Macroeconomic Dynamics, **6**, 2002, 457–475. Printed in the United States of America.
DOI: 10.1017.S1365100501010033

ARTICLES

IS THERE A GOLDEN RULE FOR THE STOCHASTIC SOLOW GROWTH MODEL?

KLAUS REINER SCHENK–HOPPÉ
Institute for Empirical Research in Economics, University of Zurich

This paper analyzes the dependence of average consumption on the saving rate in a one-sector neoclassical Solow growth model with production shocks and stochastic rates of population growth and depreciation where arbitrary ergodic processes are considered. We show that the long-run behavior of the stochastic capital intensity, and hence average consumption along any sample path, is uniquely determined by a random fixed point that depends continuously on the saving rate. This result enables us to prove the existence of a golden-rule saving rate that maximizes average consumption per capita. We also show that the golden-rule path is dynamically efficient. The results are illustrated numerically for Cobb–Douglas and CES production functions.

Keywords: Stochastic Solow Model, Golden Rule, Random Fixed Points, Random Dynamical Systems

1. INTRODUCTION

The existence of a golden-rule saving rate for the Solow growth model is a standard textbook problem; see, for example, Phelps (1966) and Barro and Sala-i-Martin (1995). The analogous question for the *stochastic* Solow model, however, has not been studied in the literature so far to the author's best knowledge. If productivity, population growth, or depreciation is subject to exogenous perturbations, the analysis of the descriptive Solow growth model requires methods different from those applied in the deterministic case. A first analysis of this stochastic growth model is due to Mirman (1972, 1973). Using the framework of Markov processes, he could prove existence and uniqueness of stationary probability measures (Markov equilibria) for the Solow model with independent and identically distributed (i.i.d.) production shocks.

In the past 25 years, our understanding of the dynamical properties of nonlinear stochastic dynamical systems has improved dramatically. This opens the door to

Research was supported in part by Deutsche Forschungsgemeinschaft while the author was at the Department of Economics, University of Bielefeld. I am grateful to an anonymous referee whose comments have helped to distinctly improve this paper. Address correspondence to: Klaus Reiner Schenk–Hoppé, Institute for Empirical Research in Economics, University of Zurich, Blümlisalpstrasse 10, CH–8006 Zürich, Switzerland; e-mail: klaus@iew.unizh.ch.

© 2002 Cambridge University Press 1365-1005/02 $9.50

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

the study of stochastic economic models and problems that have not been tractable before. In this paper, we apply random dynamical systems theory to the study of the stochastic Solow model. This theory, which is comprehensively presented in the monograph by Arnold (1998), provides a coherent mathematical description of nonlinear dynamical systems with randomness. An increasing number of methods for the analysis of stochastic systems is being developed using this framework.

Here, we employ the concept of a random fixed point to prove the existence of a golden-rule saving rate for the stochastic Solow model in which the productivity and the rates of population growth and depreciation are subject to perturbations stemming from arbitrary ergodic processes. This approach has been used by Schenk–Hoppé and Schmalfuss (2001) to describe the long-run behavior of the capital stock in the stochastic Solow model. We apply their main result on the existence of a unique globally attracting random fixed point.

Because of the presence of exogenous shocks, consumption in each period depends on the past realization of shocks and the initial capital stock. We first prove that average consumption along each sample path of the stochastic Solow growth model is well defined and independent of the particular realization of the sequence of shocks and of the initial capital stock. Therefore, the average consumption of an infinitely lived representative agent is independent of the prevailing history of perturbations. Since the shocks are assumed to be ergodic processes, average consumption is equal to the mean of consumption at the random fixed point. A central result of this paper then ensures that the average consumption is a continuous function of the saving rate. A direct consequence of these findings is the existence of a golden-rule saving rate yielding maximal average consumption. We also prove a dynamic efficiency result that ensures the importance of the golden rule in the stochastic Solow growth model. Our results are illustrated numerically for Cobb–Douglas and CES production function.

The remainder of the paper is organized as follows. Section 2 introduces the stochastic Solow model and explains the model of the parameter fluctuations. In Section 3, the long-run behavior of the stochastic Solow model is analyzed using the concept of a random dynamical system. Section 4 contains the main result of this paper on the existence of a golden-rule saving rate. Dynamic efficiency of the golden-rule path is proved in Section 5. Two examples are studied in Section 6.

2. MODEL

In this paper, we analyze the generalization of the neoclassical Solow (1956) growth model given by

$$k_{t+1} = h_s(\theta^t\omega, k_t) := \frac{[1-\delta(\theta^t\omega)]k_t + s\xi(\theta^t\omega)f(k_t)}{1+n(\theta^t\omega)}. \tag{1}$$

The stochastic parameters depreciation rate $\delta(\theta^t\omega)$, population growth rate $n(\theta^t\omega)$, and production shock $\xi(\theta^t\omega)$ are assumed to be ergodic processes. This models exogenously determined stationary fluctuations of the respective parameters.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

Fluctuations in the population growth rate represent changes in the birth and death rates as well as labor migration. A discussion on exogenous stochastic depreciation rates is provided by Ambler and Paquet (1994). The saving rate s is assumed to be a deterministic parameter, fixed for all times; f is a neoclassical production function. At each period of time, the capital intensity k_{t+1} in the next period is a random variable whose state is determined by the realization of the stochastic parameters $\delta(\theta^t\omega)$, $n(\theta^t\omega)$, and $\xi(\theta^t\omega)$, and by the capital intensity k_t in the current period.

Throughout the paper we impose the following conditions on the parameters:

(i) $\delta(\omega) \in [\delta_{\min}, \delta_{\max}] \subset [0, 1]$
(ii) $n(\omega) \in [n_{\min}, n_{\max}] \subset]-1, \infty[$
(iii) $\xi(\omega) \in [\xi_{\min}, \infty[\subset]0, \infty[$, and $\mathbf{E}\xi < \infty$
(iv) $\delta_{\max} + n_{\max} > 0$

Under conditions (i)–(iv), the stochastic rate of population growth is in a compact interval and there is a lower bound on the production shock. The production function satisfies an Inada-type condition:

(v) $f : \mathbf{R}_+ \to \mathbf{R}_+$ is increasing, strictly concave, and continuously differentiable; $0 \le \lim_{k\to\infty} f'(k) < (\delta_{\max} + n_{\max})/\xi_{\min}$. If $f(0) = 0$, then $f'(0) = \infty$. [No condition is needed if $f(0) > 0$.]

Using a mathematical description of the stochastic parameters from ergodic theory, we can model stochastic processes as dynamical systems. Let $(\Omega, \mathcal{F}, \mathbf{P})$ denote a probability space, and let $\theta : \Omega \to \Omega$ be a bi-measurable map (i.e., θ is measurable and θ^{-1} exists and is measurable) such that θ is ergodic with respect to $\mathbf{P}$ (i.e., for all $F \in \mathcal{F}$ such that $\theta F = F$, $\mathbf{P}(F) \in \{0, 1\}$). Then the family of mappings $\theta^t : \Omega \to \Omega$, $t \in \mathbf{Z}$ forms a measurable flow; that is, $\theta^0 = \mathrm{id}_\Omega$, $\theta^t = \theta \circ \cdots \circ \theta$ (t-times) for $t > 0$, and $\theta^{-t} = \theta^{-1} \circ \cdots \circ \theta^{-1}$ for $-t < 0$. The tuple $(\Omega, \mathcal{F}, \mathbf{P}, (\theta^t)_{t\in\mathbf{Z}})$ is called an *ergodic dynamical system*. For each measurable function $g : \Omega \to \mathbf{R}$, $g(\theta^t\omega)$ is an ergodic process defined for all times $t \in \mathbf{Z}$. In particular, any real-valued ergodic process has such a representation; cf. Arnold (1998, Appendix A.1). The canonical representation is as follows: Let ξ_t be an ergodic process with values in a Borel set $X \subset \mathbf{R}$. Define the sample-path space $\Omega = X^{\mathbf{Z}}$, the Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$, and let $\mathbf{P}$ denote the probability measure that is defined by the finite-dimensional distributions of ξ_t. (Existence follows from Kolmogorov's fundamental theorem.) For instance, if ξ_t is an i.i.d. process with distribution λ, then $\mathbf{P} = \lambda^{\mathbf{Z}}$ is the product measure. The mapping θ is defined as $\theta^t\omega(\cdot) := \omega(t + \cdot)$. The sequence of random variables $\Omega \to X$, $\omega \mapsto \theta^t\omega(0) = \omega(t)$ is the process we started with.

