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SPECTRAL GAP FOR SOME INVARIANT LOG-CONCAVE PROBABILITY MEASURES

NOLWEN HUET

Abstract. We show that the conjecture of Kannan, Lovász, and Simonovits on isoperimetric properties of convex bodies and log-concave measures is true for log-concave measures of the form $\rho(|x|_B)\,dx$ on $\mathbb{R}^n$ and $\rho(t, |x|_B)\,dx$ on $\mathbb{R}^{1+n}$, where $|x|_B$ is the norm associated to any convex body B already satisfying the conjecture. In particular, the conjecture holds for convex bodies of revolution.

§1. *Introduction.* Let $K \subset \mathbb{R}^n$ be a convex body. We denote the uniform probability measure on K by μ_K. We say that K is isotropic if:

- its barycenter $\int x\, d\mu_K(x)$ is 0;
- $\int \langle x, \theta\rangle^2\, d\mu_K(x)$ is constant over $\theta \in \mathbb{S}^{n-1}$.

This means that the covariance matrix of μ_K is a multiple of the identity. In this case,

$$\text{for all } \theta \in \mathbb{S}^{n-1}, \quad \int \langle x, \theta\rangle^2\, d\mu_K(x) = \frac{\mathrm{E}_{\mu_K}|X|^2}{n} = \mathrm{E}_{\mu_K}({X_1}^2).$$

Here $\langle\cdot,\cdot\rangle$ stands for the euclidean scalar product of $\mathbb{R}^n$ and $|\cdot|$ the associated euclidean norm; $\mathbb{S}^{n-1}$ is the euclidean sphere of radius 1, E_μ is the expectation under the measure μ, and X_1 the first coordinate of X. Let us note that other authors require more, namely $\mathrm{Vol}(K) = 1$ in [20] or $\int \langle x, \theta\rangle^2\, d\mu_K = 1$ in [11], but we do not. For such isotropic convex bodies, Kannan *et al* state in [11] a conjecture (which we call here the KLS conjecture) on a certain isoperimetric property, which can be formulated by results of Maz'ya [17, 18] and Cheeger [9], and also Ledoux [15], as follows.

KLS CONJECTURE FOR CONVEX BODIES. *There exists a universal constant C such that, for every $n \geq 1$ and every isotropic convex body $K \subset \mathbb{R}^n$, the Poincaré constant $C_{\mathrm{P}}(\mu_K)$ of μ_K is bounded from above by $C\mathrm{E}_{\mu_K}({X_1}^2) = C\mathrm{E}_{\mu_K}|X|^2/n$.*

Let us recall that the Poincaré constant $C_{\mathrm{P}}(\mu)$ of a measure μ is the best constant C such that, for every smooth function f,

$$\mathrm{Var}_\mu f \leq C \int |\nabla f|^2\, d\mu,$$

where ∇f is the gradient of f and $\mathrm{Var}_\mu f = \int (f - \int f\, d\mu)^2\, d\mu$ is its variance under μ. So the conjecture tells us that the Poincaré inequality for convex bodies is tight for linear functions, up to a universal constant.

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To date, this conjecture is known to be true for ℓ^p-balls with $p \geq 1$ (see [8] for the euclidean case $p = 2$, [24] for $p \in [1, 2]$ and [14] for $p \geq 2$), for the hypercube (see [10] or [6]), and for the regular simplex (see [1]). Moreover, as the Poincaré inequality is stable under tensorization [5], an arbitrary product of convex sets satisfying the KLS conjecture also satisfies the conjecture. Quite obviously, it is stable under dilation, too.

One can also extend the definition of isotropy and the conjecture to any log-concave measure μ on $\mathbb{R}^n$ by just replacing μ_K by μ in the statements. Recall that a measure μ on $\mathbb{R}^n$ is log-concave if it satisfies the following Brunn–Minkowski inequality for all compact sets A, B and real $\lambda \in [0, 1]$:

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}.$$

Equivalently μ is absolutely continuous with respect to the Lebesgue measure on an affine space of dimension $m \leq n$ and its density's logarithm is a concave function (see [7]). Then the KLS conjecture becomes the following.

