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MODELING OCCURRENCES OF OBJECTS IN RELATIONS

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Abstract. We study the logical structure of 'real' relations, and in particular the notion of occurrences of objects in a state. We start with formulating a number of principles for occurrences and defining corresponding mathematical models. These models are analyzed to get more insight in the formal properties of occurrences. In particular, we prove uniqueness results that tell us more about the possible logical structures relations might have.

§1. Introduction. Relations are sometimes simply identified with sets of tuples of certain objects. For mathematical relations this is obviously appropriate since they *are* sets of tuples. But what about other relations, like, for example, the love relation? The idea that the state of Albert's loving Karen would be nothing but a tuple seems perverse.

According to Fine (2000), the standard view of relations is that the constituents of a relation always come in a certain order. But then any relation also has a converse relation. For example, the relation *is older than* has the converse relation *is younger than*. Now consider the state of Thomas's being older than Nick. If—in a certain context—you regard this as exactly the same state as Nick's being younger than Thomas, and you regard this state as a relational complex of a single underlying relation, then the question is, of *which* relation?

On an alternative view of relations any relation comes with orderless positions or argument-places to which objects can be assigned. For example, for the love relation we have the positions *lover* and *beloved*. Unfortunately, this alternative view is also problematic. Fine (2000, pp. 16–17) raises two objections. First, positions seem ontologically excessive, and second, he regards positions unsuitable for strictly symmetric relations, like the adjacency relation, since switching positions of arguments would give a different state. (See Leo, 2008b, for a more detailed discussion of argument-places).

A very promising view of relations introduced by Fine is based on the notion of substitution. According to this *antipositionalist view* the states of a relation form a network in which substituting objects of a state by other objects yields another state. For example, substituting Roos for Albert and Bas for Karen in the state of Albert's loving Karen, results in the state of Roos's loving Bas.

In Leo (2008a) we defined models that represent different views of relations. In particular we defined *substitution models*. We argued that substitution models adequately model a large class of 'real' relations. However, we also noted that these substitution models have certain limitations. For example, they have no typed domains for the objects. As a consequence, they are not accurate for a relation like 'drinks', because "Mo drinks tea" corresponds in a natural way to a state, but "tea drinks Mo" does not. We do not consider

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this limitation to be a serious one, since typed domains can be incorporated into the models in a straightforward way.

Substitution models as they were defined might have a more serious shortcoming. It could be argued that the substitution mechanism of the models is perhaps not refined enough, since objects are not explicitly substituted for individual *occurrences* of objects, but only for objects in a global sense. For example, for the state of Narcissus's loving Narcissus, the model did not allow for a substitution resulting in the state of Echo's loving Narcissus. Now, if occurrences of objects are a *basic* notion for relations, then this is a serious limitation of our substitution models.

But what exactly are occurrences? The notion of occurrences does not have the reputation of being crystal clear. Mates (1972, p. 49) even called it a "woolly notion." Occurrences can be considered in different contexts, for example, in expressions (Wetzel, 1993; Janssen & Visser, 2004; Kracht, 2007). In this paper we try to get a better grip on the nature of occurrences of objects in the logical space of 'real' relations by developing mathematical models for relations.[1]

Our approach is as follows. First, we formulate an initial set of principles for occurrences. Then, in light of these principles, we define mathematical models in which occurrences have a constitutive function. We perform a technical investigation of these models to get a better understanding of the formal properties of occurrences. Section 6 is the more philosophical part of the paper. There we try to say more about the nature of occurrences and consider the question whether, for 'real' relations, models with a refined substitution mechanism for occurrences of objects are in a relevant sense more complete or adequate than models with an undifferentiated substitution mechanism. In Section 6 we also briefly discuss the idea of explicitly distinguishing between states 'out there' and relational complexes, and to allow single states to figure as relational complexes in more than one relation. On a first reading of this paper, it might not be a bad idea to peek ahead to this section.

§2. Basic principles for occurrences. In his paper 'The Problem of *De Re* Modality', Fine (1989) proposes the development of a *general theory of constituent structure*, where the basic structure of the entities involved is given by the operation of substitution. Fine considers the following syntactic notions to be basic: *occurrence of*, *occurrence in*, and *substitution*. He gives the following example of a basic principle for these notions:

> One basic principle, for example, is that if F' is the result of substituting E' for the occurrence e of E within F, then there is an occurrence e' of E' within F' such that the result of substituting any expression E'' for e' within F' is identical to the result of substituting E'' directly for e in F.

The objective of this paper is not to develop a general theory of constituent structure. We restrict ourselves to the structure of relations. We postulate the following basic principles for occurrences of objects in relations:

P-1 Each relation has a nonempty set of states.

P-2 Each state of a relation has exactly one set of occurrences of objects.

[1] By the *logical space* of a relation we mean the totality of the substitutional interconnections in which the states of a relation stand to each other.

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P-3 Each occurrence is an occurrence of exactly one object.

P-4 Each occurrence has a type that corresponds to a domain of objects.

P-5 Objects of the same type can simultaneously be substituted for occurrences of objects in a state.

P-6 Any substitution of objects for occurrences in a state results in exactly one state of the same relation.

P-7 For any occurrence of an object in a state it makes for some substitution a difference for the resulting state which object is substituted for it.

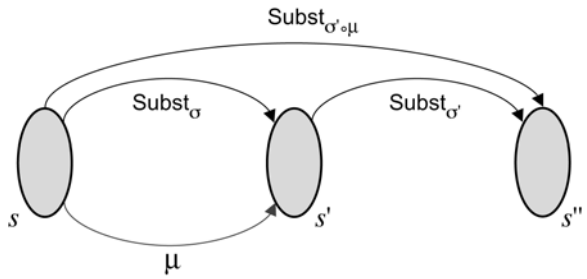
P-8 *Composition principle*: if a substitution in state s results in s', then there is a mapping μ from the occurrences in s to the occurrences in s' such that

(a) μ maps each occurrence α in s to an occurrence of the object substituted for α and

(b) any substitution in s' gives the same result as when we substitute in s for each occurrence α the object that is substituted in s' for $\mu(\alpha)$.

It should be noted that what is at stake here is substitution of occurrences of objects in states of relations and not of occurrences of terms in expressions denoting states.

We may depict the composition principle P-8 as follows:



I do not claim that the principles given above are in any sense complete. For example, one might perhaps like to add that different states have no occurrence in common. Also from these principles it cannot always be deduced uniquely how many occurrences a state has. They do not tell us whether the state of Narcissus' loving Narcissus has one or two occurrences.

Principle P-8 may seem too weak. The mapping μ is surjective by P-7 and P-8, but could we not also demand that μ is injective? Unfortunately, for certain relations this seems problematic, since substitution might perhaps result in a *coalescence* of occurrences. This is illustrated by the following three exemplary situations.

1. Consider the variably polyadic relation of *waiting for the bus*. Take the state of Janneke and Vincent's waiting. If we substitute Janneke for Vincent, we expect to get—if anything—a state with only one occurrence.
2. Consider the ternary relation $\mathfrak{R}$ where $\mathfrak{R}abc$ is the state that *a loves b and b loves c*. Assume that the order of the conjuncts is irrelevant. Now suppose that a, b, c are three different objects, and that each has one occurrence in the state $\mathfrak{R}abc$. Substituting in this state a for the occurrence of c gives the state $\mathfrak{R}aba$. Simultaneously substituting in the state $\mathfrak{R}abc$, the objects b, a, b for the occurrences of a, b, c gives the state $\mathfrak{R}bab$, which appears to be the same state as the state $\mathfrak{R}aba$. It follows

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by Principle P-8 that the state $\Re aba$ has only one occurrence of a and only one occurrence of b. So, μ cannot be injective (see Figure 1).

3. As a last example of a possible coalescence consider the conjunction $\Re$ & $\Re$ of a relation $\Re$ with itself. We may envision that the occurrences of each state s & s in the conjunction relation correspond one-to-one to the occurrences of s itself in $\Re$.

There may be arguments that can be used to reject a possible coalescence of occurrences. Against a coalescence for the ternary love relation it may, for example, be objected that substitution is a more subtle operation than we just seemed to suppose, and that substituting a for the occurrence of c in the state $\Re abc$ does *not* give the same result as substituting b, a, b for a, b, c in the state $\Re abc$. In Section 6 we will discuss such an alternative view in more detail.

In the next sections we will study the principles from a technical perspective by defining and analyzing mathematical models based on them.

§3. Modeling occurrences. We define two types of *substitution frames* to model the logical space of relations. In the first type, occurrences play no role and substitution is defined for objects. In the second type, we have a more refined substitution mechanism working on occurrences of objects.

3.1. Undifferentiated substitution frames. We call frames with substitution defined for objects *undifferentiated substitution frames*. Since this type of frames is extensively discussed in Leo (2008a), we give the definition without further explanation:

DEFINITION 3.1. *An* undifferentiated substitution frame *is a triple* $\mathcal{F} = \langle S, O, \Sigma \rangle$, *where* S *is a nonempty set of states,* O *is a nonempty set of objects, and* Σ *is a function from* $S \times O^O$ *to* S *such that*

1. *for all* $s \in S$, $\Sigma(s, \mathsf{id}_O) = s$,
2. *for all* $s \in S$ *and* $\delta, \delta' \in O^O$, $\Sigma(\Sigma(s, \delta), \delta') = \Sigma(s, \delta' \circ \delta)$.

For convenience, we will often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$. Further, we will also often write $f \cdot g$ for $g \circ f$. With this notation, Σ is such that for all $s \in S$ and for all $\delta, \delta' \in O^O$, $s \cdot \mathsf{id}_O = s$, and $(s \cdot \delta) \cdot \delta' = s \cdot (\delta \cdot \delta')$.

We now give a definition of the objects of a state. Roughly put, they are the objects for which it makes a difference for the resulting state which objects are substituted for them.

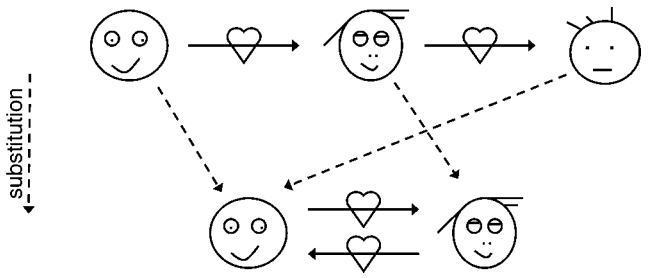


Fig. 1. Coalescence of occurrences.

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DEFINITION 3.2. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame. We call $A \subseteq O$ an* object-domain of $s \in S$ *if for every $\delta, \delta' : O \to O$,*

$$\delta =_A \delta' \;\Rightarrow\; s \cdot \delta = s \cdot \delta'.^2$$

We define the core objects of *s as:*

$$\mathsf{Core\text{-}ob}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the objects of *s. We denote this set as $\mathsf{Ob}_{\mathcal{F}}(s)$. If $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\mathsf{Ob}_{\mathcal{F}}(s)$ undefined.*

For each substitution frame we can define its degree as a cardinal number:

DEFINITION 3.3. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame. For a state s in S, we define the* object-degree of *s as:*

$$\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) = glb\,\{|A| \mid A \text{ is an object-domain of } s\}.$$

The object-degree of *$\mathcal{F}$ we define as:*

$$\mathsf{ob\text{-}degree}_{\mathcal{F}} = lub\,\{\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) \mid s \in S\}.$$

Here $|A|$ denotes as usual the cardinality of A, "glb" denotes the greatest lower bound, and "lub" denotes the least upper bound. Note that the degree of s and the degree of $\mathcal{F}$ always exist and are indeed cardinal numbers.

3.2. Differentiated substitution frames. In this subsection we define *differentiated substitution frames* that allow for a fine-grained account of substitution in states. We also discuss the adequacy of these frames, and present some basic properties.

3.2.1. Definition We give a definition of *differentiated substitution frames* that is based on the principles for occurrences given in Section 2:

DEFINITION 3.4. *A* differentiated substitution frame *is a tuple $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, Oc is a set of occurrences, Π is a mapping from Oc to O, and Θ is a function from $S \times O^{\mathsf{Oc}}$ to S such that*

1. *for all $s \in S$, $\Theta(s, \Pi) = s$,*
2. *for all $s \in S$ and $\sigma : \mathsf{Oc} \to O$ there is a mapping $\mu : \mathsf{Oc} \to \mathsf{Oc}$ such that*
   - *(a) $\Pi \circ \mu = \sigma$,*
   - *(b) for all $\sigma' : \mathsf{Oc} \to O$, $\Theta(\Theta(s, \sigma), \sigma') = \Theta(s, \sigma' \circ \mu)$.*

We say that μ corresponds to *$(s, \sigma) \mapsto \Theta(s, \sigma)$. For $\alpha \in \mathsf{Oc}$, we say that α is an* occurrence of *$\Pi(\alpha)$.*

Note that 2(a) implies that Π is surjective.

We will often write $s \cdot_{\mathcal{G}} \sigma$ or $s \cdot \sigma$ for $\Theta(s, \sigma)$. With this notation, Θ is such that (1) $s \cdot \Pi = s$, and (2) $\mu \cdot \Pi = \sigma$ and $(s \cdot \sigma) \cdot \sigma' = s \cdot (\mu \cdot \sigma')$.

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[2] For functions f, g we say that $f =_X g$ if $f \restriction X = g \restriction X$, that is, f restricted to X is equal to g restricted to X.

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If $s \cdot \sigma = s'$, then we denote the corresponding *transition* as $s \rightarrow^{\sigma} s'$.