In (1), the stochastic parameters are defined over a common ergodic dynamical system. However, note that these processes may have arbitrary correlation and, in particular, may be independent.

Let us briefly discuss the implications of conditions (i)–(v) for the deterministic Solow model, which is obtained from (1) by choosing the ergodic processes $\delta(\omega) \equiv \delta$, $n(\omega) \equiv n$, and $\xi(\omega) \equiv \xi$. Then (1) becomes

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

$$k_{t+1} = h_s(k_t) := \frac{(1-\delta)\,k_t + s\,\xi f(k_t)}{1+n}, \tag{2}$$

the Solow model with production function $\xi f(k)$.

If $n+\delta > 0$, $s\xi > 0$, and f satisfies condition (v), then (2) has a unique positive fixed point, denoted by $\bar{k}_s(\xi, n, \delta)$, which is globally asymptotically stable. This result is obtained as follows: Fix any saving rate $s > 0$. Then condition (v) implies $h_s(k) > k$ for all sufficiently small $k > 0$, and $h_s(k) < k$ for all sufficiently large $k > 0$. Further, $h_s(k)$ is continuous in k, increasing, and strictly concave. From these observations, one can straightforwardly conclude existence, uniqueness, and global asymptotic stability of $\bar{k}_s(\xi, n, \delta)$. For $s = 0$, we define $\bar{k}_0(\xi, n, \delta) := 0$, which is the unique steady state in this case.

The following result is straightforward from conditions (i)–(v) and the preceding discussion.

LEMMA 1. *For each* $0 \le s \le 1$, *the set*

$$G(s) := [\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max}), \infty[$$

is forward invariant; *that is*, $h_s[\omega, G(s)] \subset G(s)$ *for all* ω; $\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})$ *denotes the unique positive steady state of* (2) *with respective parameters.*

This result ensures that, for any initial capital intensity larger than or equal to $\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})$, the economy will never (i.e., for no feasible sequence of shocks and at no subsequent period of time) experience a capital intensity less than $\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})$.

We also have to make an assumption on the expected value of the contraction rate of the maps $h_s(\omega, k)$ at the left-hand boundary of the forward invariant set $G(s)$. This condition ensures convergence of sample paths of capital intensities. (**E** denotes the expected value with respect to **P**.)

(vi) There exists a $c > 0$ such that

$$\sup_{0 \le s \le 1} \mathbf{E} \log h'_s[\omega, \bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})] \le -c < 0.$$

By the defining equation (1), we have

$$h'_s(\omega, k) = \frac{1 - \delta(\omega) + s\xi(\omega) f'(k)}{1 + n(\omega)} \quad \text{for} \quad s > 0,$$

and

$$h'_0(\omega, k) = [1 - \delta(\omega)]/[1 + n(\omega)] \quad \text{for} \quad s = 0.$$

Condition (vi) implies that the law of motion h is a contraction (in the sense that the mean value of the rate of contraction is negative) at the capital intensity $\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})$. Moreover, the rate of contraction is uniformly bounded away from zero with respect to the saving rate s. Since h is concave, this implies that h is contracting on average (uniformly in s) at any larger capital intensity. In the

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

deterministic case, condition (vi) is equivalent to asymptotic stability of the fixed point $\bar{k}_s(\xi, n, \delta)$ for each $0 \le s \le 1$; but this is obviously satisfied under conditions (i)–(v) according to the preceding discussion.

3. THE LONG-RUN BEHAVIOR

In this section we present the main result on the long-run behavior of the stochastic Solow model (1). We need to introduce the notion of a random dynamical system, which is the key concept in the further analysis. A random dynamical system consists of two ingredients: (1) an ergodic dynamical system as a model of the perturbation and (2) a dynamical system that is coupled to this ergodic dynamical system. Fix any sample path of the stochastic parameters, that is, fix any $\omega \in \Omega$, and let k be an initial value for the capital intensity. Denote by $\varphi_s(t, \omega, k)$ the state of the system at time t, which is determined by (1). That is, φ_s is generated by iterations of the random map $k \mapsto h_s(\omega, k)$ [denoted by $h_s(\omega)$ for short]: $\varphi_s(t, \omega) = h_s(\theta^{t-1}\omega) \circ \cdots \circ h_s(\omega)$ for $t > 0$, and $\varphi_s(0, \omega) = \mathrm{id}$. In particular, the time-one map $\varphi_s(1, \omega) = h_s(\omega)$. Since $h_s(\omega)$ is invertible, one has $\varphi_s(-t, \omega) = h_s(\theta^{-t}\omega) \circ \cdots \circ h_s(\theta^{-1}\omega)$ for $-t < 0$. Further, each map $h_s(\omega)$ is continuously differentiable and thus $\varphi_s(t, \omega) : \mathbf{R}_+ \to \mathbf{R}_+$ is continuously differentiable for all (t, ω).

The family of maps $\varphi_s(t, \omega, k)$ defined by (1) forms a *random dynamical system* on $\mathbf{R}_+$ over the ergodic dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, (\theta^t)_{t\in\mathbf{Z}})$ modeling the perturbation. That is, $\varphi_s : \mathbf{Z} \times \Omega \times \mathbf{R}_+ \to \mathbf{R}_+$, $(t, \omega, k) \mapsto \varphi_s(t, \omega, k)$ is a measurable mapping such that $\varphi_s(0, \omega) = \mathrm{id}_{\mathbf{R}_+}$ and $\varphi_s(t + u, \omega) = \varphi_s(t, \theta^u\omega) \circ \varphi_s(u, \omega)$ for all $u, t \in \mathbf{Z}$ and all $\omega \in \Omega$. These properties replace the flow property of a deterministic dynamical system that is generated by the iteration of a map.

The notions "almost surely" (a.s.) and "almost all" refer to the measure $\mathbf{P}$ of the ergodic dynamical system, which models the exogenous perturbation.

THEOREM 1. *Suppose conditions* (i)–(vi) *are satisfied. Then the random dynamical system generated by the stochastic Solow model* (1) *possesses a unique positive random fixed point $k_s^\star(\omega)$ for each $s > 0$, that is, $\Omega \to \mathbf{R}_{++}$, $\omega \mapsto k_s^\star(\omega)$ is a random variable such that a.s.*

$$k_s^\star(\theta\omega) = h_s\left[\omega, k_s^\star(\omega)\right]. \tag{3}$$

The random fixed point $k_s^\star(\omega)$ has the following properties:

(i) *The random fixed point $k_s^\star(\omega)$ is globally asymptotically stable*; *that is, for all $k > 0$,*

$$\lim_{t\to\infty} \left|\varphi_s(t, \omega, k) - k_s^\star(\theta^t\omega)\right| = 0 \quad a.s. \tag{4}$$

(ii) *The random fixed point $k_s^\star(\omega)$ is measurable with respect to the past $\mathcal{F}^-$, which is defined as the σ-algebra $\mathcal{F}^- = \sigma[\omega \mapsto \varphi_s(u, \theta^{-t}\omega) \mid 0 \le t \le u]$.*

(iii) *The random fixed point $k_s^\star(\omega)$ is tempered*; *that is*, $\limsup_{t\to\infty}[\log^+ k_s^\star(\theta^t\omega)]/t = 0$ *a.s. Further, any tempered random variable $y : \Omega \to \mathbf{R}_{++}$ converges to $k_s^\star(\omega)$ in the sense that $\lim_{t\to\infty} \varphi_s[t, \theta^{-t}\omega, y(\theta^{-t}\omega)] = k_s^\star(\omega)$ a.s.*

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

(iv) *If* $\mathbf{E}k_s^\star < \infty$, *the empirical law of each sample path* $\varphi_s(t, \omega, k)$ *with* $k > 0$ *is equal to* $k_s^\star \mathbf{P}$; *that is*, $\lim_{T\to\infty} 1/T \sum_{t=0}^{T} 1_B[\varphi_s(t, \omega, k)] = k_s^\star \mathbf{P}(B)$ *a.s. for each measurable set* B. *Here* $1_B : \mathbf{R}_+ \to \{0, 1\}$ *denotes the indicator map*; *that is*, $1_B(k) = 1$ *if and only if* $k \in B$.

