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RESEARCH ARTICLE

Scanning the moduli of smooth hypersurfaces

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Abstract

We study the locus of smooth hypersurfaces inside the Hilbert scheme of a smooth projective complex variety. In the spirit of scanning, we construct a map to a continuous section space of a projective bundle, and show that it induces an isomorphism in integral homology in a range of degrees growing with the ampleness of the hypersurfaces. When the ambient variety is a curve, this recovers a result of McDuff about configuration spaces. We compute the rational cohomology of the section space and exhibit a homological stability phenomenon for hypersurfaces with first Chern class going to infinity. For simply connected varieties, the rational cohomology is shown to agree with the stable cohomology of a moduli space of hypersurfaces, with a peculiar tangential structure, as studied by Galatius and Randal-Williams.

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1. Introduction

A hypersurface, also known as an effective Cartier divisor, in a smooth projective complex variety X is the zero locus $V(s)$ of a nonzero global section $s \in H^0(X, \mathcal{L})$ of an algebraic line bundle $\mathcal{L}$:

$$V(s) := \{x \in X \mid s(x) = 0\} \subset X.$$

Such a hypersurface is said to be smooth when the derivative $ds(x) \neq 0$ for all $x \in V(s)$. Smooth hypersurfaces obtained from sections of $\mathcal{L}$ are parameterised by the complement of the discriminant inside the complete linear system $|\mathcal{L}| = \mathbb{P}\big(H^0(X, \mathcal{L})\big)$, a classical object dating back to the early days of complex geometry and whose homology was studied more recently in [34, 1]. From a topological point of view, it is however more natural to consider a moduli of smooth hypersurfaces obtained by fixing a "degree" (i.e., Chern class) rather than a linear equivalence class. Technically, this moduli space is constructed by allowing variations of the line bundle and considering the functor of relative effective Cartier divisors introduced by Grothendieck in [18]. We shall recall in Section 3 how to precisely define this functor and explicitly construct a scheme representing it. For now, let us only record the main consequences: for certain cohomology classes $\alpha \in \mathrm{NS}(X) \subset H^2(X; \mathbb{Z})$ in the Néron–Severi group of X, there is a connected smooth complex algebraic variety

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \cong \{Z \subset X \text{ smooth hypersurface with } c_1(\mathcal{O}_X(Z)) = \alpha\}$$

which we call the *moduli space of smooth hypersurfaces* with fixed Chern class. It is a parameter space whose points are smooth complex hypersurfaces embedded inside X. Furthermore, recording the linear equivalence class of a hypersurface produces the *Abel–Jacobi morphism*

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \mathrm{Pic}^{\alpha}(X)$$

to a component of the Picard scheme which parameterises isomorphism classes of line bundles on X with first Chern class α. In particular, the fibre of this map above an isomorphism class of a line bundle $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$ is the complement of the discriminant inside the linear system $|\mathcal{L}|$.

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In the present work, we investigate the topology of $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$ using tools from algebraic topology. To state our results, we write $d(\alpha)$ for the largest integer d such that all line bundles with first Chern class α are d-jet ample, and $\mathbb{P}(J^1\mathcal{O}_X)$ for the projectivisation of the first jet bundle of the structure sheaf $\mathcal{O}_X$ (we review this notion in Section 2.2).

Theorem 1.1 (See Theorem 4.6 for a precise version). *Let X be a smooth projective complex variety and let $\alpha \in \mathrm{NS}(X)$ be ample enough. Taking the first jet yields a map*

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0}\big(\mathbb{P}(J^1\mathcal{O}_X)\big)$$

which induces an isomorphism in integral homology onto the path component that it hits, in degrees $* < \frac{d(\alpha)-3}{2}$.

Remark 1.2. Under our ampleness assumption the space $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$ is connected (see Remark 3.16). Furthermore, $\pi_0\big(\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))\big) \cong H^2(X;\mathbb{Z})$ and the jet map hits the component corresponding to $\alpha \in \mathrm{NS}(X) \subset H^2(X;\mathbb{Z})$.

We recall in Section 2 the notion of jet ampleness, and explain in Appendix A how to estimate the number $d(\alpha)$. Among other things, we show that given any integer $M \geq 0$ and classes $\alpha, \beta \in \mathrm{NS}(X)$ with β ample, we have that $d(\alpha + k\beta) \geq M$ for large enough $k \gg 0$. (See Proposition A.7.) In particular, the degree range of our main theorem can be arbitrarily big. In the remainder of this introduction, we describe applications of our main theorem and connect our results to the existing literature on moduli spaces of manifolds.

1.1. Rational computations and stability

A main advantage of our main theorem resides in the fact that the homotopy type of spaces of continuous sections can be approached by purely homotopical methods. This is particularly effective if one is willing to look at the rational information only. Using tools from rational homotopy theory, we show:

Theorem 1.3 (See Theorem 6.5 for a precise version). *Let n be the complex dimension of X. Let $\alpha \in H^2(X;\mathbb{Z})$ and denote by $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ the component hit by the jet map from $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$. The rational cohomology ring of that section space is isomorphic to the cohomology of the following commutative differential graded algebra:*

$$\mathrm{Sym}^*_{\mathrm{gr}}\big(\mathbb{Q}z \oplus H_1(X;\mathbb{Q}) \oplus H_*(X;\mathbb{Q})[1]\big),$$
$$\text{with } d(z) = 0,\ d(H_1(X;\mathbb{Q})) = 0, \text{ and } d(x) = \varphi(x) \text{ for } x \in H_*(X;\mathbb{Q})[1].$$

Here $\mathrm{Sym}^*_{\mathrm{gr}}$ denotes the free graded commutative algebra on a graded vector space, $[1]$ increases the grading by one, and $\mathbb{Q}z$ is a one-dimensional vector space generated by z in degree 2. The differential is encoded by a morphism $\varphi\colon H_*(X;\mathbb{Q}) \to \mathrm{Sym}^*_{\mathrm{gr}}(\mathbb{Q}z \oplus H_1(X;\mathbb{Q}))$ which can be computed explicitly in terms of the Chern classes of $J^1\mathcal{O}_X$ and α.

To emphasise the words "can be computed explicitly" in the statement above, we have worked out completely the formulas in Example 6.6 in the case of X being a torus. Let us also mention another application of our main theorem in the form of a rational homological stability phenomenon. To state it, we write $\mathbb{P}^n_{\mathbb{Q}}$ for the rationalisation of $\mathbb{P}^n$ and define

$$\mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}}) \subset \mathrm{Map}(X, \mathbb{P}^n_{\mathbb{Q}})$$

to be the subspace of continuous maps f such that $f^*(c_1(\mathcal{O}_{\mathbb{P}^n}(1))) = \alpha \in H^2(X;\mathbb{Z})$.

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Theorem 1.4 (See Corollary 6.8). *Let X be a smooth projective complex variety whose tangent bundle is a topologically trivial complex vector bundle. Let $\alpha \in \mathrm{NS}(X)$ be ample and assume that $d(\alpha) \geq 1$. Then, for any integer $k \geq 1$, there is a map*

$$\mathcal{M}_{\mathrm{hyp}}^{k\alpha} \longrightarrow \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

inducing an isomorphism in rational homology in the range of degrees $* < \frac{k \cdot d(\alpha) - 3}{2}$. In particular, the rational homology stabilises as $k \to \infty$.

Remark 1.5. By a theorem of Wang [35], if the tangent bundle of X is holomorphically trivial then X is an abelian variety. If we only require topological triviality as in the theorem above, other examples exist such as products of abelian varieties with curves [31].

1.2. Configuration spaces on curves

On an algebraic curve, the first Chern class of a line bundle is simply its degree under the identification $\deg\colon H^2(X;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$. The vanishing locus of a nonzero global section of a line bundle of degree $d > 0$ is a set of d points counted with multiplicity. In fact, such a section is nonsingular precisely when these d points are distinct. In that case, the moduli space of hypersurfaces is the classically studied unordered configuration space and our main theorem recovers parts of a result of McDuff [25, Theorem 1.1], who studied more generally configuration spaces on any smooth manifold via scanning methods:

Theorem 1.6 (See Theorem 7.2). *Let X be a smooth projective complex curve of genus g, and denote by T^+X the fibrewise one-point compactification of the tangent bundle of X. Let $\alpha \in H^2(X;\mathbb{Z})$ be such that $\deg \alpha > 2g - 2$. The jet map*

$$\mathrm{UConf}_{\deg \alpha}(X) \cong \mathcal{M}_{\mathrm{hyp}}^{\alpha} \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \cong \Gamma^{\alpha}_{\mathcal{C}^0}(T^+X)$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{\deg \alpha - 2g - 3}{2}$.

1.3. Characteristic classes and moduli spaces of manifolds

Let $H = V(s) \subset X$ be a hypersurface defined by a nonsingular section s of a line bundle $\mathcal{L}$ on X. In the series of papers [14, 12, 11], Galatius and Randal-Williams have investigated the homology of classifying spaces of diffeomorphism groups of manifolds. In this article, we have tried to compare their results to ours in the case where the manifold under investigation is H. Deferring the technical details to Section 8.2, we describe here the main contents.

Let n be the complex dimension of X, so that H is of real dimension $2n - 2$. A tangential structure for $(2n-2)$-manifolds is a fibration $\theta\colon B \to BO(2n-2)$ which classifies a vector bundle $\theta^*\gamma_{2n-2}$ of real rank $2n-2$, obtained by pulling back the universal bundle γ_{2n-2} on $BO(2n-2)$. A θ-structure on H is then defined as a bundle map $\hat{\ell}\colon TH \to \theta^*\gamma_{2n-2}$ which is a fibrewise linear isomorphism. Given such a structure we get a homotopy class of a lift of the map τ_H classifying the tangent bundle of H

$$\begin{array}{ccc} & & B \\ & \overset{\ell}{\nearrow} & \downarrow \theta \\ H & \xrightarrow[\tau_H]{} & BO(2n-2) \end{array}$$

where we have written ℓ for the map of spaces underlying $\hat{\ell}$. For instance, if $\theta\colon BSO(2n-2) \to BO(2n-2)$ is the natural map, a θ-structure amounts to the choice of an orientation. Writing $\mathrm{Bun}^{\theta}(H)$

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for the space of bundle maps $TH \to \theta^*\gamma_{2n-2}$, the moduli space classifying smooth H-bundles with θ-structure is defined as the homotopy orbit construction

$$\mathcal{M}^\theta(H) = \mathrm{Bun}^\theta(H) /\!\!/ \mathrm{Diff}(H)$$

and we denote by $\mathcal{M}^\theta(H, \hat{\ell}) \subset \mathcal{M}^\theta(H)$ the path component of $\hat{\ell}$.

One could wonder which tangential structure is the most natural on H (e.g., H is at least canonically oriented as a complex manifold). In the last section of this article, we find a reasonable candidate and show:

Theorem 1.7 (See Theorem 8.12 and Corollary 8.16 for precise versions). *Let X be a simply connected smooth complex projective variety of dimension $n \geq 4$ and $\mathcal{L}$ be a very ample line bundle on it with $\alpha = c_1(\mathcal{L})$. There is a map $\theta\colon B \to BO(2n-2)$ depending solely on $\mathcal{L}$ such that a hypersurface H defined by a nonsingular section of $\mathcal{L}$ naturally inherits a θ-structure $\hat{\ell}$. For this tangential structure, the map classifying the universal bundle*

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \mathcal{M}^\theta(H, \hat{\ell})$$

induces an isomorphism in rational cohomology in the stable ranges.

1.4. Outline

In Section 2 we recall known properties about the Picard scheme, jet bundles, jet ampleness, and the topology of smooth hypersurfaces. In Section 3 we define precisely the moduli space $\mathcal{M}^\alpha_{\mathrm{hyp}}$. In Sections 4 and 5 we state and prove our main theorem. The rest of the paper is dedicated to applications. We make rational computations in Section 6 and describe the relation to scanning and characteristic classes of manifold bundles in Sections 7 and 8. Finally, we have assembled in Appendix A various results concerning jet ampleness.

1.5. The proof strategy for the main theorem

As explained in the introduction, we have a sequence of spaces

$$|\mathcal{L}| \setminus \Delta \longrightarrow \mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \mathrm{Pic}^\alpha(X) \tag{1}$$

where $\Delta \subset |\mathcal{L}|$ is the discriminant hypersurface. It turns out not to be a fibration, but only a microfibration. On the other hand, we have an actual fibration

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \longrightarrow B\big(\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times\big) \tag{2}$$

obtained by modding out by the constant functions $\mathbb{C}^\times \subset \mathrm{Map}(X, \mathbb{C}^\times)$ and delooping the $\mathrm{Map}(X, \mathbb{C}^\times)$-principal fibration

$$\mathrm{Map}(X, \mathbb{C}^\times) \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0) \longrightarrow \Gamma^0_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{L})) \cong \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

sending a section of $J^1\mathcal{L} \setminus 0$ to its projectivisation. We observe the weak homotopy equivalences

$$B\big(\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times\big) \simeq K(H^1(X; \mathbb{Z}), 1) \simeq \mathrm{Pic}^\alpha(X)$$

and have proved in the earlier work [1] that the jet map

$$|\mathcal{L}| \setminus \Delta \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times$$

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induces an isomorphism in homology in a range of degrees. In essence, the proof consists in comparing with two (micro)fibrations (1) and (2): we will leverage the homotopy (resp. homology) equivalence of their bases (resp. fibres) to obtain a homology equivalence between their total spaces.

2. The Picard scheme, jet bundles, and hypersurfaces

2.1. The Picard scheme

We begin with recollections on line bundles on smooth projective complex varieties and their moduli. A more precise, and much more general, account of this standard material can be found in Kleiman's notes [22]. In what follows, X is a connected smooth projective complex variety.

Definition 2.1. The *absolute Picard group* of a complex scheme T is the set $\mathrm{Pic}_{\mathrm{abs}}(T)$ of isomorphism classes of algebraic line bundles on T equipped with the group law given by the tensor product.

Definition 2.2. The Picard functor of X

$$T \longrightarrow \mathrm{Pic}_{\mathrm{abs}}(X \times T)/\mathrm{Pic}_{\mathrm{abs}}(T)$$

from complex schemes to abelian groups is represented by a scheme $\mathrm{Pic}(X)$ called the *Picard scheme* of X (relative to $\mathrm{Spec}(\mathbb{C})$).

Remark 2.3. The analytification of $\mathrm{Pic}(X)$, sometimes called the *Picard space* in this article, is the group of isomorphism classes of holomorphic line bundles on X.

Lemma 2.4 (Compare [22, Exercise 4.3]). *There exists a (nonunique) algebraic line bundle $\mathcal{P}$ on $\mathrm{Pic}(X) \times X$ satisfying the following property: given any complex scheme T and a line bundle $\mathcal{L}$ on $T \times X$, there exists a unique morphism $h\colon T \to \mathrm{Pic}(X)$ such that*

$$\mathcal{L} \cong ((h \times 1)^{*}\mathcal{P}) \otimes f^{*}\mathcal{N}$$

for $f\colon T \times X \to T$ the projection map and $\mathcal{N}$ some line bundle on T.

Definition 2.5. Any choice of a line bundle $\mathcal{P}$ as in Lemma 2.4 will be called a *Poincaré line bundle* on X.

The Picard scheme $\mathrm{Pic}(X)$ only parameterises *isomorphism classes* of line bundles on X. One should think of the choice of a Poincaré line bundle as making compatible choices of *representatives* of those isomorphism classes.

We will need a further decomposition of the Picard scheme into components. To introduce it, we let $\mathcal{O}_X(1)$ be a very ample line bundle on X and write $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and $n \in \mathbb{Z}$. Given a polynomial $P \in \mathbb{Q}[x]$, let $\mathrm{Pic}^{P}(X)(-) \subset \mathrm{Pic}(X)(-)$ be the subfunctor of the Picard functor whose T-points are represented by the line bundles $\mathcal{L}$ on $X \times T$ such that

$$\chi(X_t, \mathcal{L}_t^{-1}(n)) = P(n) \qquad \text{for all } t \in T$$

where X_t and $\mathcal{L}_t$ denote the base change to t.

Proposition 2.6 (Compare [22, Theorems 4.8 and 6.20]). *The Picard functor $\mathrm{Pic}^{P}(X)(-)$ is represented by a complex quasi-projective scheme denoted $\mathrm{Pic}^{P}(X)$. The Picard scheme is the disjoint union of the $\mathrm{Pic}^{P}(X)$ when P runs over all polynomials P.*

Passing to complex points, the picture is vastly simplified by Hodge theory as recalled in the following two results.

Definition 2.7. The *Néron–Severi* group of X, denoted $\mathrm{NS}(X)$, is the image of the morphism $c_1\colon \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ sending an isomorphism class of a line bundle to its first Chern class.

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Proposition 2.8. *The Picard space* $\mathrm{Pic}(X)$ *is a disjoint union of connected components indexed by the Néron–Severi group*

$$\mathrm{Pic}(X) = \bigsqcup_{\alpha \in \mathrm{NS}(X)} \mathrm{Pic}^{\alpha}(X).$$

Each space $\mathrm{Pic}^{\alpha}(X)$ *is a torus, noncanonically isomorphic to* $H^1(X;\mathcal{O}_X)/H^1(X;\mathbb{Z})$*, and parameterises isomorphism classes of holomorphic line bundles on* X *with Chern class* α*.*

2.2. Jet bundles

We recall the definition of jet bundles in algebraic geometry and explain the construction of the jet evaluation map which will be used throughout this article. The main reference for this section is [17, Section 16.7]. In this section only, the full generality offered by schemes will be convenient, so we momentarily work in this setting.