Since the only type of frames we will discuss in this paper are differentiated and undifferentiated substitution frames, we will often just call them *frames* when it is clear from the context what type of frame we are talking about.

With any differentiated substitution frame we can associate a category with objects all states in S and with morphisms all triples $\langle s, \mu, s' \rangle$ with $\mu : \mathsf{Oc} \rightarrow \mathsf{Oc}$ satisfying Conditions 2(a) and 2(b) of the definition of a differentiated substitution frame for some $\sigma : \mathsf{Oc} \rightarrow O$ with $s \cdot \sigma = s'$.

Note that differentiated and undifferentiated substitution frames are both defined in the language of second-order logic. In Appendix C we will consider some alternative definitions of frames that also fulfill the principles in the previous section.

3.2.2. Adequacy of the definition For reasons of simplicity we deliberately chose to let our definition of a differentiated substitution frame deviate slightly from the principles in Section 2. The main differences are:

1. Substitution frames do not have typed domains for the objects.
2. The definition does not imply that every state has exactly one set of occurrences.
3. The definition is not completely consistent with the composition principle P-8.

Ad 1. As we remarked in the introduction, typed domains can easily be incorporated into our models. For our purposes, however, they do not really play a role.

Ad 2. In the next subsection we will define occurrences of states in accordance with Principle P-7. However, as Example 3.7 will show, our definition of differentiated substitution frames is too liberal, since occurrences are not always defined for every state. So, Principle P-2 is not always fulfilled. But I think this should not be considered to be a serious weakness of our models. Allowing certain borderline cases might be illuminating. It might even be that Principle P-2 turns out to be just too strict for certain 'real' relations.

Ad 3. The function μ corresponding to a transition $s \rightarrow^{\sigma} s'$ might map the occurrences of s to a proper superset of the occurrences of s'. Although I do not have an example of a 'real' relation where this is the case, I see no compelling reasons to exclude the possibility.

A further point to note is that in our models states may share the same occurrences. In our analysis we will take care to explicitly mention if any of our results depend on this.

3.2.3. Basic properties We define occurrences of states as follows:

DEFINITION 3.5. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *be a differentiated substitution frame. Then we call* $X \subseteq \mathsf{Oc}$ *an* occurrence-domain of $s \in S$ *if for all* $\sigma, \sigma' : \mathsf{Oc} \rightarrow O$,

$$\sigma =_X \sigma' \Rightarrow s \cdot \sigma = s \cdot \sigma'.$$

We define the core occurrences of s *as:*

$$\mathsf{Core\text{-}oc}_{\mathcal{G}}(s) = \bigcap \{X \mid X \text{ is an occurrence-domain of } s\}.$$

If $\mathsf{Core\text{-}oc}_{\mathcal{G}}(s)$ *is an occurrence-domain, then we call this set the* occurrences of s, *and we denote it as* $\mathsf{Oc}_{\mathcal{G}}(s)$. *If* $\mathsf{Core\text{-}oc}_{\mathcal{G}}(s)$ *is not an occurrence-domain, then we leave* $\mathsf{Oc}_{\mathcal{G}}(s)$ *undefined.*

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DEFINITION 3.6. *We say that a transition* $s \to^{\sigma''} s''$ *is a* composition of $s \to^{\sigma} s'$ and $s' \to^{\sigma'} s''$ *if* $\sigma'' =_X \mu \cdot \sigma'$ *for some occurrence-domain* X *of* s *and some mapping* μ *corresponding to* $s \to^{\sigma} s'$.

It may happen that $\mathsf{Core\text{-}oc}(s)$ is not an occurrence-domain:

EXAMPLE 3.7. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *be a differentiated substitution frame with* O *an infinite set,* $\mathsf{Oc} = O$, $\Pi = \mathsf{id}_O$, S *the set of subsets of* O *modulo a finite difference, that is,*

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, *and* Θ *defined by*

$$\widehat{A} \cdot \sigma = \widehat{\sigma[A]}.^3$$

Θ *is well-defined, since for any* $A, B \subseteq O$, *if* $\widehat{A} = \widehat{B}$, *then* $\widehat{\sigma[A]} = \widehat{\sigma[B]}$. *Further,* $\mathcal{G}$ *is a substitution frame, since*

1. $\widehat{A} \cdot \Pi = \widehat{\mathsf{id}_O[A]} = \widehat{A}$ *and*
2. $(\widehat{A} \cdot \sigma) \cdot \sigma' = \widehat{\sigma[A]} \cdot \sigma' = \widehat{\sigma'[\sigma[A]]} = \widehat{(\sigma \cdot \sigma')[A]} = \widehat{A} \cdot (\sigma \cdot \sigma')$.

It is not difficult to see that for any $A \subseteq O$, $\mathsf{Core\text{-}oc}(\widehat{A})$ *is the empty set, but if* A *is infinite, then the empty set is not an occurrence-domain of* $\widehat{A}$.

For undifferentiated substitution frames, a similar example can be given that shows $\mathsf{Core\text{-}ob}(s)$ is not always an object-domain (Leo, 2008a, Example 3.16).

LEMMA 3.8. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *be a differentiated substitution frame. For every* $s \in S$, *the occurrence-domains of* s *form a (possibly nonproper) filter on* Oc.

Proof. To prove that the occurrence-domains of s are closed under finite intersection, let X and X' be occurrence-domains of s. Let $\sigma, \sigma' : \mathsf{Oc} \to O$ be such that $\sigma =_{X \cap X'} \sigma'$. Define

$$\sigma''(\alpha) = \begin{cases} \sigma(\alpha) & \text{if } \alpha \in X - X', \\ \sigma(\alpha) = \sigma'(\alpha) & \text{if } \alpha \in X \cap X', \\ \sigma'(\alpha) & \text{if } \alpha \in X' - X. \end{cases}$$

Then $\sigma'' =_X \sigma$ and $\sigma'' =_{X'} \sigma'$. So, $s \cdot \sigma = s \cdot \sigma'' = s \cdot \sigma'$. Thus $X \cap X'$ is an occurrence-domain of s.
It is trivial that the occurrence-domains of s are upward closed.
Since an occurrence-domain may be empty, we may have a nonproper filter. □

DEFINITION 3.9. *For a differentiated substitution frame* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$, *we call* $A \subseteq O$ *an* object-domain of $s \in S$ *if for some occurrence-domain* X *of* s,

$$A = \Pi[X].$$

We define the core objects of s *as:*

$$\mathsf{Core\text{-}ob}_{\mathcal{G}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

[3] $f[X] = \{f(x) \mid x \in X\}$, the image of X under f.

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If $\mathsf{Core\text{-}ob}(s)$ *is an object-domain, then we call this set the* objects of s*, and we denote it as* $\mathsf{Ob}(s)$*. If* $\mathsf{Core\text{-}ob}(s)$ *is not an object-domain, then we leave* $\mathsf{Ob}(s)$ *undefined.*

LEMMA 3.10. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ *be a differentiated substitution frame. For every* $s \in S$*, the object-domains of* s *form a (possibly nonproper) filter on* O*.*

Proof. To prove that the object-domains of s are closed under finite intersection, let A and A' be object-domains of s. Then $A = \Pi[X]$ and $A' = \Pi[X']$ for some occurrence-domains X, X'. So, $A \cap A' = \Pi[X] \cap \Pi[X'] \supseteq \Pi[X \cap X']$. By Lemma 3.8, $X \cap X'$ is an occurrence-domain. It follows that $A \cap A'$ is an object-domain of s.
By the surjectivity of Π and the upward closedness of occurrence-domains, the object-domains of s are also upward closed.
Since an object-domain may be empty, we may have a nonproper filter. □

In the next lemma we show that (i) the objects of the core-occurrences of a state form a subset of its core-objects and (ii) if the occurrences of a state exist, then the objects of the state also exist, and are exactly the objects of the occurrences of the state.

LEMMA 3.11. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ *be a differentiated substitution frame. For every state* $s \in S$*,*

(i) $\Pi[\mathsf{Core\text{-}oc}(s)] \subseteq \mathsf{Core\text{-}ob}(s)$.

(ii) If $\mathsf{Oc}(s)$ *exists, then* $\mathsf{Ob}(s)$ *exists and* $\Pi[\mathsf{Oc}(s)] = \mathsf{Ob}(s)$.

Proof. By Definition 3.5 and Definition 3.9,

$$\begin{aligned}\Pi(\mathsf{Core\text{-}oc}(s)) &= \Pi\Big[\bigcap\{X \mid X \text{ is an occurrence-domain of } s\}\Big]\\ &\subseteq \bigcap\{\Pi[X] \mid X \text{ is an occurrence-domain of } s\}\\ &= \mathsf{Core\text{-}ob}(s).\end{aligned}$$

To prove the second claim, assume $\mathsf{Oc}(s)$ exists. Then $\mathsf{Oc}(s)$ is an occurrence-domain of s. So, $\Pi[\mathsf{Oc}(s)]$ is an object-domain of s. Thus $\mathsf{Core\text{-}ob}(s) \subseteq \Pi[\mathsf{Oc}(s)]$. By the first claim of the lemma, we also have the reverse inclusion. So, $\mathsf{Core\text{-}ob}(s) = \Pi[\mathsf{Oc}(s)]$. Because $\Pi[\mathsf{Oc}(s)]$ is an object-domain, the claim follows. □

The inclusion of Lemma 3.11 may be a proper inclusion, as the next example shows.

EXAMPLE 3.12. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ *be a differentiated substitution frame with* $O = \{a, b\}$*,* $S = \{s_1, s_2\}$*,* $\mathsf{Oc} = O \times \omega$*,* $\Pi(x, n) = x$*, and* Θ *defined by*

$$s \cdot \sigma = \begin{cases} s_1 & \text{if } s = s_1, \text{ and both } \sigma^{-1}(a) \text{ and } \sigma^{-1}(b) \text{ are infinite,}\\ s_2 & \text{otherwise.}\end{cases}$$

We see that $\mathsf{Ob}(s_1) = \{a, b\}$*, but* $\mathsf{Core\text{-}oc}(s_1) = \varnothing$ *and* $\mathsf{Oc}(s_1)$ *is undefined.*

The next lemma expresses the core-occurrences of a state in terms of single substitutions:

LEMMA 3.13. *If in* $\mathsf{Core\text{-}oc}(s)$ *the number of occurrences of each object is finite, then*

$$\mathsf{Core\text{-}oc}(s) = \{\alpha \mid \exists b \in O\ [\, s \cdot \Pi[\alpha \mapsto b] \neq s\,]\}.^4$$

[4] $f[a \mapsto b]$ denotes the function defined by $f[a \mapsto b](x) = b$ if $x = a$; $f(x)$ otherwise.

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Proof. If O has just one element, then it is obviously true. So, assume that O has at least two elements. Consider $\alpha_0 \in \mathsf{Core\text{-}oc}(s)$. Let $a = \Pi(\alpha_0)$ and b be some object in O with $b \neq a$. Define $\sigma_0 = \Pi[\alpha_0 \mapsto b]$. By the definition of differentiated substitution frames, there is a mapping $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ such that

$$\mu_0 \cdot \Pi = \sigma_0 \text{ and for all } \sigma : \mathsf{Oc} \to O,\ (s \cdot \sigma_0) \cdot \sigma = s \cdot (\mu_0 \cdot \sigma).$$

Assume that $A = \{\alpha \in \mathsf{Core\text{-}oc}(s) \mid \Pi(\alpha) = a\}$ is finite. Then there is an $\alpha_1 \in A$ such that $\alpha_1 \notin \mu_0[\mathsf{Core\text{-}oc}(s)]$. It follows that $s \cdot \sigma_0 \neq s$.
The inclusion in the other direction is obvious. □

In the lemma we cannot drop the finiteness condition, as the next example shows.

EXAMPLE 3.14. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame with $O = \{a, b\}$, $S = \{s_0, s_1, s_2, \ldots, s_\infty\}$, $\mathsf{Oc} = O \times \omega$, $\Pi(x, n) = x$, and Θ defined by*

$$s \cdot \sigma = \begin{cases} s_0 & \text{if } s = s_n \text{ with } n \in \omega \text{ and } \sigma(a, 0) = \sigma(b, 0), \\ s_n & \text{if } s = s_n \text{ with } n \in \omega \text{ and } \sigma(a, 0) \neq \sigma(b, 0), \\ s_{|\sigma^{-1}(x)|} & \text{if } s = s_\infty \text{ and } \sigma^{-1}(x) \text{ is finite,} \\ s_\infty & \text{otherwise.}^5 \end{cases}$$

Then $\mathsf{Oc}(s_\infty) = O \times \omega$, but for any $b \in O$ and $\alpha \in \mathsf{Oc}$, $s_\infty \cdot \Pi[\alpha \mapsto b] = s_\infty$.

The notion of *occurrence-degree* and *object-degree* are defined as follows:

DEFINITION 3.15. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame. For a state s in S, we define the* occurrence-degree of s *as:*

$$\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = glb\ \{|X| \mid X \text{ is an occurrence-domain of } s\}.$$

The occurrence-degree of $\mathcal{G}$ *we define as:*

$$\mathsf{oc\text{-}degree}_{\mathcal{G}} = lub\ \{\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) \mid s \in S\}.$$

The object-degrees $\mathsf{ob\text{-}degree}_{\mathcal{G}}(s)$ *and* $\mathsf{ob\text{-}degree}_{\mathcal{G}}$ *are defined in a similar way by starting with the object-domains of s.*

In Section 4 we will see that the occurrence-degree of a frame can be much higher than its object-degree.