For $s = 0, k_0^\star(\omega) \equiv 0$ *is a unique globally asymptotically stable random fixed point, possessing all above properties.*

The proof of Theorem 1 can be found in the Appendix.

A random fixed point is a random variable that is defined over the ergodic dynamical system that models the exogenous perturbation. For each sample path of the perturbation, that is, for each fixed $\omega \in \Omega$, the random fixed point defines a sample path in the state space via $t \mapsto k_s^\star(\theta^t\omega)$. Equation (3) says that the random fixed point $k_s^\star(\omega)$ generates a stationary solution to the stochastic Solow model in the sense that $k_s^\star(\theta^{t+1}\omega) = h_s[\theta^t\omega, k_s^\star(\theta^t\omega)]$ for all $t \in \mathbf{Z}$. If the stochastic variables are trivial, that is, constants, then a random fixed point is simply a fixed point of the corresponding deterministic system. In this case, we reobtain from Theorem 1 the results already discussed for the deterministic Solow model.

The stability result (i) says that the evolution of the capital intensity is uniquely determined by the random fixed point. More precisely, for almost any sample path of the stochastic variables, the sample path of any capital intensity with positive initial value converges toward the sample path of the random fixed point; cf. (4). The interpretation of this result is straightforward. Any economic policy that causes a temporary change in the law of motion of the economy (1) and that leads to a different capital intensity cannot have a long-run effect. After abandoning this policy, the capital intensity converges again toward the sample path of the random fixed point.

Result (ii) is not only of technical interest. It asserts that the state of the random fixed point at each period of time can be constructed from the sole knowledge of the past; that is, consumers do not have to correctly anticipate future events to determine its state.

Result (iii) is a technical result that is needed in the following: Temperedness of the random fixed point tells us that almost any sample path $t \mapsto k_s^\star(\theta^t\omega)$ grows subexponentially fast. According to a result from O'Brien (1982), the only alternative would be that almost any sample path grows faster than $\exp(at)$ for any finite $a > 0$. See Arnold (1998, sect. 4.1.1), for a thorough discussion of this topic. The contraction property, $\lim_{t\to\infty} \varphi_s[t, \theta^{-t}\omega, y(\theta^{-t}\omega)] = k_s^\star(\omega)$ for any tempered positive random variable y, provides information on the "attraction strength" of the random fixed point $k_s^\star(\omega)$. $\varphi_s[t, \theta^{-t}\omega, y(\theta^{-t}\omega)]$ is the capital intensity at time zero of an economy with capital intensity $y(\theta^{-t}\omega)$ at time $-t < 0$, that is, in the past. Therefore, any sequence of economies, indexed by t, which has the property that each economy t starts at time $-t$ with a capital intensity not exceeding $y(\theta^{-t}\omega)$, leads to a sequence of capital intensities at time zero that converges toward $k_s^\star(\omega)$.

Result (iv) is a statement on the statistical property of each sample path of capital intensities. If agents average over observed capital intensities, they obtain a

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

stable feature of its long-run distribution: regardless of the period of time at which the measurement is started, the empirical measure converges to the probability measure $k_s^\star \mathbf{P}$. If the random variables $[\xi(\theta^t\omega), n(\theta^t\omega), \delta(\theta^t\omega)]$, $t \in \mathbf{Z}$ are i.i.d., property (iv) ensures that $k_s^\star \mathbf{P}$ is a Markov equilibrium. The Markov equilibrium is stable in the sense that it is equal to the empirical measure for almost any sample path. This renders the corresponding results in Mirman (1972, 1973).

The following result is central to ensure existence of a golden rule in the next section. The proof is contained in the Appendix.

PROPOSITION 1. *The random fixed point* $k_s^\star(\omega)$, *defined in Theorem* 1, *is continuous and strictly increasing in the saving rate* $s \in [0, 1]$.

4. EXISTENCE OF THE GOLDEN RULE

We are now in a position to study the influence of the saving rate on the consumption in the stochastic Solow model. Analogously to the deterministic case, we ask whether there is a saving rate such that average consumption per capita is maximal.

For any sample path of the stochastic variables, consumption in the stochastic Solow model at any period t is given by $(1-s)\xi(\theta^t\omega) f[\varphi(t, \omega, k)]$. We know from Theorem 1 that the long-run behavior of an economy with a fixed saving rate s is uniquely determined by the sample path $t \mapsto k_s^\star(\theta^t\omega)$. Therefore, average consumption also will be determined by the random fixed point. Let us assume $\mathbf{E}\xi f(k_1^\star) < \infty$. Then, by Theorem 1(i) and continuity of f, the ergodic theorem yields that a.s.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} (1-s)\xi(\theta^t\omega) f[\varphi_s(t, \omega, k)] = (1-s)\mathbf{E}\left\{\xi(\omega) f\left[k_s^\star(\omega)\right]\right\} \quad (\mathbf{5})$$

for all $k > 0$. This implies that the average consumption along almost all sample paths is well defined and equal to the expected value of the consumption $c_s^\star(\omega) := (1-s)\xi(\omega) f[k_s^\star(\omega)]$ at the random fixed point $k_s^\star(\omega)$. Obviously, $c_1^\star(\omega) \equiv 0$, and $c_0^\star(\omega) \equiv 0$ by Theorem 1. For the deterministic Solow model, $c_s^\star(\omega)$ is equal to the consumption at the nontrivial steady state.

We are now in a position to state the main result of this paper.

THEOREM 2. *Suppose conditions* (i)–(vi) *are satisfied. If* $\mathbf{E}\xi f(k_1^\star) < \infty$, *then there exists a golden rule. That is, there is a saving rate* $s^{\text{gold}} \in [0, 1]$ *such that average consumption,* $\mathbf{E}c_s^\star = (1-s)\mathbf{E}\xi f(k_s^\star)$, *is maximal for* $s = s^{\text{gold}}$.

Theorem 2 extends the well-known result on the existence of a golden rule saving rate for the Solow model to the stochastic case. In the proof, which is relegated to the Appendix, it is verified that $\mathbf{E}c_s^\star$ depends continuously on the saving rate. The applicability of this result is fostered by the next two corollaries in which sufficient conditions for $\mathbf{E}\xi f(k_1^\star) < \infty$ are given that are simple to check for specific parameters.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

COROLLARY 1. *Suppose conditions* (i)–(vi) *are satisfied. Then there exists a golden rule if*

(*a*) f *is bounded or*
(*b*) $\delta_{\min} + n_{\min} > 0$, $\xi(\omega) \in [\xi_{\min}, \xi_{\max}]$, *and* $\lim_{k\to\infty} f'(k) < (\delta_{\min} + n_{\min})/\xi_{\max}$.

If the only stochastic perturbations are i.i.d. production shocks, then we obtain the following result.

COROLLARY 2. *Suppose conditions* (i)–(vi) *are satisfied*, ξ *is an i.i.d. process, and* δ *and* n *are constants. Then, the following holds. If* $\lim_{k\to\infty} f'(k) < (\delta + n)/\mathbf{E}\xi$, *then there exists a golden rule.*

The golden-rule saving rate has to be determined numerically in general because of the lack of a characterization via first-order conditions. However, for a certain class of production functions, we can prove that the golden rule for the stochastic model coincides with the one for the associated deterministic model. In this case, the golden rule is determined by an analytical expression. The result applies to the case of a Cobb–Douglas technology; see Section 6.

