epsilon) = \epsilon.$$

THEOREM 4·7. *Let G be a non-discrete metrisable locally compact group, let μ_0 be a given self-adjoint positive measure in $M(G)$ and let N be a given $w^\sim$-neighbourhood of μ_0. Then there is in N a positive self-adjoint measure μ with independent powers.*

Proof. Let $(\mathscr{T}_0, \tau_0)$ be the open template obtained from μ_0 and N as in Proposition 2·5. Let $N_1, N_2, \ldots$ be a basic set of neighbourhoods of e in G. Suppose we have arrived at $(\mathscr{T}_n, \tau_n)$ $(n \geqslant 0)$: then we construct $(\mathscr{T}_{n+1}, \tau_{n+1})$ as in Proposition 4·6, with N_{n+1} for N and $(n+1)^{-1}$ for ϵ. In this way we obtain a sequence of templates as in Theorem 4·5. Taking, for $n \geqslant 1$, any self-adjoint measure μ_n subordinate to $(\mathscr{T}_n, \tau_n)$ (in particular, the canonical measure as described in §2, if we wish) we obtain a sequence that is $w^\sim$-convergent to a limit μ say. This limit is evidently subordinate to $(\mathscr{T}_n, \tau_n)$ for all n. By Theorem 4·5 μ has independent powers. Since each μ_n is self-adjoint, the same is true of μ.

THEOREM 4·8. *If G is a non-discrete metrisable locally compact group then $M(G)$ is asymmetric.*

Proof. By Theorem 4·7 there is in $M(G)$ a self-adjoint measure with independent powers (take e.g. $\mu_0 = \delta$, $N = M(G)$). As already pointed out at the beginning of this section, this is enough to establish the required asymmetry.

We now sketch an alternative approach to the last two theorems.

PROPOSITION 4·9. *Let N_n be a basic set of neighbourhoods of e in G. For each n, let $(\mathscr{T}_n, \tau_n)$ be a closed symmetric template with property X and subordinate to N_n. Let μ be a measure subordinate to $(\mathscr{T}_n, \tau_n)$ for each n. Then the support of μ has property X.*

Proof. Let $\{x_1, \ldots, x_l\}$ be an irredundant subset of the support of μ. Then the points x_i belong to disjoint subsets of $\mathscr{T}_n$ for some n and hence $\{x_1, \ldots, x_l\}$ has property X.

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THEOREM 4·10. *Let μ be a positive continuous measure concentrated on a set with property X. Then μ has independent powers.*

Proof. In view of the continuity of μ, we may, for each $h > 0$, choose a finite partition $\{H_1, \ldots, H_{l(h)}\}$ of a set H on which μ is concentrated in such a way that the sets H_i are symmetric and the restrictions $\mu_i = \mu|_{H_i}$ have mass less than h. Now $\{H_1, \ldots, H_{l(h)}\}$ has property X by Lemma 3·5 so that the measures $(\mu^r)'$ (defined as before for the decomposition $\mu = \Sigma\mu_i$) are pairwise singular for varying r. Hence for any polynomial $p(\mu) = \sum_{r=0}^{k} c_r\mu^r$, we have

$$\left\| \sum_{r=0}^{k} c_r(\mu^r)' \right\| = \sum_{r=0}^{k} |c_r| \, \|(\mu^r)'\|,$$

and the proof is completed along the lines of the proof of Theorem 4·5.

5. *The asymmetry of $M_0(G)$ for metrisable G*

The construction we have described for self-adjoint independent-power measures n $M(G)$ is well-adapted for the inclusion of extra conditions. In particular, one may mpose conditions on the templates and measures sufficient to ensure that the limit imeasure μ is in $M_0(G)$. This will establish the asymmetry of $M_0(G)$.

In distinction to the abelian case, the appropriate definition of $M_0(G)$ in the general non-abelian case is not quite immediate, and the required properties are substantially harder to establish. There are several possible starting-points, one of which we now describe.

Let U be a continuous unitary representation of G on a Hilbert space H. It is well-known (see, for example ((5), §29)) that there is an essentially 1–1 correspondence between such representations of G and continuous $\sim$-representations T of $L_1(G)$ ($\cong M_a(G)$, the absolutely continuous measures). More precisely, if we adjoin a unit 1 to $L_1(G)$ we may consider representations T such that the subspace of H

$$N = \{\xi : T(\mu)\,\xi = 0, \forall \mu \in L_1(G)\},$$

is $\{0\}$. We say that such a T 'does not contain the degenerate representation'. Then ((5), theorem 1, pp. 376–377) there is a 1–1 correspondence between all such U and all such T, given by

$$T(\lambda 1 + \mu) = \lambda I + \int_G U(t)\, d\mu(t),$$

where λ is a complex scalar and I is the identity operator on H.

Now, given such a U, one can evidently define a corresponding representation T of $M(G)$ by the formula

$$T(\mu) = \int_G U(t)\, d\mu(t),$$

where now μ is an arbitrary measure in $M(G)$, no longer necessarily absolutely continuous.

There are, even in the abelian case, many more $\sim$-representations of $M(G)$ than are given by the above formula: take for example $T_d(\mu) = T(\mu_d)$ where T is as above and μ_d is the discrete part of μ.

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Denote by $\mathscr{T}_c = \mathscr{T}_c(G)$ the representations of G or of $M(G)$ that are, or arise, as above, from continuous unitary representations of G. We should strictly write $(T, U: H) \in \mathscr{T}_c$, but on occasions we shall simply write $T \in \mathscr{T}_c$ or $U \in \mathscr{T}_c$, as the case may be.

If we write, for arbitrary $\mu \in M(G)$

$$\|\mu\|' = \sup_{T \in \mathscr{T}_c} \|T(\mu)\|,$$

then $\|\mu\|'$ is a norm on $M(G)$. It is clear that $\|\mu\|'$ is a semi-norm on $M(G)$: the fact that $\|\mu\|' = 0 \Rightarrow \mu = 0$ results from the following. If $\mu \neq 0$ then there exists $f \in C_{00}(G)$ with $\int f(t)\, d\mu(t) \neq 0$. We define, for $g \in L_1(G)$, $\mu_g \in M_a(G)$ by $\int \phi d\mu_g = \int \phi(t)\, g(t)\, dt$, where dt denotes left Haar measure. Thus if $f^\dagger(t) = f(t^{-1})$, then $\mu\mu_{f^\dagger}$ is a measure in $M_a(G)$ corresponding to a function that is not identically zero (non-zero at e), and continuous, so $\mu\mu_{f^\dagger}$ is non-zero. Since $M_a(G)$ is known to be semi-simple ((5), §28, 2, VII (p. 375)) there is thus a $T \in \mathscr{T}_c$ with $T(\mu\mu_{f^\dagger})$ non-zero, which implies $T(\mu)$ is non-zero. So if $\mu \neq 0$, $\|T(\mu)\| > 0$ for at least one $T \in \mathscr{T}_c$.