3.3. Underlying frames. There is a natural embedding of the undifferentiated substitution frames in the class of differentiated substitution frames, and, conversely, each differentiated substitution frame also has an underlying undifferentiated substitution frame:

DEFINITION 3.16. *Let $\mathcal{F} = \langle S, O, \Sigma\rangle$ be an undifferentiated substitution frame and let $\mathcal{G} = \langle S', O', \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame. We say that $\mathcal{F}$* underlies *$\mathcal{G}$ if $S = S'$, $O = O'$, and for every $s \in S$ and $\delta : O \to O$,*

$$s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s \cdot_{\mathcal{F}} \delta.$$

We call $\mathcal{G}$ a basic refinement *of $\mathcal{F}$ if $\mathcal{F}$ underlies $\mathcal{G}$, $\mathsf{Oc} = O$ and $\Pi = \mathsf{id}_O$.*

---

[5] $|X|$ denotes as usual the cardinality of X.

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Note that in a basic refinement, different states may have occurrences in common. To prevent this, we could alternatively have defined for a basic refinement the set of occurrences as $S \times O$.

THEOREM 3.17. *(i) Each undifferentiated substitution frame has a unique basic refinement. (ii) Each differentiated substitution frame has a unique underlying undifferentiated substitution frame. (iii) If $\mathcal{F}$ is an undifferentiated substitution frame, then the underlying undifferentiated substitution frame of the basic refinement of $\mathcal{F}$ is $\mathcal{F}$ itself.*

Proof.

(i) This follows immediately from the definition of a basic refinement and the definition of a differentiated substitution frame.

(ii) Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be defined by $s \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{G}} (\Pi \cdot \delta)$. Then by Condition 1 of the definition of a differentiated substitution frame $s \cdot_{\mathcal{F}} \mathsf{id}_O = s \cdot_{\mathcal{G}} (\Pi \cdot \mathsf{id}_O) = s$. By Condition 2 of the definition of a differentiated substitution frame:

$$\begin{aligned}(s \cdot_{\mathcal{F}} \delta) \cdot_{\mathcal{F}} \delta' &= (s \cdot_{\mathcal{G}} (\Pi \cdot \delta)) \cdot_{\mathcal{G}} (\Pi \cdot \delta') \\ &= s \cdot_{\mathcal{G}} (\mu \cdot \Pi \cdot \delta') \qquad \text{for some } \mu \text{ with } \mu \cdot \Pi = \Pi \cdot \delta \\ &= s \cdot_{\mathcal{G}} (\Pi \cdot \delta \cdot \delta') \\ &= s \cdot_{\mathcal{F}} (\delta \cdot \delta').\end{aligned}$$

So, $\mathcal{F}$ is an undifferentiated substitution frame. Its uniqueness follows immediately from the definition of an underlying undifferentiated substitution frame.

(iii) This also follows immediately from the definitions. □

In Appendix B we will define categories for undifferentiated and differentiated substitution frames and show that there is an adjunction between them.

LEMMA 3.18. *Let $\mathcal{G}$ be a basic refinement of $\mathcal{F} = \langle S, O, \Sigma \rangle$. Then for every $s \in S$, the occurrence-domains of s in $\mathcal{G}$ are the same as the object-domains of s in $\mathcal{G}$ and the same as the object-domains of s in $\mathcal{F}$.*

Proof. The lemma follows immediately from the definitions. □

We may use this lemma to deduce certain results about undifferentiated substitution frames from results about differentiated substitution frames. For example, from Lemma 3.10 it follows that also for undifferentiated substitution frames the object-domains of s form a filter on O, and from Lemma 3.13 we get for undifferentiated substitution frames a characterization of the core of s. But of course, these results could also be obtained more directly.

The next lemma shows that refining a frame does not change the object-domains of its states.

LEMMA 3.19. *If $\mathcal{F}$ underlies $\mathcal{G}$, then any state has in $\mathcal{F}$ the same object-domains as in $\mathcal{G}$.*

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ underly $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$. Let A be an object-domain of s in $\mathcal{F}$. If O consists of one object, then $\varnothing$ is an object-domain of s in $\mathcal{G}$, and so is A. Therefore we may assume that O contains (at least) two different objects a and b. Let

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$\sigma_1, \sigma_1' : \mathsf{Oc} \to O$ be such that $\sigma_1 =_{\Pi^{-1}[A]} \sigma_1'$. Define

$$\sigma_2(\alpha) = \begin{cases} \sigma_1(\alpha) = \sigma_1'(\alpha) & \text{if } \Pi(\alpha) \in A, \\ \sigma_1(\alpha) & \text{if } \Pi(\alpha) = a, \\ \sigma_1'(\alpha) & \text{if } \Pi(\alpha) = b, \\ \Pi(\alpha) & \text{otherwise.} \end{cases}$$

We will show that $s \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} \sigma_1$. Then, by symmetry of the definition of σ_2, we also have $s \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} \sigma_1'$, and thus A is also an object-domain of s in $\mathcal{G}$.

Define $\delta_0 : O \to O$ by

$$\delta_0(d) = \begin{cases} d & \text{if } d \in A, \\ a & \text{otherwise.} \end{cases}$$

Define $\sigma_0 = \Pi \cdot \delta_0$. Then $s \cdot_{\mathcal{G}} \sigma_0 = s \cdot_{\mathcal{F}} \delta_0 = s$. Because $\mathcal{G}$ is a differentiated substitution frame, there is a function $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ such that $\mu_0 \cdot \Pi = \sigma_0$, and for any $\sigma : \mathsf{Oc} \to O$, $(s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma)$. So,

$$s \cdot_{\mathcal{G}} \sigma_2 = (s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma_2) = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma_1) = (s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma_1 = s \cdot_{\mathcal{G}} \sigma_1.$$

Conversely, let A be an object-domain of s in $\mathcal{G}$. Then there is an occurrence-domain X of s such that $A = \Pi[X]$. So, for any $\delta, \delta' : O \to O$ with $\delta =_A \delta'$, we have $\Pi \cdot \delta =_X \Pi \cdot \delta'$, and thus $s \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s \cdot_{\mathcal{G}} (\Pi \cdot \delta') = s \cdot_{\mathcal{F}} \delta'$. It follows that A is also an object-domain of s in $\mathcal{F}$. □

As a direct consequence of the lemma, if $\mathcal{F}$ underlies $\mathcal{G}$, then for any state s, $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s) = \mathsf{Core\text{-}ob}_{\mathcal{G}}(s)$, and $\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) = \mathsf{ob\text{-}degree}_{\mathcal{G}}(s)$.

§4. Refining occurrences. An undifferentiated substitution frame may underly various differentiated substitution frames. In establishing for a given relation the most adequate one, our principles in Section 2 seem to leave us with a lot of choices. In this section we investigate a natural (pre)ordering of frames. A consequence of the main result to be presented here, is that for a large class of relations *unique* maximally differentiated substitution frames exist.

We define refinements of differentiated substitution frames as follows:

DEFINITION 4.1. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *and* $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta' \rangle$ *be differentiated substitution frames. We say that* $\mathcal{G}$ *is a* refinement of $\mathcal{G}'$ *if* $S = S'$, $O = O'$, *and for every* $s \in S$ *there is a function* $\tau : \mathsf{Oc} \to \mathsf{Oc}'$ *such that*

1. $\tau \cdot \Pi' = \Pi$,
2. *for every* $\sigma : \mathsf{Oc}' \to O$, $s \cdot_{\mathcal{G}} (\tau \cdot \sigma) = s \cdot_{\mathcal{G}'} \sigma$.

We say that $\mathcal{G}$ *and* $\mathcal{G}'$ *are* equally refined *if for every* $s \in S$ *the function* τ *is injective on some occurrence-domain* X *of* s *in* $\mathcal{G}$.

We call a refinement a proper *refinement, if the frames are not equally refined.*

We also call $\mathcal{G}$ a refinement of an undifferentiated frame $\mathcal{F}$ if $\mathcal{G}$ is a refinement of the basic refinement of $\mathcal{F}$.

Note that $\mathcal{G}$ is a refinement of an undifferentiated substitution frame $\mathcal{F}$ iff $\mathcal{F}$ underlies $\mathcal{G}$. Also note that the relation *is equally refined* is an equivalence relation.

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For an undifferentiated substitution frame $\mathcal{F}$ we will call $\mathcal{G}$ a proper refinement, if $\mathcal{G}$ is a proper refinement of the basic refinement of $\mathcal{F}$.

LEMMA 4.2. *Differentiated substitution frames with all states of finite occurrence-degree are refinements of each other iff they are equally refined.*

Proof. The lemma follows immediately from the definitions. □

I am not sure whether the claim of the lemma is also true for all frames of infinite occurrence-degree. But refinements—modulo equal refinedness—obviously form a pre-order. In the next subsections we investigate the structure of this preorder in more detail.

4.1. Spurious refinements. Perhaps contrary to what one would expect, an undifferentiated substitution frame for a relation like the love relation has (many) proper refinements:

THEOREM 4.3. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame with O finite, and* $\mathsf{oc\text{-}degree}_{\mathcal{G}}$ *also finite, but unequal zero. Then for any infinite cardinal κ, there is a refinement $\mathcal{G}'$ of $\mathcal{G}$ with* $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.

Proof. Let κ be an infinite cardinal. Define the filter F on κ by $A \in F$ iff $|\kappa - A| < \kappa$. Let U be an ultrafilter on κ extending F. (Such an ultrafilter exists by the Axiom of Choice.) Then for any $A \in U$, $|A| = \kappa$. Further, since O is finite, for any $\tau \in O^{\kappa}$, there is exactly one $a \in O$ such that $\{\lambda \mid \tau(\lambda) = a\} \in U$.

It follows that we can make a proper refinement of $\mathcal{G}$ by replacing each occurrence by κ occurrences of the same object. More precisely, define $\mathcal{G}' = \langle S, O, \mathsf{Oc}', \Pi', \Theta'\rangle$ with

1. $\mathsf{Oc}' = \mathsf{Oc} \times \kappa$,
2. $\Pi' = \mathsf{proj}_1|_{\mathsf{Oc}'} \cdot \Pi$,
3. $s \cdot_{\mathcal{G}'} \sigma = s \cdot_{\mathcal{G}} \overline{\sigma}$ with $\overline{\sigma}$ such that $\forall \alpha \in \mathsf{Oc},\ \{\lambda \mid \sigma(\alpha, \lambda) = \overline{\sigma}(\alpha)\} \in U$.

We will show that (i) $\mathcal{G}'$ is a differentiated substitution frame, (ii) $\mathcal{G}'$ is a refinement of $\mathcal{G}$, and (iii) $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.

(i) We show that $\mathcal{G}'$ satisfies Conditions 1 and 2 of the definition of a differentiated substitution frame.

1. $s \cdot_{\mathcal{G}'} \Pi' = s$ because $\{\lambda \mid \Pi'(\alpha, \lambda) = \Pi(\alpha)\} = \kappa \in U$.
2. Consider any $s \in S$, and $\sigma : \mathsf{Oc}' \to O$. Let $\overline{\sigma}$ be as in the definition of $\mathcal{G}'$, let $\overline{\mu} : \mathsf{Oc} \to \mathsf{Oc}$ correspond to $s \to^{\overline{\sigma}} s \cdot \overline{\sigma}$, and let $\mu : \mathsf{Oc}' \to \mathsf{Oc}'$ be such that

   (a) $\mu \cdot_{\mathcal{G}'} \Pi' = \sigma$,
   (b) $\mu(\alpha, \lambda) = (\overline{\mu}(\alpha), \lambda)$ if $\sigma(\alpha, \lambda) = \overline{\sigma}(\alpha)$.

For any $\sigma' : \mathsf{Oc}' \to O$, let $\overline{\sigma}'$ be such that $\forall \alpha \in \mathsf{Oc}, \{\lambda \mid \sigma'(\alpha, \lambda) = \overline{\sigma}'(\alpha)\} \in U$. Then

$$\begin{aligned}(s \cdot_{\mathcal{G}'} \sigma) \cdot_{\mathcal{G}'} \sigma' &= (s \cdot_{\mathcal{G}} \overline{\sigma}) \cdot_{\mathcal{G}} \overline{\sigma}' \\ &= s \cdot_{\mathcal{G}} (\overline{\mu} \cdot \overline{\sigma}') \\ &= s \cdot_{\mathcal{G}'} (\mu \cdot \sigma'),\end{aligned}$$

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where the last equation can be proved as follows. Define for each $\alpha \in \mathsf{Oc}$ the following sets:

$$A_\alpha = \{\lambda \mid \mu(\alpha, \lambda) = (\overline{\mu}(\alpha), \lambda)\},$$
$$B_\alpha = \{\lambda \mid \sigma'(\alpha, \lambda) = \overline{\sigma}'(\alpha)\},$$
$$C_\alpha = \{\lambda \mid \sigma'(\mu(\alpha, \lambda)) = \overline{\sigma}'(\overline{\mu}(\alpha))\}.$$

Then $A_\alpha \cap B_{\overline{\mu}(\alpha)} \subseteq C_\alpha$. Because $A_\alpha, B_{\overline{\mu}(\alpha)} \in U$, and U is a filter, we have $C_\alpha \in U$, from which it follows that $s \cdot_{\mathcal{G}'} (\mu \cdot \sigma') = s \cdot_{\mathcal{G}} (\overline{\mu} \cdot \overline{\sigma}')$. Thus $\mathcal{G}'$ is a differentiated substitution frame.