KLS CONJECTURE FOR LOG-CONCAVE MEASURES. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. If μ is isotropic then $C_{\mathrm{P}}(\mu) \leq C\mathrm{E}_\mu({X_1}^2) = C\mathrm{E}_\mu|X|^2/n$, where C is a universal constant, independent of n.*

This conjecture seems rather difficult to tackle. It has only been checked in cases where some additional structure is involved. For instance, the conjecture holds for any log-concave product measure or more generally for any product of measures satisfying the KLS conjecture [5], which generalizes the case of the hypercube. In addition, the conjecture was confirmed by Bobkov for spherically symmetric log-concave measures in [3], which generalizes the case of the euclidean ball. Let us note also the existence of dimension-dependent bounds for the Poincaré constant of isotropic log-concave measures,

$$C_{\mathrm{P}}(\mu) \leq C\mathrm{E}_\mu|X|^2 \leq Cn\mathrm{E}_\mu({X_1}^2), \tag{1}$$

whenever μ is an isotropic log-concave measure on $\mathbb{R}^n$. This follows from a theorem proved by Kannan *et al* [11] thanks to their localization lemma, and also deduced by Bobkov [2] from his isoperimetric inequality for log-concave measures.

THEOREM 1 (Kannan *et al* [11], Bobkov [2]). *Let $X = (X_1, \ldots, X_n)$ be a log-concave random vector in $\mathbb{R}^n$ of law μ. Then, there exists a universal constant C such that*

$$C_{\mathrm{P}}(\mu) \leq C\mathrm{E}_\mu|X - \mathrm{E}(X)|^2 = C\sum_{i=1}^{n} \mathrm{Var}_\mu(X_i).$$

This was improved by Bobkov [4] to

$$C_{\mathrm{P}}(\mu) \leq C(\mathrm{Var}|X|^2)^{1/2}.$$

Using Klartag's power-law estimates [12] for $\mathrm{Var}|X|^2$, this yields (cf. [19]) the following improvement of the bound (1): for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$

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such that

$$C_P(\mu) \leq C_\varepsilon n^{1-(1/5)+\varepsilon} \mathrm{E}({X_1}^2).$$

If, moreover, μ is uniform on an unconditional convex set, i.e. invariant under coordinate reflections, Klartag [13] shows that

$$C_P(\mu) \leq C(\log n)^2 \mathrm{E}({X_1}^2).$$

In §2, we deal with the case of log-concave measures symmetric with respect to the norm associated to a convex body B satisfying the KLS conjecture. We encompass the result of Bobkov on spherically symmetric log-concave measures. Bobkov's proof relied on the tensorization of the radial measure with the uniform measure on the sphere. He used the following property of the law of the radius.

THEOREM 2 (Bobkov [3]). *Let ν be a probability measure on $\mathbb{R}_+$ defined by*

$$\nu(dr) = r^{n-1}\rho(r)\mathbb{1}_{\mathbb{R}_+}(r)\,dr$$

with ρ log-concave. Then

$$\mathrm{Var}_\nu(r) \leq \frac{\mathrm{E}_\nu(r^2)}{n}.$$

Here, a natural idea would be to use the polar representation $X = R\theta$, where $R = |X|_B$ is the norm of X, and the distribution of θ is the cone measure on ∂B. However, in the non-euclidean case, the differentiation on ∂B is more difficult to handle. So we choose to decompose X into SU where $S \in \mathbb{R}_+$ and U is uniform on B. Using the same method, we study convex bodies of revolution in §3, as well as more general log-concave measures of the form $\rho(x_0, |x_1|_{B_1}, \ldots, |x_k|_{B_k})\,dx$, with convex bodies B_i satisfying the KLS conjecture. To simplify the statements, we use the following definition.

Definition. An isotropic convex body B (respectively an isotropic log-concave probability μ) is said to satisfy the KLS conjecture with constant C if $C_P(\mu_B) \leq C\mathrm{E}_{\mu_B}(|X|^2)/n$ (respectively $C_P(\mu) \leq C\mathrm{E}_\mu(|X|^2)/n$).

If X, Y are random variables and μ is a probability measure, we will denote $X \sim Y$ when the two random variables have the same distribution, and $X \sim \mu$ when μ is the law of X. We close this introduction by stating a very useful theorem due to Milman (see [19, Theorem 2.4] where a stronger result is stated).

THEOREM 3 (Milman [19]). *Let μ be a log-concave probability measure. If there exists $C > 0$ such that, for every smooth function f,*

$$\mathrm{Var}_\mu(f) \leq C\|\|\nabla f\|\|_\infty^2,$$

then $C_P(\mu) \leq cC$, where $c > 0$ is a universal constant.