We now define $M_0(G)$ to be that subset of $M(G)$ consisting of measures that can be approximated arbitrarily closely in the $\|\cdot\|'$ norm by measures in $M_a(G)$. That is, $\mu \in M_0(G)$ if and only if, given $\epsilon > 0$, there exists a $\nu \in M_a(G)$ with $\|\mu - \nu\|' < \epsilon$.

Notice that we do not have here anything corresponding to the definition of $M_0(G)$ for abelian G as consisting of measures 'whose transform tends to zero at infinity', although, of course, $\|\mu\|'$ becomes the Fourier–Stieltjes norm

$$\|\hat{\mu}\|_\infty = \sup\{|\hat{\mu}(\gamma)| : \gamma \in G^\wedge\}$$

in the abelian case.

It may also be remarked that we have made no use of irreducible representations in our definition of $M_0(G)$. If we had chosen to do so, we could have restricted attention to irreducible representations in $\mathscr{T}_c$ and the resulting definition of $M_0(G)$ would have been equivalent to that given above.

As mentioned above, there are various alternative definitions of $M_0(G)$. One such is to say that $\mu \in M_0(G)$ if and only if $\|L_t\mu - \mu\|' \to 0$ as $t \to e$, where L_t is the left translation operator defined by

$$\int f d(L_t\mu) = \int f(tu)\, d\mu(u).$$

The equivalence of this with the definition given is proved in ((2), Theorems 3 and 4). Other definitions involve the norm $\|\cdot\|_\rho$ derived from the regular representation instead of $\|\cdot\|'$.

We turn next to a relatively minor technicality. We wish to say that the limit-measure μ of a sequence of absolutely continuous measures μ_n is in $M_0(G)$ when we know that μ is the $w^\sim$-limit of μ_n, and the sequence μ_n is Cauchy in the norm $\|\cdot\|'$.

LEMMA 5·1. *If $\mu_n \to \mu$ in the $w^\sim$-topology and $\|\mu_n - \mu_m\|' \to 0$ as $m, n \to \infty$ then it follows that $\|\mu_n - \mu\|' \to 0$ as $n \to \infty$.*

Proof. We prove first that since $\|\mu\| = \lim \|\mu_n\|$ there is, given $\epsilon > 0$, a compact set K such that

$$\int_{G \setminus K} d|\mu_n| < \epsilon \|\mu_n\|, \quad \text{for all } n.$$

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We begin by choosing a compact set K_0 such that

$$\int_{G\setminus K_0} d|\mu| < \epsilon\|\mu\|/4.$$

Let $f \in C_0(G)$, $|f| \leqslant 1$, $\left|\int_G f d\mu\right| > (1-\epsilon/4)\|\mu\|$; then

$$\left|\int_{K_0} f d\mu\right| > (1-\epsilon/2)\|\mu\|.$$

Since $\mu_n \to \mu$ in the $w^\sim$-topology, there exists n_1 such that

$$\left|\int_{K_0} f d\mu_n\right| > (1-3\epsilon/4)\|\mu\| \quad \text{for} \quad n \geqslant n_1,$$

and there exists n_2 such that

$$(1-3\epsilon/4)\|\mu\| > (1-\epsilon)\|\mu_n\| \quad \text{for} \quad n \geqslant n_2.$$

So if $n_0 = \max(n_1, n_2)$ then for $n \geqslant n_0$ we have

$$\int_{K_0} d\mu_n| > \left|\int_{K_0} f d\mu_n\right| > (1-\epsilon)\|\mu_n\|,$$

i.e.

$$\int_{G\setminus K_0} d|\mu_n| < \epsilon\|\mu_n\|.$$

Now for each r, $1 \leqslant r \leqslant n_0$, there exists K_r such that

$$\int_{G\setminus K_r} d|\mu_r| < \epsilon\|\mu_r\|,$$

and so if we take $K = K_0 \cup K_1 \cup \ldots \cup K_{n_0}$ it is clear that

$$\int_{G\setminus K} d|\mu_n| < \epsilon\|\mu_n\| \quad \text{for all } n.$$

It follows that if f is any bounded continuous function then

$$\int f(t)\, d\mu_n(t) \to \int f(t)\, d\mu(t) \quad \text{as} \quad n \to \infty.$$

Now take the particular function $(U(t)\xi|\eta)$, where U is a continuous unitary representation of G on H, and $\xi, \eta \in H$. We have

$$(T(\mu)\xi|\eta) = \int_G (U(t)\xi|\eta)\, d\mu(t),$$

and it follows that

$$(T(\mu_n)\xi|\eta) \to (T(\mu)\xi|\eta) \quad \text{as} \quad n \to \infty.$$

Now let $(T, U : H)$ be arbitrary in $\mathscr{T}_c$, and ξ, η arbitrary in H. We choose an arbitrary $\epsilon > 0$. We have

$$\begin{aligned} &|(T(\mu_n)\xi|\eta) - (T(\mu)\xi|\eta)| \\ &\qquad \leqslant |(T(\mu_n)\xi|\eta) - (T(\mu_m)\xi|\eta)| + |(T(\mu_m)\xi|\eta) - (T(\mu)\xi|\eta)| \\ &\qquad \leqslant \|T(\mu_n) - T(\mu_m)\|\,\|\xi\|\,\|\eta\| + |(T(\mu_m)\xi|\eta) - (T(\mu)\xi|\eta)|. \end{aligned}$$

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We may as well suppose that $\|\xi\| = \|\eta\| = 1$ – if not, we can easily normalise. Choose first an n_0 such that

$$\|\mu_n - \mu_m\|' < \epsilon/3 \quad \text{for} \quad m, n \geqslant n_0.$$

Next, for given T, ξ, η choose m so large ($> n_0$) that

$$|(T(\mu_m)\,\xi|\eta) - (T(\mu)\,\xi|\eta)| < \epsilon/3.$$

Then for any T, any ξ, any η we have

$$|(T(\mu_n)\,\xi|\eta) - (T(\mu)\,\xi|\eta)| < 2\epsilon/3,$$

and hence

$$\sup_{T\in\mathscr{T}_c}\sup_{\xi}\sup_{\eta}|(T(\mu_n-\mu)\,\xi|\eta)| < \epsilon,$$

i.e.

$$\sup_{T\in\mathscr{T}_c}\|T(\mu_n-\mu)\| = \|\mu_n-\mu\|' < \epsilon.$$

provided only that $n \geqslant n_0$. This establishes the required result.

LEMMA 5·2. *Let $\nu_1, \ldots, \nu_k$ be absolutely continuous measures. Then the functions $U(t)\,T(\nu_r)\,\xi$ form an equicontinuous set for $(T, U : H) \in \mathscr{T}_c$, $\xi \in H$, $\|\xi\| \leqslant 1$, $1 \leqslant r \leqslant k$.*

Proof.

$$\begin{aligned}\|U(t)\,T(\nu_r)\,\xi - U(t')\,T(\nu_r)\,\xi\| &\leqslant \|U(t)\,T(\nu_r) - U(t')\,T(\nu_r)\| \\ &\leqslant \|L_t\,\nu_r - L_{t'}\,\nu_r\|\end{aligned}$$

(where L_t is the left translation operator)

$$< \epsilon \quad \text{for} \quad 1 \leqslant r \leqslant k,$$

provided $t'^{-1}t \in N$ for suitably chosen N.