(ii) To prove that $\mathcal{G}'$ is a refinement of $\mathcal{G}$, define $\tau : \mathsf{Oc}' \to \mathsf{Oc}$ by $\tau(\alpha, \lambda) = \alpha$. Then clearly τ is surjective and $\Pi' = \tau \cdot \Pi$. Further, $s \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}'} (\tau \cdot \sigma)$, because for each $\alpha \in \mathsf{Oc}$, $\{\lambda \mid \sigma(\tau(\alpha, \lambda)) = \sigma(\alpha)\} = \kappa \in U$.

(iii) Since $\mathsf{oc\text{-}degree}_{\mathcal{G}}$ is finite, and κ is infinite, we have $\mathsf{oc\text{-}degree}_{\mathcal{G}'} \leq \kappa$. Since $\mathsf{oc\text{-}degree}_{\mathcal{G}} \neq 0$, there is a $s_0 \in S$ and $\sigma_0 : \mathsf{Oc} \to O$ such that $s_0 \cdot_{\mathcal{G}} \sigma_0 \neq s_0$. Let X be an occurrence-domain of s_0 in $\mathcal{G}'$. Define $\sigma_0' : \mathsf{Oc}' \to O$ by

$$\sigma_0'(\alpha, \lambda) = \begin{cases} \Pi(\alpha) & \text{if } (\alpha, \lambda) \in X, \\ \sigma_0(\alpha) & \text{otherwise.} \end{cases}$$

Then $\sigma_0' =_X \Pi'$. So, $s_0 \cdot_{\mathcal{G}'} \sigma_0' = s_0$. Now suppose $|X| < \kappa$. Then, by the definition of U, for any $\alpha \in \mathsf{Oc}$, $\{\lambda \mid (\alpha, \lambda) \in \mathsf{Oc}' - X\} \in U$, and thus $s_0 \cdot_{\mathcal{G}'} \sigma_0' = s_0 \cdot_{\mathcal{G}} \sigma_0 \neq s_0$. So, we have a contradiction. It follows that $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$. □

What about frames with an infinite number of objects? Do they also always have proper refinements? By a rather straightforward modification of the proof of the previous theorem we can show that *any* frame with at least one transition from a state to a different one has a proper refinement—provided that there are arbitrary large measurable cardinals:

THEOREM 4.4. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *be a differentiated substitution frame with* $\mathsf{oc\text{-}degree}_{\mathcal{G}} \neq 0$. *Then for any measurable cardinal* $\kappa > \mathsf{max}(|O|, \mathsf{oc\text{-}degree}_{\mathcal{G}})$, *there is a refinement* $\mathcal{G}'$ *of* $\mathcal{G}$ *with* $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.

Proof. By definition, an uncountable cardinal κ is measurable iff there is a nonprincipal κ-complete ultrafilter on κ, where a filter is called κ-complete if it is closed under intersection of less than κ sets. See for example, Jech (2006, p. 127) or Kanamori (2005, p. 26).

Now let κ be a measurable cardinal greater than $\mathsf{max}(|O|, \mathsf{oc\text{-}degree}_{\mathcal{G}})$, and let U be a nonprincipal κ-complete ultrafilter over κ. Then for any $A \in U$, $|A| = \kappa$. Further, since $\kappa > |O|$, for any $\tau \in O^\kappa$, there is exactly one $a \in O$ such that $\{\lambda \mid \tau(\lambda) = a\} \in U$.

Define $\mathcal{G}' = \langle S, O, \mathsf{Oc}', \Pi', \Theta' \rangle$ exactly as in the proof of previous theorem. We may prove that $\mathcal{G}'$ is a differentiated substitution frame and that $\mathcal{G}'$ is a refinement of $\mathcal{G}$ in the same way as we did in (i) and (ii) of the proof of the previous theorem.

To prove that $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$, we first note that since $\mathsf{oc\text{-}degree}_{\mathcal{G}} \leq \kappa$, and κ is infinite, we have $\mathsf{oc\text{-}degree}_{\mathcal{G}'} \leq \kappa$. Then we may follow (iii) of the proof of the previous theorem to complete the proof. □

For 'real' relations the ultrarefinements constructed in the proof of the last two theorems do not seem to be adequate, since the states do not have a well-defined set of occurrences. Probably no metaphysical significance should be given to the existence of these spurious refinements.

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4.2. Normal refinements. In this section we focus on frames of finite occurrence-degree, which, as a consequence of Lemma 4.2, form—modulo equal refinedness—a partial order. In particular we want to investigate whether two given frames have a supremum. We begin with a negative result:

EXAMPLE 4.5. *Let* $\mathcal{F} = \langle S, O, \Sigma \rangle$ *be an undifferentiated substitution frame with* $O = \{a, b\}$, $S = \{\{a\}, \{b\}, \{a, b\}\}$, *and* Σ *defined by*

$$s \cdot \delta = \delta[s].$$

Then $\mathcal{F}$ *underlies a frame* $\mathcal{G}$ *with for each* $s \in S$, $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = 2$. *But for any natural number* $k > 2$, $\mathcal{F}$ *also underlies a frame* $\mathcal{G}'$ *in which the state* $\{a, b\}$ *has* k *occurrences for* a *and* k *occurrences for* b. *Since in* $\mathcal{G}'$ *the other states necessarily have only one occurrence, it follows immediately that* $\mathcal{G}$ *and* $\mathcal{G}'$ *have no refinement of finite occurrence-degree in common. Some extra thought shows that they also have no refinement of infinite occurrence-degree in common.*

In the example, the frame $\mathcal{G}'$ has the property of having an occurrence-degree that is greater than its object-degree. Although we can imagine that this might be the case for certain 'real' relations, we expect that in 'normal' cases the occurrence-degree and object-degree will be the same:

DEFINITION 4.6. *We call a frame or refinement* normal *if its occurrence-degree is the same as its object-degree. We call a normal frame or refinement* maximal *if it has no proper, normal refinement.*

The next theorem will be very useful in evaluating possible structures of relations:

THEOREM 4.7. *Let* Ψ *be a nonempty collection of normal refinements of a common substitution frame of finite object-degree. Then there is a least common refinement of the frames in* Ψ, *which is also normal and unique, modulo equal refinedness.*

Proof. We will construct a normal frame $\mathcal{G}^* = \langle S, O, \mathsf{Oc}^*, \Pi^*, \Theta^* \rangle$ of which we will show that it is a unique least common refinement of the frames in Ψ.

(i) *Construction of* $\mathcal{G}^*$:
Choose a frame $\mathcal{G}_0 = \langle S, O, \mathsf{Oc}_0, \Pi_0, \Theta_0 \rangle$ in Ψ.
Define $\mathsf{Oc}^* = \mathsf{Oc}_0 \times O$, and $\Pi^*(\alpha, a) = a$. For the definition of Θ^* we need some preparations:
Let $\rightarrow^*$ be the transitive closure of the relation $\rightarrow$ defined by:

$$s \rightarrow s' \text{ if } s \cdot_{\mathcal{G}} \sigma = s' \text{ for some } \mathcal{G} \in \Psi \text{ and some } \sigma.$$

For each $s \in S$ define

$$\Delta_s = \{s' \in S \mid s \rightarrow^* s' \ \&\ s' \rightarrow^* s\}.$$

In each Δ_s choose s_0 with $\mathsf{ob\text{-}degree}_{\mathcal{G}_0}(s_0)$ maximal. Because the frames in Ψ are normal refinements of a common frame, we have for each $\mathcal{G} \in \Psi$, $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0) = \mathsf{ob\text{-}degree}_{\mathcal{G}_0}(s_0)$. It follows that for any $\mathcal{G} \in \Psi$ and any $s \in S$ there is a transition from s_0 to s in $\mathcal{G}$ iff there is a transition from s_0 to s in $\mathcal{G}_0$. Thus, for every $s_1 \in \Delta_s$, we may choose σ_1 such that $s_0 \cdot_{\mathcal{G}_0} \sigma_1 = s_1$. Now define for an arbitrary $\sigma : \mathsf{Oc}^* \rightarrow O$,

$$s_1 \cdot_{\mathcal{G}^*} \sigma = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \sigma)$$

with $\tau_1 : \mathsf{Oc}_0 \rightarrow \mathsf{Oc}^*$ defined by $\tau_1(\alpha) = (\alpha, \sigma_1(\alpha))$. This completes the definition of $\mathcal{G}^*$.

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(ii) *Proof that $\mathcal{G}^*$ is a differentiated substitution frame*:
We will show that $\mathcal{G}^*$ fulfills the two conditions of Definition 3.4. Let $\mathcal{G}_0$, s_0, s_1, σ_1, τ_1 be as in the definition of $\mathcal{G}^*$. Then:

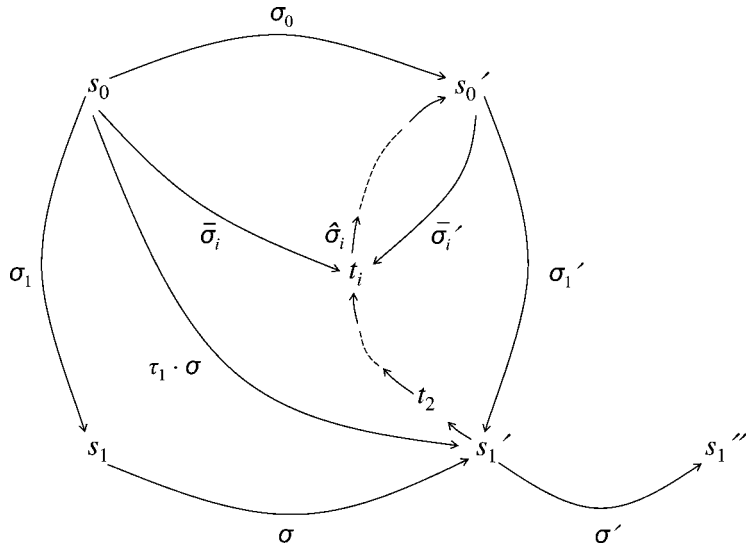
1. $s_1 \cdot_{\mathcal{G}^*} \Pi^* = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \Pi^*) = s_0 \cdot_{\mathcal{G}_0} \sigma_1 = s_1$.
2. Consider any $\sigma : \mathsf{Oc}^* \to O$. We have to show that there is a μ such that
   (a) $\mu \cdot \Pi^* = \sigma$,
   (b) for all $\sigma' : \mathsf{Oc}^* \to O$, $(s_1 \cdot_{\mathcal{G}^*} \sigma) \cdot_{\mathcal{G}^*} \sigma' = s_1 \cdot_{\mathcal{G}^*} (\mu \cdot \sigma')$.

Let $s_1' = s_1 \cdot_{\mathcal{G}^*} \sigma$, and let s_0', σ_1', τ_1' be related entities of s_1' in the construction of $\mathcal{G}^*$. As we noted before, for any $\mathcal{G} \in \Psi$ and any $s \in S$ there is a transition from s_0 to s in $\mathcal{G}$ iff there is a transition from s_0 to s in $\mathcal{G}_0$. So, because there is a transition from s_0 to s_1' in $\mathcal{G}_0$, and because there is a path from s_1' to s_0' of transitions in frames of Ψ, there is a transition σ_0 from s_0 to s_0' in $\mathcal{G}_0$. Then for any corresponding mapping $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ we have:

$$
\begin{aligned}
(s_1 \cdot_{\mathcal{G}^*} \sigma) \cdot_{\mathcal{G}^*} \sigma' &= s_0' \cdot_{\mathcal{G}_0} (\tau_1' \cdot \sigma') \\
&= (s_0 \cdot_{\mathcal{G}_0} \sigma_0) \cdot_{\mathcal{G}_0} (\tau_1' \cdot \sigma') \\
&= s_0 \cdot_{\mathcal{G}_0} (\mu_0 \cdot \tau_1' \cdot \sigma') \\
&= s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \sigma') \qquad \text{with } \tau_1 \cdot \tau_1^* = \mathsf{id}_{\mathsf{Oc}_0} \\
&= s_1 \cdot_{\mathcal{G}^*} (\tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \sigma').
\end{aligned}
$$

This proves Condition 2b.
Condition 2a requires some extra work. Let $X = \{(\alpha, \sigma_1(\alpha)) \mid \alpha \in \mathsf{Oc}_{\mathcal{G}_0}(s_0)\}$. Showing that $\tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \Pi^* =_X \sigma$ for some appropriate μ_0, will prove 2a. Because $\tau_1(\alpha) = (\alpha, \sigma_1(\alpha))$, $(\tau_1 \cdot \tau_1^*)(\alpha) = \alpha$ and $\tau_1' \cdot \Pi^* = \sigma_1'$, this is equivalent to showing that $\mu_0 \cdot \sigma_1' =_{\mathsf{Oc}_{\mathcal{G}_0}(s_0)} \tau_1 \cdot \sigma$ for some appropriate μ_0.

Choose a path $s_1' = t_1 \to^{\widehat{\sigma}_1} t_2 \to^{\widehat{\sigma}_2} \cdots \to^{\widehat{\sigma}_{n-1}} t_n = s_0'$ of transitions in frames of Ψ such that for each transition $t_i \to^{\widehat{\sigma}_i} t_{i+1}$ in $\mathcal{G}$ the mapping $\widehat{\sigma}_i \restriction \mathsf{Oc}_{\mathcal{G}}(t_i)$ is injective.