Let $f : Y \to S$ be a morphism of schemes, $\Delta : Y \to Y \times_S Y$ be the diagonal and $\mathcal{I}$ be its ideal sheaf. We let $p_i : Y \times_S Y \to Y$ be the two projections for $i = 1, 2$.

Definition 2.9. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Its *relative r-jet bundle* is defined by

$$J^r_{Y/S}\mathcal{F} := (p_1)_*\big(\mathcal{O}_{Y\times_S Y}/\mathcal{I}^{r+1} \otimes p_2^*\mathcal{F}\big).$$

The two projections p_i give two morphisms of sheaves of rings $\mathcal{O}_Y \to J^r_{Y/S}\mathcal{O}_Y$. We choose the one given by p_1 to define an $\mathcal{O}_Y$-module structure. The other one, induced by p_2, is denoted by

$$d^r_{Y/S} : \mathcal{O}_Y \to J^r_{Y/S}\mathcal{O}_Y$$

and is called the *jet map*. In particular $J^r_{Y/S}\mathcal{F} = J^r_{Y/S}\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$ where $J^r_{Y/S}\mathcal{O}_Y$ is seen as a right $\mathcal{O}_Y$-module via $d^r_{Y/S}$. We will also write $d^r_{Y/S} : \mathcal{F} \to J^r_{Y/S}\mathcal{F}$ for the tensor of the jet map with $\mathcal{F}$.

The fibre of the jet bundle at a closed point $y \in Y$ with maximal ideal sheaf $\mathfrak{m}$ is $(J^r_{Y/S}\mathcal{F})|_y \cong \mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F}$. Intuitively, the jet map should be thought of as taking the r-th Taylor expansion of a function. In particular, as the Leibniz rule for differentiation shows, it is not a morphism of $\mathcal{O}_Y$-modules when $r > 0$. On the contrary, taking the derivative of a function commutes with multiplication by a constant. At the level of the relative jet bundles, the functions on S play the role of the scalars, and this fact is expressed by the following:

Lemma 2.10. *The pushforward of the jet map*

$$f_* d^r_{Y/S} : f_*\mathcal{F} \longrightarrow f_* J^r_{Y/S}\mathcal{F}$$

is a morphism of $\mathcal{O}_S$*-modules.*

Proof. The claim can be checked locally on an affine cover, where it follows from a simple computation. □

Definition 2.11. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. The *fibrewise jet evaluation map* is the composition of the pushforward of the jet map followed by the counit:

$$f^* f_*\mathcal{F} \longrightarrow f^* f_* J^r_{Y/S}\mathcal{F} \longrightarrow J^r_{Y/S}\mathcal{F}.$$

To explain the definition above, we assume that Y and S are complex varieties for the rest of this section. As we will alternate between two points of view on vector bundles (as sheaves or as spaces over the base), it will sometimes be convenient to be explicit about which viewpoint is adopted:

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Definition 2.12. Let $\mathcal{F}$ be a vector bundle (i.e., a locally free sheaf of $\mathcal{O}_Y$-modules of finite rank) on a complex variety Y. The total space of the associated *geometric vector bundle* is

$$\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}_Y\left(\mathrm{Sym}^*(\mathcal{F}^\vee)\right)^{\mathrm{an}}$$

where $(-)^{\mathrm{an}}$ denotes the analytification functor.

Suppose now that $\mathcal{F}$ is a vector bundle on Y such that $f_*\mathcal{F}$ is also a vector bundle on S. As sets, we have an identification

$$\mathbb{V}(f_*\mathcal{F}) = \{(s,\sigma) \mid s \in S,\ \sigma \in H^0(Y_s; \mathcal{F}|_{Y_s})\}$$

where $Y_s = f^{-1}(s) \subset Y$ is the fibre above s. In particular, when $S = \mathrm{Spec}\,\mathbb{C}$ is a point, this is the space of global sections $H^0(Y;\mathcal{F})$. In general, it should be thought of as a space of fibrewise sections.

Lemma 2.13. *Under the above assumptions, the counit map $f^*f_*\mathcal{F} \to \mathcal{F}$ induces the evaluation map*

$$\begin{aligned}\mathbb{V}(f^*f_*\mathcal{F}) \cong \mathbb{V}(f_*\mathcal{F}) \times_S Y &\longrightarrow \mathbb{V}(\mathcal{F})\\ ((s,\sigma),y) &\longmapsto \sigma(y)\end{aligned}$$

on geometric realisation.

Proof. This follows by unpacking the definitions. □

Summarising the situation, we see that the fibrewise jet evaluation map

$$\mathbb{V}(f_*\mathcal{F}) \times_S Y \longrightarrow \mathbb{V}(J^r_{Y/S}\mathcal{F})$$

is given above a point $s \in S$ by

$$\begin{aligned}H^0(Y_s; \mathcal{F}|_{Y_s}) \times Y &\longrightarrow \mathbb{V}(J^r\mathcal{F}|_{Y_s})\\ (\sigma, y) &\longmapsto j^r\sigma(y) \in \mathcal{F}|_{Y_s}\big/\mathfrak{m}^{r+1}\mathcal{F}|_{Y_s}\end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal sheaf of $y \in Y_s$. In other words, it takes the Taylor expansion of σ at y up to order r.

2.3. Jet ampleness

Having now defined jet bundles, we state the crucial definition of jet ampleness of a line bundle on a smooth projective complex variety X.

Definition 2.14 (Compare [2]). Let $k \geq 0$ be an integer. Let $x_1, \ldots, x_t$ be t distinct points in X and $(k_1, \ldots, k_t)$ be a t-uple of positive integers such that $\sum_i k_i = k+1$. Denote by $\mathcal{O}_X$ the structure sheaf of X and by $\mathfrak{m}_i$ the maximal ideal sheaf corresponding to x_i. We regard the tensor product $\otimes_{i=1}^t \mathfrak{m}_i^{k_i}$ as a subsheaf of $\mathcal{O}_X$ under the multiplication map $\otimes_{i=1}^t \mathfrak{m}_i^{k_i} \to \mathcal{O}_X$. We say that a line bundle $\mathcal{L}$ is *k-jet ample* if the evaluation map

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma\left(\mathcal{L} \otimes \left(\mathcal{O}_X/\otimes_{i=1}^t \mathfrak{m}_i^{k_i}\right)\right) \cong \bigoplus_{i=1}^t \Gamma\left(\mathcal{L} \otimes \left(\mathcal{O}_X/\mathfrak{m}_i^{k_i}\right)\right)$$

is surjective for any $x_1, \ldots, x_t$ and $k_1, \ldots, k_t$ as above.

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Example 2.15. Being 0-jet ample corresponds to being spanned by global sections. Furthermore, 1-jet ampleness is the usual notion of very ampleness. On a curve, a line bundle is k-jet ample whenever it is k-very ample. However, on higher dimensional varieties, a k-jet ample line bundle is also k-very ample but the converse is not true in general.

The following proposition is the main tool to produce line bundles having a very high degree of jet ampleness.

Proposition 2.16 (See [2, Proposition 2.3]). *If $\mathcal{A}$ and $\mathcal{B}$ are respectively a- and b-jet ample line bundles, then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is $(a+b)$-jet ample.*

Definition 2.17. Let X be a smooth projective complex variety and $\alpha \in H^2(X;\mathbb{Z})$. We write $d(X,\alpha)$ for the largest integer $d \geq -1$ such that all line bundles on X with first Chern class equal to α are d-jet ample. (By convention, we declare that being (-1)-jet ample is an empty condition.)

We refer to Appendix A for how to compute $d(X,\alpha)$ in some special cases. Given an integer d, we also explain in Proposition A.7 how to find an α such that $d(X,\alpha) \geq d$.

2.4. The topology of hypersurfaces

It is well known that all smooth degree d complex hypersurfaces in $\mathbb{P}^n$ are diffeomorphic. As a way of justifying the study of the moduli space of hypersurfaces of a given Chern class, we observe that such hypersurfaces are also all diffeomorphic, provided the Chern class is ample enough. First, recall that ampleness is a numerical property:

Theorem 2.18 (Nakai–Moishezon criterion). *A line bundle $\mathcal{L}$ on a proper scheme over a field is ample if and only if $\int_Y c_1(\mathcal{L})^{\dim Y} > 0$ for every integral subscheme $Y \subset X$.*

Definition 2.19. A Chern class $\alpha \in \mathrm{NS}(X)$ is called *ample* if it satisfies the Nakai–Moishezon criterion.

We recall the following classical definition which is central in our work:

Definition 2.20. A global section $s \in \Gamma(X,\mathcal{L})$ of a line bundle $\mathcal{L}$ on a smooth projective complex variety X is called *nonsingular* if for all $x \in X$ we have $(s(x), ds(x)) \neq 0$.

Remark 2.21. Given a nonsingular section $s \in \Gamma(X,\mathcal{L})$, its vanishing locus

$$V(s) := \{x \in X \mid s(x) = 0\} \subset X$$

is a smooth hypersurface.

Any hypersurface H can be seen as a Weil divisor, hence a Cartier divisor (X is smooth), and therefore has an attached line bundle $\mathcal{O}_X(H)$. If $H = V(s)$ with $s \in \Gamma(X,\mathcal{L})$, then $\mathcal{O}_X(H) \cong \mathcal{L}$. The following bit of language will be convenient:

Definition 2.22. The *Chern class of a hypersurface H* is the first Chern class of its associated line bundle $c_1(\mathcal{O}_X(H))$.

Proposition 2.23. *Let X be a smooth projective complex variety with canonical sheaf K_X. Let $\alpha \in \mathrm{NS}(X)$ be a Chern class ample enough such that:*

1. *the class $\alpha - c_1(K_X)$ is ample;*
2. *for any line bundle $\mathcal{L}$ of Chern class α, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ consisting of the nonsingular global sections is nonempty.*

Then all the smooth hypersurfaces of Chern class α are diffeomorphic to one another.

Remark 2.24. Let us make some remarks on the two assumptions of the proposition above. Let $\mathcal{L}$ be a very ample line bundle on X. Then $K_X^{-1} \otimes \mathcal{L}^{\otimes k}$ is very ample for $k \gg 0$ big enough, and $\alpha = c_1(\mathcal{L}^{\otimes k})$

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satisfies the first assumption. Furthermore, by Bertini theorem, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ is dense. The second assumption is thus satisfied as soon as all line bundles of Chern class α are very ample. We explain how to arrange this in Appendix A.

Proof. The proof is an adaptation in families of the classical proof for a single linear system. Let $\mathrm{Pic}^{\alpha}(X)$ be the connected component of the Picard space classifying isomorphism classes of line bundles of Chern class α, and let $\mathcal{P}$ be a Poincaré line bundle on $\mathrm{Pic}^{\alpha}(X) \times X$. For $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$, we write $\mathcal{P}_{[\mathcal{L}]}$ for the line bundle on X which represents the isomorphism class $[\mathcal{L}]$. Let $p\colon \mathrm{Pic}^{\alpha}(X) \times X \to \mathrm{Pic}^{\alpha}(X)$ be the projection. By cohomology and base change [20, Theorem III.12.11], the sheaf $p_*\mathcal{P}$ is a vector bundle provided that $H^1(X, \mathcal{P}_{[\mathcal{L}]}) = 0$ for all $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$. This follows by the Kodaira vanishing theorem and the assumption that $\alpha - c_1(K_X)$ is ample. Let

$$\mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}} \subset \mathbb{V}(p_*\mathcal{P}) = \{([\mathcal{L}], s) \mid [\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X),\ s \in \Gamma(\mathcal{P}_{[\mathcal{L}]})\}$$

be the subset of those sections that are nonsingular. It is the complement of the (fibrewise) discriminant which has complex codimension at least 1 by our second assumption on the ampleness of α. Hence $\mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}}$ is connected. The incidence variety

$$\{([\mathcal{L}], s, x) \in \mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}} \times X \mid s(x) = 0\} \longrightarrow \mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}}$$

is a smooth fibre bundle by Ehresmann's lemma. The base is connected and therefore all the fibres are diffeomorphic. □

3. The moduli of hypersurfaces

In this section, we precisely define our main object of interest in this paper: the moduli of smooth hypersurfaces. From now on, we adopt the following conventions:

- X is a connected smooth complex projective variety;
- $\mathrm{Pic}(X)$ is its associated Picard scheme or space (see Definition 2.2);
- $p\colon \mathrm{Pic}(X) \times X \to \mathrm{Pic}(X)$ is the first projection;
- $\mathcal{P}$ is a choice, once and for all, of a Poincaré line bundle (see Definition 2.5);
- if $\mathcal{L}$ is a line bundle on X, we write $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ for the subspace of nonsingular sections.

3.1. Moduli functors and the Hilbert scheme

In this subsection, we define the moduli functor of smooth hypersurfaces in X and show that it is representable by an open subscheme of the Hilbert scheme of X. Under further assumptions on the ampleness of the first Chern class, we give an explicit construction of that scheme as an open subscheme of a projective bundle over the Picard scheme. Although the techniques we will use are known to algebraic geometers working with moduli, we have chosen to include those results for two main reasons: firstly to convince topologists that our space $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$ (introduced in the introduction, and properly defined in this section) really is a moduli space with a universal property, and secondly to uncover precisely which assumptions on the first Chern class α are needed to construct the moduli space as a subspace of a projective bundle.

The reader only interested in $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$ as a mere topological space can jump to the next subsection where we provide a point set model for the analytification of the moduli scheme, which will be thereafter used throughout the article.

In algebraic geometry, families of hypersurfaces are more commonly known as *relative effective Cartier divisors*. (See (33, Tag 056P).) Following [22, Part 3], we recall the definition of their moduli functor:

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Definition 3.1. Let $P \in \mathbb{Q}[x]$ be a polynomial. The *moduli functor of effective divisors* with Hilbert polynomial P is the functor

$$\begin{aligned}\mathfrak{M}^P : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} &\longrightarrow \mathsf{Set}\\ T &\longmapsto \left\{ Z \subset X \times T \;\middle|\; \begin{array}{l} Z \to T \text{ is flat and proper and for all } t \in T \\ \text{the ideal sheaf } \mathcal{I}_{Z_t} \text{ is a line bundle and is} \\ \text{such that } \chi(X_t, \mathcal{I}_{Z_t}^{-1}(n)) = P(n) \end{array} \right\}\end{aligned}$$

We define $\mathfrak{M}^{\mathrm{sm},P} \subset \mathfrak{M}^P$ to be the subfunctor[1] where Z_t is furthermore required to be smooth over the residue field $\mathrm{Spec}(\kappa(t))$ at $t \in T$.

When $\mathrm{Pic}(X)$ is a discrete space, families of hypersurfaces are represented by the more classical linear systems of divisors. We give an example to orient the reader:

Example 3.2 (Compare [22, Definition 3.12 and Theorem 3.13]). Let $X = \mathbb{P}^n$ and let $P' \in \mathbb{Q}[x]$ be the Hilbert polynomial of a hypersurface of degree $d \geq 1$. Such a hypersurface has ideal sheaf $\mathcal{O}(-d)$ with Hilbert polynomial $P(m) = \chi(\mathbb{P}^n, \mathcal{O}(d+m)) = \chi(\mathbb{P}^n, \mathcal{O}(m+2d)) - P'(m+2d)$.[2] One can show that the Hilbert scheme is in this case the complete linear system

$$\mathrm{Hilb}^{P'}(\mathbb{P}^n) = |\mathcal{O}(d)| = \mathbb{P}\big(H^0(\mathbb{P}^n, \mathcal{O}(d))\big).$$

Therefore $\mathrm{Hilb}^{P'}(\mathbb{P}^n)(-) = \mathfrak{M}^P(-)$ is represented by a complex projective space of complex dimension $\dim_{\mathbb{C}} H^0(\mathbb{P}^n, \mathcal{O}(d)) - 1$. The subfunctor $\mathfrak{M}^{\mathrm{sm},P}$ is represented by the complement of the discriminant hypersurface.

In general, we have the following general representability result:

Theorem 3.3. *The subfunctor $\mathfrak{M}^P(-) \subset \mathrm{Hilb}(X)(-)$ is represented by a union of connected components of the Hilbert scheme* $\mathrm{Hilb}(X)$.