LEMMA 5·3. *Let $(\mathscr{T}_0, \tau_0)$ be a given symmetric open template, μ_0 the canonical measure on $(\mathscr{T}_0, \tau_0)$. Let $(\phi_i)_{i\in I}$ be an equicontinuous set of functions on G to Hilbert spaces H_i, and let $\epsilon > 0$ be given. Then there is an open symmetric refinement $(\mathscr{T}, \tau)$ of $(\mathscr{T}_0, \tau_0)$ such that if μ is any measure subordinate to $(\mathscr{T}, \tau)$ we have*

$$\left\|\int_G \phi_i(t)\,d\mu_0(t) - \int_G \phi_i(t)\,d\mu(t)\right\| < \epsilon, \quad \forall i \in I.$$

Proof. Suppose without loss of generality that $\|\mu_0\| = 1$. Let N be an open symmetric neighbourhood of e such that

$$t_1^{-1}t_2 \in N \Rightarrow \|\phi_i(t_1) - \phi_i(t_2)\| < \epsilon/2 \quad \text{for} \quad t_1, t_2 \in \operatorname{supp}\mathscr{T}_0, \quad \forall i \in I.$$

Cover $\operatorname{supp}\mathscr{T}_0$ by sets of the form $\{xN, Nx : x \in \operatorname{supp}\mathscr{T}_0\}$ and let $\mathscr{S}$ be a minimal finite symmetric refinement, say

$$\mathscr{S} = \{S_1, \ldots, S_k, S_{k+1}, S_{k+1}^{-1}, \ldots, S_l, S_l^{-1}\}$$

where the sets S_i, S_i^{-1} are all distinct but not in general disjoint. Each S_i has a non-void intersection with some T_j or $T_j^{-1} \in \mathscr{T}_0$. Suppose that j is fixed and take the cases $T_j \neq T_j^{-1}$, $T_j = T_j^{-1}$ separately.

(1) $T_j \neq T_j^{-1}$ (so that $T_j \cap T_j^{-1} = \varnothing$, by our conventions about templates). By relabelling the sets of $\mathscr{S}$ if necessary suppose that $\{S_1, \ldots, S_m\}$ is a minimal cover of T_j. Write $W_1 = S_1$, $W_2 = S_2 \setminus S_1, \ldots, W_m = S_m \setminus (S_1 \cup \ldots \cup S_{m-1})$. Choose $x_r \in T_j \cap W_r$. Then choose an open symmetric neighbourhood N' of e such that $x_1 N', \ldots, x_m N', N'x_1^{-1}, \ldots, N'x_m^{-1}$ have disjoint closures and $x_r N' \subset S_r \cap T_j$ for all r.

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(2) $T_j = T_j^{-1}$. Again, suppose $\{S_1, \ldots, S_m\}$ is a minimal symmetric cover of T_j. Let W_r and x_r be as before: choose N' such that $x_1 N', \ldots, x_n N'$, and any inverses $N' x_r^{-1}$ with $x_r^{-1} \neq x_r$, are all disjoint and $x_r N'$, $N' x_r^{-1}$ are in $T_j \cap S_r$.

In any case let $(\mathscr{T}, \tau)$ be the open symmetric template with sets

$$x_r N',\ N' x_r^{-1}\,(x_r \neq x_r^{-1}), \quad x_r N' \cap N' x_r^{-1}\,(x_r^{-1} = x_r)$$

where
$$x_r = x_r(j), \quad N' = N'(j).$$

and τ defined appropriately on the disjoint collections corresponding to varying j:

$$\tau(x_r N') = \mu_0(T_j \cap S_r \setminus (S_1 \ldots S_{r-1})).$$

Let μ be any measure subordinate to $(\mathscr{T}, \tau)$. Then we have

$$\begin{aligned}
\left\| \int_G \phi_i(t)\, d\mu_0(t) - \int_G \phi_i(t)\, d\mu(t) \right\| &= \left\| \Sigma \left(\int_{T_j \cap W_r} \phi_i(t)\, d\mu_0(t) - \int_{x_r N'} \phi_i(t)\, d\mu(t) \right) \right\| \\
&\leqslant \sum_{j,r} \left(\left\| \int_{T_j \cap W_r} \phi_i(t)\, d\mu_0(t) - \phi_i(x_r)\, \tau(x_r N') \right\| \right. \\
&\quad \left. + \left\| \phi_i(x_r)\, \tau(x_r N') - \int_{x_r N'} \phi_i(t)\, d\mu(t) \right\| \right) \\
&< \sum_{j,r} (\epsilon/2 + \epsilon/2)\, \tau(x_r N') \\
&= \epsilon
\end{aligned}$$

as required.

LEMMA 5·4. *Let $\nu_1, \ldots, \nu_k$ be given measures in $M_a(G)$. Let $(\mathscr{T}_0, \tau_0)$ be an open symmetric template, μ_0 the canonical measure on $(\mathscr{T}_0, \tau_0)$. Let $\epsilon > 0$ be given. Then there is an open symmetric refinement $(\mathscr{T}, \tau)$ of $(\mathscr{T}_0, \tau_0)$ such that for any measure μ subordinate to $(\mathscr{T}, \tau)$ we have*

$$\|(\mu_0 - \mu)\, \nu_r\|' < \epsilon, \quad 1 \leqslant r \leqslant k.$$

Proof. Let $(T, U : H)$ be arbitrary in $\mathscr{T}_c$ and let ξ be arbitrary in H, $\|\xi\| \leqslant 1$. We have

$$T[(\mu_0 - \mu)\, \nu_r]\, \xi = \int U(t)\, T(\nu_r)\, \xi\, d(\mu_0 - \mu)\,(t),$$

and by Lemma 5·2 the functions $U(t)\, T(\nu_r)\, \xi$ are equicontinuous. Next apply Lemma 5·3 with this equicontinuous set instead of $(\phi_i)_{i \in I}$ and $\frac{1}{2}\epsilon$ instead of ϵ. It follows that there is an open symmetric refinement $(\mathscr{T}, \tau)$ of $(\mathscr{T}_0, \tau_0)$ such that for any μ subordinate to $(\mathscr{T}, \tau)$ we have

$$\|T[(\mu_0 - \mu)\, \nu_r]\, \xi\| < \tfrac{1}{2}\epsilon,$$

and the lemma follows on taking suprema, first over $\xi \in H$, $\|\xi\| \leqslant 1$ and then over $(T, U : H) \in \mathscr{T}_c$.