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For any $\mathcal{G} \in \Psi$, there is a natural one-to-one correspondence π between the occurrences of s_0 in $\mathcal{G}$ and in $\mathcal{G}_0$, namely $\pi(\alpha) = \alpha'$ iff $\Pi(\alpha) = \Pi_0(\alpha')$. For convenience, we will simply identify the occurrences of s_0 in $\mathcal{G}$ and $\mathcal{G}_0$. For s_0' we will do similarly. Now define transitions $s_0 \to^{\overline{\sigma}_i} t_i$ as follows:

1. $s_0 \to^{\overline{\sigma}_1} t_1$ is the transition $s_0 \to^{\tau_1 \cdot \sigma} t_1$,
2. $s_0 \to^{\overline{\sigma}_{i+1}} t_{i+1}$ is a composition of $s_0 \to^{\overline{\sigma}_i} t_i$ and $t_i \to^{\widehat{\sigma}_i} t_{i+1}$.

Also define transitions $s_0' \to^{\overline{\sigma}'_i} t_i$:

1. $s_0' \to^{\overline{\sigma}'_1} t_1$ is the transition $s_0' \to^{\sigma'_1} t_1$,
2. $s_0' \to^{\overline{\sigma}'_{i+1}} t_{i+1}$ is a composition of $s_0' \to^{\overline{\sigma}'_i} t_i$ and $t_i \to^{\widehat{\sigma}'_i} t_{i+1}$.

Because for each transition $t_i \to^{\widehat{\sigma}_i} t_{i+1}$ in $\mathcal{G}$ the mapping $\widehat{\sigma}_i \restriction \mathsf{Oc}_{\mathcal{G}}(t_i)$ is injective, it follows that if transition $s_0 \to^{\overline{\sigma}_{i+1}} t_{i+1}$ is a composition of $s_0 \to^{\sigma_0} s_0'$ and $s_0' \to^{\overline{\sigma}'_{i+1}} t_{i+1}$, then $s_0 \to^{\overline{\sigma}_i} t_i$ is a composition of $s_0 \to^{\sigma_0} s_0'$ and $s_0' \to^{\overline{\sigma}'_i} t_i$. So, choosing transition $s_0 \to^{\sigma_0} s_0'$ as a composition of $s_0 \to^{\overline{\sigma}_n} t_n$ and an inverse of $s_0' \to^{\overline{\sigma}'_n} t_n$ gives by induction that $s_0 \to^{\overline{\sigma}_1} t_1$ is a composition of $s_0 \to^{\sigma_0} s_0'$ and $s_0' \to^{\overline{\sigma}'_1} t_1$. Thus, $\mu_0 \cdot \sigma'_1 =_{\mathsf{Oc}_{\mathcal{G}}(s_0)} \tau_1 \cdot \sigma$.

(iii) *Proof that $\mathcal{G}^*$ is a refinement of each frame in Ψ:*
Let s_0, s_1, σ_1 be as in the construction of $\mathcal{G}^*$. Let $\mu_1 : \mathsf{Oc}_0 \to \mathsf{Oc}_0$ correspond to $s_0 \to^{\sigma_1} s_1$ in $\mathcal{G}_0$. Let $\tau : \mathsf{Oc}^* \to \mathsf{Oc}_0$ be a function with

$$\tau(\alpha, a) = \begin{cases} \mu_1(\alpha) & \text{if } a = \sigma_1(\alpha), \\ \alpha' \text{ with } \Pi_0(\alpha') = a & \text{otherwise.} \end{cases}$$

Then

1. $\tau \cdot \Pi_0 = \Pi^*$, because

$$\Pi_0(\tau(\alpha, a)) = \begin{cases} \Pi_0(\mu_1(\alpha)) = \sigma_1(\alpha) = \Pi^*(\alpha, a) & \text{if } a = \sigma_1(\alpha), \\ a = \Pi^*(\alpha, a) & \text{otherwise.} \end{cases}$$

2. for every $\sigma : \mathsf{Oc}_0 \to O$,
$$s_1 \cdot_{\mathcal{G}^*} (\tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}_0} (\mu_1 \cdot \sigma) = s_1 \cdot_{\mathcal{G}_0} \sigma.$$

So, $\mathcal{G}^*$ is a refinement of $\mathcal{G}_0$.
To see that $\mathcal{G}^*$ is also a refinement of any other frame $\mathcal{G}$ in Ψ, observe that if in the construction of $\mathcal{G}^*$ we would have chosen $\mathcal{G}$ instead of $\mathcal{G}_0$, then we would have obtained a frame equally refined as $\mathcal{G}^*$. Thus it follows that $\mathcal{G}^*$ is a refinement of each frame in Ψ.

(iv) *Proof that $\mathcal{G}^*$ is a unique least common refinement of the frames in Ψ:*
Let $\mathcal{G}^\diamond = \langle S, O, \mathsf{Oc}^\diamond, \Pi^\diamond, \Theta^\diamond \rangle$ be an arbitrary common refinement of the frames in Ψ. Let s_0, s_1, σ_1 be as in the construction of $\mathcal{G}^*$. Then there is a mapping $\tau^\diamond : \mathsf{Oc}^\diamond \to \mathsf{Oc}^*$ with $\tau^\diamond \cdot \Pi^* = \Pi^\diamond$ and $\tau^\diamond(\alpha) \in \mathsf{Oc}_{\mathcal{G}^*}(s_0)$ for any $\alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_0)$. Let $\mu_1 : \mathsf{Oc}^\diamond \to \mathsf{Oc}^\diamond$ be a bijection corresponding to $s_0 \to^{\tau^\diamond \cdot \sigma_1} s_1$ in $\mathcal{G}^\diamond$, and let $\mu_2 : \mathsf{Oc}^* \to \mathsf{Oc}^*$ correspond to $s_0 \to^{\sigma_1} s_1$ in $\mathcal{G}^*$. Further, let $\tau : \mathsf{Oc}^\diamond \to \mathsf{Oc}^*$ be a

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function with

$$\tau(\alpha) = \begin{cases} (\mu_1^{-1} \cdot \tau^\diamond \cdot \mu_2)(\alpha) & \text{if } \alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_1), \\ \alpha' \text{ with } \Pi^*(\alpha') = \Pi^\diamond(\alpha) & \text{otherwise.} \end{cases}$$

Then

1. $\tau \cdot \Pi^* = \Pi^\diamond$, because $(\tau^\diamond \cdot \mu_2 \cdot \Pi^*)(\alpha) = (\tau^\diamond \cdot \sigma_1)(\alpha) = (\mu_1 \cdot \Pi^\diamond)(\alpha)$ if $\alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_0)$.
2. $s_1 \cdot_{\mathcal{G}^\diamond} (\tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}^\diamond} (\tau^\diamond \cdot \mu_2 \cdot \sigma) = s_0 \cdot_{\mathcal{G}^*} (\mu_2 \cdot \sigma) = s_1 \cdot_{\mathcal{G}^*} \sigma$.

So, $\mathcal{G}^\diamond$ is a refinement of $\mathcal{G}^*$, and thus, because we assumed $\mathcal{G}^\diamond$ to be an arbitrary common refinement of the frames in Ψ, we see that $\mathcal{G}^*$ is not only a least common refinement, but also unique, modulo equal refinedness. □

In the theorem we assumed that the object-degree of the frames is finite. I am not sure whether this condition could be dropped.

REMARK 4.8. *We can also prove that for any nonempty collection of normal refinements of a common substitution frame of finite object-degree, there is a greatest common subrefinement, which is also normal and unique, modulo equal refinedness. The construction of this subrefinement is analogous to the construction of $\mathcal{G}^*$ in Theorem 4.7, only here we use, instead of $\rightarrow^*$, the transitive closure of the relation $\rightarrow_1$ defined by:*

$$s \rightarrow_1 s' \text{ if } s \cdot_{\mathcal{G}} \sigma = s' \text{ for all } \mathcal{G} \in \Psi \text{ and some } \sigma.$$

The proof that $\mathcal{G}^*$ is indeed the greatest common subrefinement and unique is relatively simple.

It follows that for a substitution frame of finite object-degree, the normal refinements—modulo equal refinedness—form a complete lattice.

A direct consequence of Theorem 4.7 is the following uniqueness result:

COROLLARY 4.9. *A normal substitution frame of finite object-degree has—modulo equal refinedness—a unique maximal normal refinement.*

4.3. Coalescence-free refinements. Of special interest are frames in which no coalescence of occurrences takes place:

DEFINITION 4.10. *We call a frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$* coalescence-free *if each transition $s \rightarrow^\sigma s'$ has a corresponding $\mu : \mathsf{Oc} \rightarrow \mathsf{Oc}$ that is injective on an occurrence-domain of s.*

Note that coalescence-free normal frames are maximally refined.

We will characterize coalescence-free frames in terms of their underlying undifferentiated frames. We will restrict ourselves to cases where the underlying frame is of finite object-degree and *simple*:

DEFINITION 4.11. *We call a frame $\mathcal{F} = \langle S, O, \Sigma \rangle$* simple *if there is a state s_0 such that*

$$S = \{s_0 \cdot \delta \mid \delta : O \rightarrow O\}.$$

We call s_0 an initial state.

Similarly, we call a frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ simple *if there is a state s_0 such that*

$$S = \{s_0 \cdot \sigma \mid \sigma : \mathsf{Oc} \rightarrow O\}.$$

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We define a class of frames whose transitions from an initial state s_0 are unique, modulo loops of s_0:

DEFINITION 4.12. *We call a simple frame* $\mathcal{F} = \langle S, O, \Sigma \rangle$ *of finite object-degree a* loop-initial frame *if for any initial state* s_0*, and any* $\delta_1, \delta_2 \in O^O$*,*

$$s_0 \cdot \delta_1 = s_0 \cdot \delta_2 \;\Rightarrow\; \exists \delta_0 \,[\, \delta_2 =_{\mathsf{Ob}(s_0)} \delta_0 \cdot \delta_1 \text{ and } s_0 \cdot \delta_0 = s_0 \,].$$

THEOREM 4.13. *Let* $\mathcal{F}$ *be a simple undifferentiated substitution frame of finite object-degree. Then* $\mathcal{F}$ *has a coalescence-free normal refinement iff* $\mathcal{F}$ *is a loop-initial frame .*

Proof. Assume $\mathcal{F}$ has a coalescence-free normal refinement $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$. Let s_0 be an initial state of $\mathcal{F}$, and let δ_1, δ_2 be such that $s_0 \cdot_{\mathcal{F}} \delta_1 = s_0 \cdot_{\mathcal{F}} \delta_2$. Then, because $\mathcal{G}$ is a refinement of $\mathcal{F}$, $s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_1) = s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_2)$.

For $i = 1, 2$ let $\mu_i : \mathsf{Oc} \to \mathsf{Oc}$ correspond to the transition $\Pi \cdot \delta_i$ from s_0. Since all states have the same, finite occurrence-degree, we may assume that the mappings μ_i are bijective. Clearly, there is a mapping $\delta_0 : O \to O$ such that $\Pi \cdot \delta_0 =_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \Pi$. So,

$$\begin{aligned}
s_0 &= s_0 \cdot_{\mathcal{G}} (\mu_1 \cdot \mu_1^{-1} \cdot \Pi) \\
&= (s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_1)) \cdot_{\mathcal{G}} (\mu_1^{-1} \cdot \Pi) \\
&= (s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_2)) \cdot_{\mathcal{G}} (\mu_1^{-1} \cdot \Pi) \\
&= s_0 \cdot_{\mathcal{G}} (\mu_2 \cdot \mu_1^{-1} \cdot \Pi) \\
&= s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_0) \\
&= s_0 \cdot_{\mathcal{F}} \delta_0.
\end{aligned}$$

Further,

$$\begin{aligned}
\Pi \cdot \delta_0 \cdot \delta_1 &=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \Pi \cdot \delta_1 \\
&=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \mu_1 \cdot \Pi \\
&=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \Pi \\
&=_{\mathsf{Oc}(s_0)} \Pi \cdot \delta_2.
\end{aligned}$$

So, $\delta_0 \cdot \delta_1 =_{\mathsf{Ob}(s_0)} \delta_2$. We have proved that $\mathcal{F}$ is a loop-initial frame.

Conversely, assume $\mathcal{F} = \langle S, O, \Sigma \rangle$ is a loop-initial frame. Define $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ as follows:

1. Define $\mathsf{Oc} = S \times O$.
2. Choose an initial state s_0 of $\mathcal{F}$.
3. For each $s \in S$ choose one $\delta_s \in O^O$ such that $s_0 \cdot_{\mathcal{F}} \delta_s = s$.
4. Define $\Pi(s, d) = \delta_s(d)$.
5. Define $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta$ with for all $d \in O$, $\delta(d) = \sigma(s, d)$.

We prove that $\mathcal{G}$ is a differentiated substitution frame by showing that $\mathcal{G}$ fulfills the two conditions of Definition 3.4.

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(1) $s \cdot_{\mathcal{G}} \Pi = s_0 \cdot_{\mathcal{F}} \delta_s = s$.

(2) Consider any $s \in S$ and $\sigma : \mathsf{Oc} \to O$. Then $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta_{s \cdot_{\mathcal{G}} \sigma}$. Also we have $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta$ with for all $d \in O$, $\delta(d) = \sigma(s, d)$. Because $\mathcal{F}$ is a loop-initial frame, $\delta =_{\mathsf{Ob}(s_0)} \delta_0 \cdot \delta_{s \cdot_{\mathcal{G}} \sigma}$ and $s_0 \cdot_{\mathcal{F}} \delta_0 = s_0$ for some δ_0. Define a function $\mu : \mathsf{Oc} \to \mathsf{Oc}$ with $\mu(s, d) = (s \cdot_{\mathcal{G}} \sigma, \delta_0(d))$, and for any other state $s' \in S$, $\Pi(\mu(s', d)) = \sigma(s', d)$. Then

$$\begin{aligned} \Pi(\mu(s, d)) &= \Pi(s \cdot_{\mathcal{G}} \sigma, \delta_0(d)) \\ &= \delta_{s \cdot_{\mathcal{G}} \sigma}(\delta_0(d)) \\ &= \sigma(s, d). \end{aligned}$$

So, $\mu \cdot \Pi = \sigma$.