Proof. By [23, Theorem 1.13], $\mathfrak{M}^P(-)$ is represented by an open subscheme U of the Hilbert scheme such that the inclusion $U \subset \mathrm{Hilb}(X)$ is universally closed (this uses that X is smooth over $\mathrm{Spec}(\mathbb{C})$). As the Hilbert scheme is separated (because projective), U must be a union of connected components. □

Our goal is now to recall an explicit description of the scheme representing $\mathfrak{M}^P(-)$ under some ampleness conditions. For the remainder of this section, we fix a polynomial $P \in \mathbb{Q}[x]$, denote by $\mathrm{Pic}^P(X) \subset \mathrm{Pic}(X)$ the subscheme defined in Proposition 2.6, and still write $\mathcal{P}$ for the restriction of the chosen Poincaré line bundle to it. Recall that $p\colon \mathrm{Pic}^P(X) \times X \to \mathrm{Pic}^P(X)$ denotes the first projection. Let

$$\mathcal{K} := \ker\big(p^* p_* \mathcal{P} \to \mathcal{P}\big)$$

be the kernel sheaf of the counit of $p^* \dashv p_*$. From the monoidality of p^* and standard properties of the relative Proj construction, we have an isomorphism

$$\underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\, p^*(p_*\mathcal{P})^\vee\big) \cong \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \times X.$$

Using the surjection of sheaves $p^*(p_*\mathcal{P})^\vee \twoheadrightarrow \mathcal{K}^\vee$ we thus obtain a closed immersion

$$\underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\, \mathcal{K}^\vee\big) \subset \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \times X.$$

[1] Recall that smoothness is preserved under base change.

[2] The difference between the two Hilbert polynomials is an artefact of our conventions, which we have chosen to be consistent with the existing literature and the Abel–Jacobi morphism.

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Definition 3.4. We define the *universal family* to be the morphism

$$\mathcal{U} := \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\,\mathcal{K}^\vee\big) \longrightarrow \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) =: \mathcal{M}$$

obtained by projecting onto the first coordinate.

Theorem 3.5. *Assume that $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$. Then the universal family $\mathcal{U} \to \mathcal{M}$ represents the moduli functor $\mathfrak{M}^P(-)$.*

Proof. This is explained in [4, Proposition 8.2.7]. To translate to the notation in that book: take f to be the structure morphism $X \to \mathrm{Spec}(\mathbb{C})$, T to be $\mathrm{Pic}^P(X)$, and $\mathcal{L}$ to be $\mathcal{P}$. The flatness assumption on $\mathcal{L}$ is implied by our assumption using cohomology and base change [20, Theorem III.12.11]. □

Definition 3.6. The morphism of functors

$$\mathfrak{M}^P(T) \longrightarrow \mathrm{Pic}^P(X)(T), \quad (Z \subset X \times T) \longmapsto [\mathcal{I}_Z^{-1}]$$

is represented by the projection morphism

$$\mathcal{M} = \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \longrightarrow \mathrm{Pic}^P(X)$$

which is usually called the *Abel–Jacobi* morphism.

We now explain how to obtain a scheme representing $\mathfrak{M}^{\mathrm{sm},P}(-)$ by removing the discriminant locus from $\mathcal{M}$. We assume that the evaluation morphism $p^*p_*\mathcal{P} \to \mathcal{P}$ is surjective. Recall the jet evaluation morphism from Definition 2.11 and denote by $\mathcal{Q}$ the cokernel of its dual morphism:

$$\mathcal{Q} := \mathrm{coker}\Big(\big(J^1_{\mathrm{Pic}^P(X)\times X/\mathrm{Pic}^P(X)}\mathcal{P}\big)^\vee \longrightarrow p^*(p_*\mathcal{P})^\vee\Big).$$

The composition

$$\mathcal{P}^\vee \longrightarrow p^*(p_*\mathcal{P})^\vee \longrightarrow \mathcal{Q}$$

of the dual of the evaluation morphism followed by the projection, is the zero morphism. Indeed it factors through $(J^1\mathcal{P})^\vee$ via the dual of the projection morphism from the first jet bundle onto the zeroth jet bundle $J^0\mathcal{P} = \mathcal{P}$. We thus obtain an induced surjective morphism $\mathcal{K}^\vee \twoheadrightarrow \mathcal{Q}$.

Theorem 3.7. *Assume that the counit morphism $p^*p_*\mathcal{P} \to \mathcal{P}$ is surjective, and that $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$. Then the moduli functor $\mathfrak{M}^{\mathrm{sm},P}(-)$ is represented by an open subscheme of $\mathcal{M}$, hence of the Hilbert scheme of X. More precisely, let $\pi\colon \mathcal{U} \to \mathcal{M}$ be the universal family and*

$$\mathcal{Z} := \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\,\mathcal{Q}\big) \hookrightarrow \mathcal{U}$$

be the closed subscheme determined by the surjection $\mathcal{K}^\vee \twoheadrightarrow \mathcal{Q}$. Then $\mathfrak{M}^{\mathrm{sm},P}(-)$ is represented by $\mathcal{M} \setminus \pi(\mathcal{Z})$.

Remark 3.8. The sheaf $\mathcal{Q}$ is only a coherent sheaf of $\mathcal{O}_{\mathrm{Pic}^P(X)\times X}$-modules and $\mathcal{Z}$ is therefore not a vector bundle in general. Nonetheless, if we furthermore assume that the jet evaluation morphism is surjective (e.g., if all line bundles parameterised by $\mathrm{Pic}^P(X)$ are very ample), then $\mathcal{Q}$ is the dual of the kernel of a surjective morphism of locally free sheaves, hence itself locally free.

Proof. Smoothness is an open condition: the projection π is flat and of finite presentation, so (33, Tag 01V9) applies and $\mathfrak{M}^{\mathrm{sm},P}(-)$ is seen to be represented by an open subscheme of $\mathcal{M}$. The subscheme $\mathcal{M} \setminus \pi(\mathcal{Z}) \subset \mathcal{M}$ is open because π is proper. It represents the moduli functor as smoothness can be checked locally using the Jacobian criterion. □

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We close this section with some general remarks about our assumptions in Theorem 3.7. We show that, although stated scheme-theoretically, they can be checked after analytification.

Lemma 3.9. *Let $\mathcal{P}^{\mathrm{an}}$ denote the analytic sheaf associated to $\mathcal{P}$. If the counit morphism $p^*p_*\mathcal{P}^{\mathrm{an}} \to \mathcal{P}^{\mathrm{an}}$ is surjective, then the same is true before analytification.*

Proof. This follows from exactness of the analytification of sheaves functor. □

Lemma 3.10. *Let $P \in \mathbb{Q}[x]$ be a polynomial and suppose that $H^1(X, \mathcal{L}) = 0$ for all $[\mathcal{L}] \in \mathrm{Pic}^P(X)(\mathbb{C})$. Then $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$.*

Proof. By upper semicontinuity of cohomology, the subscheme

$$\left\{t \in \mathrm{Pic}^P(X) \mid H^1(X_t, \mathcal{P}_t) = 0\right\} \subset \mathrm{Pic}^P(X)$$

is open. Assume that its complement is nonempty. Then it contains a complex point: it is locally of finite type over $\mathrm{Spec}(\mathbb{C})$ and Hilbert's Nullstellensatz applies. This cannot be the case by assumption. □

We finally comment on the relation between Hilbert polynomials and Chern classes. For a holomorphic line bundle $\mathcal{L}$ on X, recall the Hirzebruch–Riemann–Roch theorem giving an equality

$$\chi(X, \mathcal{L}) = \int_X \mathrm{ch}(\mathcal{L})\,\mathrm{td}(X)$$

where $\mathrm{ch}(-)$ is the Chern character and $\mathrm{td}(X)$ is the Todd class of X. In particular, if $Z \subset X$ is an effective Cartier divisor, its Hilbert polynomial only depends on the first Chern class of its associated line bundle $\mathcal{O}_X(Z)$. We thus obtain a numerical criterion:

Lemma 3.11. *Let $P \in \mathbb{Q}[x]$ and let C be the collection of Chern classes*

$$C := \left\{c_1(\mathcal{L}) \mid [\mathcal{L}] \in \mathrm{Pic}^P(X)(\mathbb{C})\right\}.$$

If $\alpha - c_1(K_X)$ is ample for all $\alpha \in C$, then $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$.

Proof. This follows from Lemma 3.10 whose assumption is verified by the Kodaira vanishing theorem. □

3.2. A convenient point-set model

In this section, we unravel the result of Theorem 3.7 and give an explicit point-set model for the moduli space of smooth hypersurfaces.

We begin with notations which we will use throughout the rest of the article. If $[\mathcal{L}] \in \mathrm{Pic}(X)$ is an isomorphism class of a line bundle, we write $\mathcal{P}_{[\mathcal{L}]}$ for the representative of that isomorphism class given by the restriction of $\mathcal{P}$ to $X \cong \{[\mathcal{L}]\} \times X \subset \mathrm{Pic}(X) \times X$. If $\alpha \in \mathrm{NS}(X)$ we recall from Proposition 2.8 that $\mathrm{Pic}^\alpha(X) \subset \mathrm{Pic}(X)$ denotes the connected component parameterising line bundles of Chern class α, and we will write $\mathcal{P}_\alpha$ for the restriction of the Poincaré line bundle to that component.

Definition 3.12. The first jet bundle of $\mathcal{P}$ relative to the projection p (see Definition 2.9) is denoted

$$J^1_p\mathcal{P} := J^1_{\mathrm{Pic}(X)\times X/\mathrm{Pic}(X)}\mathcal{P}.$$

When restricted to $\mathrm{Pic}^\alpha(X)$ for some $\alpha \in \mathrm{NS}(X)$, we will write $J^1_p\mathcal{P}_\alpha := J^1_{\mathrm{Pic}^\alpha(X)\times X/\mathrm{Pic}^\alpha(X)}\mathcal{P}_\alpha$.

Lemma 3.13. *Let K_X be the canonical sheaf of X. Let $\alpha \in \mathrm{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. Then $p_*\mathcal{P}_\alpha$ is a vector bundle and the fibrewise jet evaluation map gives a map of vector bundles*

$$p^*p_*\mathcal{P}_\alpha \longrightarrow J^1_p\mathcal{P}_\alpha.$$

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As sets, the geometric realisations are given by

$$\mathbb{V}(p^*p_*\mathcal{P}_\alpha) = \mathbb{V}(p_*\mathcal{P}_\alpha) \times X = \left\{([\mathcal{L}], x, s) \mid [\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ x \in X,\ s \in \Gamma(\mathcal{P}_{[\mathcal{L}]})\right\}$$

and

$$\mathbb{V}(J^1_p\mathcal{P}_\alpha) = \left\{([\mathcal{L}], x, v) \mid [\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ x \in X,\ v \in J^1\mathcal{P}_{[\mathcal{L}]}|_x\right\}.$$

Under these identifications, the jet evaluation map is given by

$$\mathrm{jev}\colon ([\mathcal{L}], x, s) \longmapsto ([\mathcal{L}], x, j^1 s(x)).$$

Proof. The fact that $p_*\mathcal{P}_\alpha$ is a vector bundle follows directly from cohomology and base change and the Kodaira vanishing theorem under the assumption that $\alpha - c_1(K_X)$ is ample. The rest of the lemma follows from the results recalled in Section 2.2. □

Recall from Theorem 3.7 the scheme $\mathcal{M} \setminus \pi(\mathcal{Z})$ representing the moduli functor of smooth hypersurfaces. After analytification, we may restrict the Abel–Jacobi map

$$\left(\mathcal{M} \setminus \pi(\mathcal{Z})\right)^{\mathrm{an}} \longrightarrow \mathrm{Pic}(X)^{\mathrm{an}} \tag{3}$$

to the connected component $\mathrm{Pic}^\alpha(X) \subset \mathrm{Pic}(X)$ (we will from now on drop the superscript "an"), recalled in Proposition 2.8, provided that α is ample enough:

Definition 3.14. Let $\alpha \in \mathrm{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. The *moduli of smooth hypersurfaces in X of Chern class* α is defined to be the preimage of $\mathrm{Pic}^\alpha(X)$ under the Abel–Jacobi map (3). In other terms, we have a homeomorphism:

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \cong \left(\mathbb{V}(p_*\mathcal{P}_\alpha) \setminus \mathrm{proj}(\mathrm{jev}^{-1}(0))\right)/\mathbb{C}^\times$$

where the scalars act fibrewise over $\mathrm{Pic}(X)$, and $\mathrm{proj}\colon \mathbb{V}(p^*p_*\mathcal{P}) \to \mathbb{V}(p_*\mathcal{P})$ is the projection induced by p.

Remark 3.15. As sets, we have an identification

$$\mathcal{M}^\alpha_{\mathrm{hyp}} = \{([\mathcal{L}], [s]) \mid [\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ [s] \in \Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^\times\}.$$

That is, it will be technically convenient to think of a smooth hypersurface as a tuple consisting of a line bundle $\mathcal{L}$ and a nonsingular global section of it (up to isomorphism and scaling action). However, the name of moduli space is justified by the previous section: we have a homeomorphism

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \cong \{Z \subset X \text{ smooth hypersurface with } c_1(\mathcal{O}_X(Z)) = \alpha\} \subset \mathrm{Hilb}(X)^{\mathrm{an}}.$$

Remark 3.16. If $\alpha - c_1(K_X)$ is ample and $d(X, \alpha) \geq 0$, then by Theorem 3.7 the space $\mathcal{M}^\alpha_{\mathrm{hyp}}$ is a Zariski open subset of a projective bundle on the connected space $\mathrm{Pic}^\alpha(X)$, hence itself connected. This is the only situation we consider in this paper.

4. Statement of the main theorem

In this section, we construct a topological counterpart to the moduli of smooth hypersurfaces described in Definition 3.14. We then state our main theorem comparing the two objects.

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4.1. A topological counterpart

We begin with some generalities about the topology of continuous section spaces. Let $E \to A \times B$ be a fibre bundle on a topological space $A \times B$. We denote a point of E as a tuple (a, b, e) where $a \in A$, $b \in B$ and $e \in E|_{(a,b)}$ is in the fibre. All mapping spaces are given the compact open topology.

Definition 4.1. The space of *fibrewise sections* of E over A is defined to be the subspace

$$\begin{aligned}\Gamma_{\mathcal{C}^0,\mathrm{fib}}(E \to A) := \{(a, s) \mid a \in A,\ s \in \Gamma_{\mathcal{C}^0}(E|_{a\times B})\} &\hookrightarrow \mathrm{Map}(B, E)\\ (a, s) &\mapsto [b \mapsto (a, b, s(b))].\end{aligned}$$

Postcomposition with the projection maps $E \to A \times B \to A$ gives a continuous map $\mathrm{Map}(B, E) \to \mathrm{Map}(B, A)$ which, when restricted to fibrewise sections, yields the projection map

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(E \to A) \longrightarrow A, \quad (a, s) \longmapsto a.$$

In particular, this projection map is continuous.

Remark 4.2. Let Z be a topological space. A continuous map $Z \to \Gamma_{\mathcal{C}^0,\mathrm{fib}}(E \to A)$ is the same datum as a continuous map $f\colon Z \times B \to E$ over B such that $\mathrm{proj} \circ f(z, -)\colon B \to E \to A$ is constant for any $z \in Z$.

Remark 4.3. When A and B are smooth projective complex varieties, one can modify the definition above by using the spaces of holomorphic maps instead of the whole mapping spaces. In fact, assume that $E = \mathbb{V}(\mathcal{E})$ is also a vector bundle and that $\pi_*\mathcal{E}$ is a vector bundle, where $\pi\colon A \times B \to A$ is the first projection. Then the holomorphic fibrewise section space is exactly $\mathbb{V}(\pi_*\mathcal{E})$. For good measure, let us notice that there are a priori two natural topologies on holomorphic mapping spaces: the analytic topology inherited from the Hom scheme, and the compact open topology from the continuous mapping space. Fortunately they coincide, as shown (in much greater generality) by Douady in [8].

To lighten the notation, and as no confusion can arise, we will from now on drop the symbol $\mathbb{V}(-)$ when considering continuous sections of a vector bundle.

Definition 4.4. Taking fibrewise, over $\mathrm{Pic}(X)$, continuous global sections of $J^1_p\mathcal{P}$ which are never vanishing, we obtain the space

$$\begin{aligned}\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P} \setminus 0) &:= \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P} \setminus 0 \to \mathrm{Pic}(X))\\ &= \left\{([\mathcal{L}], s) \mid [\mathcal{L}] \in \mathrm{Pic}(X),\ s \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}]} \setminus 0)\right\}.\end{aligned}$$

The group $\mathbb{C}^\times$ acts by multiplying the sections by scalars and we let $\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P} \setminus 0)/\mathbb{C}^\times$ be the quotient for that action.

4.2. The main theorem

By Remark 4.3, the fibrewise jet map followed by the inclusion of the space of holomorphic sections inside continuous sections gives rise to a continuous map

$$j^1\colon \mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times.$$

Denote by $\mathbb{P}(J^1\mathcal{O}_X)$ the projectivisation of the first jet bundle of $\mathcal{O}_X$ on X. We will make use of the following:

Proposition 4.5 (Compare [5, Lemma 2.5 and Section 3 p. 73]). *The connected components of the section space $\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ are in one-to-one correspondence with $H^2(X;\mathbb{Z})$. For a given Chern class α, the associated connected component $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ consists of those sections s such that the pullback $s^*\mathcal{O}(1)$ of the tautological bundle has Chern class α.*

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It follows from the proposition that if $\mathcal{L}$ is a line bundle with Chern class α, the quotient map

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0) \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{L})) \cong \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

has image inside the connected component $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. Here we have used that $J^1\mathcal{L} \cong J^1\mathcal{O}_X \otimes \mathcal{L}$ and the fact that the projectivisation of a vector bundle is invariant under tensoring with a line bundle.