THEOREM 3. *Suppose conditions* (i)–(vi) *are satisfied, and* $\mathbf{E}\xi f(k_1^\star) < \infty$. *If there exists a continuously differentiable function* $g : [0, 1] \to [0, 1]$ *such that, for all* $k > 0$,

$$sf[g(s)k] = g(s)f(k), \tag{6}$$

then any golden rule fulfills

$$\frac{\partial}{\partial s}\left[\frac{1-s}{s}g(s)\right] = 0. \tag{7}$$

If (7) *has a unique solution, then this solution is the unique golden rule.*

If the production function has the "weighted" homogeneous-of-degree-one property (6), the random fixed point $k_s^\star(\omega)$ depends on the saving rate in a very particular way: We have $k_s^\star(\omega) = g(s)k_1^\star(\omega)$; see the proof of Theorem 3 in the Appendix. We also obtain a simple formula for average consumption: $\mathbf{E}c_s^\star(\omega) = [(1-s)/s]g(s)\mathbf{E}\{\xi(\omega)f[k_1^\star(\omega)]\}$. From this equation the characterization of the golden rule given by (7) is straightforward. The proof also leads to the following remark.

Remark 1. (i) Under the assumptions of Theorem 3, the golden rule depends only on parameters of the production function.

(ii) The function $g(s)$ in Theorem 3 is strictly increasing by Proposition 1. Further, $g(0) = 0$ and $g(1) = 1$.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

5. DYNAMIC EFFICIENCY

We derive a dynamic efficiency result for the stochastic Solow model which reveals the importance of the golden rule for that model. It is useful to recall the dynamic efficiency result for the deterministic model first.

In the absence of shocks, consumption is strictly decreasing in all capital stocks that exceed the capital intensity $k^{\star}_{s^{\text{gold}}}$, the fixed point that is assigned to the golden-rule saving rate s^{gold} in the deterministic case. This observation immediately implies that any feasible sample path of capital intensities $\tilde{k}_t$ with $\tilde{k}_t \geq k^{\star}_{s^{\text{gold}}}$ and $\tilde{k}_t > k^{\star}_{s^{\text{gold}}}$ for at least one t cannot be dynamically efficient: Increasing consumption in each period in which $\tilde{k}_t > k^{\star}_{s^{\text{gold}}}$ such that the next period's capital stock is not lower than $k^{\star}_{s^{\text{gold}}}$ raises consumption in these periods but does not harm consumers in all other periods.

In the stochastic case, *average consumption*, that is, consumption averaged along sample paths of the capital intensity, is considered, and the golden rule is defined as the saving rate s^{gold} for which average consumption is maximal. Note that, in the deterministic case, average consumption along each sample path is equal to consumption at the steady state. Since all sample paths along which consumption only differs at most in finitely many periods yield the same average consumption, our dynamic inefficiency result can only cover the case in which consumption is altered in infinitely many periods. This motivates the following definition.

DEFINITION 1. *A stochastic process of capital intensities $\tilde{k}(\omega)$ is called dynamically inefficient if average consumption along almost any feasible sample path $t \mapsto \tilde{k}(\theta^t\omega)$ is strictly lower than average consumption associated with the golden-rule saving rate*; *that is*,

$$\mathbf{E}\tilde{c}(\omega) < \mathbf{E}c^{\star}_{s^{\text{gold}}}(\omega),$$

where $\tilde{c}(\omega) := [1-\delta(\omega)]\tilde{k}(\omega) + \xi(\omega) f[\tilde{k}(\omega)] - [1+n(\omega)]\tilde{k}(\theta\omega)$. *A sample path of capital intensities $\tilde{k}(\omega)$ is called feasible if* $\tilde{c}(\omega) \geq 0$ *a.s.*

By the ergodic theorem, average consumption along sample paths is equal to the expected value of consumption at the random fixed point for any period of time almost surely. Therefore, average consumption along any dynamically inefficient stochastic process of capital intensities is strictly lower than along almost any sample path at the golden-rule saving rate.

Let us first show existence of stochastic processes with feasible sample paths along which capital intensities are greater than those associated with the golden rule. We prove that there is a constant $\bar{\varepsilon} > 0$ such that, for all processes $\varepsilon(\omega)$ with $0 \leq \varepsilon(\omega) \leq \bar{\varepsilon}$ for all ω and $\mathbf{E}\varepsilon(\omega) > 0$, the sample paths $t \mapsto k^{\star}_{s^{\text{gold}}}(\theta^t\omega) + \varepsilon(\theta^t\omega)$ are feasible. Obviously it suffices to show that there is an $\bar{\varepsilon} > 0$ such that, for all ω,

$$k^{\star}_{s^{\text{gold}}}(\theta\omega) + \bar{\varepsilon} \leq \frac{[1-\delta(\omega)]k^{\star}_{s^{\text{gold}}}(\omega) + \xi(\omega) f\left[k^{\star}_{s^{\text{gold}}}(\omega)\right]}{1+n(\omega)}.$$

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

Using (1) and the random fixed-point property, this equation is equivalent to

$$\bar{\varepsilon} \leq \frac{(1 - s^{\text{gold}})\xi(\omega) f\left[k^{\star}_{s^{\text{gold}}}(\omega)\right]}{1 + n(\omega)}.$$

We readily obtain the existence result from this relation, noting that

$$0 < \frac{(1 - s^{\text{gold}})\xi_{\min} f[\bar{k}_{s^{\text{gold}}}(\xi_{\min}, n_{\max}, \delta_{\max})]}{1 + n_{\max}} \leq \frac{(1 - s^{\text{gold}})\xi(\omega) f\left[k^{\star}_{s^{\text{gold}}}(\omega)\right]}{1 + n(\omega)}$$

by conditions (i)–(v).

We can now state the main result of this section; see the Appendix for the proof.

THEOREM 4. *Suppose conditions* (i)–(vi) *are satisfied*, $\mathbf{E}\xi f(k^{\star}_1) < \infty$, *and*

$$\mathbf{E}\left\{\delta(\omega) + n(\omega) - \xi(\omega) f'\left[k^{\star}_{s^{\text{gold}}}(\omega)\right]\right\} \geq 0. \tag{8}$$

Each policy that changes the capital intensity $k^{\star}_{s^{\text{gold}}}(\omega)$ *associated with the golden rule by a feasible amount* $\varepsilon(\omega)$ *with*

(*a*) $\mathbf{E}\varepsilon(\omega) > 0$ *and*
(*b*) $\operatorname{Cov}\left\{\varepsilon(\omega), \delta(\omega) + n(\theta^{-1}\omega) - \xi(\omega) f'\left[k^{\star}_{s^{\text{gold}}}(\omega)\right]\right\} \geq 0$

leads to a dynamically inefficient stochastic process of capital intensities.

Theorem 4 states that any deviation $t \mapsto \varepsilon(\theta^t\omega)$ with $\mathbf{E}\varepsilon > 0$ from the sample path $t \mapsto k^{\star}_{s^{\text{gold}}}(\theta^t\omega)$ is disadvantageous in the sense that it leads to a dynamically inefficient outcome.

Equation (8) requires that average marginal productivity at the random fixed point associated with the golden rule should not exceed the mean value of the sum of population growth rate and depreciation. In the deterministic case, condition (8) is satisfied because $\delta + n = \xi f'(k^{\star}_{s^{\text{gold}}})$, which characterizes the golden rule via the first-order condition.

Condition (a) of Theorem 4 ensures that the feasible stochastic processes considered lead to a larger average capital intensity than that associated with the golden-rule saving rate. As already pointed out above, any sample path with $\mathbf{E}[\tilde{k}(\omega) - k^{\star}_{s^{\text{gold}}}(\omega)] = 0$ does not change average consumption and, from this point of view, does not represent a policy that is distinguishable from obeying the golden rule. However, the condition $\mathbf{E}\varepsilon(\omega) > 0$ does not rule out feasible stochastic processes with lower capital intensities than the golden-rule path at infinitely many periods in time. Such sample paths may well be dynamically inefficient.