We have $(s \cdot_{\mathcal{G}} \sigma) \cdot_{\mathcal{G}} \sigma' = s_0 \cdot_{\mathcal{F}} \delta'$ with for all $d \in O$, $\delta'(d) = \sigma'(s \cdot_{\mathcal{G}} \sigma, d)$. Further, $s \cdot_{\mathcal{G}} (\mu \cdot \sigma') = s_0 \cdot_{\mathcal{F}} \delta''$ with for all $d \in O$, $\delta''(d) = (\mu \cdot \sigma')(s, d)$. So, because $\mu(s, d) = (s \cdot_{\mathcal{G}} \sigma, \delta_0(d))$, we see that $\delta'' = \delta_0 \cdot \delta'$. Thus, because $s_0 \cdot_{\mathcal{F}} \delta_0 = s_0$, we have $(s \cdot_{\mathcal{G}} \sigma) \cdot_{\mathcal{G}} \sigma' = s \cdot_{\mathcal{G}} (\mu \cdot \sigma')$. This completes the proof that $\mathcal{G}$ is a differentiated substitution frame.

To show that $\mathcal{G}$ is a refinement of $\mathcal{F}$, we note that $s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s_0 \cdot_{\mathcal{F}} \delta'$ with for all $d \in O$, $\delta'(d) = (\Pi \cdot \delta)(s, d)$. So, $s_0 \cdot_{\mathcal{F}} \delta' = (s_0 \cdot_{\mathcal{F}} \delta_s) \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{F}} \delta$.

Further, it follows from Item 5 of the definition of $\mathcal{G}$ that $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = \mathsf{ob\text{-}degree}_{\mathcal{F}}(s_0)$, and thus that $\mathcal{G}$ is a coalescence-free normal frame. □

Note that in the second part of the proof we *choose* certain functions from O to O. But since O consists of *objects*, we may perhaps not assume that O is representable in ZFC. So, perhaps it would be more accurate to add an additional condition in the theorem. Because the object-degree of $\mathcal{F}$ is finite, it would be enough to assume that O can be totally ordered.

§5. Restricting frames. We can define operations like conjunction, disjunction and negation for undifferentiated substitution frames in a rather straightforward way. For example, the conjunction $\mathcal{F}$ & $\mathcal{F}'$ will have states s & s' with s a state of $\mathcal{F}$ and s' a state of $\mathcal{F}'$, and substitution defined by $(s \ \& \ s') \cdot \delta = s \cdot \delta \ \& \ s' \cdot \delta$. Note that the definition only requires that the following condition is satisfied:

$$s \ \& \ s' = t \ \& \ t' \ \Rightarrow \ s \cdot \delta \ \& \ s' \cdot \delta = t \cdot \delta \ \& \ t' \cdot \delta.$$

For differentiated substitution frames the situation is a bit more complicated. For example, in defining conjunction it is not immediately clear whether or not we should let the occurrences of a state s & s correspond one-to-one to the occurrences of s itself. Also, we should maybe be content with results that are unique modulo equal refinedness.

A really controversial issue is how to interpret such operations metaphysically. For example, in *The Philosophy of Logical Atomism* Russell (1956) said that when he argued that there were negative facts, it nearly produced a riot (p. 211). Russell says that on the whole he is inclined to believe that there are negative facts, but no disjunctive facts (Russell, 1956, p. 215). Armstrong (1997) rejects both negative and disjunctive facts, but he does accept conjunctive facts and totality facts. We will not pursue this issue here.

What I would like to discuss here in more detail is the notion of *restriction* for relations. Consider a frame for the love relation with states *x loving y*. If we restrict the states to those of *x loving Mo*, then we get a new frame for this restricted set of states. We get another type of restriction if we take as states only *x loves x*. More generally, we define:

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DEFINITION 5.1. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ *and* $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta'\rangle$ *be substitution frames. We say that* $\mathcal{G}$ *is a* restriction of $\mathcal{G}'$ *if* $S \subseteq S'$, $O = O'$, *and for every* $s \in S$ *there is an* $X \subseteq \mathsf{Oc}'$ *and a function* $\tau : X \to \mathsf{Oc}$ *such that*

1. $\tau \cdot \Pi =_X \Pi'$,
2. *for every* $\sigma : \mathsf{Oc} \to O$, $s \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}'} \sigma'$ *with* $\sigma' =_X \tau \cdot \sigma$ *and* $\sigma' =_{\mathsf{Oc}'-X} \Pi'$.

We call a restriction $\mathcal{G}$ *of* $\mathcal{G}'$ simple *if* $\mathcal{G}$ *is a simple substitution frame.*

We call a restriction $\mathcal{G}$ *of* $\mathcal{G}'$ maximal for S *if for any* $\mathcal{G}_1$ *with state set* S, *and such that* $\mathcal{G}$ *is a restriction of* $\mathcal{G}_1$ *and* $\mathcal{G}_1$ *is a restriction of* $\mathcal{G}'$, *the frames* $\mathcal{G}$ *and* $\mathcal{G}_1$ *are equally refined.*

Note that if $\mathcal{G}'$ is a refinement of $\mathcal{G}$, then $\mathcal{G}$ is a restriction of $\mathcal{G}'$.

The relation *restriction of* is transitive:

LEMMA 5.2. *If* $\mathcal{G}_1$ *is a restriction of* $\mathcal{G}_2$ *and* $\mathcal{G}_2$ *is a restriction of* $\mathcal{G}_3$, *then* $\mathcal{G}_1$ *is a restriction of* $\mathcal{G}_3$.

Proof. The proof follows straightforwardly from the definition of restriction. □

The relation *maximal restriction of* is not transitive:

EXAMPLE 5.3. *Let* $\mathfrak{R}xyuv$ *be the state* $x \mathrel{-\!\!\heartsuit\!\!\to} y \;\&\; u \mathrel{-\!\!\heartsuit\!\!\to} v$. *Let* $\mathcal{G}_1$ *be a frame with states* $\mathfrak{R}xyuv$ *and such that each state has four occurrences. Let* $\mathcal{G}_2$ *be a maximal restriction of* $\mathcal{G}_1$ *with states* $\mathfrak{R}xyyv$, *and such that the states* $\mathfrak{R}xyyx$ *have one occurrence of* x *and one occurrence of* y. *Now for a given* b *let* $\mathcal{G}_3$ *be a maximal simple restriction of* $\mathcal{G}_2$ *with states* $\mathfrak{R}xbbv$. *Then,* $\mathfrak{R}xbbx$ *obviously has one occurrence of* x *in* $\mathcal{G}_2$, *but this state has two occurrences of* x *in some refinement of* $\mathcal{G}_3$. *Since this refinement is also a restriction of* $\mathcal{G}_1$, *it follows that* $\mathcal{G}_3$ *is not a maximal restriction of* $\mathcal{G}_1$.

Note that the example also shows that coalescence-free frames are not closed under the restriction relation.

Also note that for any frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ and any nonempty $S' \subseteq S$, there is a restriction of $\mathcal{G}$ with state set S', namely frame $\mathcal{G}' = \langle S', O, \mathsf{Oc}, \Pi, \Theta'\rangle$ with $s \cdot_{\mathcal{G}'} \sigma = s$ for every $\sigma : \mathsf{Oc} \to O$. Because of such (degenerated) restrictions, maximal restrictions will obviously in general not be unique, modulo equal refinedness. However, if we restrict ourselves to simple restrictions, we have the following uniqueness result:

THEOREM 5.4. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ *be a normal frame of finite object-degree. Then for any* $S' \subseteq S$, *there is—modulo equal refinedness—at most one simple maximal restriction of* $\mathcal{G}$ *for* S'.

Proof. Let $\mathcal{G}_1 = \langle S', O, \mathsf{Oc}_1, \Pi_1, \Theta_1\rangle$ and $\mathcal{G}_2 = \langle S', O, \mathsf{Oc}_2, \Pi_2, \Theta_2\rangle$ be simple restrictions of $\mathcal{G}$. First we will show that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same underlying undifferentiated substitution frame. Then by Theorem 4.7, $\mathcal{G}_1$ and $\mathcal{G}_2$ have a least common refinement, which is normal and unique, modulo equal refinedness. We will show that this refinement is also a restriction of $\mathcal{G}$. Then the theorem follows since there are—modulo equal refinedness—only a finite number of normal refinements of $\mathcal{G}_1$.

(i) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the underlying undifferentiated substitution frames of $\mathcal{G}_1$ and $\mathcal{G}_2$. Choose a state $s_0 \in S'$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0)$ maximal. Then s_0 is an initial state of $\mathcal{G}_1$ and $\mathcal{G}_2$, because otherwise there would be a state $t_0 \in S'$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}}(t_0) > \mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0)$. It is not difficult to see that $\mathsf{Ob}_{\mathcal{G}_1}(s_0) = \mathsf{Ob}_{\mathcal{G}_2}(s_0)$, and that for any $\delta : O \to O$, $s_0 \cdot_{\mathcal{F}_1} \delta = s_0 \cdot_{\mathcal{F}_2} \delta$. So, for any $s \in S'$, there is a δ' such that

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$s \cdot_{\mathcal{F}_1} \delta = (s_0 \cdot_{\mathcal{F}_1} \delta') \cdot_{\mathcal{F}_1} \delta = s_0 \cdot_{\mathcal{F}_1} (\delta' \cdot \delta) = s_0 \cdot_{\mathcal{F}_2} (\delta' \cdot \delta) = (s_0 \cdot_{\mathcal{F}_2} \delta') \cdot_{\mathcal{F}_2} \delta = s \cdot_{\mathcal{F}_2} \delta$. Thus, $\mathcal{F}_1 = \mathcal{F}_2$.

(ii) Consider the construction of $\mathcal{G}^*$ in the proof of Theorem 4.7 starting with $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $s_0, s_1, \sigma_1, \tau_1$ be as in the construction of $\mathcal{G}^*$.
Further, let $X_0 \in \mathsf{Oc}_{\mathcal{G}}(s_0)$ and $\widetilde{\tau}_0 : X_0 \to \mathsf{Oc}_1$ be such that $\widetilde{\tau}_0 \cdot \Pi_1 =_{X_0} \Pi$, and for every $\sigma : \mathsf{Oc}_1 \to O$, $s_0 \cdot_{\mathcal{G}_1} \sigma = s_0 \cdot_{\mathcal{G}} \sigma'$ with $\sigma' =_{X_0} \widetilde{\tau}_0 \cdot \sigma$ and $\sigma' =_{\mathsf{Oc}'-X_0} \Pi$.
Define $\widetilde{\sigma}_1 : \mathsf{Oc} \to O$ by

$$\widetilde{\sigma}_1(\alpha) = \begin{cases} \sigma_1(\widetilde{\tau}_0(\alpha)) & \text{if } \alpha \in X_0 \\ \Pi(\alpha) & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mu}_1$ correspond to $s_0 \to^{\widetilde{\sigma}_1} s_1$. Then $\widetilde{\mu}_1$ is injective on X_0, because there is in $\mathcal{G}$ also a transition from s_1 to s_0. Now define $X_1 = \widetilde{\mu}_1[X_0]$ and $\widetilde{\tau}_1 = (\widetilde{\mu}_1 \restriction X_0)^{-1} \cdot \widetilde{\tau}_0 \cdot \tau_1$. It is straightforward to check that $\widetilde{\tau}_1 \cdot \Pi^* =_{X_1} \Pi$, and for every $\sigma : \mathsf{Oc}^* \to O$, $s \cdot_{\mathcal{G}^*} \sigma = s \cdot_{\mathcal{G}} \sigma'$ with $\sigma' =_{X_1} \widetilde{\tau}_1 \cdot \sigma$ and $\sigma' =_{\mathsf{Oc}-X_1} \Pi$. So, it follows that $\mathcal{G}^*$ is a restriction of $\mathcal{G}$. □

Other forms of restriction that never introduce a coalescence of occurrences are conceivable. But for such restrictions more than one state may have the same underlying state in the original relation. For example, if we start with the 'double' love relation of Example 5.3 then the state $x \mathrel{-\!\!\heartsuit\!\!\to} y \;\&\; y \mathrel{-\!\!\heartsuit\!\!\to} x$ may underly two states of a restriction, namely a state with two occurrences of x and one of y and another state with one occurrence of x and two of y. Also one state of a restriction may perhaps have more than one underlying state. This would, for example, be the case if we can make a further restriction to states $x \mathrel{-\!\!\heartsuit\!\!\to} y \;\&\; y \mathrel{-\!\!\heartsuit\!\!\to} x$.

§6. Back to reality. A key question here is how to determine, for any state of a relation, what its occurrences of objects are. For the love relation the state of Echo's loving Narcissus obviously has two occurrences, but what about the state of Narcissus's loving Narcissus? If this state has one occurrence, then substituting Narcissus for Echo in Echo's loving Narcissus gives a coalescence of occurrences.

If in the 'real' world no coalescence of occurrences takes place, then for many relations the logical space is straightforwardly determined. But if we may not exclude coalescence of occurrences, then the principles in Section 2 often leave us with many choices.