§2. *B-symmetric log-concave measures.* Let B be a convex body on $\mathbb{R}^n$ whose interior contains 0. We associate to B the (non-necessary symmetric) norm $|\cdot|_B$ defined by

$$|x|_B = \inf\left\{\lambda > 0,\ \frac{x}{\lambda} \in B\right\}.$$

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Then B and its boundary ∂B correspond respectively to the unit ball and the unit sphere for this norm. Let us denote respectively μ_B and σ_B as the uniform measure on B and the cone measure on ∂B normalized so as to be probability measures. Recall that the cone measure is characterized by the following decomposition formula for all measurable $f : \mathbb{R}^n \to \mathbb{R}$:

$$\int_B f \, d\mu_B = \int_0^1 n r^{n-1} \int_{\partial B} f(r\theta) \, d\sigma_B(\theta) \, dr.$$

Let μ be a probability measure on $\mathbb{R}^n$ with density $\rho(|x|_B)$ with respect to the Lebesgue measure. If ρ is log-concave and non-increasing, then μ is log-concave. In that case, we say that μ is a B-symmetric log-concave probability measure. Note that the isotropy of μ amounts to the isotropy of B.

PROPOSITION 4. *Let B be an isotropic convex body of $\mathbb{R}^n$ satisfying the KLS conjecture with constant C. Then any probability measure $\mu(dx) = \rho(|x|_B)\, dx$ with ρ log-concave and non-increasing on $\mathbb{R}^+$ satisfies the KLS conjecture with constant αC, where α is a universal constant.*

To prove this proposition, we use the following decomposition of μ.

LEMMA 5. *Let X be a random variable on $\mathbb{R}^n$ of law $\mu(dx) = \rho(|x|_B)\, dx$ with ρ log-concave and non-increasing on $\mathbb{R}^+$. Then $X \sim SU$, where U and S are independent random variables respectively on B and $\mathbb{R}^+$, of law $U \sim \mu_B$ and $S \sim -\mathrm{Vol}(B) s^n \rho'(s) \mathbb{1}_{R^+}(s)\, ds$.*

Note that as ρ is log-concave, we have that ρ is locally Lipschitz, and thereby is differentiable almost everywhere and satisfies the fundamental theorem of calculus (see, for instance, [23]).

Proof. Let us recall the classical polar decomposition: if θ and R are independent random variables respectively on ∂B and $\mathbb{R}^+$, of law $\theta \sim \sigma_B$ and $R \sim n \ \mathrm{Vol}(B) r^{n-1} \rho(r) \mathbb{1}_{R^+}(r)\, dr$, then $R\theta \sim \mu$. Let $T \sim n t^{n-1} \mathbb{1}_{[0,1]}(t)\, dt$ be independent of $\theta \sim \sigma_B$. By the same polar decomposition, $U = T\theta$ is uniform on B. So, if $S \sim -\mathrm{Vol}(B) s^n \rho'(s) \mathbb{1}_{R^+}(s)\, ds$ is independent of T and θ, it remains only to show that $ST \sim R$ to prove that $SU = (ST)\theta \sim \mu$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then

$$\begin{aligned}
\mathrm{E}(f(ST)) &= \iint f(st) n \ \mathrm{Vol}(B) t^{n-1} s^n (-\rho'(s)) \mathbb{1}_{[0,1]}(t) \mathbb{1}_{R^+}(s)\, dt\, ds \\
&= \int f(r) n \ \mathrm{Vol}(B) r^{n-1} \mathbb{1}_{R^+}(r) \left(\int_r^{+\infty} -\rho'(s)\, ds \right) dr \\
&= \int f(r) n \ \mathrm{Vol}(B) r^{n-1} \rho(r) \mathbb{1}_{R^+}(r)\, dr.
\end{aligned}$$

In the last equality, we use the fact that $-\rho'(x) \geq 0$ and $\rho(x) \to 0$ when $x \to +\infty$. Actually, as ρ is log-concave, non-increasing, and non-constant as a density, there exists $c > 0$ such that $\rho(x) \leq \mathrm{e}^{-cx}$ for x large enough. □

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We then need a kind of Poincaré inequality for S in the above representation.

LEMMA 6. *Let S be a random variable on $\mathbb{R}^+$ of law $\eta(ds) = -s^n \rho'(s)\, ds$ where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a log-concave non-increasing function. Then, for every smooth function f on $\mathbb{R}^+$,*

$$\mathrm{Var}_\eta(f) \le c \frac{\mathrm{E}(S^2)}{n} \|f'\|_\infty^2,$$

where $c > 0$ is a universal constant.