Now, let $\mathcal{L}_0$ be a chosen line bundle with Chern class α. By choosing an isomorphism of topological line bundles $\mathcal{L}_0 \cong \mathcal{P}_{[\mathcal{L}]}$ for each $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$, we obtain a map

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{L}_0)) \cong \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \tag{4}$$

which factors through $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. As any two choices of isomorphisms $\mathcal{L}_0 \cong \mathcal{P}_{[\mathcal{L}]}$ differ by a nonzero constant, we see that the map is indeed uniquely well-defined and continuous. The following is our main result:

Theorem 4.6. *Let X be a smooth projective complex variety. Let $\alpha \in \mathrm{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. The jet map*

$$j^1 \colon \mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d(X,\alpha)-3}{2}$. (See Definition 2.17.)

5. Proof of the main theorem

The proof of the main theorem is executed in two steps. In Section 5.1, we first prove:

Proposition 5.1. *Let X and α be as in Theorem 4.6. The jet map*

$$j^1 \colon \mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d(X,\alpha)-3}{2}$.

Then, in Section 5.2, we show the following:

Proposition 5.2. *The map defined in (4)*

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

is a weak homotopy equivalence.

5.1. The homology isomorphism

The jet map fits in the following diagram where the top row is its restriction to a fibre above an $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$:

$$\begin{array}{ccc}
\Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^\times & \longrightarrow & \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}]} \setminus 0)/\mathbb{C}^\times \\
\downarrow & & \downarrow \\
\mathcal{M}^{\alpha}_{\mathrm{hyp}} & \longrightarrow & \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \\
\downarrow & & \downarrow \\
\mathrm{Pic}^{\alpha}(X) & = & \mathrm{Pic}^{\alpha}(X)
\end{array}$$

The uppermost map was studied in [1] where the following result was proved:

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Theorem 5.3 (Compare [1, Corollary 8.1]). *Let $\mathcal{L}$ be a d-jet ample line bundle on a smooth projective complex variety X. Then the jet map*

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times$$

induces an isomorphism in homology in the range of degrees $* < \frac{d-1}{2}$.

Now, if both lower vertical maps were fibrations, a comparison of the associated Serre spectral sequences would prove Proposition 5.1. This is indeed the case for the map on the right-hand side. The other map is only a microfibration, which turns out to be sufficient for the argument to go through. We start by reviewing this technical notion popularised by Weiss in [36].

Definition 5.4. A map $\pi\colon E \to B$ is called a Serre microfibration if for any $k \geq 0$ and any commutative diagram

$$\begin{array}{ccc} \{0\} \times D^k & \xrightarrow{u} & E \\ \downarrow & & \downarrow \pi \\ [0,1] \times D^k & \xrightarrow[v]{} & B \end{array}$$

there exists an $\varepsilon > 0$ and a map $h\colon [0,\varepsilon] \times D^k \to E$ such that $h(0,x) = u(x)$ and $\pi \circ h(t,x) = v(t,x)$ for all $x \in D^k$ and $t \in [0,\varepsilon]$.

Remark 5.5. Any Serre fibration is a microfibration. More generally, the restriction of a Serre fibration to an open subspace of the total space is a microfibration.

Contrary to the case of fibrations, the homotopy types of the fibres of a microfibration can vary. Nonetheless, we have the very useful comparison theorem of Raptis generalising a result of Weiss:

Theorem 5.6 (Compare [29, Theorem 1.3]). *Let $p\colon E \to B$ be a Serre microfibration, $q\colon V \to B$ be a Serre fibration, and $f\colon E \to V$ a map over B. Suppose that $f_b\colon p^{-1}(b) \to q^{-1}(b)$ is n-connected for some $n \geq 1$ and for all $b \in B$. Then the map $f\colon E \to V$ is n-connected.*

In the present situation, we only have access to Theorem 5.3 which provides an isomorphism in homology, rather than on homotopy groups. The remedy chosen here is to suspend the spaces so as to obtain simply connected spaces and then apply the homology Whitehead theorem.

Definition 5.7. For a map $p\colon E \to B$, its fibrewise (unreduced) k^{th} suspension is defined to be

$$\Sigma^k_B E = \Big(E \times [0,1] \times S^{k-1}\Big) \Big/ \big((e,0,s) \sim (e,0,s') \text{ and } (e,1,s) \sim (e',1,s) \text{ if } p(e) = p(e')\big).$$

The fibre of the natural map $\Sigma^k_B p\colon \Sigma^k_B E \to B$ induced by p is the unreduced k^{th} suspension of the fibre of p (here modelled as the join with the sphere S^{k-1}):

$$(\Sigma^k_B p)^{-1}(b) = \Sigma^k p^{-1}(b), \quad \forall b \in B.$$

Lemma 5.8. *The map*

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \mathrm{Pic}^\alpha(X)$$

is a fibre bundle.

Proof. Let $U \subset \mathrm{Pic}^\alpha(X)$ be a small contractible open subset. A topological vector bundle being trivial over a contractible base, we obtain an isomorphism of vector bundles

$$\psi\colon J^1_p\mathcal{P}_\alpha|_{U\times X} \xrightarrow{\cong} U \times J^1\mathcal{P}_{[\mathcal{L}_0]}$$

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over $U \times X$, with $[\mathcal{L}_0] \in U$ a chosen basepoint. The map

$$\begin{aligned}\left(\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times\right)|_U &\longrightarrow U \times \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)\\ ([\mathcal{L}], s) &\longmapsto ([\mathcal{L}], \psi \circ s)\end{aligned}$$

is then a homeomorphism over U exhibiting the local triviality of the fibre bundle. □

We will say that a map $A \to B$ is homology m-connected if it induces an isomorphism on homology groups $H_i(A) \to H_i(B)$ for $i < m$ and a surjection when $i = m$.

Lemma 5.9. *Let $q\colon V \to B$ be a fibre bundle, and $p\colon U \to B$ be the restriction of a fibre bundle $E \to B$ to an open subset $U \subset E$. Let $f\colon U \to V$ be a map over B and suppose that for every $b \in B$, the restriction to the fibre*

$$f_b\colon p^{-1}(b) \longrightarrow q^{-1}(b)$$

is homology m-connected. Then $f\colon U \to V$ is homology m-connected.

Proof. For any $b \in B$, the suspension of f on the fibre

$$\Sigma^2 f_b\colon \Sigma^2 p^{-1}(b) \longrightarrow \Sigma^2 q^{-1}(b)$$

induces an isomorphism in homology in degrees $* \leq m+1$ and a surjective morphism in degree $* = m+2$. As both spaces are simply connected, the homology Whitehead theorem implies that this map is $(m+2)$-connected. We would like to apply Theorem 5.6 to $\Sigma^2_B f$, but $\Sigma^2_B U \subset \Sigma^2_B E$ is not open and it is unclear if $\Sigma^2_B U \to B$ is a microfibration. We resolve the issue by enlarging slightly the space to a homotopy equivalent one. More precisely, let

$$W = \left(\Sigma^2_B U\right) \cup \left((E \times (0.5, 1] \times S^1))/\sim\right) \subset \Sigma^2_B E,$$

and denote by E_b, W_b, U_b the fibres of the respective spaces above a point $b \in B$. Using in each fibre the homotopy equivalence $\left((E_b \times (0.5,1] \times S^1)/\sim\right) \simeq S^1$ given by collapsing gives a homotopy equivalence

$$(W, W_b) \xrightarrow{\simeq} (\Sigma^2_B U, \Sigma^2 U_b)$$

for all $b \in B$. Now, the fibrewise suspension of the fibre bundles $E \to B$ and $V \to B$ are fibre bundles. As $W \subset \Sigma^2_B E$ is open, the restriction $W \to B$ is a microfibration. Applying Theorem 5.6 to the composite

$$W \xrightarrow{\simeq} \Sigma^2_B U \xrightarrow{\Sigma^2_B f} \Sigma^2_B V$$

and using that the first map is a homotopy equivalence, we obtain that $\Sigma^2_B f\colon \Sigma^2_B U \to \Sigma^2_B V$ is $(m+2)$-connected. Hence it is homology $(m+2)$-connected. Comparing the Mayer–Vietoris sequences of the fibrewise suspensions finally shows that $f\colon U \to B$ is homology m-connected. □

Proof of Proposition 5.1. By Definition 3.14, the map $\mathcal{M}^\alpha_{\mathrm{hyp}} \to \mathrm{Pic}^\alpha(X)$ is the restriction of the projective bundle $\mathbb{P}(p_*\mathcal{P}_\alpha) \to \mathrm{Pic}^\alpha(X)$ to the open subset $\mathcal{M}^\alpha_{\mathrm{hyp}}$. Using Theorem 5.3 and Lemma 5.8, we can apply Lemma 5.9 to conclude. □

5.2. *The homotopy type of the space of fibrewise sections*

To prove that the map of Proposition 5.2 is a weak equivalence, we will extend it to a morphism between fibrations with equal total spaces, and then compare the homotopy fibres. (See (7) below for the precise

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diagram.) For this latter part, we shall need to work with convenient point-set models for the homotopy fibres, and thus we begin by some basic algebraic topology to fix the notation.

For a pointed space (A, a), we let $P(A, a) = \mathrm{Map}_*(([0,1], 0), (A, a))$ be the space of paths in A starting at a. We will write cst_* for the constant loop based at a point $*$. For a map of pointed spaces $\pi\colon (E, e_0) \to (B, b_0)$, we write

$$H\pi := \{(e, \gamma) \in E \times P(B, b_0) \mid \gamma(1) = \pi(e)\}$$

for its homotopy fibre.

5.2.1. The homotopy fibre of a homotopy fibre
Let $\pi\colon (E, e_0) \to (B, b_0)$ be a fibration between pointed spaces, $F = \pi^{-1}(b_0)$ be the fibre, and $\Omega_{b_0}B$ be the loop space of B based at b_0. Writing $i\colon F \hookrightarrow E$ for the inclusion, it is well known that Hi and $\Omega_{b_0}B$ are homotopy equivalent:

Lemma 5.10. *Let $\gamma\colon [0,1] \to B$ be a loop based at b_0. Let $\alpha\colon [0,1] \to E$ be a lift of that loop starting at e_0. The map on connected components*

$$\begin{aligned} \pi_0(\Omega_{b_0}B) &\longrightarrow \pi_0(Hi) \\ [\gamma] &\longmapsto [(\alpha(1), \alpha)] \end{aligned}$$

is a bijection.

Proof. It follows from basic manipulations and [7, Note 4.7.1]. □

5.2.2. Homotopy fibre of a principal bundle
Assume now that $\pi\colon (E, e_0) \to (B, b_0)$ is a principal $G = \pi^{-1}(b_0)$-bundle. Let $\alpha\colon [0,1] \to B$ be a path from b_0 to a point b_1, and choose a point $e_1 \in \pi^{-1}(b_1)$. We may choose a lift of the path α to a path $\beta\colon [0,1] \to E$ such that $\pi \circ \beta = \alpha$, and define $e_1 = \beta(1)$. As the action of G on $\pi^{-1}(b_1)$ is free and transitive, there exists a unique $g_1 \in G$ such that $g_1 \cdot e_1 = e_1'$ (where $\cdot$ denotes the action).

Lemma 5.11. *We keep the notation as above. Then the points (e_1', α) and $(g_1 \cdot e_0, \mathrm{cst}_{e_0})$ are in the same connected component of the homotopy fibre $H\pi$.*

Proof. It follows from a direct verification. □

5.2.3. The proof of Proposition 5.2 the homotopy equivalence
For concreteness, we start by fixing basepoints. Let $[\mathcal{L}_0] \in \mathrm{Pic}^{\alpha}(X)$. We will write $\mathcal{P}_{[\mathcal{L}_0]} = \mathcal{L}_0$ for brevity. We also pick a section $s_0 \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)$. We will use these as basepoints, as well as the images $[s_0] \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times$ and $\mathbb{P}s_0 \in \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$.

Pointwise multiplication of maps gives $\mathrm{Map}(X, \mathbb{C}^\times)$ the structure of a topological group. By [5, Proposition 2.6], there is a principal $\mathrm{Map}(X, \mathbb{C}^\times)$-bundle:

$$\mathrm{Map}(X, \mathbb{C}^\times) \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0) \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)). \tag{5}$$

There is also the subgroup $\mathbb{C}^\times \subset \mathrm{Map}(X, \mathbb{C}^\times)$ of the constant functions, and modding out fibrewise gives a principal bundle:

$$\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)). \tag{6}$$

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We obtain a commutative diagram of pointed spaces where each row is a fibration sequence

$$\begin{array}{ccccc} F_1 & \longrightarrow & \big(\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times,\ [s_0]\big) & \longrightarrow & \Big(\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times,\ [s_0]\Big) \\ \downarrow & & \| & & \downarrow \\ F_2 & \longrightarrow & \big(\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times,\ [s_0]\big) & \longrightarrow & \Big(\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)),\ \mathbb{P}s_0\Big) \end{array} \tag{7}$$

and the spaces F_1 and F_2 are defined as the respective homotopy fibres. Using the 5-lemma and the long exact sequence of homotopy groups associated to a fibration, Proposition 5.2 follows directly from the next lemma.

Lemma 5.12. *Using the notation as above, the map induced on the homotopy fibres $F_1 \to F_2$ is a homotopy equivalence.*

Proof. We already know that $F_1 \simeq \Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)$ (cf. Lemma 5.8) and $F_2 \simeq \mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times$ (cf. (Eq. 6)), which are both homotopy equivalent to the discrete space $H^1(X;\mathbb{Z})$. Therefore we only need to verify that the map $F_1 \to F_2$ induces a bijection on the set of connected components. We have a diagram of sets

$$\begin{array}{ccc} \pi_0\big(\Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)\big) & \xrightarrow[\cong]{(1)} & \pi_0(F_1) \\ \vdots & & \downarrow (2) \\ \pi_0\big(\mathrm{Map}(X,\mathbb{C}^\times)/\mathbb{C}^\times\big) & \xrightarrow[(3)]{\cong} & \pi_0(F_2) \end{array}$$

where the right vertical map is induced from $F_1 \to F_2$, the top map (1) is explained in Lemma 5.10, the bottom map (3) is induced by the inclusion of the fibre inside the homotopy fibre, and the dotted arrow is defined by composition. It suffices to show that this last arrow is a bijection.

To do so, we go through the composition and use the explicit descriptions of the maps recalled in the last section. Let $\gamma \in \Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)$ be a loop. Using the fibre bundle of Lemma 5.8

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \mathrm{Pic}^\alpha(X)$$

we choose a lift to a path

$$\alpha\colon [0,1] \to \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times$$

starting at $[s_0]$ and ending at some $[s_1']$. The map (1) is given according to Lemma 5.10 by

$$\pi_0\big(\Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)\big) \longrightarrow \pi_0(F_1), \quad [\gamma] \longmapsto [([s_1'], \alpha)].$$

Write $\mathbb{P}\alpha$ for the image of α in $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. Applying (2) then gives

$$[([s_1'], \alpha)] \longmapsto [([s_1'], \mathbb{P}\alpha)].$$

Using the fibration (6)

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

we lift the path $\mathbb{P}\alpha$ to a path β in $\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}_0 \setminus 0)/\mathbb{C}^\times$ starting at $[s_0]$ and ending at some point $[s_1]$. Using the principal bundle structure of (6), there is a unique class of a map $[\varphi_1] \in \mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times$ such that

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$[\varphi_1 \cdot s_1] = [s_1']$. By Lemma 5.11

$$[([s_1'], \mathbb{P}\alpha)] = [([\varphi_1 \cdot s_1], \mathbb{P}\alpha)] = [([\varphi_1 \cdot s_0], \mathbb{P}\mathrm{cst}_{s_0})].$$

From the homotopy equivalence given by the inclusion of the strict fibre inside the homotopy fibre of the bundle (6):

$$\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times \xrightarrow{\simeq} F_2, \quad [\varphi] \longmapsto [([\varphi \cdot s_0], \mathbb{P}\mathrm{cst}_{s_0})]$$

we see that the inverse of the map (3) then sends

$$[([\varphi_1 \cdot s_0], \mathbb{P}\mathrm{cst}_{s_0})] \longmapsto [\varphi_1].$$

We see first that it is surjective: indeed the composition of (2) and the inverse of (3) is surjective as any $[\varphi]$ is the image of $[([\varphi \cdot s_0], \mathrm{cst}_{[s_0]})]$. Secondly we check that it is compatible with the group structures: on the source given by composition of loops, and on the target given by multiplication of maps. This is enough to finish the proof as both groups are isomorphic to $H^1(X; \mathbb{Z})$, so that any surjective morphism is in fact an isomorphism.