We are inclined to think that there is just one love relation, one adjacency relation, one similarity relation, and so forth. But this might perhaps be disputed. Why would we exclude the possibility that there is a love relation where the state of Narcissus's loving Narcissus has only one occurrence of Narcissus, and another relation where Narcissus's loving Narcissus has two occurrences of him? Maybe we should not even exclude the possibility of a love relation with both states, one with two occurrences of him and another with one. Could it be that there simply *is* no deeper metaphysical fact that determines the right choice?

These considerations might engender the uncomfortable feeling that we could easily get stuck with an abundance of relations for the same states of affairs 'out there' in reality. However, what we want is a clear, uncomplicated canonical view on the logical structure of relations.

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In our mathematical analysis we have arrived at a few results that provide more insight with regard to the possibilities:

(i) By Corollary 4.9, any normal substitution frame of finite object-degree has a unique maximal normal refinement.[6]

(ii) By Theorem 5.4, any normal substitution frame of finite object-degree has for each subset of states at most one simple maximal restriction.

We might formulate the first result in terms of relations as follows. Suppose we modeled the logical space of a given relation by a substitution frame for which the maximum number of occurrences of its states is finite and equal to the maximum number of objects of its states. Suppose further that the 'real' occurrences of the relation are a maximal refinement of the occurrences of the frame. Then the logical space of the relation is in fact uniquely determined by the original frame.

It seems reasonable to postulate that the logical space for restrictions of relations is maximal, *if unique* in an appropriate sense. By Theorem 5.4, we know that this applies to relations whose logical space corresponds to a simple restriction of a normal frame of finite object-degree. What makes this and the previous uniqueness result of metaphysical interest is that they probably apply to a large class of 'real' relations. Encouraged by these uniqueness results we might go a step further and postulate the following *maximality principle*:

P-x The occurrences of every relation are always maximally refined.

If this principle is true, then that would be very nice, since it would make the notion of occurrences more accessible. Let us test its viability by examining some objections that could be made:

Objection 1: Example 4.5 does not support the maximality principle. The example shows a substitution frame whose refinements have no refinement in common. So, in this case there is no unique maximum. Furthermore, it is an open question whether all normal substitution frame of *infinite* object-degree have a unique maximal normal refinement.

Objection 2: A conjunction of a relation with itself will introduce a coalescence of occurrences, if the occurrences of a state $s \ \& \ s$ correspond one-to-one with the occurrences of s. If so, then this would obviously contradict the maximality principle.

Objection 3: Relations whose states have a *set-like* character are straightforward counterexamples to the maximality principle. Take the relation of *waiting for the bus*. If we may substitute Janneke for Vincent in the state of Janneke and Vincent's waiting for the bus, the resulting state will have only one occurrence, and therefore will not be maximally refined.

These objections seem strong, but there may be escape routes available:

Against Objection 1 it might be argued that the frame of the example is farfetched and probably does not correspond to a 'real' relation. As a safer alternative, we could restrict the maximality principle to relations whose logical space corresponds to normal frames of finite degree.

[6] More precisely we should say: unique up to equal refinedness.

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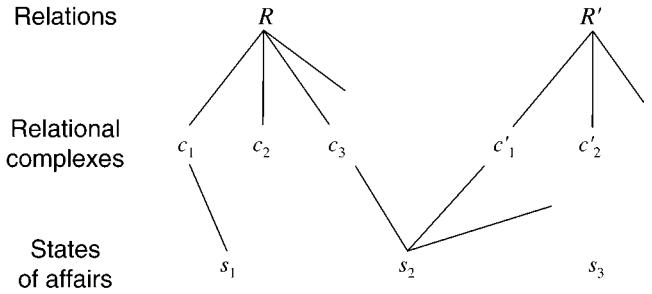


Fig. 2. Should we distinguish relational complexes from states of affairs?

Some people will reject Objection 2, because they will deny that there are conjunctive relations. But let us suppose that conjunctive relations exist. Then why would the occurrences of s & s correspond one-to-one to the occurrences of s? I don't see a convincing argument. Even if s & s and s are identical states, then there could still be a way out, namely by arguing that not states, but *relational complexes* have occurrences. A relational complex could be conceived of as a structured perspective on a state 'out there', where we give up the idea that there is a one-to-one correspondence or even identity between states of affairs and relational complexes[7] (see Figure 2). With this approach, there seems to be no compelling reason why there should be a one-to-one correspondence between occurrences of different relational complexes corresponding to the same state.

Maintaining both states 'out there' and relational complexes, however, has as potential drawback that it might give rise to an inflation of ontology. A more detailed analysis is needed to address this issue. Note, by the way, that in our formal analysis we also allowed states to belong to more than one frame: in the definition of a restriction of a frame (Definition 5.1) the states of a restriction of a frame $\mathcal{G}$ are a subset of the states of $\mathcal{G}$ itself. If this is not acceptable, then an alternative definition of a restriction might be given that takes *hiding* and *merging* of occurrences as primitive operations.

With respect to Objection 3 two different replies are possible. First, one could argue that the objects of states with a set-like character are sets themselves, not the members of the sets. For the given example, this would mean that in fact we should substitute the set consisting of Janneke for the set consisting of Janneke and Vincent. Alternatively, one could argue that certain substitutions in states with a set-like character are highly dubious, since it is in some cases unclear what exactly the result should be. In the example, it may be more natural to leave substituting Janneke for Vincent undefined. The interrelatedness of the states may more adequately be expressed by the operation of *subtraction*. It seems natural to say that *subtracting* Vincent from the state of Janneke and Vincent's waiting for the bus results in the state of Janneke's waiting for the bus. If we follow this road, then we need a weaker version of Principle P-6:

P-6′ Any substitution of objects for occurrences in a state results in *at most* one state of the same relation.

In addition, we should formulate a principle for subtracting occurrences similar to P-6′ and a composition principle for subtractions.

[7] Such a one-to-one correspondence is often taken for granted. For example, Russell (1984, p. 80) says: "there is certainly a one-one correspondence of complexes and facts."

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As an aside, note that Principle P-6′ also has the advantage that it allows us to keep meaningless or (conceptually) impossible states out of our ontology. Substituting, for example, *a* for *b* in *a's being adjacent to b* might be problematic. If you do not accept the existence of conceptually impossible states, then it could be better to leave the result of this substitution undefined.

To conclude this discussion, I think that a maximality principle like principle P-x might be the right choice for getting close to capturing the essence of occurrences. But to make such a principle really plausible, we need to conduct a more profound study of the notion of states, of substitution and probably of other operations, like subtraction. Also a follow-up of our formal analysis will be needed where partial substitution and subtraction functions are taken into account.

A view on relations without coalescence of occurrences still seems an attractive alternative, because of its simplicity. We already countered two arguments against it, namely a coalescence in relations with set-like states, and coalescence introduced by a conjunction of relations. In Section 2 we gave a third objection against a coalescence-free account. We considered the ternary relation $\mathfrak{R}$ where $\mathfrak{R}abc$ is the state of *a's loving b & b's loving c*. This relation has the peculiar property that $\mathfrak{R}bab$ and $\mathfrak{R}aba$ are identical states. As a consequence, this state can have only one occurrence of *a* and one occurrence of *b*, and thus in a transition from $\mathfrak{R}abc$ to this state we get coalescence of occurrences. How can we counter this objection?

Again relational complexes may be the solution. We could argue that there are *two* relational complexes corresponding to the state of *a's loving b & b's loving a*, namely one relational complex with two occurrences of *a* and one with two occurrences of *b*. Only in this case we would have to accept that one state 'out there' can have more than one relational complex within the *same* relation. This approach looks quite natural if we regard the relation $\mathfrak{R}$ as a restriction of a quaternary relation with states like *a's loving b & c's loving d*, because then one of the relational complexes corresponding to *a's loving b & b's loving a* in $\mathfrak{R}$ could be taken as the result of merging two occurrences of *a* and the other relational complex as the result of merging two occurrences of *b*. For any relation with a comparable symmetry, a similar 'solution' could be given. We also regard this issue as something that requires further investigation.

There is one final issue of metaphysical importance that I would like to discuss, namely the question: Do we really *need* occurrences?

Occurrences are definitely a very useful part of our representation of reality, but that in some cases the best way to express certain properties of states is in terms of occurrences does not force us to any ontological commitment to them. So, do we have a compelling reason to assume that *for relations* occurrences are ontologically basic?

Let us assume, for the moment, that the maximality principle is valid, and in addition that for every 'real' relation the underlying undifferentiated substitution frame has a unique maximal refinement. Would this not imply that a complete account of relations could be given in terms of undifferentiated substitutions? If so, then a parsimonious undifferentiated substitution mechanism might be enough for the ontology of relations.

I only see one strong argument in favor of occurrences being basic for relations, namely that some relations might just not have enough objects for an occurrence-free account of relations. Consider a relation whose occurrence-degree is larger than its object-degree. Then undifferentiated substitution might not provide enough information about the interconnection of the states. We might have such a situation, for example, with a conjunction $\mathfrak{R}\ \&\ \mathfrak{R}\ \&\ \cdots\ \&\ \mathfrak{R}$, where $\mathfrak{R}$ is a binary relation with a finite number of objects.

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Probably such situations do not only occur in complex relations, but also in what we would regard as elementary relations. An example of such an elementary relation could maybe be something like a relation with states of *a's loving b more than c* in a mini-world with two inhabitants.

My conclusion is that we have good reason to consider occurrences of relations to be a primitive notion. Further, I think our analysis has brought us closer to revealing the essence of the logical structure of relations. Although some major issues are still open, the way we have articulated them may very well contribute to their solution.

§7. Acknowledgments. I am grateful to Kit Fine for illuminating discussions about coalescence of occurrences. Further, I thank Albert Visser and the referee for valuable comments on the manuscript.

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Appendix A: Occurrences and roles. An object can fulfill one or more roles in a state. For example, in the amatory relation an object can play the role of *lover* or the role of *beloved*. In this appendix we investigate how roles are related to occurrences.

We only define roles for simple substitution frames of finite degree. The definition will be such that

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1. any object of any initial state fulfills exactly one role,
2. if s_0 is an initial state, $a_0 \in \mathsf{Ob}(s_0)$, and $a \in \mathsf{Ob}(s)$, then a fulfills in s the role of a_0 in s_0 iff there is a δ such that $s = s_0 \cdot \delta$ and $\delta(a_0) = a$.

It is not clear how to generalize the definition for arbitrary undifferentiated substitution frames.

DEFINITION A.1. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple undifferentiated substitution frame of finite object-degree. Then for any initial state $s_0 \in S$ and $a_0 \in \mathsf{Ob}(s_0)$, we define the* role of a_0 in s_0 *as*

$$\mathsf{Role}(s_0, a_0) = \{(s, a) \mid \exists\delta\, [\, s \cdot \delta = s_0 \;\&\; a \in \mathsf{Ob}(s) \;\&\; \delta(a) = a_0\,]\}.$$

Further, we define the roles of $\mathcal{F}$ *as*

$$\mathsf{Roles}_{\mathcal{F}} = \{\mathsf{Role}(s_0, a_0) \mid s_0 \in S \textit{ is an initial state} \;\&\; a_0 \in \mathsf{Ob}(s_0)\}.$$

For arbitrary $s \in S$, $a \in \mathsf{Ob}(s)$, we say that a in *s* fulfills role $\rho \in \mathsf{Roles}_{\mathcal{F}}$ *if for some s_0, a_0 with $\rho = \mathsf{Role}(s_0, a_0)$ there is a mapping $\delta : O \to O$ such that*

$$s = s_0 \cdot \delta \;\textit{ and }\; a = \delta(a_0).$$

It is easy to see that if $\mathcal{F}$ is an n-ary simple substitution frame, then $\mathcal{F}$ has at most n roles. Objects sometimes fulfill more than one role in certain states. For example, if $\mathcal{F}$ models the amatory relation, then in the state where Narcissus loves himself, he fulfills both roles of the model. It is also possible that an n-ary model with $n > 1$, has only one role. This is, for example, the case for cyclic models.

For differentiated substitution frames of finite occurrence-degree we can define roles in a similar way:

DEFINITION A.2. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a simple differentiated substitution frame of finite occurrence-degree. Then for any initial state $s_0 \in S$ and $\alpha_0 \in \mathsf{Oc}(s_0)$, we define the* role of α_0 in s_0 *as*

$$\mathsf{Role}(s_0, \alpha_0) = \Big\{(s, \alpha) \mid \exists\sigma\, [\, s \cdot \sigma = s_0 \;\&\; \alpha \in \mathsf{Oc}(s) \;\&\; \mu(\alpha) = \alpha_0 \textit{ with } \mu \textit{ corresponding to } s \to^{\sigma} s_0\,]\Big\}.$$

Further, we define the roles of $\mathcal{G}$ *as*

$$\mathsf{Roles}_{\mathcal{G}} = \{\mathsf{Role}(s_0, \alpha_0) \mid s_0 \in S \textit{ is an initial state} \;\&\; \alpha_0 \in \mathsf{Oc}(s_0)\}.$$

For arbitrary $s \in S$, $\alpha \in \mathsf{Oc}(s)$, we say that a in *s* fulfills role $\rho \in \mathsf{Roles}_{\mathcal{G}}$ *if for some s_0, α_0 with $\rho = \mathsf{Role}(s_0, \alpha_0)$ there is a mapping $\sigma : \mathsf{Oc} \to O$ such that*

$$s = s_0 \cdot \sigma \;\textit{ and }\; \mu(\alpha_0) = \alpha \textit{ for a corresponding } \mu.$$

Consider again the relation $\mathfrak{R}$ where $\mathfrak{R}abc$ is the state that *a loves b & b loves c*. Let $\mathcal{G}$ be a differentiated substitution frame for it with $\mathsf{oc\text{-}degree}_{\mathcal{G}} = 3$. Then because the state $\mathfrak{R}aba$ is identical with the state $\mathfrak{R}bab$, a and b each fulfill three roles, but a and b each have only one occurrence in $\mathfrak{R}bab$.