Proof. By integration by parts, we see that

$$-\int_0^{+\infty} r^n \rho'(r)\, dr = \int_0^{+\infty} n r^{n-1} \rho(r)\, dr = 1.$$

So $\nu(dr) = n r^{n-1} \rho(r)\, dr$ is a probability measure, and by Theorems 1 and 2, if $R \sim \nu$, then there exists a universal constant $c > 0$ such that

$$C_{\mathrm{P}}(\nu) \le c\, \mathrm{Var}(R) \le c \frac{\mathrm{E}(R^2)}{n}.$$

Remark. As pointed out by the referee, Theorem 1 is not really necessary here to derive the first inequality since we are in one dimension. For instance, the inequality is true with $c = 12$ by [**2**, Corollary 4.3].

Let f be a smooth function on $\mathbb{R}^+$ such that $\mathrm{E}(f(R)) = 0$ and $\|f'\|_\infty < \infty$. We perform again an integration by parts:

$$\begin{aligned}
\mathrm{E}(f^2(S)) &= -\int_0^{+\infty} f^2(r) r^n \rho'(r)\, dr \\
&= \int_0^{+\infty} (f^2(r) n r^{n-1} + 2 f(r) f'(r) r^n) \rho(r)\, dr \\
&= \mathrm{E}(f^2(R)) + \frac{2}{n} \mathrm{E}(R f(R) f'(R)).
\end{aligned}$$

In the second equality, we use the fact that $f^2(r) r^n \rho(r) \to 0$ when $r \to +\infty$ since f is assumed Lipschitz and since ρ decays exponentially. The Poincaré inequality for R leads to

$$\mathrm{E}(f^2(R)) \le c \frac{\mathrm{E}(R^2)}{n} \|f'\|_\infty^2.$$

Moreover,

$$\begin{aligned}
\mathrm{E}(R f(R) f'(R)) &\le \|f'\|_\infty \sqrt{\mathrm{E}(R^2)} \sqrt{\mathrm{E}(f^2(R))} \\
&\le \|f'\|_\infty \sqrt{\mathrm{E}(R^2)} \sqrt{c \frac{\mathrm{E}(R^2)}{n} \|f'\|_\infty^2} \\
&= \sqrt{\frac{c}{n}} \mathrm{E}(R^2) \|f'\|_\infty^2.
\end{aligned}$$

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Thus,

$$\mathrm{E}(f^2(S)) \le (c + 2\sqrt{c/n})\frac{\mathrm{E}(R^2)}{n}\|f'\|_\infty^2.$$

Now, for any smooth function g on $\mathbb{R}^+$, we set $f = g - \mathrm{E}(g(R))$. As $\mathrm{Var}_\eta(g) \le \mathrm{E}(g(S) - a)^2$ for every $a \in \mathbb{R}$, the following holds:

$$\mathrm{Var}_\eta(g) \le (c + 2\sqrt{c/n})\frac{\mathrm{E}(R^2)}{n}\|g'\|_\infty^2.$$

To conclude, let us remark that

$$\mathrm{E}(R^2) = \int_0^{+\infty} nr^{n+1}\rho(r)\,dr = -\frac{n}{n+2}\int_0^{+\infty} r^{n+2}\rho'(r)\,dr = \frac{n}{n+2}\mathrm{E}(S^2).$$

□

We can now prove Proposition 4 by tensorization.

Proof of Proposition 4. Let $\mu(dx) = \rho(|x|_B)\,dx$ be a probability measure with ρ log-concave non-increasing. Let S and U be independent random variables respectively on $\mathbb{R}^+$ and uniform on B, such that $X = SU \sim \mu$ as in Lemma 5. Let f be a smooth function on $\mathbb{R}^n$. By Lemma 6, there exists a universal constant $c > 0$ such that

$$\begin{aligned}\mathrm{E}_S(f^2(SU)) &\le [\mathrm{E}_S(f(SU))]^2 + c\frac{\mathrm{E}(S^2)}{n}\max_{s\ge 0}(\langle\nabla f(sU), U\rangle^2)\\ &\le [\mathrm{E}_S(f(SU))]^2 + c\frac{\mathrm{E}(S^2)}{n}|U|^2|||\nabla f|||_\infty^2. \qquad (2)\end{aligned}$$