We check the compatibility with the group structures using the notations as above. Let $\gamma_1, \gamma_2 \in \Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)$ be two loops. As above, choose lifts α_1 and α_2 starting at $[s_0]$ and ending at $[s_1']$ and $[s_2']$ respectively. Then lift $\mathbb{P}\alpha_1$ and $\mathbb{P}\alpha_2$ to paths β_1, β_2 starting at $[s_0]$ and ending at $[s_1]$ and $[s_2]$ respectively. Write $[\varphi_1 \cdot s_1] = [s_1']$ and $[\varphi_2 \cdot s_2] = [s_2']$, for unique $[\varphi_1]$, $[\varphi_2]$. As we have shown above, the composition of (1), (2) and the inverse of (3) maps

$$[\gamma_1] \longmapsto [\varphi_1], \quad [\gamma_2] \longmapsto [\varphi_2].$$

Let us now consider the concatenation of loops $\gamma_1 * \gamma_2$. First, observe that there exists a unique $[\varphi]$ such that $[\varphi \cdot s_0] = [s_1']$. Therefore the loop $\gamma_1 * \gamma_2$ can be lifted to the path $\alpha_3 := \alpha_1 * (\varphi \cdot \alpha_2)$ ending at $[s_3'] := [\varphi \cdot s_2']$. Then we may lift $\mathbb{P}\alpha_3 := \mathbb{P}(\alpha_1 * (\varphi \cdot \alpha_2)) = \mathbb{P}(\alpha_1 * \alpha_2)$ to the path $\beta_1 * (\varphi_1^{-1} \cdot \varphi \cdot \beta_2)$ ending at $[s_3]$. We write $[\varphi_3 \cdot s_3] = [s_3']$ for a unique $[\varphi_3]$. Once again, the composition of (1), (2) and the inverse of (3) maps

$$[\gamma_1 * \gamma_2] \longmapsto [\varphi_3].$$

Now observe that, as $[s_3]$ is the end point of $\beta_1 * (\varphi_1^{-1} \cdot \varphi \cdot \beta_2)$, we have

$$[s_3] = [\varphi_1^{-1} \cdot \varphi \cdot s_2].$$

Therefore

$$[\varphi_3 \cdot s_3] = [\varphi_3 \cdot \varphi_1^{-1} \cdot \varphi \cdot s_2].$$

But $[\varphi_3 \cdot s_3] = [s_3'] = [\varphi \cdot s_2'] = [\varphi \cdot \varphi_2 \cdot s_2]$. By uniqueness

$$[\varphi_3 \cdot \varphi_1^{-1} \cdot \varphi] = [\varphi \cdot \varphi_2],$$

hence $[\varphi_3] = [\varphi_1 \cdot \varphi_2]$, which achieves the proof. □

6. Rational computations and stability

In this part, we show how Theorem 4.6 can be used to make explicit computations of the rational cohomology of $\mathcal{M}_{\mathrm{hyp}}^\alpha$. Assuming that the underlying variety X is topologically parallelisable, we will also exhibit a phenomenon of homological stability.

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We will first recall a general strategy, dating back to Haefliger [19], to compute the cohomology of continuous section spaces. In Theorem 6.5 below, we provide a commutative differential graded algebra (CDGA) computing the rational cohomology of the section space of the projective bundle. We hope that this will convince the reader that the homotopical approach taken in this paper may be useful in practical computations. We will freely use the notations and results from rational homotopy theory. A textbook account can be found in [9]. In particular, we follow an established tradition in the field and write $\Lambda(-)$ as a shorthand for the free graded commutative algebra $\mathrm{Sym}^*_{\mathrm{gr}}(-)$. We let n be the complex dimension of X.

6.1. Haefliger's tower of section spaces

Although Theorem 4.6 provides an integral homology isomorphism, we will mainly be interested in the rational cohomology groups for computational reasons. Fibrewise rationalisation (denoted $(-)_{f\mathbb{Q}}$) yields a fibration

$$\mathbb{P}^n_{\mathbb{Q}} \longrightarrow \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}} \longrightarrow X. \tag{8}$$

By [26, Theorem 5.3], the natural map $\mathbb{P}(J^1\mathcal{O}_X) \to \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}$ induces a map on section spaces

$$\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$$

which is a rationalisation when restricted to a connected component on the source and target. (Beware the connected components of the source are naturally indexed by $H^2(X;\mathbb{Z})$, while those of the target are naturally indexed by $H^2(X;\mathbb{Q})$.) We apply the general strategy described in [19, Section 1.3] to compute the rational homotopy type of the section space $\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$. The fibration (8) admits a Moore–Postnikov decomposition of the form

$$\begin{array}{ccccc}
K(\mathbb{Q},2n+1) & \longrightarrow & Y_2 \simeq \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}} & & \\
 & & \downarrow{\scriptstyle p_2} & & \\
K(\mathbb{Q},2) & \longrightarrow & Y_1 & \xrightarrow{k_1} & K(\mathbb{Q},2n+2) \\
 & & \downarrow{\scriptstyle p_1} & & \\
 & & Y_0 = X & \xrightarrow{k_0} & K(\mathbb{Q},3)
\end{array}$$

where each $p_i\colon Y_i \to Y_{i-1}$ is a principal fibration classified by the k-invariant k_{i-1}. The latter were computed by Møller:

Lemma 6.1 (Compare [27, Lemma 2.1]). *The k-invariant k_0 is trivial. In particular $Y_1 \simeq X \times K(\mathbb{Q},2)$. Writing $z \in H^2(K(\mathbb{Q},2);\mathbb{Q})$ for the generator, k_1 corresponds to the cohomology class*

$$\sum_{i=0}^{n+1} (-1)^i c_i(J^1\mathcal{O}_X) \otimes z^{n+1-i} \in H^*(X;\mathbb{Q}) \otimes H^*(K(\mathbb{Q},2);\mathbb{Q}).$$

Let $s \in \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$ with $\alpha \in H^2(X;\mathbb{Q})$. The map $p_2 \circ s$ is a section of $Y_1 \to X$, and we denote by $\Gamma_1 \subset \Gamma_{\mathcal{C}^0}(Y_1 \to X)$ its connected component. As k_0 is trivial by Lemma 6.1, there is a homotopy equivalence

$$\Gamma_{\mathcal{C}^0}(Y_1 \to X) \simeq \mathrm{Map}(X, K(\mathbb{Q},2)) \simeq K(\mathbb{Q},2) \times K(H^1(X;\mathbb{Q}),1) \times H^2(X;\mathbb{Q}).$$

and Γ_1 corresponds to the connected component indexed by α.

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Lemma 6.2 (Compare [27, Lemma 2.2]). *Let Ψ be the composite*

$$\Psi\colon K(\mathbb{Q},2)\times K(H^1(X;\mathbb{Q}),1)\times X\simeq \Gamma_1\times X\xrightarrow{\mathrm{ev}} Y_1\xrightarrow{k_1} K(\mathbb{Q},2n+2). \tag{9}$$

Let $z\in H^2(K(\mathbb{Q},2);\mathbb{Q})$ be the generator. Let $\{x_j\}$ be a basis of $H^1(X;\mathbb{Z})$ and let $\{x'_j\}$ be the dual basis of $H^1(K(H^1(X;\mathbb{Z}),1);\mathbb{Z})\cong H^1(X;\mathbb{Z})^\vee$. The morphism induced in cohomology Ψ^* sends the generator $\chi\in H^{2n+2}(K(\mathbb{Q},2n+2);\mathbb{Q})$ to the class:

$$\Psi^*(\chi)=\sum_{i=0}^{n+1}(-1)^i\left(1\otimes 1\otimes c_i(J^1\mathcal{O}_X)\right)\cup\left(z\otimes 1\otimes 1+1\otimes 1\otimes\alpha+\sum_j 1\otimes x'_j\otimes x_j\right)^{n+1-i}.$$

Let $\overline{k_1}\colon \Gamma_1\to K(\mathbb{Q},2n+2)^X$ be the adjoint of the map (9). There is a homotopy equivalence (see [19, Section 1])

$$K(\mathbb{Q},2n+2)^X\simeq\prod_{i=2}^{2n+2}K(H^{2n+2-i}(X;\mathbb{Q}),i). \tag{10}$$

Lemma 6.3 (Compare [19]). *Let φ_i be the map to the i-th factor of the product:*

$$\varphi_i\colon \Gamma_1\xrightarrow{\overline{k_1}} K(\mathbb{Q},2n+2)^X\longrightarrow K(H^{2n+2-i}(X;\mathbb{Q}),i).$$

The morphism induced in cohomology is given explicitly by:

$$\begin{aligned}\varphi_i^*\colon H^{2n+2-i}(X;\mathbb{Q})^\vee\cong H^i(K(H^{2n+2-i}(X;\mathbb{Q});\mathbb{Q})&\longrightarrow H^i(\Gamma_1;\mathbb{Q})\\ y'&\longmapsto y'\cap\Psi^*(\chi).\end{aligned}$$

Here, for $w\otimes y\in H^*(\Gamma_1)\otimes H^*(X)$ and $y'\in H^*(X)^\vee$, we write $y'\cap(w\otimes y)=y'(y)w$.

Proposition 6.4 (Compare [19]). *There is a fibration*

$$K(\mathbb{Q},2n+1)^X\longrightarrow\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}})\longrightarrow\Gamma_1$$

pulled back from the path space fibration over $K(\mathbb{Q},2n+2)^X$ via the map $\overline{k_1}$.

Theorem 6.5. *Let z and $\{x'_j\}$ be as in Lemma 6.2. Let $\{y'_{ik}\}$ be a basis of the rational cohomology of (10) where $y'_{ik}\in H^{2n+2-i}(X;\mathbb{Q})^\vee$ is in degree i. The rational cohomology of $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}})$ is given by the cohomology of the following commutative differential graded algebra:*

$$\Lambda\big(z,\ x'_j,\ s^{-1}y'_{ik}\big),\quad d(z)=0,\ d(x'_j)=0,\ d(s^{-1}y'_{ik})=\varphi_i^*(y'_{ik})$$

where z is in degree 2, each x'_j is in degree 1, each $s^{-1}y'_{ik}$ is in degree $i-1$, and φ_i^* is given as in Lemma 6.3.

Proof. By Proposition 6.4, there is a homotopy pullback square:

$$\begin{array}{ccc}\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}}) & \longrightarrow & *\\ \downarrow & & \downarrow\\ \Gamma_1 & \xrightarrow{\overline{k_1}} & K(\mathbb{Q},2n+2)^X\end{array}$$

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By the Eilenberg–Moore theorem, the cohomology of the pullback is given by the derived tensor product

$$\Lambda(z, x'_j) \otimes^{\mathbb{L}}_{\Lambda(y'_{ik})} \mathbb{Q}$$

which can be computed by choosing $\Lambda(y'_{ik}) \to \left(\Lambda(s^{-1}y'_{ik}, y'_{ik}),\ d(s^{-1}y'_{ik}) = y'_{ik}\right) \simeq \mathbb{Q}$ as a cofibrant replacement. □

Example 6.6. Let X be a smooth curve ($n = 1$) of genus 1 (i.e., a torus). It is a framed manifold, hence its jet bundle has trivial Chern classes. Write a, b for the standard basis of $H^1(X;\mathbb{Z})$ such that $a^2 = b^2 = 0$ and $u = ab$ generates $H^2(X;\mathbb{Z})$. Let a', b' be the dual basis. Let $\alpha = k \cdot u$ for some $k \in \mathbb{Z}$. With the notations of Lemma 6.2 we have

$$\begin{aligned}\Psi^*(\chi) &= \left(z \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha + 1 \otimes a' \otimes a + 1 \otimes b \otimes b'\right)^2 \\ &= (2k(z \otimes 1) - 2(1 \otimes a'b')) \otimes u + 2(z \otimes a' \otimes a) + 2(z \otimes b' \otimes b) + z^2 \otimes 1 \otimes 1.\end{aligned}$$

The morphisms φ_i^* of Lemma 6.3 are given by

$$\begin{aligned}\varphi_2^*\colon u' &\longmapsto u' \cap \Psi^*(\chi) = 2k(z \otimes 1) - 2(1 \otimes a'b') \\ \varphi_3^*\colon a' &\longmapsto a' \cap \Psi^*(\chi) = 2(z \otimes a') \\ b' &\longmapsto b' \cap \Psi^*(\chi) = 2(z \otimes b') \\ \varphi_4^*\colon 1 &\longmapsto 1 \cap \Psi^*(\chi) = z^2 \otimes 1.\end{aligned}$$

Therefore the cohomology of $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \simeq \mathrm{Map}_{\alpha}(X, \mathbb{P}^1)$ is given by the cohomology of the CDGA:

$$\Lambda(z, a', b', y_1, y_2, y'_2, y_3), \quad d(y_1) = 2kz - 2a'b',\ d(y_2) = 2za',\ d(y'_2) = 2zb',\ d(y_3) = z^2$$

where the indices on the last four variables indicate their degrees. (See [27, Section 3] for related computations.)

6.2. Homological stability

Despite the formula given in Theorem 6.5, we do not know in general whether the cohomology varies with α or not. Nonetheless, when X is topologically parallelisable, we can make the following qualitative remark:

Proposition 6.7. *Let X be a smooth projective complex variety such that Ω^1_X is a topologically trivial vector bundle, and let $\alpha \in H^2(X;\mathbb{Q})$. Then there is a homotopy equivalence*

$$\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}}) \simeq \Gamma^{k\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}})$$

for any nonzero rational number $k \in \mathbb{Q}^{\times}$.

Proof. As X is topologically parallelisable, the jet bundle $J^1\mathcal{O}_X$ is topologically trivial. Hence the section space is the mapping space into the fibre:

$$\Gamma^{k\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\,\mathbb{Q}}) \simeq \mathrm{Map}_{k\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

where the subscript $k\alpha$ on the right-hand side indicates the connected component of the maps which pull back the generator in cohomology to $k\alpha$. Postcomposing with a self map of $\mathbb{P}^n_{\mathbb{Q}}$ of degree $1/k$ gives a homotopy equivalence

$$\mathrm{Map}_{k\alpha}(X, \mathbb{P}^n_{\mathbb{Q}}) \simeq \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}}).$$

□

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Corollary 6.8. *Let X be a smooth projective complex variety which is topologically parallelisable. Let $\alpha \in \mathrm{NS}(X)$ be ample, and assume that $d(X,\alpha) \geq 1$ (see Definition 2.17). Then, for any integer $k \geq 1$, there is a map*

$$\mathcal{M}^{k\alpha}_{\mathrm{hyp}} \longrightarrow \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

inducing an isomorphism in rational cohomology in the range of degrees $* < \frac{k \cdot d(X,\alpha)-3}{2}$. In particular, the rational cohomology stabilises as $k \to \infty$.

Remark 6.9. The rational homotopy type of the mapping space $\mathrm{Map}(X, \mathbb{P}^n_{\mathbb{Q}})$ can be easily computed without the results of the last section. In [3, Theorem 1.4], Berglund gives an explicit L_∞-algebra model whose underlying graded $\mathbb{Q}$-vector space is given by

$$H^*(X;\mathbb{Q}) \otimes \mathbb{Q} \cdot \{u, w\}$$

where $H^i(X)$ sits in degree $-i$, and u, w are respectively in degrees 1 and $2n$. (This uses that X is a formal space.) For the connected component corresponding to $\alpha \in H^2(X;\mathbb{Q})$, the associated Maurer–Cartan element is $\tau = \alpha \otimes u$. In particular, the only possibly nonvanishing brackets are given by

$$[x_1 \otimes u, \ldots, x_r \otimes u]_\tau = \pm \frac{(n+1)!}{(n+1-r)!} (\alpha^{n+1-r} x_1 x_2 \cdots x_r) \otimes w.$$

In fact, in the case of the torus, the Chevalley–Eilenberg complex associated to this L_∞-algebra is the CDGA given in Example 6.6.

7. Scanning and configuration spaces on curves

In this section, we explain how the present article fits into the general philosophy of scanning maps in topology. In Theorem 7.2, we recover a special case of a result of McDuff about the homology of configuration spaces of points on a curve.

7.1. Scanning

We begin with a brief and intuitive exposition of the general idea behind scanning. Suppose given $M \subset N$, a d-dimensional submanifold of an n-dimensional manifold. We can try to see what M looks like by looking locally at each point of N. One can imagine looking through a magnifying glass: either we are far from M and see nothing, or close to M and see a first-order approximation of M, that is, a tangent space, together with a small vector from the center of the lens to M. To formalise this intuition, recall the tautological quotient bundle over the Grassmannian of d-dimensional planes in $\mathbb{R}^n$:

$$\mathbb{R}^{n-d} \longrightarrow \gamma^{\perp}_{d,n} := \{(H, v) \mid H \in \mathrm{Gr}(d, \mathbb{R}^n),\ v \in \mathbb{R}^n/H\} \longrightarrow \mathrm{Gr}(d, \mathbb{R}^n).$$

One thinks of a point $(H, v) \in \gamma^{\perp}_{d,n}$ as a d-dimensional plane together with a normal vector. The Thom space $\mathrm{Gr}(d, \mathbb{R}^n)^{\gamma^{\perp}_{d,n}}$ is obtained by one-point compactifying the total space. This construction can be done fibrewise to the tangent bundle TN of N, and we denote by $\mathrm{Gr}(d, TN)^{\gamma^{\perp}_{d,n}}$ the resulting bundle over N. The submanifold M then gives a section

$$N \longrightarrow \mathrm{Gr}(d, TN)^{\gamma^{\perp}_{d,n}}$$

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obtained by sending a point far away from M to the point at infinity (in the Thom space), and sending a point $x \in N$ close to a point $y \in M$ to the tangent space $T_yM \subset T_yN$ together with the vector pointing from y to x. Of course this requires to be made precise, for example, by choosing a tubular neighbourhood of M inside N. In many cases, this idea can be implemented in families to obtain a map from a parameter space of submanifolds to a section space.

