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SURVEY

Dynamics of random transformations: smooth ergodic theory

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Abstract. A selection of basic results in smooth ergodic theory and in the thermodynamic formalism of dynamical systems generated by compositions of *random* maps is reviewed in this article.

1. *Introduction*

Motions of particles in a phase space can often be modeled by iterations of a single transformation from the phase space to itself (for the discrete time case), with the transformation standing for the law of the motions. This results in a 'classical' dynamical system which can be considered 'deterministic' in the sense that the particles move exactly according to the same law all the time. In practical situations or in computer simulations, however, a system is usually affected by external perturbations or roundoff errors which in many cases are of great uncertainty and hence may be treated as random. Such a consideration leads to the theory of random dynamical systems (RDSs). In this article we will mainly restrict ourselves to RDSs generated by applying at each time step a transformation chosen randomly from a given family according to some probability distribution, and refer the reader to the comprehensive book by Arnold [**6**] for more general (especially continuous time) frameworks. The need to consider such RDSs was first pointed out by Ulam and von Neumann [**160**] in 1945, but the significance of the subject received its real recognition much later around 1980 when it was discovered that solutions of stochastic differential equations (SDEs) can be realized as stochastic flows of diffeomorphisms and their time discretization provides a rich source of random transformations (see Arnold [**6**] for a historical review of this topic). Ergodic theory concerning Lyapunov exponents and the corresponding invariant manifolds of stochastic flows of diffeomorphisms was essentially initiated by L. Arnold who suggested in 1981 extending Ruelle's work [**139**] to the case of such flows, and this was accomplished by

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Carverhill [**45**]. It should be noticed that an earlier work by Ruelle [**141**] contains a similar but infinite-dimensional program suitable for applications to the Navier–Stokes time evolution. In the present article we will not cover this program and confine ourselves to the finite-dimensional case. Now there have appeared a large number of works on invariant manifolds and the Lyapunov spectrum, as well as their applications to applied sciences such as engineering, of stochastic flows of diffeomorphisms arising from SDEs. Here we refer to Elworthy [**58**], Arnold [**6**], Arnold *et al* [**7**] and references therein for these works. In the present article we will touch upon such specific flows and their time discretization only in some places and will mainly deal with the general (discrete time) random mappings. The first systematic treatment of ergodic theory of independent and identically distributed (i.i.d.) random transformations was elaborated in Kifer [**85**]. Trying to get rid of the i.i.d. conditions, as advocated by L. Arnold, has led to developments of stationary random transformations. The literature on this subject is still growing rapidly, and we will not try (it also seems impossible) to cover it completely (in some cases we will just give the references containing the final results and refer to them for earlier ones). The aim of this article is to provide a basic sketch of notions and results on this subject, with the presentation being more orientated towards intended applications to physics, and specifically to statistical mechanics. Although we have tried to include some of the more recent developments, the text is not intended for specialists: we will present the simplest possible version of the results, referring to the original literature for the full statements. It can be easily noticed that many results of this theory are extensions of the corresponding ones for deterministic systems and some of these extensions are rather parallel. In most of these cases we will try to focus on the component brought about by the randomness, without paying much attention to the works in the deterministic case (for the majority of which we refer to the nice review by Eckmann and Ruelle [**59**]).

Topics mainly reviewed in this article include a partial random version of the so-called Pesin theory, the dimension aspects, the thermodynamic formalism and random hyperbolic systems. Stochastic stability will be discussed briefly. A large part of our presentation is devoted to i.i.d. random transformations. This particular case is closely related to the culture of Markov processes, especially to solutions of the classical SDEs, and in this case some interesting results were obtained specifically, among which we particularly mention here Ledrappier and Young's entropy formula and dimension formula [**101, 102**] which show a sharp contradistinction to the picture in the deterministic case (see §2 of the present article). As alluded to above, almost all of the many significant results on stochastic flows arising from SDEs, including for example large deviations (see Baxendale and Strook [**24**]), Furstenberg–Khasminskii formulae for Lyapunov exponents (see Arnold [**6**, Ch. 6] for a systematic review and account) and the geometric aspects (Elworthy [**58**]) etc., will not be covered here. Another topic not covered is the topological classification theory of RDSs, for which we refer to Nguyen [**116**]. Besides the work of Furstenberg and Kifer [**63**] mentioned later, there are numerous highly interesting results on random matrix products as well as on their non-trivial applications (see, for example, Cohen *et al* [**48**], Bougerol and Lacroix [**37**]; for more recent work see, for example, Goldsheid and Guivarc'h [**64**] with references therein; see Luenberger [**105**] for an extension to random nonlinear maps and see Ambroladze and Wallin [**4**] for connections with random iterations of Möbius

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transformations). This subject is worthy of a review of its own and is not discussed in this article. Finally, we remark that a number of papers have appeared on random iterations of complex maps and we refer to Brück *et al* [**18**] and Ambroladze [**3**] for examples.