LEMMA 5·5. *Let $\epsilon > 0$ be given; then there exists $\delta > 0$ such that if A and B are any self-adjoint operators on any Hilbert space satisfying the conditions*

$$\|A\| \leqslant \alpha, \quad \|B\| \leqslant \beta, \quad \|AB\| < \delta\alpha\beta$$

then
$$\|A + B\| < (1 + \epsilon) \max(\alpha, \beta).$$

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Proof. For a somewhat stronger result, with $\delta = \epsilon^2$, see (8), Lemma 27.

COROLLARY 1. *If A, B are self-adjoint and satisfy $\|AB\| < \delta \|A\| \|B\|$ then*

$$\|A+B\| < (1+\epsilon)\max(\|A\|, \|B\|).$$

COROLLARY 2. *If A, B are self-adjoint and $AB = 0$ then $\|A+B\| \leqslant \max(\|A\|, \|B\|)$ (and hence, by a further elementary argument, $\|A+B\| = \max(\|A\|, \|B\|)$).*

COROLLARY 3. *If $\epsilon > 0$ is given there exists $\delta > 0$ such that if μ, ν are self-adjoint measures such that*

$$\|\mu\nu\|' < \delta\|\mu\|'\|\nu\|'$$

then

$$\|\mu+\nu\|' < (1+\epsilon)\max(\|\mu\|', \|\nu'\|).$$

Proof. Given ϵ choose δ so that $\|AB\| < \delta\alpha\beta$ implies

$$\|A+B\| < (1+\tfrac{1}{2}\epsilon)\max(\alpha, \beta).$$

For any $T \in \mathscr{T}_c$ we have

$$\|T(\mu)T(\nu)\| \leqslant \|\mu\nu\|' < \delta\|\mu\|'\|\nu\|'$$

so that

$$\|T(\mu)+T(\nu)\| < (1+\tfrac{1}{2}\epsilon)\max(\|\mu\|', \|\nu\|').$$

But this implies at once that

$$\|\mu+\nu\|' = \sup_{T\in\mathscr{T}_c}\|T(\mu)+T(\nu)\| \leqslant (1+\tfrac{1}{2}\epsilon)\max(\|\mu\|', \|\nu\|') < (1+\epsilon)\max(\|\mu\|', \|\nu\|').$$

We are now ready to embark on the main construction. We start with a given symmetric template $(\mathscr{T}_0, \tau_0)$ that may be chosen (by Proposition 2·5) so as to produce a measure that is within a given $w^\sim$-neighbourhood N of a given positive measure μ_0. If we are not concerned with proximity to a given measure μ_0, then $(\mathscr{T}_0, \tau_0)$ may be chosen arbitrarily: for example, $\mathscr{T}_0$ may be taken to consist of a single symmetric neighbourhood T of e, with $\tau(T) = 1$.

We proceed to refine our template by 'stages', each stage comprising many sub-stages. The general procedure is similar to that of ((3) or (1)), but there are substantial differences of detail. More precisely, we shall prove

THEOREM 5·6. *Let μ_0 be a given positive self-adjoint measure and let N be a given $w^\sim$-neighbourhood of μ_0. Then there is in N a positive self-adjoint measure μ that has independent powers and is moreover in $M_0(G)$.*

Proof. We fix in advance a convergent product $\Pi(1+\epsilon_r)$ with $\epsilon_r > 0$, and a basic set $N_1, N_2, N_3, \ldots$ of neighbourhoods of e in G with $N_r \supset N_{r+1}$. Let $(\mathscr{T}_0, \tau_0)$ be the template associated with μ_0 and N by Proposition 2·5; we may assume without loss of generality that $\|\mu_0\| = 1$. In describing the stages and sub-stages of the refinement process it will be convenient to list the sub-stages of stage n as $1, 2, \ldots p(n)$ and write

$$s(n, m) = p(0)+p(1)+p(n-1)+m;$$

i.e. $s(n, m)$ is the cumulative total of sub-stages required to reach stage n, sub-stage m.

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Suppose that we have arrived at stage n of the process, having completed stage $n-1$ if $n \geqslant 1$, and that $(\mathscr{T}_n, \tau_n)$ is a symmetric open template satisfying the conditions:

(i) $(\tau_n)_{\max} \leqslant 1/(n+1)^2$,

(ii) if $n \geqslant 1$, $\mathscr{T}_n$ is subordinate to N_n,

(iii) if $n \geqslant 1$, $(\mathscr{T}_n, \tau_n)$ is a refinement of $(\mathscr{T}_{n-1}, \tau_{n-1})$.

Let σ_n be the canonical measure on $(\mathscr{T}_n, \tau_n)$: since $\|\mu_0\| = 1$, we have $\|\sigma_n\| = 1$ for all n.

Let $\mathscr{S}_n$ be the collection of all subsets of $\mathscr{T}_n$ that are symmetric and are generated by $n+1$ sets of $\mathscr{T}_n$: i.e. $\mathscr{S}_n$ consists of all collections of the form

$$\{T_1, \ldots, T_k, T_{k+1}, T_{k+1}^{-1}, \ldots, T_{n+1}, T_{n+1}^{-1}\}$$

where $T_i^{-1} = T_i$ for $1 \leqslant i \leqslant k$, $T_i \in \mathscr{T}_n$ $(1 \leqslant i \leqslant n+1)$. $\mathscr{S}_n$ is not empty, in view of our assumption that $\|\mu_0\| = 1$, and condition (i). Let the elements of $\mathscr{S}_n$ be listed in arbitrary order as $\mathscr{S}_{n,1}, \ldots, \mathscr{S}_{n,p(n)}$. We shall list the corresponding templates in stage n as

$$(\mathscr{T}_n, \tau_n) = (\mathscr{T}_{n,1}, \tau_{n,1}), (\mathscr{T}_{n,2}, \tau_{n,2}), \ldots, (\mathscr{T}_{n,p(n)}, \tau_{n,p(n)}),$$
$$(\mathscr{T}_{n,p(n)+1}, \tau_{n,p(n)+1}) = (\mathscr{T}_{n+1}, \tau_{n+1}) = (\mathscr{T}_{n+1,1}, \tau_{n+1,1}).$$

At the mth sub-stage of stage n we shall obtain $(\mathscr{T}_{n,m+1}, \tau_{n,m+1})$ by modifying that part of the template $(\mathscr{T}_{n,m}, \tau_{n,m})$ that is subordinate to $\mathscr{S}_{n,m}$. We write $(\mathscr{T}_{n,m}^{(1)}, \tau_{n,m}^{(1)})$ for the restriction of $(\mathscr{T}_{n,m}, \tau_{n,m})$ to $\mathscr{S}_{n,m}$, and $(\mathscr{T}_{n,m}^{(2)}, \tau_{n,m}^{(2)})$ for the complement of $(\mathscr{T}_{n,m}^{(1)}, \tau_{n,m}^{(1)})$ in $(\mathscr{T}_{n,m}, \tau_{n,m})$.