We are now ready to give an interpretation of the jet map of Theorem 4.6 in the spirit of scanning. We take X and α as in the assumptions of that theorem. We also denote by $n = \dim_{\mathbb{C}} X$ the complex dimension of X. The main observation is the following:

Lemma 7.1. *For integers d, m, let $\mathrm{Gr}(d, \mathbb{C}^m)$ be the Grassmannian of complex d-dimensional planes in $\mathbb{C}^m$ and $\gamma^{\perp}_{d,m}$ be the tautological quotient bundle. There is a homeomorphism*

$$\begin{aligned} \gamma^{\perp}_{d,m} &\xrightarrow{\cong} \mathrm{Gr}(d+1, \mathbb{C}^m \oplus \mathbb{C}) \setminus \mathrm{Gr}(d+1, \mathbb{C}^m) \\ (H, v) &\longmapsto (H, 0) \oplus (v, 1) \end{aligned}$$

where $\mathrm{Gr}(d+1, \mathbb{C}^m)$ is embedded inside $\mathrm{Gr}(d+1, \mathbb{C}^m \oplus \mathbb{C})$ via $P \mapsto (P, 0)$.

When V is an n-dimensional complex vector space, the tautological quotient bundle over $\mathrm{Gr}(n-1, V)$ is homeomorphic to $\mathbb{P}(V \oplus \mathbb{C}) \setminus \{*\}$. Hence its Thom space is $\mathbb{P}(V \oplus \mathbb{C})$. Using this observation and the isomorphism $J^1\mathcal{O}_X \cong \Omega^1_X \oplus \mathcal{O}_X$ as smooth complex vector bundles, we see that

$$\mathbb{P}(J^1\mathcal{O}_X) \cong \mathrm{Gr}(n-1, \Omega^1_X)^{\gamma^{\perp}_{n-1,n}} \cong \mathrm{Gr}(n-1, TX)^{\gamma^{\perp}_{n-1,n}}.$$

Under these identifications, the jet map

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0}\big(\mathrm{Gr}(n-1, TX)^{\gamma^{\perp}_{n-1,n}}\big)$$

is very close to the general idea of scanning described above. Indeed, given a smooth hypersurface $V(s) \subset X$, $x \mapsto s(x)$ vanishes on the hypersurface and thus records in some sense the distance to the hypersurface, an analogue of the normal vector. Moreover, when $s(x)$ vanishes the derivative $ds(x)$ is nonzero and we can identify the tangent space at $x \in V(s)$ as the kernel $\ker(ds(x))$.

7.2. Configuration spaces on curves

Let us now describe the case $n = 1$ in more detail. The variety X is then a curve, and for $\alpha \in \mathrm{NS}(X) \subset H^2(X; \mathbb{Z})$ we write $\deg \alpha \in \mathbb{Z}$ for its image under the degree isomorphism $\deg\colon H^2(X; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ induced by the complex orientation. A hypersurface of Chern class α is simply an unordered configuration of $\deg \alpha$ points and we have a homeomorphism

$$\begin{aligned} \mathcal{M}^{\alpha}_{\mathrm{hyp}} &\xrightarrow{\cong} \mathrm{UConf}_{\deg \alpha}(X) \\ ([\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X),\ [s] \in \Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^{\times}) &\longrightarrow V([s]). \end{aligned}$$

There is also an identification

$$\mathbb{P}(J^1\mathcal{O}_X) \cong \mathrm{Gr}(0, TX)^{\gamma^{\perp}_{0,1}} \cong T^+X$$

with the fibrewise one-point compactification of the tangent bundle. In [25], McDuff studied a scanning map on configuration spaces of points on a manifold, that is, spaces of 0-dimensional submanifolds. In the present work, we instead study (complex) codimension 1 submanifolds. On a curve these agree and we recover a special case of McDuff's theorem, although our scanning map is now more algebraic in nature:

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Theorem 7.2. *Let X be a smooth projective complex curve of genus g. Let $\alpha \in H^2(X;\mathbb{Z})$ be such that $\deg \alpha > 2g-2$. The jet map*

$$\mathrm{UConf}_{\deg \alpha}(X) \cong \mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \cong \Gamma^{\alpha}_{\mathcal{C}^0}(T^+X)$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{\deg \alpha - 2g - 3}{2}$.

Proof. This is a direct consequence of Theorem 4.6. To verify the assumption on the ampleness of $\alpha - c_1(K_X)$, recall that the canonical divisor has degree $2g-2$ and that a line bundle of positive degree is ample. The final bound is obtained by computing $d(X,\alpha) = \deg \alpha - 2g$ using Riemann–Roch as explained in Lemma A.1. □

Remark 7.3. In [25], McDuff does not provide any concrete range for the homology isomorphism. However, the range $* \leq \alpha/2$ was later proved by Segal in [32, Appendix A]. In particular, it is better than the one given in the present article.

8. Characteristic classes and manifold bundles

In this section, we comment on the stable rational cohomology of $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$. Our main motivation is trying to relate it to the stable cohomology of moduli spaces of manifolds as investigated by Galatius and Randal-Williams [14, 12, 11]. None of this section uses the new results of this article, and will in fact be deduced entirely from [1]. Nonetheless, we think that it fits naturally with the "moduli space point of view" adopted in this paper.

We will assume that $\dim_{\mathbb{C}} X = n \geq 4$ and that the fundamental group of X is trivial to apply the results of [13]. In particular this implies that the Picard variety is a point and $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times = \mathcal{M}^{\alpha}_{\mathrm{hyp}}$ for $\alpha = c_1(\mathcal{L})$. We also choose a very ample line bundle $\mathcal{L}$ on X.

8.1. Recollections on stable classes

We will shortly recall from [1] the geometric interpretation of the stable classes in the rational cohomology of $\Gamma_{\mathrm{ns}}(\mathcal{L})$. As we are here mostly interested in the quotient by the scalars $\mathbb{C}^\times$, we first make two observations.

Lemma 8.1. *Let $\mathcal{L}$ be a 2-jet ample line bundle on X. Then there exists a homogeneous polynomial*

$$\Delta : \Gamma(\mathcal{L}) \longrightarrow \mathbb{C}$$

such that $\Gamma_{\mathrm{ns}}(\mathcal{L}) = \Delta^{-1}(\mathbb{C}^\times)$. In other words, the complement of $\Gamma_{\mathrm{ns}}/\mathbb{C}^\times \subset \mathbb{P}(\Gamma(\mathcal{L}))$ is a hypersurface given by the vanishing of the discriminant polynomial Δ.

Proof. The general theory of discriminants is explained in [15], see, for example, page 15 for the definition of the discriminant polynomial. In general, the subspace of singular sections of $\mathcal{L}$ has codimension at least 1, and exactly 1 in most cases, see, for example, [15, Corollary 1.2]. We show that when $\mathcal{L}$ is 2-jet ample, we are indeed in the latter case. Consider the incidence variety

$$R := \{(x,H) \in X \times \mathbb{P}(\Gamma(\mathcal{L})) \mid x \in H \text{ is a singular point}\}$$

and its two projections $p_1 \colon R \to X$ and $p_2 \colon R \to \mathbb{P}(\Gamma(\mathcal{L}))$. The set of singular hypersurfaces is $p_2(R)$. By 1-jet ampleness of $\mathcal{L}$, p_1 is a vector bundle. This implies that R is irreducible of dimension $\dim \mathbb{P}(\Gamma(\mathcal{L})) - 1$. In [21, Proposition 3.4], Katz shows that the locus of $(x,H) \in R$ where x is a nondegenerate singular point of H is open. By 2-jet ampleness, it is nonempty, hence dense as R is irreducible. Now [21, Proposition 3.5] shows that p_2 is birational. Therefore $p_2(R)$ has codimension 1. □

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Lemma 8.2. *Suppose that $\mathcal{L}$ is 2-jet ample. Then there is an isomorphism of $H^*(\Gamma_{ns}(\mathcal{L});\mathbb{Q})$-modules:*

$$H^*(\Gamma_{ns}(\mathcal{L});\mathbb{Q}) \cong H^*(\Gamma_{ns}(\mathcal{L})/\mathbb{C}^\times;\mathbb{Q}) \otimes H^*(\mathbb{C}^\times;\mathbb{Q}).$$

Proof. Let $\Delta : \Gamma(\mathcal{L}) \to \mathbb{C}$ be the discriminant provided by Lemma 8.1 so that $\Gamma_{ns}(\mathcal{L}) = \Gamma(\mathcal{L}) \setminus \Delta^{-1}(0)$. There is a fibre bundle

$$\mathbb{C}^\times \longrightarrow \Gamma_{ns}(\mathcal{L}) \longrightarrow \Gamma_{ns}(\mathcal{L})/\mathbb{C}^\times$$

and for any fibre the map $\mathbb{C}^\times \hookrightarrow \Gamma_{ns}(\mathcal{L}) \xrightarrow{\Delta} \mathbb{C}^\times$ is of degree $\deg(\Delta) \neq 0$, hence an isomorphism on rational cohomology. The lemma then follows by the Leray–Hirsch theorem. □

Consider the universal bundle of hypersurfaces:

$$V(s) \longrightarrow U(\mathcal{L}) := \{(s,x) \in \Gamma_{ns}(\mathcal{L}) \times X \mid s(x) = 0\} \xrightarrow{\pi} \Gamma_{ns}(\mathcal{L}).$$

At each point $(s,x) \in U(\mathcal{L})$, the derivative $ds(x)$ is nonzero, thus giving a map

$$j : U(\mathcal{L}) \longrightarrow \Omega^1_X \otimes \mathcal{L} \setminus 0.$$

For $\mathcal{L}$ ample enough such that the Euler class of $\Omega^1_X \otimes \mathcal{L}$ is non zero, the cohomology of the target of j in degrees $* \geq 2n$ is

$$H^{*\geq 2n}(\Omega^1_X \otimes \mathcal{L} \setminus 0;\mathbb{Z}) \cong H^{*\geq 1}(X;\mathbb{Z})[2n-1],$$

where $[2n-1]$ indicates a shift of degrees.

Proposition 8.3 (Compare [1, Proposition 8.6]). *Let $\mathcal{L}$ be a d-jet ample line bundle on a smooth projective complex variety X. Suppose furthermore that the Euler class of $\Omega^1_X \otimes \mathcal{L}$ is nonzero. Then the morphism given by pulling back along j and integrating along the fibres of π*

$$\pi_! \circ j^* : \Lambda(H^{*\geq 2n}(\Omega^1_X \otimes \mathcal{L} \setminus 0;\mathbb{Q})) \longrightarrow H^*(\Gamma_{ns}(\mathcal{L});\mathbb{Q}) \longrightarrow H^*(\Gamma_{ns}(\mathcal{L})/\mathbb{C}^\times;\mathbb{Q})$$

is an isomorphism in the range of degrees $* < \frac{d-1}{2}$. (The second arrow is obtained from Lemma 8.2 by projecting away from the second tensor factor.)

Proof. Let us write $g : \Gamma_{ns}(\mathcal{L}) \times X \to J^1\mathcal{L} \setminus 0$ for the jet map and $p : \Gamma_{ns}(\mathcal{L}) \times X \to X$ for the projection. Then [1, Proposition 8.6] shows that

$$p_! \circ g^* : \Lambda(H^{*\geq 2n+2}(J^1\mathcal{L} \setminus 0;\mathbb{Q})) \longrightarrow H^*(\Gamma_{ns}(\mathcal{L})/\mathbb{C}^\times;\mathbb{Q})$$

is an isomorphism in the range $* < \frac{d-1}{2}$. There is a pullback square

$$\begin{array}{ccc} \Omega^1_X \otimes \mathcal{L} \setminus 0 & \xhookrightarrow{\iota} & J^1\mathcal{L} \setminus 0 \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle g} \\ U(\mathcal{L}) & \xhookrightarrow{i} & \Gamma_{ns}(\mathcal{L}) \times X \end{array}$$

where ι and g are transverse, as one sees directly from a local computation using that $\mathcal{L}$ is globally generated. Thus $i_! \circ j^* = g^* \circ \iota_!$. Furthermore, $\pi = p \circ i$ so that $\pi_! = p_! \circ i_!$. The proposition now follows from the fact that $\iota_!$ sends $H^{*\geq 2n}(\Omega^1_X \otimes \mathcal{L} \setminus 0;\mathbb{Q})$ isomorphically onto $H^{*\geq 2n+2}(J^1\mathcal{L} \setminus 0;\mathbb{Q})$. □

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8.2. Comparison with diffeomorphisms

Let us fix a nonsingular section $s \in \Gamma_{\mathrm{ns}}(\mathcal{L})$ and write $H := V(s)$ for the associated hypersurface. From the point of view of the Hilbert scheme developed in Section 3, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$ classifies algebraic bundles with fibres equivalent to H (as divisors) and embedded in X. Seeing H only as a smooth oriented real manifold, one can consider the classifying space $B\mathrm{Diff}^{\mathrm{or}}(H)$ of its group of orientation preserving diffeomorphisms. By definition, this latter space classifies fibre bundles with fibre H and structure group $\mathrm{Diff}^{\mathrm{or}}(H)$. In particular, there is a map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow B\mathrm{Diff}^{\mathrm{or}}(H) \tag{11}$$

classifying the universal bundle

$$U(\mathcal{L})/\mathbb{C}^\times := \{([s], x) \in \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \times X \mid s(x) = 0\} \xrightarrow{\pi} \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times. \tag{12}$$

One could wonder if the map (11) induces an isomorphism in rational cohomology in a range of degrees. This was shown to be false when $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(d)$ by Randal-Williams [28] (and it will also follow from our results below). On the other hand, one could alter the situation by considering diffeomorphisms preserving other kinds of tangential structures (see Remark 8.4 below, or [13, Definition 2.1, and Section 4.5]): we have picked an orientation, but could have chosen an almost complex structure, or a spin structure in some cases, etc. We construct below a tangential structure θ such that the map classifying the universal bundle

$$\Gamma_{\mathrm{ns}}(\mathcal{L}) \longrightarrow B\mathrm{Diff}^{\theta}(H)$$

is "very close" to being a rational homology isomorphism in a range of degrees.

Choose maps $[TX]\colon X \to BU(n)$ and $[\mathcal{L}]\colon X \to BU(1)$ classifying respectively the tangent bundle of X and $\mathcal{L}$ as topological complex vector bundles. Let B be the space defined by the following homotopy pullback square:

$$\begin{array}{ccc} B & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle ([TX],\,[\mathcal{L}])} \\ BU(n-1)\times BU(1) & \xrightarrow[(\text{“}\oplus\text{”},\,\mathrm{pr}_2)]{} & BU(n)\times BU(1) \end{array} \tag{13}$$

where "$\oplus$"$\colon BU(n-1)\times BU(1) \to BU(n)$ classifies taking the direct sum of vector bundles. We will adopt the convention that a tangential structure is a space over $BO(2n-2)$ (See [13, Section 4.5] for how it relates to other definitions.) In this language, the *tangential structure* θ simply means the map:

$$\theta\colon B \longrightarrow BU(n-1)\times BU(1) \xrightarrow{\mathrm{pr}_1} BU(n-1) \longrightarrow BO(2n-2).$$

Remark 8.4. Up to replacing θ by a fibration $B \simeq \overline{B} \to BO(2n-2)$, a θ-structure on a $(2n-2)$-manifold M is the datum of a lift of the map classifying the tangent bundle:

$$\begin{array}{ccc} & & \overline{B} \\ & \nearrow & \downarrow{\scriptstyle \theta} \\ M & \xrightarrow[[TM]]{} & BO(2n-2) \end{array}$$

By the universal property of the homotopy pullback, this amounts to providing two maps

$$M \longrightarrow BU(n-1)\times BU(1) \quad \text{and} \quad M \longrightarrow X$$

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which become homotopic after further composing to $BU(n) \times BU(1)$ and such that $M \to BU(n-1)$ classifies the tangent bundles of M. In other words, this is the data of a map $\iota\colon M \to X$ and a complex line bundle $\mathcal{L}'$ (corresponding to $M \to BU(1)$) such that $TH \oplus \mathcal{L}' \cong \iota^* TX$ and $\iota^* \mathcal{L} \cong \mathcal{L}'$. Therefore, a θ-structure on M should be interpreted as a choice of an immersion $\iota\colon M \to X$ with normal bundle $\iota^* \mathcal{L}$.

We have chosen to construct B via the homotopy pullback (13) as it allowed us to informally understand the geometric meaning of a θ-structure. But it turns out that we can give more familiar expressions for B and the bundle classified by θ as explained in the following two lemmas.