Condition (b) of Theorem 4 restricts the feasible deviations from the golden-rule path by requiring a positive correlation between the difference of the sum of depreciation and (previous period's) population growth rate and marginal productivity. Condition (b) is obviously fulfilled for a constant positive deviation from the golden-rule path.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

6. TWO EXAMPLES

Two examples are analyzed to illustrate our theoretical results. For the standard cases of Cobb–Douglas and CES production function, we derive the golden-rule saving rate and study numerically the effect of the different stochastic parameters on average consumption. In both cases we consider three stochastic models and, for comparison, the associated deterministic model in which all parameters are set to their respective expected value. The specifications are

(a) Deterministic: $\xi = 1$, $n = 0$, and $\delta = 1/2$.
(b) Production shocks: $\xi \in \{3/4, 5/4\}$, $n = 0$, and $\delta = 1/2$.
(c) Stochastic population growth: $\xi = 1$, $n \in \{\pm 7/100\}$, and $\delta = 1/2$.
(d) Stochastic depreciation: $\xi = 1$, $n = 0$, and $\delta \in \{1/3, 2/3\}$.

In each model, the stochastic parameter is assumed to be an ergodic process that follows a two-state irreducible Markov chain with transition probabilities $p_{11} = 0.9$, $p_{12} = 0.1$, $p_{21} = 0.1$, and $p_{22} = 0.9$. The stationary measure assigns equal probability to both states.

6.1. Cobb–Douglas Production Function

$$f(k) = Ak^{\alpha} \quad \text{with} \quad A > 0, \quad \text{and} \quad 0 < \alpha < 1. \tag{9}$$

It is obvious that condition (v) follows from conditions (i)–(iv). A simple sufficient condition for (vi), which allows us to check that our results apply in the cases (a)–(d), can be derived as follows: First, note that, for all $s > 0$,

$$\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max}) = [(\delta_{\max} + n_{\max})/(sA\xi_{\min})]^{1/(\alpha-1)} \tag{10}$$

and therefore $f'[\bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})] = \alpha(\delta_{\max} + n_{\max})/(s\xi_{\min})$. As a consequence, one obtains for each $s > 0$,

$$h'_s[\omega, \bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})] = \frac{1 - \delta(\omega) + \alpha\frac{\xi(\omega)}{\xi_{\min}}(\delta_{\max} + n_{\max})}{1 + n(\omega)}.$$

Since the right-hand side of the last equation is independent of s, and for $s = 0$, $\bar{k}_0(\xi_{\min}, n_{\max}, \delta_{\max}) = 0$ and $h'_0(\omega, 0) = [1 - \delta(\omega)]/[1 + n(\omega)]$, condition (vi) turns out to be equivalent to

$$\mathbf{E}\log\frac{1 - \delta(\omega) + \alpha\frac{\xi(\omega)}{\xi_{\min}}(\delta_{\max} + n_{\max})}{1 + n(\omega)} < 0 \tag{11}$$

Increasing the right-hand side by interchanging $\mathbf{E}$ and log and replacing $n(\omega)$ with $n_{\min}$, we find a sufficient condition for (11):

$$\alpha < \frac{\xi_{\min}}{\mathbf{E}\xi}\frac{\mathbf{E}\delta(\omega) + n_{\min}}{\delta_{\max} + n_{\max}}. \tag{12}$$

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

In the deterministic case, that is, if all ergodic processes are constants, (12) reduces to $\alpha < 1$ (which is no additional restriction). It is noteworthy that $\alpha \le [\xi_{\min}(\mathbf{E}\delta(\omega) + n_{\min})]/[\mathbf{E}\xi(\delta_{\max} + n_{\max})]$ implies (11) [and thus is a sufficient condition for (vi)] if at least one ergodic process is nondegenerated. This follows from the fact that, in this case, interchanging $\mathbf{E}$ and log strictly increases the left-hand side of (11).

Summarizing, we obtain the following lemma.

LEMMA 2. *The stochastic Solow model with Cobb–Douglas production function satisfies condition* (vi) *for all* α *that fulfill* (12).

Let us next determine the golden-rule saving rate. For the deterministic Solow model, we find by (10) that

$$c_s = (1-s)\xi A[\bar{k}_s(\xi, n, \delta)]^\alpha = (1-s)s^{\frac{\alpha}{1-\alpha}}\xi A\left(\frac{\delta + n}{A\xi}\right)^{\frac{\alpha}{\alpha-1}}. \tag{13}$$

Solving $(\partial/\partial s)c_s = 0$, we readily obtain the well-known result that c_s is maximal for $s = \alpha$. Note that the golden rule $s^{\text{gold}} = \alpha$ is independent of the parameters δ, n, and ξ.

The fact that the golden rule depends only on the production function is a hint that Theorem 3 might apply; see Remark 1. Equation (13) leads to the "guess" $g(s) = s^{\frac{1}{1-\alpha}}$. We need to check first that $sf[g(s)k] = g(s)f(k)$ for all $k > 0$; cf. condition (6). This is equivalent to

$$sA\left(s^{\frac{1}{1-\alpha}}k\right)^\alpha = s^{\frac{1}{1-\alpha}}Ak^\alpha,$$

which is obviously true for all $k > 0$. It is straightforward to check that $(\partial/\partial s)(1-s)g(s)/s = 0$ has a unique interior solution. Summarizing, we obtain that the golden rules in both deterministic and stochastic cases coincide. We may state the following corollary.

COROLLARY 3. *Consider the stochastic Solow model with Cobb–Douglas production function. Suppose conditions* (*i*)–(*vi*) *are satisfied and* $\mathbf{E}\xi f(k_1^\star) < \infty$. *Then the* (*unique*) *golden-rule saving rate is given by* $s^{\text{gold}} = \alpha$, *i.e.* $\mathbf{E}c_s^\star$ *is maximal for* $s = \alpha$.

We now fix the parameters of the Cobb–Douglas production function; let $A = 10$ and $\alpha = 3/4$. It is straightforward to check that, for these parameter values, conditions (i)–(v) and the sufficient condition (12) for (vi), see Lemma 2, are satisfied in each case (a)–(d). Since $\delta_{\min} + n_{\min} > 0$ and $0 = \lim_{k\to\infty} f'(k) < (\delta_{\min} + n_{\min})/\xi_{\max}$, Corollary 1(b) ensures that $\mathbf{E}\xi f(k_1^\star) < \infty$. Summarizing, we have found that a nontrivial random fixed point exists for all positive saving rates, Theorem 1, that $\mathbf{E}c_s^\star(\omega) < \infty$ for all s, and that $s^{\text{gold}} = \alpha$, Corollary 1.

Table 1 lists the average consumption at the golden-rule saving rate in all four cases. The approximation of average consumption per capita $\mathbf{E}c_s^\star$ with saving rate

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

Table 1. Golden rule saving rate, numerically approximated average consumption per capita at the golden rule, and average consumption per capita relative to the deterministic case (a)

Case	Golden rule	Average consumption	Average consumption relative to (a) (%)
(a)	0.75	8,437.5	100.0
(b)	0.75	9,404.0	111.5
(c)	0.75	8,883.4	105.3
(d)	0.75	10,203.6	120.9

$s = \alpha = 0.75$ has been carried out as follows: First, a sample path of length 1,000 is calculated to approximate the random fixed point $k_s^\star(\theta^t\omega)$. Second, a sample path of length 1,000,000 is calculated and average consumption is calculated according to the convergence result (5). The numerical procedure has been implemented in C [Schenk–Hoppé (2001)].

Table 1 documents that average consumption at the golden rule is higher in all stochastic cases (b)–(d) than in the deterministic case (a). If production shocks occur, case (b), the average consumption is more than 11% higher than in the deterministic case (a). Fluctuations of the depreciation rate, case (d), increase average consumption by more than 20%. A stochastic population growth rate, case (c), leads to a 5.3% increase in average consumption.