Let $\sigma_{n,m}$ be the canonical measure on $(\mathscr{T}_{n,m}, \tau_{n,m})$, $\sigma_{n,m}^{(1)}$ the canonical measure on $(\mathscr{T}_{n,m}^{(1)}, \tau_{n,m}^{(1)})$ and $\sigma_{n,m}^{(2)}$ the canonical measure on $(\mathscr{T}_{n,m}^{(2)}, \tau_{n,m}^{(2)})$, so that

$$\sigma_{n,m} = \sigma_{n,m}^{(1)} + \sigma_{n,m}^{(2)}.$$

We shall choose a symmetric refinement $(\mathscr{T}_{n,m}^{(3)}, \tau_{n,m}^{(3)})$ of $(\mathscr{T}_{n,m}^{(1)}, \tau_{n,m}^{(1)})$ in a way to be described in detail below, and then take $\sigma_{n,m}^{(3)}$ to be the canonical measure on $(\mathscr{T}_{n,m}^{(3)}, \tau_{n,m}^{(3)})$. Finally we define

$$(\mathscr{T}_{n,m+1}, \tau_{n,m+1}) = (\mathscr{T}_{n,m}^{(3)}, \tau_{n,m}^{(3)}) \cup (\mathscr{T}_{n,m}^{(2)}, \tau_{n,m}^{(2)})$$

and

$$\sigma_{n,m+1} = \sigma_{n,m}^{(3)} + \sigma_{n,m}^{(2)}.$$

By the assumptions that $\mathscr{T}_{n,m}^{(1)}$ is generated by $n+1$ sets of $\mathscr{T}_n$, and hence contains at most $2n+2$ sets of $\mathscr{T}_n$, and that $(\tau_n)_{\max} \leqslant \dfrac{1}{(n+1)^2}$ it follows that

$$\sum_{T \in \mathscr{T}_{n,m}} \tau_{n,m}(T) \leqslant \frac{(2n+2)}{(n+1)^2} = \frac{2}{n+1}$$

and hence that

$$\|\sigma_{n,m+1} - \sigma_{n,m}\| = \|\sigma_{n,m}^{(3)} - \sigma_{n,m}^{(1)}\| \leqslant \frac{2}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

For the next stage in the argument it will be convenient to index the sequences $(\mathscr{T}_{n,m}, \tau_{n,m})$, $\sigma_{n,m}$ by the single suffix $s = s(n,m)$ as defined above. We recall that $\Pi(1+\epsilon_r)$ is a convergent product, with $\epsilon_r > 0$. Let δ_r be the real number corresponding

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to ϵ_r whose existence is assured by Lemma 5·5, Corollary 3, such that if μ and ν are self-adjoint measures then

$$\|\mu\nu\|' < \delta_r \|\mu\|' \|\nu\|' \quad \text{implies} \quad \|\mu+\nu\|' < (1+\epsilon_r)\max(\|\mu\|', \|\nu\|').$$

Now let s be fixed for the moment and take in Lemma 5·4 $\sigma_s - \sigma_r$ $(1 \leqslant r < s)$ instead of $\nu_1, \ldots, \nu_k$; $(\mathscr{T}_s^{(1)}, \tau_s^{(1)})$ instead of $(\mathscr{T}_0, \tau_0)$, and $\delta_s s^{-2} \min_{1\leqslant r<s} \|\sigma_s - \sigma_r\|'$ instead of ϵ. By Lemma 5·4 we can find an open symmetric refinement $(\mathscr{T}_s^{(4)}, \tau_s^{(4)})$ of $(\mathscr{T}_s^{(1)}, \tau_s^{(1)})$ such that for every measure $\sigma_s^{(4)}$ subordinate to $(\mathscr{T}_s^{(4)}, \tau_s^{(4)})$ we have, for $1 \leqslant r < s$,

$$\|(\sigma_s^{(4)} - \sigma_s^{(1)})(\sigma_s - \sigma_r)\|' < \delta_s . s^{-2} . \min_{1\leqslant r<s} \|\sigma_s - \sigma_r\|'.$$

We shall presently choose an open symmetric refinement $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$ of $(\mathscr{T}_s^{(4)}, \tau_s^{(4)})$, and take $\sigma_s^{(3)}$ to be the canonical measure on $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$; but in the meantime we show that for any choice of a self-adjoint measure $\sigma_s^{(4)}$ subordinate to $(\mathscr{T}_s^{(4)}, \tau_s^{(4)})$ if σ_{s+1} is defined to be $\sigma_s^{(4)} + \sigma_s^{(2)}$ then the sequence σ_s converges to its limit μ in the sense of the $\|\cdot\|'$ norm. (The fact that σ_s converges to μ in the $w^\sim$-topology follows from the general properties of the templates $(\mathscr{T}_n, \tau_n)$.)

We make the inductive hypothesis that for each $r \geqslant 1$ we have, for $s > r$,

$$\|\sigma_s - \sigma_r\|' \leqslant \prod_{m=r}^{s-1} (1+\epsilon_m)\left(\max(\|\sigma_{r+1} - \sigma_r\|', \ldots, \|\sigma_s - \sigma_{s-1}\|') + \frac{1}{r} - \frac{1}{s-1}\right) = q(r,s) \quad \text{say.}$$

Then there are two cases to consider:

(i) $$\|\sigma_{s+1} - \sigma_s\|' = \|\sigma_s^{(4)} - \sigma_s^{(1)}\|' < \frac{1}{s^2}.$$

In this case,

$$\|\sigma_{s+1} - \sigma_r\|' \leqslant \|\sigma_s - \sigma_r\|' + \frac{1}{s^2},$$

and it is then routine to verify that the right side of this is not greater than $q(r, s+1)$.

(ii) $\|\sigma_{s+1} - \sigma_s\|' = \|\sigma_s^{(4)} - \sigma_s^{(1)}\|' \geqslant 1/s^2$. In this case the inequality

$$\|(\sigma_s^{(4)} - \sigma_s^{(1)})(\sigma_s - \sigma_r)\|' < \delta_s s^{-2} \min_{1\leqslant r<s} \|\sigma_s - \sigma_r\|'$$

implies

$$\|(\sigma_{s+1} - \sigma_s)(\sigma_s - \sigma_r)\|' < \delta_s \|\sigma_{s+1} - \sigma_s\|' \|\sigma_s - \sigma_r\|' \quad (1 \leqslant r < s).$$

Taking in Lemma 5·5, Corollary 3, $\sigma_{s+1} - \sigma_s$ for μ, $\sigma_s - \sigma_r$ for ν, we conclude that

$$\|\sigma_{s+1} - \sigma_r\|' < (1+\epsilon_s)\max(\|\sigma_{s+1} - \sigma_s\|', \|\sigma_s - \sigma_r\|').$$

By the inductive hypothesis this right side does not exceed

$$(1+\epsilon_s)\max(\|\sigma_{s+1} - \sigma_s\|', q(r,s)) \leqslant \max\left\{(1+\epsilon_s)\|\sigma_{s+1} - \sigma_s\|',\right.$$

$$\left.\prod_{r}^{s-1} (1+\epsilon_m)\left(\max(\|\sigma_{r+1} - \sigma_r\|', \ldots \|\sigma_s - \sigma_{s-1}\|') + \frac{1}{r} - \frac{1}{s-1}\right)\right\} \leqslant q(r, s+1),$$

as is easily verified.