Lemma 8.5. *There is a homotopy equivalence above X*

$$B \simeq \Omega^1_X \otimes \mathcal{L} \setminus 0.$$

Proof. We will use the following explicit point set models:

$$\begin{aligned} BU(j) &:= \{P \subset \mathbb{C}^\infty \mid P \text{ is a } j\text{-dimensional plane}\},\\ \gamma_j &:= \{(P, v) \mid P \in BU(j),\ v \in P\},\\ \gamma_j^\vee &:= \{(P, \varphi) \mid P \in BU(j),\ \varphi\colon P \to \mathbb{C} \text{ linear map}\}, \end{aligned}$$

for the classifying space, and its tautological vector bundle and the dual of it. Recall that the classical fibration sequence

$$\mathbb{C}^n \setminus 0 \longrightarrow BU(n-1) \longrightarrow BU(n)$$

can be modelled by the sphere bundle of the dual tautological bundle using the homeomorphism

$$\gamma_n^\vee \setminus 0 \cong BU(n-1), \quad (P, \varphi\colon P \twoheadrightarrow \mathbb{C}) \mapsto \ker(\varphi).$$

In fact, we can likewise model the map

$$BU(n-1) \times BU(1) \xrightarrow{(\text{“}\oplus\text{”}, \mathrm{pr}_2)} BU(n) \times BU(1)$$

by the sphere bundle $\gamma_n^\vee \boxtimes \gamma_1 \setminus 0$. Indeed, we may write

$$\gamma_n^\vee \boxtimes \gamma_1 \setminus 0 \cong \{(P, L, \varphi \otimes v) \mid P \in BU(n),\ L \in BU(1),\ \varphi \otimes v \in P^\vee \otimes L \setminus 0\}$$

and use the homeomorphism

$$\gamma_n^\vee \boxtimes \gamma_1 \setminus 0 \cong BU(n-1) \times BU(1), \quad (P, L, \varphi \otimes v) \mapsto (\ker(\varphi \otimes v), L).$$

Under these identifications, and fixing an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$, one can check that the map “ $\oplus$ ” is simply the direct sum $\oplus$:

$$(P, L, \varphi \otimes v) \longmapsto \ker(\varphi \otimes v) \oplus L \subset \mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty.$$

Hence, by pulling back along $([TX], [\mathcal{L}])$, one sees that $B \simeq \Omega^1_X \otimes \mathcal{L} \setminus 0$. □

Lemma 8.6. *Let $q\colon \Omega^1_X \otimes \mathcal{L} \setminus 0 \to X$ be the projection. The virtual vector bundle $q^*(TX - \mathcal{L})$ is in fact the genuine vector bundle $\theta^*\gamma$ classified by the map $\theta\colon \Omega^1_X \otimes \mathcal{L} \setminus 0 \to BO(2n-2)$.*

Proof. We will use the homeomorphism

$$\Omega^1_X \otimes \mathcal{L} \setminus 0 \cong \{(x, \varphi) \mid x \in X,\ \varphi\colon TX|_x \twoheadrightarrow \mathcal{L}|_x \text{ surjective linear map}\}$$

given by identifying a nonzero vector of $(\Omega^1_X \otimes \mathcal{L})|_x$ with a surjective linear map. As one sees from the point set models described in the proof of Lemma 8.5, the pullback vector bundle $\theta^*\gamma$ classified by θ is

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equivalent to the one whose fibre above a point (x, φ) is given by the kernel of φ. Writing out the vector bundles

$$q^*TX = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1 \otimes \mathcal{L} \setminus 0,\ v \in TX|_x\},$$
$$q^*\mathcal{L} = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1 \otimes \mathcal{L} \setminus 0,\ v \in \mathcal{L}|_x\},$$

we identify $\theta^*\gamma$ as the kernel of the morphism of vector bundles

$$q^*TX \longrightarrow q^*\mathcal{L},\ (x, \varphi, v) \longmapsto (x, \varphi, \varphi(v)).$$

We thus obtain the short exact sequence of vector bundles

$$0 \longrightarrow \theta^*\gamma \longrightarrow q^*TX \longrightarrow q^*\mathcal{L} \longrightarrow 0$$

which proves the lemma. □

Let $H = V(s)$ be a smooth hypersurface with $s \in \Gamma_{\mathrm{ns}}(\mathcal{L})$. Using nonsingularity, we obtain a map $\ell\colon H \to \Omega^1_X \otimes \mathcal{L} \setminus 0$ given by $\ell(x) = ds(x)$.

Lemma 8.7. *The map $\ell\colon H \to \Omega^1_X \otimes \mathcal{L} \setminus 0$ is $(n-1)$-connected.*

Proof. The inclusion $\iota\colon H \hookrightarrow X$ factors as

$$H \xrightarrow{\ell} \Omega^1_X \otimes \mathcal{L} \setminus 0 \longrightarrow X$$

where the second map is the projection map of the bundle, hence $(2n-1)$-connected. Therefore it suffices to show that $\iota\colon H \to X$ is $(n-1)$-connected. But this is precisely the Lefschetz hyperplane theorem. □

Recall from [13, Section 4.5] that a *θ-structure* on a manifold M is a bundle map (fibrewise linear isomorphism) $TM \to \theta^*\gamma$, where γ is the universal bundle above $BO(2n-2)$. In the proposition below, we observe that each $\ell\colon H \to \Omega^1_X \otimes \mathcal{L} \setminus 0$ underlies a bundle map $\hat{\ell}\colon TH \to \theta^*\gamma$, and that these naturally assemble when varying H in the universal bundle.

Proposition 8.8. *The universal bundle $U(\mathcal{L}) \to \Gamma_{\mathrm{ns}}(\mathcal{L})$ admits the structure of a smooth fibre bundle with θ-structure over ℓ in each fibre.*

Proof. Let $T_vU(\mathcal{L})$ be the vertical tangent bundle of the universal bundle. By definition of a bundle with θ-structure, we have to provide the horizontal maps in the following diagram to construct a vector bundle map

$$\begin{array}{ccc} T_vU(\mathcal{L}) & \longrightarrow & \theta^*\gamma \\ \downarrow & & \downarrow \\ U(\mathcal{L}) & \longrightarrow & \Omega^1_X \otimes \mathcal{L} \setminus 0 \end{array}$$

which restricts to a linear isomorphism in each fibre. Using the notation from the proof of Lemma 8.6, we write

$$\Omega^1_X \otimes \mathcal{L} \setminus 0 = \{(x, \varphi) \mid x \in X,\ \varphi\colon TX|_x \twoheadrightarrow \mathcal{L}|_x \text{ surjective linear map}\}$$

and

$$\theta^*\gamma = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1_X \otimes \mathcal{L} \setminus 0,\ v \in \ker(\varphi)\}.$$

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Differentiating a nonsingular section $s\colon X \to \mathcal{L}$ yields a short exact sequence of vector bundles

$$0 \longrightarrow TV(s) \longrightarrow TX \xrightarrow{ds} \mathcal{L} \longrightarrow 0$$

which shows that $\ker(ds(x)) = TV(s)|_x$. In particular, we have a point set model for the vertical tangent bundle given by

$$T_vU(\mathcal{L}) = \{(s,x,v) \mid s \in U(\mathcal{L}),\ x \in V(s),\ v \in \ker(ds(x))\}.$$

Hence, taking

$$U(\mathcal{L}) \longrightarrow \Omega^1_X \otimes \mathcal{L} \setminus 0, \quad (s,x) \longmapsto (x, ds(x))$$

and

$$T_vU(\mathcal{L}) \longrightarrow \theta^*\gamma, \quad (s,x,v) \longmapsto (x, ds(x), v)$$

gives the wanted square. □

Let us now look at hypersurfaces of higher degree. For every integer $d \geq 1$, we pick a section $s_d \in \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})$ and write $H_d = V(s_d) \subset X$ for the associated hypersurface. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes d}$ in the diagram (13), we obtain spaces $B_d \simeq \Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0$. We write $\theta_d\colon B_d \to BO(2n-2)$ for the tangential structure and $\hat{\ell}_d\colon TH_d \to \theta_d^*\gamma$ for the tangential structure on H_d induced from the inclusion inside X (as constructed in the proof of Proposition 8.8). Let $\mathrm{Bun}^{\theta_d}(H_d)$ denote the space of bundle maps $TH_d \to \theta_d^*\gamma$, and write

$$\mathcal{M}^{\theta_d}(H_d) = \mathrm{Bun}^{\theta_d}(H_d) /\!\!/ \mathrm{Diff}(H_d)$$

for the Borel construction. As explained in [13, Section 2.2], this is a moduli space classifying smooth H_d-bundles with θ_d-structure. We let $\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) \subset \mathcal{M}^{\theta_d}(H_d)$ be the connected component of $\hat{\ell}_d$. Work of Galatius and Randal-Williams provides the following:

Theorem 8.9 (Compare [13, Theorem 4.5]). *Using the notations as above, let $\alpha = c_1(\mathcal{L})$ and $N := \int_X \alpha^{n+1} \neq 0$. There is a map*

$$\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) \longrightarrow \Omega^\infty MT\theta_d \simeq \Omega^\infty\big(\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0\big)^{q^*\mathcal{L}^{\otimes d} - q^*TX}$$

which, when regarded as a map onto the path component that it hits, induces an isomorphism in integral homology in degrees $* \leq \frac{1}{3}Cd^{n+1} + O(d^n)$, for some constant C depending on n and satisfying $\frac{1}{2} \cdot \frac{13}{15}N \leq C$.

Proof. The connectivity assumption of [13, Theorem 4.5] is verified in Lemma 8.7. The identification of the Thom spectrum is given by Lemma 8.6. The range given in [13, Theorem 4.5] depends on the genus of $(H_d, \hat{\ell}_d)$ in the sense of [13, Definition 3.1]. Although it is a priori not obvious how to compute that genus, it can be estimated using [13, Theorem 3.4] (here we use that X and hence B_d is assumed to be simply connected). Finally, the paragraph [13, Remark 5.6] shows how to obtain the explicit bound. In particular, as $d \to \infty$, so does the genus. □

A major advantage of that theorem resides in the fact that the rational homotopy of the infinite loop space is easily calculated:

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Lemma 8.10. *Suppose that the Euler class of $\Omega^1_X \otimes \mathcal{L}^{\otimes d}$ does not vanish. Then there is an isomorphism of commutative rings*

$$H^*\big(\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d); \mathbb{Q}\big) \cong \Lambda\big(H^{2n-1}(X)[1] \oplus H^1(X)[2] \oplus H^2(X)[3] \oplus \cdots \oplus H^{2n}(X)[2n+1]\big)$$
$$=: \Lambda\big(H^{2n-1}(X)[1]\big) \otimes \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big)$$

where $H^i(X)[j]$ denotes the graded $\mathbb{Q}$-vector space $H^i(X;\mathbb{Q})$ placed in degree j.

Proof. This is a well-known computation in rational homotopy theory using the Thom isomorphism. See for example [13, Remark 4.2]. □

By Proposition 8.8, the universal bundle is pulled back along a map

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d). \tag{14}$$

Our work describes the stable rational cohomology of the domain, whereas Galatius and Randal-Williams compute the one of the codomain. We will shortly reveal the relation between these two rings of characteristic classes. First, as argued as the beginning of this section, it is more geometrically natural to consider the quotient of the domain of (14) by the group $\mathbb{C}^\times$. We have not yet seen a counterpart to this action on the codomain, so we explain how to proceed in the following:

Proposition 8.11. *Suppose that $\mathcal{L}^{\otimes d}$ is 2-jet ample, and denote by $\Delta\colon \Gamma(\mathcal{L}^{\otimes d}) \to \mathbb{C}$ the discriminant polynomial (see Lemma 8.1). Let $\mu_{\deg\Delta} \subset \mathbb{C}^\times$ be the cyclic subgroup of the $(\deg\Delta)^{th}$ roots of unity. Then there is a homeomorphism*

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times \cong \Delta^{-1}(1)/\mu_{\deg\Delta},$$

and an action of $\mu_{\deg\Delta}$ on $\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)$ such that the map (14) fits into a commutative diagram:

$$\begin{array}{ccc} \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) & \longrightarrow & \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) \\ \downarrow & & \downarrow \\ \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times & \longrightarrow & \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg\Delta} \end{array}$$

where both vertical arrows are the quotient maps for the actions and the bottom arrow is the induced map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \cong \Delta^{-1}(1)/\mu_{\deg\Delta} \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg\Delta}$$

on the quotients from $\Delta^{-1}(1) \subset \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \to \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)$.

Proof. The classical theory of global Milnor fibrations provides a commutative diagram whose top two rows and leftmost two columns are fibre bundles:

$$\begin{array}{ccccc} \mu_{\deg\Delta} & \hookrightarrow & \Delta^{-1}(1) & \longrightarrow & \Delta^{-1}(1)/\mu_{\deg\Delta} \\ \downarrow & & \downarrow & & \downarrow \cong \\ \mathbb{C}^\times & \hookrightarrow & \Gamma_{\mathrm{ns}}(\mathcal{L}) & \longrightarrow & \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \\ \downarrow & & \downarrow \Delta & & \\ \mathbb{C}^\times/\mu_{\deg\Delta} & \xrightarrow{\cong} & \mathbb{C}^\times & & \end{array}$$

The inverse of the homeomorphism

$$\Delta^{-1}(1)/\mu_{\deg\Delta} \xrightarrow{\cong} \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$$

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is explicitly given by $s \mapsto \Delta(s)^{-1/\deg\Delta} \cdot s$. This proves the first claim. For the second, recall that we have defined:

$$\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) = \mathrm{Bun}^{\theta_d}(H_d, \hat{\ell}_d) /\!\!/ \mathrm{Diff}(H_d)$$

where $\mathrm{Bun}^{\theta_d}(H_d, \hat{\ell}_d)$ denotes the connected component of $\hat{\ell}_d$ in the space of bundle maps. Recall also from the proof of Proposition 8.8 the point set model:

$$\theta^*\gamma = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1_X \otimes \mathcal{L} \setminus 0,\ v \in \ker(\varphi)\}.$$

The group $\mathbb{C}^\times$ acts fibrewise on the vector bundle $\Omega^1_X \otimes \mathcal{L}$, thus on $\theta^*\gamma$ via

$$\lambda \cdot (x, \varphi, v) = (x, \lambda \cdot \varphi, v) \quad \text{for } \lambda \in \mathbb{C}^\times,$$

and therefore acts on $\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)$ by postcomposition on bundle maps. This action can be restricted to the cyclic subgroup $\mu_{\deg\Delta} \subset \mathbb{C}^\times$. Finally the existence of the claimed commutative square follows by inspection. □

We are now ready to summarise the relation between the two rings of characteristic classes in the main result of this section:

Theorem 8.12. *Let X be a simply connected smooth projective complex variety. Let $i \geq 0$ be an integer and let $d \gg 0$ be big enough so that*

$$H^*(\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d); \mathbb{Q}) \cong \Lambda\big(H^{2n-1}(X)[1]\big) \otimes \Lambda\big(H^{\bullet>0}(X)[\bullet + 1]\big)$$

and

$$H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times; \mathbb{Q}) \cong \Lambda\big(H^{\bullet>0}(X)[\bullet + 1]\big)$$

in degrees $* \leq i$. The map classifying the universal bundle

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)$$

induces a ring morphism in rational cohomology with the following properties in cohomological degrees $* \leq i$:

(i) *Its restriction to $\Lambda\big(H^{\bullet>0}(X)[\bullet + 1]\big) \subset H^*(\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d); \mathbb{Q})$ is injective.*
(ii) *Its restriction to $\Lambda\big(H^{2n-1}(X)[1]\big) \subset H^*(\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d); \mathbb{Q})$ is zero.*

In particular, its image in degrees $* \leq i$ is the subring $H^*(\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times; \mathbb{Q}) \subset H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}); \mathbb{Q})$, and the map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg\Delta}$$

of Proposition 8.11 induces an isomorphism in rational cohomology in degrees $* \leq i$.

Proof. Recall from [13, Section 3.1] that the characteristic classes of θ_d-bundles are given by integration along the fibres. From the similar description given in Proposition 8.3, we see that, in degrees $* \leq i$, a basis of

$$H^{\bullet>0}(X)[\bullet + 1] \subset H^*(\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d); \mathbb{Q})$$

is sent to a basis of

$$H^{\bullet>0}(X)[\bullet + 1] \subset H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}); \mathbb{Q})$$

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under the morphism induced by the map classifying the universal bundle. This proves the first point. To prove the second, we recall that the image of an element $w \in H^{2n-1}(X)$ is the fibre integration

$$\pi_!(i^*w) \in H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}); \mathbb{Q})$$

where π is the universal bundle, and $i\colon U(\mathcal{L}^{\otimes d}) \hookrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \times X \to X$ is the map $(f, x) \mapsto x$. From the commutative diagram

$$\begin{array}{ccc} U(\mathcal{L}^{\otimes d}) & \xrightarrow{\iota} & \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \times X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) & = & \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \end{array}$$

we compute that

$$\pi_!(i^*w) = (\mathrm{pr}_1)_!(\iota_!(1) \cup 1 \otimes w) = (\mathrm{pr}_1)_!(1 \otimes c_1(\mathcal{L}^{\otimes d}) \cup w) = 0.$$

The identification of the subring corresponding to the image follows from Lemma 8.2. Finally, to prove the last claim, it remains to show that the quotient map

$$\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg \Delta}$$

induces an isomorphism in rational cohomology. As the action is free, the quotient is the homotopy orbit space and we thus have a fibration:

$$\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg \Delta} \longrightarrow B\mu_{\deg \Delta}.$$

The monodromy action of $\pi_1(B\mu_{\deg \Delta}) = \mu_{\deg \Delta}$ on the cohomology of the fibre is trivial as it can be extended to the connected group $\mathbb{C}^\times$. A finite group has trivial rational cohomology, here $H^*(B\mu_{\deg \Delta}; \mathbb{Q}) = \mathbb{Q}$, and the result follows. □

Remark 8.13. By general theory, a map from a space T to the homotopy orbit space $\mathcal{M}^\theta(H, \hat{\ell}) /\!\!/ \mu_{\deg \Delta}$ is given by a principal $\mu_{\deg \Delta}$-bundle $P \to T$ and an equivariant map $P \to \mathcal{M}^\theta(H, \hat{\ell})$. From that point of view, the map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \mathcal{M}^\theta(H, \hat{\ell})/\mu_{\deg \Delta}$$

is given by the datum of a θ-structure on the pullback of the universal bundle along the étale cover

$$\Delta^{-1}(1) \longrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$$

with Galois group $\mu_{\deg \Delta}$.