Since the golden rule is the same in all four cases, the gain in average consumption has to be due to the nonlinearity of the Solow model or, in case (c), to slower population growth. In case (c), the average population growth rate is $0.9975 \approx \exp\{0.5[\log(0.93) + \log(1.07)]\} < 1$. Thus the population size decreases in case (c) but it is constant in all other cases. However, this has only a minor effect on the increase in average consumption because the difference in the population growth rates is approximately 0.25% and thus accounts only for less than 5% of the increase in average consumption. Therefore, the increase in consumption is caused by the nonlinearity of the law of motion, which is due to the strict concavity of the production function. It is important to point out that, for the same perturbation with transition probability p_{ij} replaced with $1 - p_{ij}$, $i, j = 1, 2$, that is, for a fast fluctuating process, average consumption in the stochastic cases is lower than in the deterministic case.

Summarizing, we may conclude that our numerical results provide evidence on the importance of taking into account stochastic parameter fluctuations. Replacing a stochastic parameter with its expected value would lead to erroneous data on average consumption. Table 1 illustrates that a nonneglectable error would be made.

Finally, we briefly discuss the applicability of the results on dynamic efficiency given in Section 5. The task is to find conditions under which the crucial

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

assumption of Theorem 4, equation (8), is fulfilled. This assumption requires $\mathbf{E}\{\delta(\omega)+n(\omega)-\xi(\omega)f'[k^\star_{s^{\text{gold}}}(\omega)]\}\geq 0$. We have to resort to numerical simulations to check this condition [Schenk–Hoppé (2001)]. Our numerical results indicate that condition (8) is satisfied in the following cases: Deterministic case (a), for all $0<\alpha<1$; production shocks (b), for all $\alpha\leq 0.4$; and stochastic population growth (c), for all $0<\alpha<1$. In case (d), stochastic depreciation, the condition is not fulfilled for any α.

6.2. CES Production Function

$$f(k)=(1-A+Ak^\alpha)^{1/\alpha} \quad \text{with} \quad 0\neq\alpha<1 \quad \text{and} \quad 0<A<1 \qquad (\mathbf{14})$$

We first derive a sufficient condition for the indispensable condition (vi), which ensures applicability of our results. Elementary calculations show that $\bar{k}_s(\xi_{\min},n_{\max},\delta_{\max})=(1-A)^{1/\alpha}\{[(\delta_{\max}+n_{\max})/(s\xi_{\min})]^\alpha-A\}^{-1/\alpha}$ and hence $f'[\bar{k}_s(\xi_{\min},n_{\max},\delta_{\max})]=A[(\delta_{\max}+n_{\max})/(s\xi_{\min})]^{1-\alpha}$. From these expressions, we find that $h'_s[\omega,\bar{k}_s(\xi_{\min},n_{\max},\delta_{\max})]=\{1-\delta(\omega)+s^\alpha\xi(\omega)A[(\delta_{\max}+n_{\max})/\xi_{\min}]^{1-\alpha}\}/[1+n(\omega)]$ for all $s>0$. Therefore, condition (vi) is fulfilled if

$$\mathbf{E}\log\frac{1-\delta(\omega)+\xi(\omega)A\left(\frac{\delta_{\max}+n_{\max}}{\xi_{\min}}\right)^{1-\alpha}}{1+n(\omega)}<0.$$

Analogously to the Cobb–Douglas case, we get the sufficient condition for (vi):

$$A<\frac{\xi_{\min}^{1-\alpha}}{\mathbf{E}\xi}\frac{\mathbf{E}\delta+n_{\min}}{(\delta_{\max}+n_{\max})^{1-\alpha}}. \qquad (\mathbf{15})$$

The Inada condition (v) $\lim_{k\to\infty}f'(k)=\lim_{k\to\infty}A[(1-A)k^{-\alpha}+A]^{(1-\alpha)/\alpha}=A^{1/\alpha}$ is satisfied if $A<((\delta_{\max}+n_{\max})/\xi_{\min})^\alpha$. No condition is needed at the capital intensity zero because $f(0)=(1-A)^{1/\alpha}>0$.

We now fix the parameters of the CES production function: $A=1/2$ and $\alpha=1/2$. Analogously to the Cobb–Douglas case, one can check that, for these parameter values, conditions (i)–(v) and the sufficient condition (15) for (vi) are satisfied in cases (a)–(d). Further, it holds $\mathbf{E}\xi f(k_1^\star)<\infty$. From this we may conclude that a nontrivial random fixed point exists for all positive saving rates, that $\mathbf{E}c_s^\star(\omega)<\infty$ for all s, and that a golden rule exists.

Applying the same approximation procedure as in the preceding case, we obtain the average consumption per capita $\mathbf{E}c_s^\star$ for each fixed saving rate $s\in[0,1]$ in cases (a)–(d). The results are collected in Table 2.

Average consumption is higher in the stochastic cases than in the deterministic case. However, although the ranking of the cases according to maximal average consumption is as in the Cobb–Douglas case, the relative gain is only about half that in the Cobb–Douglas case. The main difference in the CES production function is that the golden-rule saving rate depends on the particular case.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

Table 2. Stochastic Solow model with CES production function: Numerically approximated golden-rule saving rate, average consumption per capita in absolute values, and average consumption per capita relative to the deterministic case (a)

Case	Golden rule	Average consumption	Average consumption relative to (a) (%)
(a)	0.5	0.5	100.0
(b)	0.53	0.528	105.6
(c)	0.52	0.51	102.0
(d)	0.56	0.55	110.0

The observations made in our case study for the CES production function can be summarized as follows: The stochastic fluctuation of any parameter of the Solow growth model (1) increases average consumption for any saving rate. The golden-rule saving rate is higher as well.

Let us close this section with a short remark on dynamic efficiency in cases (a)–(d). Numerical results indicate that condition (8) of Theorem 4 is satisfied in all cases (b)–(d). In the CES example, we therefore derive the strongest dynamic efficiency result that is possible by Theorem 4.

REFERENCES

Ambler, S. & A. Paquet (1994) Stochastic depreciation and the business cycle. *International Economic Review* 35, 101–116.

Arnold, L. (1998) *Random Dynamical Systems*. Berlin: Springer-Verlag.

Barro, R.J. & X. Sala-i-Martin (1995) *Economic Growth*. New York: McGraw-Hill.

Mirman, L.J. (1972) On the existence of steady state measures for one sector growth models with uncertain technology. *International Economic Review* 13, 271–286.

Mirman, L.J. (1973) The steady state behavior of a class of one sector growth models with uncertain technology. *Journal of Economic Theory* 6, 219–242.

O'Brien, G.L. (1982) The occurrence of large values in stationary sequences. *Probability and Related Fields* 61, 347–353.

Phelps, E.S. (1966) *Golden Rules of Economic Growth*. New York: Norton.

Schenk–Hoppé, K.R. (2001) Stochastic Solow model. Software, http://www.iew.unizh.ch/home/klaus/numerics.

Schenk–Hoppé, K.R. & B. Schmalfuss (2001) Random fixed points in a stochastic Solow growth model. *Journal of Mathematical Economics* 36, 19–30.

Schmalfuss, B. (1996) A random fixed point theorem based on Lyapunov exponents. *Random & Computational Dynamics* 4, 257–268.

Schmalfuss, B. (1998) A random fixed point theorem and the random graph transformation. *Journal of Mathematical Analysis and Applications* 225, 91–113.

Solow, R.M. (1956) A contribution to the theory of economic growth. *Quarterly Journal of Economics* 70, 65–94.

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

APPENDIX

Proof of Theorem 1. Assertions (i), (ii), and (iv) correspond to Theorem 1 of Schenk–Hoppé and Schmalfuss (2001). Assertion (iii) is immediate from Theorem 2.2 of Schmalfuss (1996). Only the case $s = 0$ is not covered in these references. However, we have

$$\varphi_0(t+1,\omega,k) = \frac{1-\delta(\theta^t\omega)}{1+n(\theta^t\omega)}\varphi_0(t,\omega,k) = \prod_{u=0}^{t}\frac{1-\delta(\theta^u\omega)}{1+n(\theta^u\omega)}k.$$

By condition (vi), this yields assertions (i)–(iv). Observe that $k_0^\star(\omega) \equiv 0$.