THEOREM A.3. *Let $\mathcal{G}$ be a simple differentiated substitution frame of finite occurrence-degree. Then each occurrence of a state of $\mathcal{G}$ fulfills exactly one role iff $\mathcal{G}$ is coalescence-free.*

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Proof. The theorem follows immediately from the definitions. □

Appendix B: Categories of substitution frames. For a better understanding of the relation between undifferentiated and differentiated substitution frames, it is useful to consider them in terms of category theory.

There are various ways in which categories of substitution frames can be defined. We give for both types of substitution frames a definition that seems to be a rather natural choice.

DEFINITION B.1. *We define* **USF** *as the category with objects all undifferentiated substitution frames, and with morphisms from* $\mathcal{F} = \langle S, O, \Sigma \rangle$ *to* $\mathcal{F}' = \langle S', O', \Sigma' \rangle$ *all pairs* $\langle f_S, f_O \rangle$ *of functions* $f_S : S \to S'$ *and* $f_O : O \to O'$ *such that*

1. f_O *is injective,*
2. $f_S(s \cdot_{\mathcal{F}} \delta) = f_S(s) \cdot_{\mathcal{F}'} \delta'$ *with* $\delta'(d') = \begin{cases} f_O(\delta(d)) & \text{if } d' = f_O(d), \\ d' & \text{otherwise.} \end{cases}$

Note that Condition 1 guarantees that the function δ' in Condition 2 is uniquely defined by the function δ.

Our category for differentiated substitution frames is somewhat more complicated:

DEFINITION B.2. *We define* **DSF** *as the category with objects all differentiated substitution frames and with morphisms from* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *to* $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta' \rangle$ *all triples* $\langle f_S, f_O, f_{\mathsf{Oc}} \rangle$ *of functions* $f_S : S \to S'$, $f_O : O \to O'$ *and* $f_{\mathsf{Oc}} : S \times X \to \mathsf{Oc}$ *with* $X = (\Pi')^{-1}[\mathsf{im}\, f_O]$, *the inverse image of* $\mathsf{im}\, f_O$ *by* Π', *such that*

1. f_O *is injective,*
2. *for each* $s \in S$ *and* $\alpha' \in X$, $f_O(\Pi(f_{\mathsf{Oc}}(s, \alpha'))) = \Pi'(\alpha')$,
3. $f_S(s \cdot_{\mathcal{G}} \sigma) = f_S(s) \cdot_{\mathcal{G}'} \sigma'$ *with* $\sigma'(\alpha') = \begin{cases} \sigma(f_{\mathsf{Oc}}(s, \alpha')) & \text{if } \alpha' \in X, \\ \Pi'(\alpha') & \text{otherwise.} \end{cases}$

The next theorem states that there is an adjunction from USF to DSF.

THEOREM B.3. *For the categories* USF *and* DSF *the functor that assigns to an undifferentiated substitution frame its basic refinement is a left-adjoint for the functor that assigns to a differentiated substitution frame its underlying undifferentiated substitution frame.*

Proof. Let $R : \mathsf{USF} \to \mathsf{DSF}$ be the functor that assigns to an undifferentiated substitution frame its basic refinement, and $U : \mathsf{DSF} \to \mathsf{USF}$ be the forgetful functor that assigns to a differentiated substitution frame its underlying undifferentiated substitution frame.

For each $\mathcal{F} \in \mathsf{USF}$ and $\mathcal{G} \in \mathsf{DSF}$, we get a bijection

$$\Phi_{\mathcal{F},\mathcal{G}} : \mathsf{DSF}(R(\mathcal{F}), \mathcal{G}) \to \mathsf{USF}(\mathcal{F}, U(\mathcal{G}))$$

by assigning $\langle f_S, f_O \rangle$ to the morphism $\langle f_S, f_O, f_{\mathsf{Oc}} \rangle$. It is not difficult to see that the family of bijections

$$\Phi : \mathsf{DSF}(R(\mathcal{F}), \mathcal{G}) \cong \mathsf{USF}(\mathcal{F}, U(\mathcal{G}))$$

is natural in $\mathcal{F}$ and $\mathcal{G}$. So, $\langle R, U, \Phi \rangle$ is an adjunction from USF to DSF. □

The theorem can also be proved by characterizing an adjunction in terms of the unit of adjunction (see, e.g., Theorem IV.1.2(i) of Mac Lane, 1998, p. 83). The unit of adjunction is in this case simply the identity natural transformation $\eta : I_{\mathsf{USF}} \dot{\to} I_{\mathsf{USF}}$.

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We should be careful when drawing conclusions from this theorem. In particular, I would not directly regard it as strong evidence of a natural relationship between undifferentiated and differentiated substitution frames. For some slightly different, but equally natural, definitions of a category of differentiated substitution frames, we do not get an adjunction.

Appendix C: Alternative substitution frames. In this appendix we present two alternative types of frames. In the first type the states have disjoint occurrence-domains, and in the second type occurrences are explicitly mapped to occurrences. Further, we discuss two reasons why our attempts to simplify our definition of differentiated substitution frames failed.

C.1 Frames with disjoint occurrence-domains. In the definition of differentiated substitution frames we did not demand that different states have disjoint occurrence-domains. One may perhaps regard this as a shortcoming, but it can rather easily be fixed by defining a differentiated substitution frame as a sextuple $\mathcal{F} = \langle S, O, \mathsf{Oc}, \Pi, \Pi_S, \Theta\rangle$ with Π_S a mapping from Oc to S that provides the states in S with disjoint occurrence-domains $\mathsf{Oc}_s = \Pi_S^{-1}(s)$. Further, Θ is a mapping from $\{(s, \sigma) \mid s \in S \;\&\; \sigma \in O^{\mathsf{Oc}_s}\}$ to S such that:

1. $\Theta(s, \Pi{\restriction}\mathsf{Oc}_s) = s$,
2. for all $s \in S$ and $\sigma : \mathsf{Oc}_s \to O$, there is a mapping $\mu : \mathsf{Oc}_s \to \mathsf{Oc}_{\Theta(s,\sigma)}$ such that
   (a) $\Pi \circ \mu = \sigma$,
   (b) for all $\sigma' : \mathsf{Oc}_{\Theta(s,\sigma)} \to O$, $\Theta(\Theta(s, \sigma), \sigma') = \Theta(s, \sigma' \circ \mu)$.

Because of the extra complexity of this definition we prefer to work with the initial definition of differentiated substitution frames, being on the alert for results that only applies to them, but not to those with disjoint occurrence-frames.

C.2 Frames explicitly mapping occurrences to occurrences. In our refined frames the correspondence between the occurrences of different states is implicit in the definition of Θ. We could make this correspondence explicit by defining frames as tuples $\langle S, O, \mathsf{Oc}, \Pi, \Psi\rangle$, where for any $s, s' \in S$, $\Psi(s, s')$ is a set of functions from Oc to Oc such that

1. $\mathsf{id}_{\mathsf{Oc}} \in \Psi(s, s)$,
2. if $\mu \in \Psi(s, s')$ and $\mu' \in \Psi(s', s'')$, then $\mu \cdot \mu' \in \Psi(s, s'')$,
3. if $\mu \in \Psi(s, s')$ and μ is a bijection, then $\mu^{-1} \in \Psi(s', s)$,
4. for all $s \in S$ and $\sigma : \mathsf{Oc} \to O$ there is exactly one $s' \in S$ and a mapping $\mu \in \Psi(s, s')$ such that $\mu \cdot \Pi = \sigma$.

There are several variants possible. For example, instead of assuming that the correspondence between the occurrences of one state and the occurrences of another one is *functional*, we could (perhaps more adequately) assume only a *relational* correspondence between the occurrences.

C.3 Failures to simplify our definition. The definition of a differentiated substitution frame perhaps looks more complex than needed. Our attempts to simplify the definition failed, because differentiated substitution frames lack certain nice properties, as we show in the next two observations.

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OBSERVATION C.1. *Not for every differentiated substitution frame and every mapping* $\mu : \mathsf{Oc} \to \mathsf{Oc}$ *we have* $(s \cdot (\mu \cdot \Pi)) \cdot \sigma = s \cdot (\mu \cdot \sigma)$.

Proof. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame without any symmetry and with a state s for which $\mathsf{Oc}(s) = \{\alpha_0, \alpha_1\}$ with $\alpha_0 \neq \alpha_1$ and $\Pi(\alpha_0) = \Pi(\alpha_1) = a$. Let $\mu : \mathsf{Oc} \to \mathsf{Oc}$ be such that $\mu(\alpha_0) = \alpha_1$ and $\mu(\alpha_1) = \alpha_0$. Then $s \cdot (\mu \cdot \Pi) = s$, but for any $b \in O$ with $b \neq a$, if $\sigma(\alpha_0) = a$ and $\sigma(\alpha_1) = b$, then $s \cdot \sigma \neq s \cdot (\mu \cdot \sigma)$. So, $(s \cdot (\mu \cdot \Pi)) \cdot \sigma \neq s \cdot (\mu \cdot \sigma)$. □

To a transition $s \to^{\sigma} s'$ more than one mapping $\mu : \mathsf{Oc} \to \mathsf{Oc}$ may correspond. The next observation shows that it is unlikely that we can always select one of them as a canonical mapping.

OBSERVATION C.2. *Not for every differentiated substitution frame, there is a representative function* $\overline{\mu} : S \times O^{\mathsf{Oc}} \to \mathsf{Oc}^{\mathsf{Oc}}$ *preserving composition, that is,*

(representative) $\overline{\mu}(s, \sigma)$ *corresponds to a transition* $s \to^{\sigma} s'$,
(preservation) $\overline{\mu}(s, \overline{\mu}(s, \sigma) \cdot \sigma') = \overline{\mu}(s, \sigma) \cdot \overline{\mu}(s \cdot \sigma, \sigma')$.

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame with an initial state s_0 for which $\mathsf{Ob}(s_0) = \{a, b, c, d\}$, and such that $s_0 \cdot \delta = s_0 \cdot \delta'$ iff $\delta' =_{\mathsf{Ob}(s_0)} (\delta_0)^i \cdot \delta$ with $i \in \{0, 1, 2, 3\}$ and

$$\delta_0 \restriction \mathsf{Ob}(s_0) = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}.$$

Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a refinement of $\mathcal{F}$ with for each state four occurrences.

Now suppose there is a function $\overline{\mu} : S \times O^{\mathsf{Oc}} \to \mathsf{Oc}^{\mathsf{Oc}}$ such that:

1. $\overline{\mu}(s, \sigma) \cdot \Pi = \sigma$,
2. $s \cdot (\overline{\mu}(s, \sigma) \cdot \sigma') = (s \cdot \sigma) \cdot \sigma'$,
3. $\overline{\mu}(s, \overline{\mu}(s, \sigma) \cdot \sigma') = \overline{\mu}(s, \sigma) \cdot \overline{\mu}(s \cdot \sigma, \sigma')$.

Let $s_1 = s_0 \cdot_{\mathcal{F}} \delta_1$ with $\delta_1 \restriction \mathsf{Ob}(s_0) = \begin{pmatrix} a & b & c & d \\ a & b & a & b \end{pmatrix}$.

Further, let $\mathsf{Oc}(s_1) = \{0, 1, 2, 3\}$ with $\Pi(0) = \Pi(2) = a$, and let

$$\sigma_1 \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ b & a & b & a \end{pmatrix}.$$

Then $s_1 \cdot \sigma_1 = s_1$. It is not difficult to see that by Properties (1) and (2) of $\overline{\mu}$:

$$\overline{\mu}(s_1, \sigma_1) \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix} \text{ or } \overline{\mu}(s_1, \sigma_1) \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

So, $\overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1, \sigma_1) \neq_{\mathsf{Oc}(s_1)} \mathsf{id}_{\mathsf{Oc}}$.

By Properties (1) and (3) of $\overline{\mu}$ we have for any $s \in S$:

$$\begin{aligned} \overline{\mu}(s, \Pi) &= \overline{\mu}(s, \overline{\mu}(s, \Pi) \cdot \Pi) \\ &= \overline{\mu}(s, \Pi) \cdot \overline{\mu}(s \cdot \Pi, \Pi) \\ &= \overline{\mu}(s, \Pi) \cdot \overline{\mu}(s, \Pi). \end{aligned}$$

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So, because for any $s \in S$, $\overline{\mu}(s, \Pi)[\mathsf{Oc}(s)] = \mathsf{Oc}(s)$, we have $\overline{\mu}(s, \Pi) =_{\mathsf{Oc}(s)} \mathsf{id}_{\mathsf{Oc}}$, and so, because $\overline{\mu}(s_1, \sigma_1) \cdot \sigma_1 =_{\mathsf{Oc}(s_1)} \Pi$, we have $\overline{\mu}(s_1, \overline{\mu}(s_1, \sigma_1) \cdot \sigma_1) =_{\mathsf{Oc}(s_1)} \mathsf{id}_{\mathsf{Oc}}$.

It follows that $\overline{\mu}(s_1, \overline{\mu}(s_1, \sigma_1) \cdot \sigma_1) \neq \overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1, \sigma_1) = \overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1 \cdot \sigma_1, \sigma_1)$, contradicting Property (3) of $\overline{\mu}$. □

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