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It is clear that if $s = r+1$ then $q(r,s) = (1+\epsilon_r)\|\sigma_{r+1} - \sigma_r\|'$, so that the assumed inequality does indeed hold for all $s > r$.

In view of the fact that the sequence of templates and associated measures $\sigma_s = \sigma_{n,m}$ has been chosen so that $\|\sigma_{n,m+1} - \sigma_{n,m}\| \to 0$ as $n \to \infty$, hence $\|\sigma_{s+1} - \sigma_s\| \to 0$ as $s \to \infty$, it follows that $\|\sigma_{s+1} - \sigma_s\|' \to 0$ as $s \to \infty$, and hence $q(r,s) \to 0$ as $r, s \to \infty$. That is, the sequence σ_s is Cauchy in the $\|\cdot\|'$ norm. If $\sigma_s \to \mu$ as $s \to \infty$ it follows, by Lemma 5·1, that $\|\sigma_s - \mu\|' \to 0$, i.e. $\mu \in M_0(G)$.

It remains to impose the conditions required in order to ensure that the limit measure μ has independent powers. We use Proposition 4·6 and choose, for each $s = s(n,m)$, a symmetric open refinement $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$ of $(\mathscr{T}_s^{(4)}, \tau_s^{(4)})$ to satisfy the conditions

(i) $(\tau_s^{(3)})_{\max} \leqslant 1/(n+2)^2$,

(ii) $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$ has Property X,

(iii) $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$ is subordinate to N_{n+1}.

Let $\sigma_s^{(3)}$ be the canonical measure on $(\mathscr{T}_s^{(3)}, \tau_s^{(3)})$. We define

$$(\mathscr{T}_{s+1}, \tau_{s+1}) = (\mathscr{T}_s^{(3)}, \tau_s^{(3)}) \cup (\mathscr{T}_s^{(2)}, \tau_s^{(2)})$$

and

$$\sigma_{s+1} = \sigma_s^{(3)} + \sigma_s^{(2)}.$$

We note first that with the above conditions (i), (ii) if $s = s(n, p(n)+1) = s(n+1, 1)$ then $(\mathscr{T}_s, \tau_s)$ does satisfy the conditions assumed for $(\mathscr{T}_n, \tau_n)$, but with n replaced by $n+1$, so that the next stage of the refinement process can proceed.

All we have established so far for a general choice of measures σ_s subordinate to $(\mathscr{T}_s, \tau_s)$ continues to hold, of course, for the special choice now made. In particular, σ_s converges in the $w^\sim$-topology to a limit μ, in view of (iii), and also $\|\sigma_{s+1} - \sigma_s\| \to 0$ as $s \to \infty$, $\|\sigma_s - \sigma_r\|' \to 0$ as $r, s \to \infty$, so that $\|\sigma_s - \mu\|' \to 0$ as $s \to \infty$.

For the independent powers property of μ it will be convenient to consider only the subsequence of templates corresponding to $m = 1$. Let $p(\mu) = \sum_{r=0}^{k} c_r \mu^r$ be a given polynomial in μ, and suppose n chosen to be $\geqslant 2k-2$. Suppose then that $\mathscr{T}_{n,1}$ is generated by $N = N(n)$ elements: $\mathscr{T}_{n,1} = \{T_1, \ldots, T_N^{-1}\}$. Let $\mu = \mu_1 + \ldots + \mu_N$ where μ_r is the restriction of μ to $T_r \cup T_r^{-1}$. Let $(\mu^r)'$ be, as before, the sum of the terms in the multinomial expansion of μ^r that are products of r distinct factors μ_i $(1 \leqslant i \leqslant N)$.

Let $h_n = \max_{1 \leqslant r \leqslant N} \|\mu_r\|$: we have here

$$h_n \leqslant 2(\tau_{n,1})_{\max} \leqslant \frac{2}{(n+1)^2}.$$

For any two distinct integers r, p with $0 \leqslant r \leqslant k$, $0 \leqslant p \leqslant k$ we see that the supports of $(\mu^r)'$ and $(\mu^p)'$ are disjoint because they are contained in restricted words of lengths r, p respectively in the sets of $\mathscr{T}_{n,1}$. For any pair of such words there is a symmetric subset of $\mathscr{T}_{n,1}$ generated by $n+1$ elements that contains all the relevant sets, i.e. all supports are contained in some template $(\mathscr{T}_{n,m}^{(3)}, \tau_{n,m}^{(3)})$ for $1 \leqslant m \leqslant p(n)$. But this template has been chosen to have Property X, so the two words in question are disjoint. Thus the supports of all the distinct powers $(\mu^r)'$ for $0 \leqslant r \leqslant k$ are disjoint. It follows, just as in Theorem 4·5, that

$$\|p(\mu)\| \geqslant \sum_{r=0}^{k} |c_r| \, \|\mu\|^r - 2 \sum_{r=0}^{k} |c_r| \, k_r h_n$$

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and since $h_n \to 0$ as $n \to \infty$ and $2\sum_{r=0}^{k} |c_r|\, k_r$ is independent of n we have

$$\|p(\mu)\| = \sum_{r=0}^{k} |c_r|\, \|\mu\|^r$$

as required. This then shows that μ has independent powers as claimed.

THEOREM 5·7. *If G is a non-discrete metrisable locally compact group then $M_0(G)$ is asymmetric.*

Proof. Combine Proposition 4·2 and Theorem 5·6.

There is another corollary of Theorem 5·6 that perhaps deserves explicit mention. This is

THEOREM 5·8. *If G is non-discrete then $M_0(G)$ contains measures that are not absolutely continuous.*

Proof. It is easy to check that if a positive self-adjoint measure μ is absolutely continuous then it cannot have independent powers, since if μ is absolutely continuous then both μ^2 and μ^4 have densities that are strictly positive throughout some neighbourhood of e and so $\|\mu^2 - \mu^4\| < \|\mu\|^2 + \|\mu\|^4$. Thus the limit measure μ produced in Theorem 5·6 cannot be absolutely continuous.

An alternative (perhaps preferable) proof of Theorem 5·8 can be given without any reference to the independent powers condition. One has simply to modify the basic construction in Theorem 5·6 so as to include the condition that the Haar measure of the template $(\mathscr{T}_s, \tau_s)$ should tend to zero as $s \to \infty$; this may evidently be arranged without difficulty. Any limit measure of such a sequence of templates is then supported on a set of Haar measure zero, and thus cannot be absolutely continuous.