As B satisfies the KLS conjecture with constant C, we can also apply the Poincaré inequality to $u \mapsto \mathrm{E}_S(f(Su))$:

$$\begin{aligned}\mathrm{E}_U[\mathrm{E}_S(f(SU))]^2 &\le [\mathrm{E}_U\mathrm{E}_S(f(SU))]^2 + C\frac{\mathrm{E}(|U|^2)}{n}\mathrm{E}_U|\mathrm{E}_S(S\nabla f(SU))|^2\\ &\le [\mathrm{E}_U\mathrm{E}_S(f(SU))]^2 + C\frac{\mathrm{E}(|U|^2)}{n}\mathrm{E}(S^2)|||\nabla f|||_\infty^2.\end{aligned}$$

So, if we take the expectation with respect to U in (2), we obtain

$$\begin{aligned}\mathrm{Var}_\mu f &\le (c + C)\frac{\mathrm{E}(S^2)\mathrm{E}(|U|^2)}{n}|||\nabla f|||_\infty^2\\ &= (c + C)\frac{\mathrm{E}(|X|^2)}{n}|||\nabla f|||_\infty^2,\end{aligned}$$

since S and U are independent, and $X = SU$. Let us remark that $C \ge 1$ as can be seen by testing the Poincaré inequality on linear functions, so that $c + C \le (1 + c)C$. We conclude by Theorem 3. □

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§3. *Convex bodies of revolution.* The same method works with convex bodies of revolution or more generally for convex bodies $K \subset \mathbb{R}^{n+1}$ defined by

$$K = \{(t, x) \in I \times \mathbb{R}^n;\ |x|_B \leq R(t)\},$$

where I is a bounded interval of $\mathbb{R}$ and $R : I \to \mathbb{R}^+$ is a concave function. Actually we show the KLS conjecture for corresponding measures $\rho(t, |x|_B)\, dt\, dx$ and more generally log-concave measures of the form $\rho(x_0, |x_1|_{B_1}, \ldots, |x_k|_{B_k})\, dx$.

Let $n_0, \ldots, n_k$ be positive integers and $N = \sum n_i$. Let $B_1, \ldots, B_k$ be isotropic convex bodies respectively of $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}$ satisfying the KLS conjecture with constant C. Let $\rho : \mathbb{R}^{n_0} \times (\mathbb{R}^+)^k \to \mathbb{R}^+$ be a smooth log-concave function such that:

(i) $\partial_j \rho \leq 0$ for all $j = 1, \ldots, k$; and

(ii) $(-1)^j \partial^j_{k-(j-1),\ldots,k} \rho \geq 0$ for all $j = 1, \ldots, k$,

where $\partial_i \rho$ is the partial derivative of $s_i \mapsto \rho(x_0, s_1, \ldots, s_k)$.

PROPOSITION 7. *With the above notations and hypotheses, if $\mu(dx) = \rho(x_0, |x_1|_{B_1}, \ldots, |x_k|_{B_k})\, dx$ defines an isotropic probability measure on $\mathbb{R}^N$, then μ satisfies the KLS conjecture with a constant depending only on C, n_0, and k.*

The above condition (ii) on ρ is not satisfactory, but we did not succeed in getting rid of it with our method of decomposition. When using the classical polar decomposition instead, we no longer need this condition, but another one arises on the B_i themselves, because of the differentiation on ∂B_i. Nevertheless, when $n_0 = k = 1$ and B_1 is an euclidean ball, the latter proposition is sufficient to deduce the following.

COROLLARY 8. *The convex bodies of revolution satisfy the KLS conjecture.*

Proof of Proposition 7. The same method as for B-symmetric measures applies. In the same way as in Lemma 5, let $(X_0, S) = (X_0, S_1, \ldots, S_k)$ and $U_1, \ldots, U_k$ be independent variables respectively on $\mathbb{R} \times (\mathbb{R}^+)^k$ and $B_1, \ldots, B_k$, with

$$(X_0, S) \sim (-1)^k \mathrm{Vol}(B_1) \cdots \mathrm{Vol}(B_k) s_1^{n_1} \cdots s_k^{n_k} \partial^k_{1,\ldots,k} \rho(x_0, s) \mathbb{1}_{(\mathbb{R}^+)^k}(s)\, dx_0\, ds,$$
$$\text{for all } i, \quad U_i \sim \mu_{B_i}.$$

Then $X = (X_0, S_1 U_1, \ldots, S_k U_k)$ is of law μ. Moreover, we can suppose without loss of generality that, for all i,

$$\mathrm{E}|U_i|^2 = 1.$$

If the above condition is not satisfied we choose $\lambda_i^{-2} = \mathrm{E}(|U_i|^2)$ such that $V_i \sim \mu_{\lambda_i B}$ satisfies $\mathrm{E}(|V_i|^2) = 1$, and replace ρ by $\rho_\lambda : (x_0, s_1, \ldots, s_k) \mapsto \rho(x_0, \lambda_1 s_1, \ldots, \lambda_k, s_k)$ which is still log-concave and non-increasing with respect to its k last variables. In this case, $\mu(dx) = \rho_\lambda(x_0, |x_1|_{\lambda_1 B_1}, \ldots, |x_k|_{\lambda_k B_k})\, dx$.