Schenk–Hoppé and Schmalfuss (2001) apply a Banach fixed-point theorem for random maps to prove existence of a random fixed point with the specified properties. This generalization of the well-known Banach theorem for random dynamical systems is due to Schmalfuss (1996, 1998). The main idea of this theorem is to consider the space of all tempered random variables with values in some set $G(\omega)$, which is forward invariant in the sense that $\varphi[t,\omega,G(\omega)] \subset G(\theta^t\omega)$ for all $t \geq 0$; cf. Lemma 1. The operator $x \mapsto \{\lim_{t\to\infty}\varphi[t,\theta^{-t}\omega,x(\theta^{-t}\omega)]\}_{\omega\in\Omega}$ is a global contraction on that space under a condition that ensures contraction on the average [condition (vi) in the present paper is a particular version of this condition]. ■

Proof of Proposition 1. We first prove that $s \mapsto k_s^\star(\omega)$ is a strictly increasing function. Since f is increasing and $f(k) > 0$ for all $k > 0$, then for all $s < s'$, $k \leq k'$,

$$h_{s'}(\omega,k') - h_s(\omega,k) = \frac{[1-\delta(\omega)](k'-k) + \xi(\omega)[s'f(k') - sf(k)]}{1+n(\omega)}$$

$$\geq \frac{(s'-s)\xi(\omega)}{1+n(\omega)}f(k) \geq \frac{(s'-s)\xi_{\min}}{1+n_{\max}}f(k).$$

Fix any $0 < s < s'$. Then the preceding estimate ensures existence of a $c(s,s') > 0$ (which is independent of k and ω) such that $h_{s'}(\omega,k') - h_s(\omega,k) \geq c(s,s')$ for all $k \in G(s)$ and $k' \geq k$; see Lemma 1 for the definition of $G(s)$.

Using $\varphi_s(t,\theta^{-t}\omega,k) = h_s(\theta^{-1}\omega)\circ\cdots\circ h_s(\theta^{-t}\omega,k)$, we find $\varphi_{s'}(t,\theta^{-t}\omega,k) - \varphi_s(t,\theta^{-t}\omega,k) \geq c(s,s')$ for all $t \geq 0$. By the contraction property of $k_s^\star(\omega)$ stated in Theorem 1, we finally obtain for $s < s'$ and arbitrarily fixed $k \in G(s)$

$$k_s^\star(\omega) = \lim_{t\to\infty}\varphi_s(t,\theta^{-t}\omega,k) < \lim_{t\to\infty}\varphi_{s'}(t,\theta^{-t}\omega,k) = k_{s'}^\star(\omega),$$

which yields $k_s^\star(\omega)$ strictly increasing in s.

We next prove continuity of $s \mapsto k_s^\star(\omega)$. Fix any ω and any $k > 0$. $k_s^\star(\omega) = \lim_{t\to\infty}\varphi_s(t,\theta^{-t}\omega,k)$ for all $s \in [0,1]$ by Theorem 1. Since $\varphi_s(t,\theta^{-t}\omega,k)$ is continuous in s for each fixed t, it suffices to show that there exists a $k > 0$ such that this convergence is uniformly in $s \in [0,1]$. Letting $\bar{k} := \bar{k}_1(\xi_{\min}, n_{\max}, \delta_{\max})$, we find

$$\sup_{s\in[0,1]}\left|\varphi_s(t,\theta^{-t}\omega,\bar{k}) - k_s^\star(\omega)\right| = \sup_{s\in[0,1]}\left|\varphi_s(t,\theta^{-t}\omega,\bar{k}) - \varphi_s\left[t,\theta^{-t}\omega,k_s^\star(\theta^{-t}\omega)\right]\right|$$

$$\leq \sup_{s\in[0,1]}\left|k_s^\star(\theta^{-t}\omega) - \bar{k}\right|\prod_{u=1}^{t}h_s'\left[\theta^{-u}\omega,\bar{k}_s(\xi_{\min},n_{\max},\delta_{\max})\right],$$

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

where we have used $\min\{\bar{k}, k_s^\star(\omega)\} \geq \bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})$ and $h_s'(\omega, k)$ decreasing in k.

Condition (vi) implies that, for any $\varepsilon > 0$, there exists a $T(\varepsilon)$ such that

$$\prod_{u=1}^{t} h_s'\left[\theta^{-u}\omega, \bar{k}_s(\xi_{\min}, n_{\max}, \delta_{\max})\right] \leq \exp[-(c-\varepsilon)t]$$

for all $t \geq T(\varepsilon)$, where the exponential rate is independent of s. Hence, for all $\varepsilon < c$ and $t \geq T(\varepsilon)$,

$$\begin{aligned}\sup_{s\in[0,1]} \left|\varphi_s(t, \theta^{-t}\omega, \bar{k}) - k_s^\star(\omega)\right| &\leq \sup_{s\in[0,1]} \left[k_s^\star(\theta^{-t}\omega) + \bar{k}\right] \exp[-(c-\varepsilon)t] \\ &= \left[k_1^\star(\theta^{-t}\omega) + \bar{k}\right] \exp[-(c-\varepsilon)t]\end{aligned}$$

where the last equality follows from $k_s^\star(\omega)$ being strictly increasing.

Finally, temperedness of $k_1^\star(\omega)$, that is, $\limsup_{t\to\infty} k_1^\star(\theta^{-t}\omega)\exp(-\delta t) = 0$ for all $\delta > 0$, yields the convergence $k_s^\star(\omega) = \lim_{t\to\infty} \varphi_s(t, \theta^{-t}\omega, k)$ uniform in $s \in [0, 1]$. ■

Proof of Theorem 2. We prove that $s \mapsto \mathbf{E}c_s^\star(\omega)$ is a continuous function; this yields the assertion. By Proposition 1 and continuity of f, the map $s \mapsto c_s^\star(\omega) = (1-s)\xi(\omega)f[k_s^\star(\omega)]$ is continuous. Proposition 1 further implies $c_s^\star(\omega) \leq \xi(\omega)f[k_1^\star(\omega)]$ for all s. The condition $\mathbf{E}\xi f(k_1^\star) < \infty$ ensures that $\mathbf{E}c_s^\star(\omega)$ exists and is uniformly bounded in s. Thus, the dominated convergence theorem applies. ■

Proof of Corollary 1. Theorem 2 yields the result, if we can show $\mathbf{E}\xi f(k_1^\star) < \infty$ under the respective assumption. Condition (a) is immediate from condition (iii), which gives integrability of $\xi(\omega)$.

Let us turn to (b). Our assumption ensures existence of a nontrivial fixed point $\bar{k}_1(\delta_{\min}, n_{\min}, \xi_{\max})$ of the deterministic model (2) with respective parameters. Further, we have $h_1(\omega, k) \leq [(1-\delta_{\min})k + \xi_{\max}f(k)]/(1+n_{\min})$. Therefore, $k_1^\star(\omega) \leq \bar{k}_1(\xi_{\max}, n_{\min}, \delta_{\min})$ is uniformly bounded. This yields the result. ■

Proof of Corollary 2. By Theorem 2, it suffices to show that $\mathbf{E}\xi f(k_1^\star) < \infty$. Taking expected values on both sides of (1), we obtain $(\delta + n)\mathbf{E}k_1^\star = \mathbf{E}\xi f(k_1^\star)$. Observe that $\delta > 0$ because, for constant δ and n, condition (vi) yields $1 - \delta + \xi_{\min}f'(\bar{k}_1) \leq \exp\{\mathbf{E}\log[1-\delta+\xi(\omega)f'(\bar{k})]\} < 1+n$ and condition (iii) gives $\xi_{\min}f'(\bar{k}_1) > 0$, where $\bar{k} := \bar{k}_1(\xi_{\min}, n_{\max}, \delta_{\max})$. It therefore remains to prove $\mathbf{E}k_1^\star < \infty$. One finds

$$h_1(\omega, k) \leq \frac{1-\delta+\xi(\omega)f'(y)}{1+n}\,k + \frac{\xi(\omega)f(y)}{1+n} \tag{A.1}$$

because concavity of f implies $f(k) \leq f(y) + f'(y)k$ for all k with an arbitrarily fixed $y > 0$. Consider the associated affine random difference equation,

$$x_{t+1} = a(\theta^t\omega)x_t + b(\theta^t\omega), \tag{A.2}$$

with

$$a(\omega) := \frac{1-\delta+\xi(\omega)f'(y)}{1+n} \quad \text{and} \quad b(\omega) := \frac{\xi(\omega)f(y)}{1+n},$$

where $y \geq \bar{k}_1(\xi_{\min}, n_{\max}, \delta_{\max})$ is arbitrarily fixed. The random dynamical system generated by (A.2) possesses the unique globally attracting random fixed point