6. *The case of non-metrisable G*

In this section we extend the results previously obtained for metrisable groups to all non-discrete locally compact groups G.

To begin with, let G be any locally compact group and let G_1 be an open subgroup of G. Any measure on G_1 may be regarded as a measure on G, by the natural embedding. More formally, let $f \in C_0(G)$ and let ϕf be the restriction of f to G_1; evidently $\phi f \in C_0(G_1)$. Let $\mu \in M(G_1)$. Then if we define ϕ' by

$$\int_G f d(\phi'\mu) = \int_{G_1} \phi f\, d\mu,$$

it is clear that ϕ' is an isometric $\sim$-isomorphism of $M(G_1)$ into $M(G)$.

LEMMA 6·1.

(i) *If μ is absolutely continuous then $\phi'\mu$ is absolutely continuous.*

(ii) *If μ has independent powers then $\phi'\mu$ has independent powers.*

(iii) *If μ is self-adjoint then $\phi'\mu$ is self-adjoint.*

(iv) *If $\mu \in M_0(G_1)$ then $\phi'\mu \in M_0(G)$.*

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Proof. (i) This is clear since Haar measure on G_1 is the restriction to G_1 of Haar measure on G (if G and hence G_1 are compact then of course the usual normalization cannot hold for both G and G_1 simultaneously).

(ii) This is clear since ϕ' is an isometry.

(iii) This is clear since ϕ' is a $\sim$-isomorphism.

(iv) To prove this, note first that if $T \in \mathscr{T}_c(G)$ and T_1 is the restriction of T to G_1 then $T_1 \in \mathscr{T}_c(G_1)$: if $\mu \in M(G_1)$ then

$$T_1(\mu) = T(\phi'\mu) = \int_{G_1} U(t)\, d\phi'\mu(t)$$
$$= \int_{G_1} U_1(t)\, d\mu(t),$$

where U_1 is the restriction to G_1 of U. It follows that if $\mu \in M_0(G_1)$ and $\epsilon > 0$ is given there is an absolutely continuous $\nu \in M(G_1)$ with

$$\sup_{T_1 \in \mathscr{T}_c(G_1)} \|T_1(\mu) - T_1(\nu)\| < \epsilon.$$

For arbitrary $T \in \mathscr{T}_c(G)$ we have (with T_1 the restriction of T to G_1)

$$\|T(\phi'\mu) - T(\phi'\nu)\| = \|T_1(\mu) - T_1(\nu)\| \leqslant \sup_{T_1 \in \mathscr{T}_c(G_1)} \|T_1(\mu) - T_1(\nu)\| < \epsilon$$

and it follows that $\sup_{T \in \mathscr{T}_c(G)} \|T(\phi'\mu) - T(\phi'\nu)\| < \epsilon$: since $\phi'\nu$ is absolutely continuous, this means that $\phi'\mu \in M_0(G)$.

COROLLARY. *If for some open subgroup G_1 of G there is in $M(G_1)$ a self-adjoint measure with independent powers that is in $M_0(G_1)$ then there is in $M(G)$ a self-adjoint measure with independent powers that is in $M_0(G)$.*

Now for any non-discrete locally compact group G we can take a compact neighbourhood N of e in G, and the group G_1 generated by N; G_1 is evidently non-discrete, open, and σ-compact. It follows that in considering the asymmetry of $M(G)$ and $M_0(G)$ we can confine our attention to non-discrete σ-compact groups G: we have in fact

PROPOSITION 6·2. *If $M(G)$, or $M_0(G)$, is asymmetric for each non-discrete locally compact σ-compact group G then the same is true for each non-discrete group G.*

Proof. This follows at once from Lemma 6·1, taking into account Proposition 4·2.

Suppose next that G is locally compact and that G_1 is a compact invariant subgroup. Let U be a continuous unitary representation of G on a Hilbert space H. Let dt denote normalized Haar measure on G_1, and write

$$P = \int_{G_1} U(t)\, dt.$$

LEMMA 6·3. *P is a projection operator on H.*

Proof. Let $t_1 \in G_1$; then

$$P = \int_{G_1} U(t_1 t)\, d(t_1 t) = U(t_1) \int_{G_1} U(t)\, dt = U(t_1)\, P,$$

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so that
$$P = \int_{G_1} P\,dt_1 = \int_{G_1} U(t_1)\,P\,dt_1 = P^2.$$

Also,
$$P = \int_{G_1} U(t^{-1})\,d(t^{-1}) = \int_{G_1} U(t^{-1})\,dt = \int_{G_1} U(t)^*\,dt$$
$$= P^*.$$

Write $H_1 = PH$.

LEMMA 6·4. *The continuous unitary representation U of G on H induces a continuous unitary representation $\dot{U}$ of G/G_1 on H_1 by*:

$$\dot{U}(\dot{t}_1) = \int_{G_1} U(t_1 t)\,dt$$

where
$$\dot{t}_1 \in G/G_1.$$

Proof. Note first that if $t_1, t_1' \in \dot{t}_1$ then

$$\int_{G_1} U(t_1 t)\,dt = \int_{G_1} U(t_1 t_1^{-1} t_1' t)\,d(t_1^{-1} t_1' t) = \int_{G_1} U(t_1' t)\,dt,$$

so that $\dot{U}(\dot{t}_1)$ is well-defined. Since U is continuous, the same is true of $\dot{U}$. Next, note that

$$\dot{U}(\dot{t}_1) = \int_{G_1} U(t_1 t)\,dt = U(t_1)\,P,$$

if $t_1 \in \dot{t}_1$ where P is as above. Then

$$(\dot{U}(\dot{t}_1)\,\dot{\xi} | \dot{U}(\dot{t}_1)\,\dot{\eta}) = (U(t_1)\,P\dot{\xi} |\, U(t_1)\,P\dot{\eta}) = (P\dot{\xi}|P\dot{\eta}) = (\dot{\xi}|\dot{\eta}),$$

for $\dot{\xi}, \dot{\eta} \in PH = H_1$, so that $\dot{U}$ is unitary on H_1. We may similarly verify $\dot{U}(\dot{e}) = I_{H_1} (= P)$, and finally

$$\dot{U}(\dot{t}_1)\,\dot{U}(\dot{t}_2) = U(t_1)\,PU(t_2)\,P = U(t_1)\,U(t_2)\,P = U(t_1 t_2)\,P$$
$$= \dot{U}(\dot{t_1 t_2})\,P = \dot{U}(\dot{t}_1 \dot{t}_2)\,P = \dot{U}(\dot{t}_1 \dot{t}_2)$$

(since $PU(t_1) = U(t_1)\,P = PU(t_1)\,P$).

If U, $\dot{U}$ are as above, let T, $\dot{T}$ be the representations of $M(G)$, $M(G/G_1)$ associated with U, $\dot{U}$ in the usual way.