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Recall now that μ is isotropic, i.e. that

$$\mathrm{E}(X) = 0$$

and

$$\text{for all } y \in \mathbb{R}^N, \quad \mathrm{E}(\langle X, y\rangle^2) = \frac{\mathrm{E}(|X|^2)}{N}|y|^2.$$

This implies that each B_i has to be itself an isotropic convex body, X_0 is an isotropic log-concave variable, and

$$\text{for all } i, \quad \frac{\mathrm{E}(S_i^2)}{n_i} = \frac{\mathrm{E}(|X_0|^2)}{n_0} = \frac{\mathrm{E}(|X|^2)}{N}, \tag{3}$$

since $\mathrm{E}(|U_i|^2) = 1$. In the same spirit as Lemma 6, one can show the following lemma.

LEMMA 9. *For any smooth function f on $\mathbb{R}^{n_0} \times (\mathbb{R}^+)^k$,*

$$\mathrm{Var}_{(X_0,S)}(f) \le c(n_0 + k^2)\frac{\mathrm{E}|X|^2}{N}|||\nabla f|||_\infty^2,$$

where $c > 0$ is a universal constant.

Let us postpone the proof of this lemma and show how to deduce the claim of the proposition. Let f be a smooth function on $\mathbb{R}^N$. Let us denote by $\nabla_i f$ the derivative of

$$x_i \in \mathbb{R}^{n_i} \mapsto f(x_0, x_1, \dots, x_k),$$

and $su = (s_1u_1, \dots, s_ku_k)$ whenever $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ and $u = (u_1, \dots, u_k)$ with $u_i \in \mathbb{R}^{n_i}$. As in the proof of Proposition 4, we apply the Poincaré inequality for (X_0, S) and then for each U_i. First we consider $(x_0, s) \mapsto f(x_0, sU)$, where $U = (U_1, \dots, U_k)$:

$$\begin{aligned}
&\mathrm{E}_{(X_0,S)}(f^2(X_0, SU)) \\
&\quad \le [\mathrm{E}_{(X_0,S)}(f(X_0, SU))]^2 + c(n_0 + k^2)\frac{\mathrm{E}(|X|^2)}{N} \\
&\qquad \times \max_{(x_0,s)\in\mathbb{R}\times(\mathbb{R}^+)^k}\left(|\nabla_0 f(x_0, sU)|^2 + \sum_{i=1}^k \langle \nabla_i f(x_0, sU), U_i\rangle^2\right) \\
&\quad \le [\mathrm{E}_{(X_0,S)}(f(X_0, SU))]^2 + c(n_0 + k^2)\frac{\mathrm{E}(|X|^2)}{N}\left(1 + \sum_i |U_i|^2\right)|||\nabla f|||_\infty^2.
\end{aligned}$$

Then, as each B_i satisfies the KLS conjecture with constant C,

$$\begin{aligned}
&\mathrm{E}_U[\mathrm{E}_{(X_0,S)}(f(X_0, SU))]^2 \\
&\quad \le [\mathrm{E}(f(X))]^2 + \sum_{i=1}^k C\frac{\mathrm{E}(|U_i|^2)}{n_i}\mathrm{E}_{U_i}|\mathrm{E}_{(X_0,S)}(S_i\nabla_i f(X_0, SU))|^2
\end{aligned}$$

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$$\leq [\mathrm{E}(f(X))]^2 + \sum_{i=1}^{k} C\frac{\mathrm{E}(|U_i|^2)}{n_i}\mathrm{E}(S_i^2)\mathrm{E}(|\nabla_i f(X)|^2)$$
$$\leq [\mathrm{E}(f(X))]^2 + C\frac{\mathrm{E}(|X|^2)}{N}\mathrm{E}(|\nabla f(X)|^2).$$