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

$$x^\star(\omega) := b(\theta^{-1}\omega) + \sum_{i=1}^{\infty} b[\theta^{-(i+1)}\omega] \prod_{j=1}^{i} a(\theta^{-j}\omega),$$

which is an upper bound for the random fixed point of the stochastic Solow model, that is, $k_1^\star(\omega) \le x^\star(\omega)$ for all ω; see Schenk–Hoppé and Schmalfuss (2001), Proof of Theorem 1 (step 4).

We will show that $\mathbf{E}x^\star(\omega) < \infty$, which yields the assertion. By the dominated convergence theorem, it suffices to show that, for the sequence of random variables, $x_N(\omega) := b(\theta^{-1}\omega) + \sum_{i=1}^{N} b[\theta^{-(i+1)}\omega] \prod_{j=1}^{i} a(\theta^{-j}\omega)$, the expected value $\mathbf{E}x_N$ is uniformly bounded. We have

$$\mathbf{E}x_N = \mathbf{E}b + \sum_{i=1}^{N} \mathbf{E}\left\{ b[\theta^{-(i+1)}\omega] \prod_{j=1}^{i} a(\theta^{-j}\omega) \right\} \le \mathbf{E}b + \frac{f(y)}{f'(y)} \sum_{i=1}^{N} \mathbf{E} \prod_{j=1}^{i+1} a(\theta^{-j}\omega)$$

$$= \mathbf{E}b + \frac{f(y)}{f'(y)} \sum_{i=1}^{N} (\mathbf{E}a)^{i+1} = \mathbf{E}b + \frac{f(y)}{f'(y)} (\mathbf{E}a)^2 \frac{1 - (\mathbf{E}a)^N}{1 - \mathbf{E}a},$$

where we have used $a(\theta^{-i}\omega)$ and $a(\theta^{-j}\omega)$ independent for all $i \ne j$ because ξ is an i.i.d. process by assumption.

By the Inada condition $\lim_{k\to\infty} f'(k) < (n+\delta)/\mathbf{E}\xi$, there exists a $y \ge \bar{k}$ such that $\mathbf{E}a = [1 - \delta + \mathbf{E}\xi f'(y)]/(1+n) < 1$. Hence, for all sufficiently large y, we have

$$\mathbf{E}x_N \le \mathbf{E}b + \frac{f(y)}{f'(y)} \frac{(\mathbf{E}a)^2}{1 - \mathbf{E}a}$$

for all N, which asserts that $\mathbf{E}x^\star < \infty$. This in turn yields $\mathbf{E}k_1^\star < \infty$ and finishes the proof. ∎

Proof of Theorem 3. Theorems 1 and 2 ensure existence of a unique nontrivial random fixed point $k_s^\star(\omega)$ and finiteness of $\mathbf{E}c_s^\star(\omega) = (1-s)\mathbf{E}\xi(\omega) f[k_s^\star(\omega)]$ for all $s \in [0,1]$.

We first show that $k_s^\star(\omega) = g(s)\, k_1^\star(\omega)$, where $k_s^\star(\omega)$ is the unique random fixed point solving (1). Replacing $k_s^\star(\omega)$ with $g(s)\, k_1^\star(\omega)$ in (1), we observe that

$$g(s)k_1^\star(\theta\omega) = \frac{[1-\delta(\omega)]g(s)k_1^\star(\omega) + s\xi(\omega) f\big[g(s)k_1^\star(\omega)\big]}{1+n(\omega)}$$

$$= g(s) \frac{[1-\delta(\omega)]k_1^\star(\omega) + \xi(\omega) f\big[k_1^\star(\omega)\big]}{1+n(\omega)}$$

by condition (6). Dividing both sides by $g(s)$ gives an equation for $k_1^\star(\omega)$, which is fulfilled by Theorem 1. Hence $k_s^\star(\omega) = g(s)k_1^\star(\omega)$ by uniqueness of the solution.

This result implies that average consumption is given by

$$\mathbf{E}c_s^\star(\omega) = (1-s)\,\mathbf{E}\Big\{\xi(\omega) f\Big[g(s)k_1^\star(\omega)\Big]\Big\} = \frac{1-s}{s} g(s) \mathbf{E}\Big\{\xi(\omega) f\Big[k_1^\star(\omega)\Big]\Big\}.$$

Since $\mathbf{E}c_s^\star$ is continuously differentiable in s, any maximizer is an interior point, and $(\partial/\partial s)\mathbf{E}c_s^\star = 0$ if and only if (7) holds, we obtain (7) as a necessary condition for the golden-rule saving rate. If the solution to (7) is unique, then (7) is also a sufficient condition. ∎

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press

Proof of Theorem 4. We have to show that $\mathbf{E}[c^{\star}_{s\text{gold}}(\omega) - c(\omega)] > 0$, where $c(\omega) := [1 - \delta(\omega)][k^{\star}_{s\text{gold}}(\omega) + \varepsilon(\omega)] + \xi(\omega) f[k^{\star}_{s\text{gold}}(\omega) + \varepsilon(\omega)] - [1 + n(\omega)][k^{\star}_{s\text{gold}}(\theta\omega) + \varepsilon(\theta\omega)]$.

It is straightforward to check that

$$\mathbf{E}\left[c^{\star}_{s\text{gold}}(\omega) - c(\omega)\right] = \mathbf{E}\Big(\delta(\omega)\varepsilon(\omega) + n(\omega)\varepsilon(\theta\omega) - \xi(\omega)\left\{f\left[k^{\star}_{s\text{gold}}(\omega) + \varepsilon(\omega)\right] - f\left[k^{\star}_{s\text{gold}}(\omega)\right]\right\}\Big).$$

By condition (vi), $f(y) - f(x) \leq f'(x)(y - x)$ with strict inequality if $y \neq x$. The condition $\mathbf{E}\varepsilon > 0$ implies that $\varepsilon(\omega) > 0$ with positive probability. Recalling that $\xi(\omega) > 0$ for all ω by condition (iii), we therefore obtain

$$\mathbf{E}\left[c^{\star}_{s\text{gold}}(\omega) - c(\omega)\right] > \mathbf{E}\left\{\delta(\omega)\varepsilon(\omega) + n(\omega)\varepsilon(\theta\omega) - \xi(\omega) f'\left[k^{\star}_{s\text{gold}}(\omega)\right]\varepsilon(\omega)\right\}.$$

Finally, rewriting the right-hand side of this expression and employing assumption (b) yields

$$\begin{aligned}\mathbf{E}\left[c^{\star}_{s\text{gold}}(\omega) - c(\omega)\right] &> \mathbf{E}\Big(\left\{\delta(\omega) + n(\theta^{-1}\omega) - \xi(\omega) f'\left[k^{\star}_{s\text{gold}}(\omega)\right]\right\}\varepsilon(\omega)\Big)\\ &\geq \mathbf{E}\Big(\left\{\delta(\omega) + n(\theta^{-1}\omega) - \xi(\omega) f'\left[k^{\star}_{s\text{gold}}(\omega)\right]\right\}\Big)\mathbf{E}\varepsilon(\omega) \geq 0,\end{aligned}$$

where the last step uses condition (8). ■

https://doi.org/10.1017/S1365100501010033 Published online by Cambridge University Press