Let ψ be the map of $C_0(G)$ defined by

$$\psi f(\dot{t}) = \int_{G_1} f(tu)\,du,$$

where du is normalized Haar measure on G_1 and $\dot{t}$ is the equivalence class of t modulo G_1. Then it is immediate that ψ maps $C_0(G)$ into $C_0(G/G_1)$. The dual map ψ' is defined by

$$\int (\psi f)\,d\mu = \int f d(\psi'\mu)$$

($f \in C_0(G)$).

LEMMA 6·5. *ψ' is an isometric $\sim$-isomorphism of $M(G/G_1)$ into $M(G)$.*

Proof. This is essentially the same as the abelian case, proved in ((7), Lemma 4).

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LEMMA 6·6. *If T is a given representation ($\in \mathscr{T}_c$) of $M(G)$ on H, ξ a given vector in H with $\|\xi\| = 1$, there is a corresponding representation $\dot{T}$ of $M(G/G_1)$ on $H_1 = PH$ and a corresponding vector $\dot{\xi} \in H_1$ with $\|\dot{\xi}\| \leqslant 1$ such that if $\mu \in M(G/G_1)$ and $\psi'\mu$ is the corresponding measure in $M(G)$, we have*

$$T(\psi'\mu)\,\xi = \dot{T}(\mu)\,\dot{\xi}.$$

Proof.

$$(T(\psi'\mu)\,\xi|\eta) = \int_G (U(t)\,\xi|\eta)\,d\psi'\mu(t) = \int_{G/G_1} (\dot{U}(\dot{t})\,\xi|\eta)\,d\mu(\dot{t})$$

$$= \int_{G/G_1} (\dot{U}(\dot{t})\,P\xi|\eta)\,d\mu(\dot{t}) = \int_{G/G_1} (\dot{U}(\dot{t})\,P\xi|P\eta)\,d\mu(\dot{t}) = (\dot{T}(\mu)\,\dot{\xi}|\dot{\eta}),$$

say, with $\dot{\xi} = P\xi, \dot{\eta} = P\eta$. It follows that if $\eta \perp H_1$ then $(T(\psi'\mu)\,\xi|\eta) = 0$, i.e. $T(\psi'\mu)\,\xi \in H_1$ and then for arbitrary $\eta \in H_1$, $\dot{\eta} = \eta$ so that $T(\psi'\mu)\,\xi = \dot{T}(\mu)\,\dot{\xi}$ as asserted.

COROLLARY. $\|T(\psi'\mu)\| = \|\dot{T}(\mu)\|$.

It is, of course, immediate that if $\dot{T}$ is a given representation of $M(G/G_1)$ on H, there is a corresponding representation T of $M(G)$ on H. It is probably most convenient to describe the relation in terms of the corresponding representations $\dot{U}$ and U of G/G_1, G respectively. If $\dot{U}$ is given $\in \mathscr{T}_c(G/G_1)$ define $U \in \mathscr{T}_c(G)$ by: $\dot{U}(\dot{t}) = U(t)$, and it is clear that U is a continuous unitary representation of G on H. Then define

$$T(\mu') = \int U(t)\,d\mu'(t),$$

for any $\mu' \in M(G)$. If μ' is related to $\mu \in M(G/G_1)$ as described above, $\mu' = \psi'\mu$, then

$$T(\mu') = \int U(t)\,d\mu'(t) = \int \dot{U}(\dot{t})\,d\mu(t)$$

$$= \dot{T}(\mu).$$

LEMMA 6·7. *If $\mu \in M(G/G_1)$ then*

$$\sup_{T \in \mathscr{T}_c(G)} \|T(\psi'\mu)\| = \sup_{\dot{T} \in \mathscr{T}_c(G/G_1)} \|\dot{T}(\mu)\|.$$

Proof. Given $T \in \mathscr{T}_c(G)$ there exists by Lemma 6·6 a corresponding $\dot{T} \in \mathscr{T}_c(G/G_1)$ with $\|T(\psi'\mu)\| = \|\dot{T}(\mu)\|$, and it follows that

$$\sup_{T \in \mathscr{T}_c(G)} \|T(\psi'\mu)\| \leqslant \sup_{\dot{T} \in \mathscr{T}_c(G/G_1)} \|\dot{T}(\mu)\|.$$

On the other hand, given $\dot{T}$ there exists T with $T(\psi'\mu) = \dot{T}(\mu)$ and so the inequality

$$\sup_{T \in \mathscr{T}_c(G)} \|T(\psi'\mu)\| \geqslant \sup_{\dot{T} \in \mathscr{T}_c(G/G_1)} \|\dot{T}(\mu)\|$$

follows, giving the required equality.

LEMMA 6·8.

(i) *If μ is absolutely continuous then $\psi'\mu$ is absolutely continuous.*
(ii) *If μ has independent powers then $\psi'\mu$ has independent powers.*
(iii) *If μ is self-adjoint then $\psi'\mu$ is self-adjoint.*
(iv) *If $\mu \in M_0(G/G_1)$ then $\psi'\mu \in M_0(G)$.*

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Proof. (i) is clear since if $\int \dot{f} d\mu = \int \dot{f}(\dot{t})\, g(\dot{t})\, d\dot{t}$ where $g \in L_1(G/G_1)$ then

$$\int f d(\psi' \mu) = \int f(t)\, h(t)\, dt$$

where $h(t) = g(\dot{t})$ if $t \in \dot{t}$, and $h \in L_1(G)$.

(ii) is immediate since ψ' is an isometry.

(iii) is immediate since ψ' is a $\sim$-isomorphism.

(iv) follows from (i) and Lemma 6·7. For, if $\mu \in M_0(G/G_1)$ then given ϵ we can find an absolutely continuous $\nu \in M(G/G_1)$ with

$$\sup_{\dot{T} \in \mathscr{T}_c(G/G_1)} \|\dot{T}(\mu - \nu)\| < \epsilon.$$

But then

$$\sup_{T \in \mathscr{T}_c(G)} \|T(\psi' \mu - \psi' \nu)\| < \epsilon,$$

and $\psi' \mu$ is absolutely continuous, so that $\psi' \mu \in M_0(G)$.

PROPOSITION 6·9. *If $M(G)$, or $M_0(G)$, is asymmetric for each metrisable non-discrete locally compact group G then the same is true for each non-discrete locally compact σ-compact group G.*

Proof. If G is non-discrete, locally compact and σ-compact we may by ((4), Lemma 4·3) choose an invariant compact subgroup G_1 of G such that G/G_1 is metrisable and non-discrete. If there is in $M(G/G_1)$, or $M_0(G/G_1)$, a self-adjoint measure μ with independent powers, the same will be true of $M(G)$, or $M_0(G)$, the measure $\psi' \mu$ having the required properties by Lemma 6·8.

THEOREM 6·10. *If G is any non-discrete locally compact topological group then $M_0(G)$ (and hence $M(G)$) is asymmetric.*

Proof. Combine Theorem 5·7, Proposition 6·2 and Proposition 6·9.

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