The second inequality comes from Cauchy–Schwarz inequality and the independence of S and U. It follows that

$$\mathrm{Var}_\mu(f) \leq \left[c(n_0+k^2)\left(1+\sum_i \mathrm{E}(|U_i|^2)\right)+C\right]\frac{\mathrm{E}(|X|^2)}{N}|||\nabla f|||_\infty^2$$
$$= [2c(n_0+k^2)(1+k)+C]\frac{\mathrm{E}(|X|^2)}{N}|||\nabla f|||_\infty^2.$$

We are done, thanks to Theorem 3. □

Proof of Lemma 9. Let $R=(R_1,\dots,R_k)$ be a random variable on $(\mathbb{R}^+)^k$ such that

$$(X_0,\,R)\sim \nu(dx_0\,dr)=\rho(x_0,\,r)\left(\prod_{i=1}^{k}\mathrm{Vol}(B_i)n_i\mathbb{1}_{\mathbb{R}^+}(r_i)r_i^{n_i-1}\,dr_i\right)dx_0.$$

Then one can see thanks to the classical polar decomposition of μ that

$$\text{for all } i,\quad R_i\sim S_i|U_i|_{B_i}.$$

In particular, and thanks to (3),

$$\text{for all } i,\quad \frac{\mathrm{E}(R_i^2)}{n_i}=\frac{n_i}{n_i+2}\frac{\mathrm{E}(S_i^2)}{n_i}=\frac{n_i}{n_i+2}\frac{\mathrm{E}(|X|^2)}{N}\leq\frac{\mathrm{E}(|X|^2)}{N}.$$

Moreover, ν is a log-concave (non-isotropic) measure and, by Theorem 1, there exists a universal constant $c>0$ such that

$$C_P(\nu)\leq c\left(\mathrm{E}(|X_0|^2)+\sum_i \mathrm{Var}(R_i)\right).$$

Now the density of R_i is proportional to

$$r_i^{n_i-1}\int \rho(x_0,\,r)\,dx_0\,dr_1\cdots dr_{i-1}\,dr_{i+1}\cdots dr_k$$

and $r_i\mapsto\int\rho(x_0,\,r)\,dx_0\,dr_1\cdots dr_{i-1}\,dr_{i+1}\cdots dr_k$ is a non-increasing function which is also log-concave by the Prékopa–Leindler theorem [**16, 21, 22**]. According to Theorem 2,

$$\mathrm{Var}(R_i)\leq\frac{\mathrm{E}(R_i^2)}{n_i}\leq\frac{\mathrm{E}(|X|^2)}{N}.$$

Using (3), we conclude that

$$C_P(\nu)\leq c(n_0+k)\frac{\mathrm{E}(|X|^2)}{N}. \tag{4}$$

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As in the proof of Lemma 6, to prove the claim it is enough to consider smooth functions f on $\mathbb{R}^{n_0} \times (\mathbb{R}^+)^k$ such that $\mathrm{E}(f(R)) = 0$ and $|||\nabla f|||_\infty < \infty$. We can also assume $f \geq 0$. If it is not the case we consider $g = |f|$ instead of f: g is differentiable almost everywhere and $|||\nabla g|||_\infty = |||\nabla f|||_\infty$. We set $Z = \mathrm{Vol}(B_1) \cdots \mathrm{Vol}(B_k)$. Then we apply k successive integrations by parts and bound the derivatives of f by $|||\nabla f|||_\infty$ each time they appear:

$$
\begin{aligned}
&\mathrm{E}(f^2(X_0, S)) \\
&\quad = Z \int f^2(x_0, r)(-1)^k r_1^{n_1} \cdots r_k^{n_k} \partial^k_{1,\dots,k} \rho(x_0, r) \mathbb{1}_{(\mathbb{R}^+)^k}(r)\, dx_0\, dr \\
&\quad = Z \int \left[f^2 + 2\frac{r_1}{n_1} f \partial_1 f \right] (-1)^{k-1} n_1 r_1^{n_1 - 1} r_2^{n_2} \cdots r_k^{n_k} \\
&\qquad \times \partial^{k-1}_{2,\dots,k} \rho(x_0, r) \mathbb{1}_{(\mathbb{R}^+)^k}(r)\, dx_0\, dr \\
&\quad \leq Z \int \left[f^2 + 2\frac{r_1}{n_1} f |||\nabla f|||_\infty \right] (-1)^{k-1} n_1 r_1^{n_1 - 1} r_2^{n_2} \cdots r_k^{n_k} \\
&\qquad \times \partial^{k-1}_{2,\dots,k} \rho(x_0, r) \mathbb{1}_{(\mathbb{R}^+)^k}(r)\, dx_0\, dr \\
&\quad = Z \int \left[f^2 + 2\frac{r_1}{n_1} f |||\nabla f|||_\infty + 2\frac{r_2}{n_2} f \partial_2 f + 2\frac{r_1 r_2}{n_1 n_2} \partial_2 f |||\nabla f|||_\infty \right] \\
&\qquad \times (-1)^{k-2} n_1 n_2 r_1^{n_1 - 1} r_2^{n_2 - 1} r_3^{n_3} \cdots r_k^{n_k} \partial^{k-2}_{3,\dots,k} \rho(x_0, r) \mathbb{1}_{(\mathbb{R}^+)^k}(r)\, dx_0\, dr \\
&\quad \leq Z \int \left[f^2 + 2\left(\frac{r_1}{n_1} + \frac{r_2}{n_2} \right) f |||\nabla f|||_\infty + 2\frac{r_1 r_2}{n_1 n_2} |||\nabla f|||^2_\infty \right] \\
&\qquad \times (-1)^{k-2} n_1 n_2 r_1^{n_1 - 1} r_2^{n_2 - 1} r_3^{n_3} \cdots r_k^{n_k} \partial^{k-2}_{3,\dots,k} \rho(x_0, r) \mathbb{1}_{(\mathbb{R}^+)^k}(r)\, dx_0\, dr \\
&\qquad \vdots \\
&\quad \leq \mathrm{E}(f^2(X_0, R)) + 2|||\nabla f|||_\infty \mathrm{E}\left(f(X_0, R) \sum_i \frac{R_i}{n_i} \right) \\
&\qquad + 2|||\nabla f|||^2_\infty \mathrm{E}\left(\sum_{i<j} \frac{R_i R_j}{n_i n_j} \right). \qquad (5)
\end{aligned}
$$

By the Cauchy–Schwarz inequality,

$$
\mathrm{E}\left(f(X_0, R) \sum_i \frac{R_i}{n_i} \right) \leq \sqrt{\mathrm{E}(f^2(X_0, R))} \sqrt{\mathrm{E}\left(\sum_i \frac{R_i}{n_i} \right)^2}, \qquad (6)
$$

and

$$
\mathrm{E}\left(\sum_i \frac{R_i}{n_i} \right)^2 \leq \left(\sum_i \frac{1}{n_i} \right) \mathrm{E}\left(\sum_i \frac{R_i^2}{n_i} \right) \leq k^2 \frac{\mathrm{E}|X|^2}{N}. \qquad (7)
$$

The Poincaré inequality (4) satisfied by ν leads to

$$
\mathrm{E}(f^2(X_0, R)) \leq c(n_0 + k) \frac{\mathrm{E}(|X|^2)}{N} |||\nabla f|||^2_\infty. \qquad (8)
$$

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We note that

$$\sum_{i<j} \frac{R_i R_j}{n_i n_j} \leq \left(\sum_i \frac{R_i}{n_i} \right)^2$$

and plug the estimates (6), (7), and (8) into the inequality (5), so that

$$\begin{aligned} \mathrm{E}(f^2(X_0, S)) &\leq (c(n_0 + k) + 2\sqrt{c(n_0 + k)}k + 2k^2) \frac{\mathrm{E}|X|^2}{N} |||\nabla f|||_\infty^2 \\ &\leq (2c(n_0 + k) + 3k^2) \frac{\mathrm{E}|X|^2}{N} |||\nabla f|||_\infty^2. \end{aligned}$$

Consequently, the assertion follows. □

Remark. In the special case of the euclidean ball, one can improve the above estimate using the classical polar decomposition $X = (R_1\theta_1, \ldots, R_k\theta_k)$ with $\theta_i \sim \sigma_{\mathbb{S}^{n_i-1}}$. Moreover, we can drop the condition (ii) of page 57. We omit the details and only state the following proposition.

PROPOSITION 10. *Let $\mu(dx) = \rho(x_0, |x_1|, \ldots, |x_k|)\, dx$ be an isotropic log-concave probability measure on $\mathbb{R}^N$, with $x_i \in \mathbb{R}^{n_i}$. Then there exists a universal constant c such that*

$$C_{\mathrm{P}}(\mu) \leq c(n_0 + k) \frac{\mathrm{E}_\mu |X|^2}{N}.$$

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Nolwen Huet,
Institut de Mathématiques de Toulouse,
UMR CNRS 5219,
Université de Toulouse,
31062 Toulouse,
France
E-mail: nolwen.huet@math.univ-toulouse.fr

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