8.2.1. Removing the quotient

Using the general machinery of [13, Section 4.3], we construct a new tangential structure θ'_d which takes into account the $\mu_{\deg \Delta}$-action. The group $\mu_{\deg \Delta}$ acts on $B_d \simeq \Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0$ through the scalar action on the bundle $\mathcal{L}^{\otimes d}$, and we have:

Lemma 8.14. *The map $\theta_d\colon B_d \to BO(2n-2)$ factors through the orbit space of the $\mu_{\deg \Delta}$-action via a map which we denote*

$$\theta'_d\colon B_d/\mu_{\deg \Delta} \longrightarrow BO(2n-2).$$

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Proof. One can argue using the point set models described in Lemma 8.6. Indeed, recall that $B_d \simeq \Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0$ can be identified with the space of pairs (x, φ) of a point $x \in X$ and a linear surjection $\varphi\colon TX|_x \twoheadrightarrow \mathcal{L}^{\otimes d}|_x$, and the map θ_d to $BO(2n-2)$ sends (x, φ) to the kernel $\ker(\varphi)$. The group $\mu_{\deg \Delta}$ acts on that space by scalar multiplication on $\mathcal{L}^{\otimes d}$, and this preserves the kernel of an epimorphism φ. □

The analogue of Proposition 8.8 also holds for this new tangential structure:

Lemma 8.15. *The universal bundle above $\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times$ admits the structure of a smooth fibre bundle with θ'_d-structure.*

Proof. A point of the universal bundle is a pair $([f], x)$ of an equivalence class $[f] \in \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times$ and a point $x \in V(f)$. We can choose a representative f of the class $[f]$ with discriminant $\Delta(f) = 1$ and take its derivative at x to define a map:

$$([f], x) \longmapsto [df(x)] \in \left(\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0\right)/\mu_{\deg \Delta}.$$

Although $df(x)$ is only well-defined up to a root of unity, its class is well-defined in the quotient. This map thus construct a θ'_d-structure on the universal bundle. □

As before, if $H_d = V(f)$ is a smooth hypersurface with $f \in \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times$, we write $\hat{\ell}'_d\colon TH_d \to \theta'^*_d\gamma$ for the tangential structure constructed in the proof of Lemma 8.15 above. Finally we obtain:

Corollary 8.16. *Let X be a simply connected smooth projective complex variety. Let i, d be as in Theorem 8.12. The map classifying the universal bundle*

$$\mathcal{M}_{\mathrm{hyp}}^{c_1(\mathcal{L}^{\otimes d})} = \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times \longrightarrow \mathcal{M}^{\theta'_d}(H_d, \hat{\ell}'_d)$$

induces an isomorphism in rational cohomology in the range $* \leq i$.

Proof. The differential map $\ell'_d\colon H_d = V(f) \to (\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0)/\mu_{\deg \Delta}$ is not $(n-1)$-connected anymore (compare with Lemma 8.7). But the composition

$$H_d \longrightarrow \Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0 \longrightarrow (\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0)/\mu_{\deg \Delta}$$

is a Moore–Postnikov $(n-1)$-stage. The group-like topological monoid of weak equivalences of $\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0$ above its quotient by $\mu_{\deg \Delta}$ is directly seen to be equivalent to

$$\mathrm{Map}(X, \mu_{\deg \Delta}) = \mu_{\deg \Delta}.$$

Hence [13, Theorem 4.5] applies (homotopy orbits and strict orbits are equivalent here as the group action is free) and shows that $\mathcal{M}^{\theta'_d}(H_d, \hat{\ell}'_d)$ and $\mathcal{M}^{\theta_d}(H_d, \hat{\ell}_d)/\mu_{\deg \Delta}$ have the same stable rational cohomology. The result then follows by Theorem 8.12. □

A. Range estimates for jet ampleness

A.1. The case of curves

When X is a curve, that is of dimension 1, one can give an explicit formula for the jet ampleness of a line bundle depending only on its degree.

Lemma A.1. *Let X be a smooth projective complex curve of genus g. Let $\mathcal{L}$ be a line bundle on X and denote by $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z}) \cong \mathbb{Z}$ its degree, that is, its first Chern class. Let $d \geq 1$ be an integer. If $c_1(\mathcal{L}) > 2g - 1 + d$ then $\mathcal{L}$ is d-jet ample.*

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Proof. As there is only a single tangent direction at each point on a curve, to show that $\mathcal{L}$ is d-jet ample it suffices to show that the restriction map

$$H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective for all subschemes $Z \subset X$ of length $d+1$. Recall the short exact sequences of sheaves

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where $\mathcal{I}_Z$ denotes the ideal sheaf of Z. From the long exact sequence in cohomology, we see that it suffices to show that

$$H^1(\mathcal{L} \otimes \mathcal{I}_Z) = 0.$$

By Serre duality, this group is isomorphic to $H^0(K_X \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z^{-1})$ where K_X is the canonical sheaf. It is now enough to show that $K_X \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z^{-1}$ has negative degree under the assumptions of the lemma. This follows by computing that $\deg \mathcal{I}_Z^{-1} = d+1$, $\deg \mathcal{L}^{-1} = -c_1(\mathcal{L})$ and $\deg K_X = 2g-2$ by Riemann–Roch. □

A.2. The case of toric varieties

When X is a smooth projective toric variety, its fundamental group is trivial, hence the Picard scheme is discrete. In that case the results of this paper are simply obtained from [1]. We nevertheless comment on how to compute the jet ampleness of a line bundle to give a sense of the difficulty of the problem.

The basic idea is as follows: if $\mathcal{L}$ is a d-jet ample line bundle on X, then so is its restriction to any rational curve on X. On such a rational curve $C \cong \mathbb{P}^1$, a line bundle is of the form $\mathcal{O}_{\mathbb{P}^1}(a)$ and is d-jet ample if and only if $a \geq d$. Now there are some distinguished curves on X, namely the ones invariant under the torus action, and it turns out to be enough to check jet ampleness on them:

Theorem A.2 (Compare [6]). *Let $\mathcal{L}$ be a line bundle on a smooth projective toric variety. Then $\mathcal{L}$ is d-jet ample if and only if $\mathcal{L} \cdot C \geq d$ for any torus invariant curve $C \subset X$.*

In [6], Di Rocco also proves two more equivalent criteria for jet ampleness in terms of convexity of the support function of $\mathcal{L}$ and Seshadri constants at each point of X. We refer to that paper for the full details. Importantly for us, the criterion shows that d-jet ampleness can be checked by a finite number of inequations.

A.3. Fujita's conjecture and jet ampleness on surfaces

Whereas Kleiman's criterion shows that ampleness is a numerical property, jet ampleness, or even just very ampleness or global generation, is a trickier question to settle. In 1985, Fujita proposed the following conjecture which remains unsolved in general:

Conjecture A.3 (Compare [10]). *Let X be a smooth projective complex variety of dimension n. Let A be an ample line bundle on X. Then $K_X + (n+1)A$ is globally generated, and $K_X + (n+2)A$ is very ample.*

In dimension 1, the conjecture follows from the Riemann–Roch theorem. In higher dimension, the approach taken for curves would require proving a Kodaira-type vanishing theorem for noninvertible sheaves. However, in dimension 2, the conjecture was solved by Reider by different means:

Theorem A.4 (Compare [30]). *Fujita's conjecture is true for $n = 2$.*

We recommend the lecture notes of Lazarsfeld [24] for a beautiful introduction to Fujita's conjecture and Reider's theorem.

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A.4. Cohomological criterion

Although effective vanishing theorems like Riemann–Roch do not exist in higher dimension, there exist alternatives that can be used to provide qualitative statements about jet ampleness. The starting observation is the following cohomological criterion:

Lemma A.5. *Let $\mathcal{L}$ be a line bundle on a smooth projective variety X of dimension n. Let $d \geq 1$ be an integer. Then $\mathcal{L}$ is d-jet ample if the cohomology groups*

$$H^1(\mathcal{L} \otimes \mathcal{I}_Z)$$

vanish for all 0-dimensional subschemes $Z \subset X$ of length $\sum_{j=0}^{d} \binom{n+j-1}{j}$, with ideal sheaf $\mathcal{I}_Z$.

Proof. By definition, $\mathcal{L}$ is d-jet ample if the evaluation map

$$H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}/\mathfrak{m}_1^{k_1} \cdots \mathfrak{m}_l^{k_l}) \cong \bigoplus_{i=1}^{l} H^0(\mathcal{L}/\mathfrak{m}_i^{k_i})$$

is surjective for all distinct closed points $x_1, \ldots, x_l$ with associated maximal ideal sheaves $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$, and all integers $k_i \geq 1$ such that $\sum k_i = d + 1$. For a closed point with ideal sheaf $\mathfrak{m}$ and $k \geq 1$ an integer, the subscheme given by the ideal $\mathfrak{m}^k$ has length $\sum_{j=0}^{k-1} \binom{n+j-1}{j}$. Therefore the subscheme Z given by the ideal sheaf $\mathfrak{m}_1^{k_1} \cdots \mathfrak{m}_l^{k_l}$ has length

$$l(Z) = \sum_{i=1}^{l} \sum_{j=0}^{k_i-1} \binom{n+j-1}{j} \leq \sum_{j=0}^{d} \binom{n+j-1}{j}.$$

Now, if Z is a subscheme with ideal sheaf $\mathcal{I}_Z$, then surjectivity of $H^0(\mathcal{L}) \to H^0(\mathcal{L} \otimes \mathcal{O}_Z)$ is implied by vanishing of the cohomology group $H^1(\mathcal{L} \otimes \mathcal{I}_Z)$ as one sees from the long exact sequence in cohomology associated to the short exact sequence of sheaves $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. □

Remark A.6. The results of [1] are stated in terms of the jet ampleness of $\mathcal{L}$, which is why we use the same phrasing in this paper. But in fact, as we are only concerned with conditions on the first order derivatives of sections, we could settle for the following ad hoc weaker notion: a line bundle $\mathcal{L}$ is d-good if the evaluation map

$$H^0(\mathcal{L}) \longrightarrow \bigoplus_{i=1}^{l} H^0(\mathcal{L}/\mathfrak{m}_i^2)$$

is surjective for all distinct closed points $x_1, \ldots, x_l$ with associated maximal ideal sheaves $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$, and $2l \leq d + 1$. We claim that the proofs of [1] go through to study $\Gamma_{\mathrm{ns}}(\mathcal{L})$ with this weaker assumption. However, as the bounds we obtain to estimate d-goodness or jet ampleness are not very explicit, we have opted for the stronger assumption of jet ampleness which is more commonly studied.

Proposition A.7. *Let X be a smooth projective complex variety, and $\alpha, \beta \in \mathrm{NS}(X)$ with α arbitrary and β ample. Then for any integer $d \geq 1$, there exists an integer $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ all line bundles of first Chern class equal to $\alpha + k\beta$ are d-jet ample.*

Proof. Let us denote by $\mathcal{B}$ an ample line bundle with first Chern class β. Let $M = \sum_{j=0}^{d} \binom{n+j-1}{j}$ and write $\mathrm{Hilb}_M(X)$ for the Hilbert scheme 0-dimensional subschemes of X of length M. We will show that there exists a $k_0 \in \mathbb{N}$ such that

$$H^1(X, \mathcal{I}_Z \otimes \mathcal{L} \otimes \mathcal{B}^{\otimes k}) = 0 \quad \forall k \geq k_0,\ \forall Z \in \mathrm{Hilb}_M(X),\ \forall \mathcal{L} \in \mathrm{Pic}^{\alpha}(X).$$

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As tensoring with $\mathcal{B}^{\otimes k}$ maps $\mathrm{Pic}^{\alpha}(X)$ isomorphically onto $\mathrm{Pic}^{\alpha+k\beta}(X)$, the proposition will follow by Lemma A.5. As $\mathrm{Hilb}_M(X) \times \mathrm{Pic}^{\alpha}(X)$ is proper (over $\mathrm{Spec}(\mathbb{C})$), we can find a finite covering

$$\mathrm{Hilb}_M(X) \times \mathrm{Pic}^{\alpha}(X) = \bigcup_{i=1}^{N} \mathrm{Spec}(A_i)$$

by spectra of $\mathbb{C}$-algebras of finite type. Let us write $X_{A_i} = X \times \mathrm{Spec}(A_i)$ and $p_i \colon X_{A_i} \to X$ for the first projection. Then $p_i^*\mathcal{B}$ is ample by (33, Tag 0892). Let $\mathcal{I}$ be the universal family on $\mathrm{Hilb}_M(X) \times X$, let $\mathcal{P}$ be a Poincaré sheaf on $\mathrm{Pic}^{\alpha}(X) \times X$, and denote by $\mathcal{I} \otimes \mathcal{P}$ the induced sheaf on $\mathrm{Hilb}_M(X) \times \mathrm{Pic}^{\alpha}(X) \times X$. Notice that it is a coherent sheaf that is flat over $\mathrm{Hilb}_M(X) \times \mathrm{Pic}^{\alpha}(X)$. Therefore its restriction to any X_{A_i} is a coherent sheaf, flat over $\mathrm{Spec}(A_i)$. By Serre's coherent vanishing, see (33, Tag 0B5U), there exists an $N_i \geq 0$ such that for all $k \geq N_i$ we have

$$R^j(p_i)_*(\mathcal{I} \otimes \mathcal{P} \otimes p_i^*\mathcal{B}^{\otimes k}) = 0 \quad \forall j > 0. \tag{A.1}$$

We claim that this implies that

$$H^j(X, \mathcal{I}_Z \otimes \mathcal{L} \otimes \mathcal{B}^{\otimes k}) = 0 \tag{A.2}$$

for all $j > 0, k \geq N_i$, and $(Z, \mathcal{L}) \in \mathrm{Spec}(A_i) \subset \mathrm{Hilb}_M(X) \times \mathrm{Pic}^{\alpha}(X)$. Assuming this claim, observe that taking $k_0 = \max N_i$ then proves the proposition. Fix such $k, Z, \mathcal{L}$. We will prove[3] the claim by downward induction on j. Let us write $y = (Z, \mathcal{L})$ for brevity. By coherent vanishing above dimension, (Eq. A.2) is true for $j > \dim X$. Suppose we have vanishing for some $j \geq 2$. By [16, p. 7.5.3], (Eq. A.1) implies that

$$R^j(p_i)_*(\mathcal{I} \otimes \mathcal{P} \otimes p_i^*\mathcal{B}^{\otimes k} \otimes p_i^*\mathcal{F})_y = 0$$

for any quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{Spec}(A_i)$. By the long exact sequence of derived functors, we see that the functor

$$\mathcal{F} \longmapsto R^{j-1}(p_i)_*(\mathcal{I} \otimes \mathcal{P} \otimes p_i^*\mathcal{B}^{\otimes k} \otimes p_i^*\mathcal{F})_y$$

is right-exact. Under the correspondence between quasi-coherent sheaves on affine scheme and modules, this can be seen as a functor from A_i-modules to abelian groups. We may thus apply [16, p. 7.2.5] to deduce a canonical isomorphism

$$R^{j-1}(p_i)_*(\mathcal{I} \otimes \mathcal{P} \otimes p_i^*\mathcal{B}^{\otimes k})_y \otimes \mathcal{F}_y \xrightarrow{\cong} R^{j-1}(p_i)_*(\mathcal{I} \otimes \mathcal{P} \otimes p_i^*\mathcal{B}^{\otimes k} \otimes p_i^*\mathcal{F})_y$$

The left-hand side vanishes by virtue of (Eq. A.1). Taking $\mathcal{F} = \kappa(y)$ (the residue field), we obtain (Eq. A.2) for $j - 1$. □

Remark A.8. For surfaces, there is a simpler and more explicit proof using Fujita's conjecture. Indeed, the image of the ample cone under the map $A \mapsto K_X + (n+2)A$ is a cone, and any line bundle having Chern class in that cone is very ample by Reider's theorem. If $\mathcal{L}$ is a very ample line bundle, the component of the Picard scheme corresponding to $K_X + (n+2)A + (d-1)\mathcal{L}$ only contains line bundles that are d-jet ample.

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[3] We have learned this argument from [22, p. 29].

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