Set-up. Let M be a C^∞ connected compact Riemannian manifold without boundary, and let $C^r(M, M)$ ($0 \le r \le \infty$ integer) be the space of C^r maps from M into itself, endowed with the usual C^r topology (see Hirsch [**73**]) and the Borel σ-algebra. We assume that the source of randomness can be modeled by a measure-preserving system $(\Omega, \mathcal{A}, P, \tau)$, that is, $(\Omega, \mathcal{A}, P)$ is a probability space consisting of configurations of the noise and a measure-preserving transformation τ on it represents the time evolution of the noise system. Sometimes we require Ω to be a Polish space (i.e. a separable topological space with a complete metric) and $\mathcal{A}$ its Borel σ-algebra. In such a case $(\Omega, \mathcal{A}, P, \tau)$ will be called a *Polish system*. This will often be assumed when we need to work with Lebesgue spaces. By a C^r *RDS* $\mathcal{F}$ over $(\Omega, \mathcal{A}, P, \tau)$ we will mean a measurable map $\mathcal{F} : \Omega \to C^r(M, M)$ and actions on M of the randomly composed maps $f(\omega, n)$, $n \in \mathbb{Z}^+$ or $\mathbb{Z}$, $\omega \in \Omega$, where, writing $f(\omega) := \mathcal{F}(\omega)$ for the sake of notation,

$$f(\omega, n) = \begin{cases} \mathrm{id}, & \text{if } n = 0, \\ f(\tau^{n-1}\omega) \circ \cdots \circ f(\tau\omega) \circ f(\omega), & \text{if } n > 0, \\ f(\tau^n\omega)^{-1} \circ \cdots \circ f(\tau^{-2}\omega)^{-1} \circ f(\tau^{-1}\omega)^{-1}, & \text{if } n < 0, \end{cases} \tag{1.1}$$

with $f(\omega, n)$ for $n < 0$ being defined when τ is measurably invertible and $f(\omega) \in \mathrm{Diff}^r(M)$ for P-a.e. ω. (A C^r RDS is clearly $C^{r'}$ for any $0 \le r' < r$ since the inclusion map from $C^r(M, M)$ into $C^{r'}(M, M)$ is continuous. *For a C^0 RDS we will allow M to be only a compact metric space*.) We are interested in dynamical behaviors of these actions for P-a.e. ω or on the average on ω. Clearly, a RDS $\mathcal{F}$ reduces to a deterministic dynamical system when $\mathcal{F}$ maps P almost all ω to one point.

I.i.d. RDSs. Let $\mathcal{F}$ be an RDS as defined above. If $\mathcal{F} \circ \tau^n$, $n \ge 0$, and $\mathcal{F} \circ \tau^n$, $n < 0$, when τ is measurably invertible, are independent (they are clearly identically distributed), then $\mathcal{F}$ is said to be *i.i.d.* An i.i.d. RDS $\mathcal{F}$ naturally induces a family of Markov processes on M with transition probabilities $P(x, \cdot)$, $x \in M$, given by

$$P(x, A) = P\{\omega : f(\omega)x \in A\} \quad \text{for Borel } A \subset M. \tag{1.2}$$

However, the inverse is clearly not true, that is, not every Markov process can be obtained in this way. The representation of Markov processes by continuous or differentiable random maps constitutes a significant research topic (some results were obtained by Kifer [**85**, Ch. I] and Quas [**130**], respectively, for representation by continuous and differentiable maps; see Dubischar [**57**] for an up-to-date survey).

Examples.

(i) *Canonical RDSs.* Take $\Omega = C^r(M, M)^{\mathbb{Z}^+}$ ($0 \le r \le \infty$ integer), with the product topology and the Borel σ-algebra (note that Ω is Polish for $0 \le r < \infty$). Let $\tau : \Omega \to \Omega$ be the left shift operator, i.e. $(\tau\omega)_n = \omega_{n+1}$ for $\omega = (\omega_n) \in \Omega$, and let P be a probability on Ω invariant under τ. The coordinate map $\mathcal{F} : \Omega \to C^r(M, M)$, $\omega \mapsto \omega_0$ then defines an RDS $\mathcal{F}$ and we will call it a *one-sided canonical RDS*.

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Specifically, if $P = \nu^{\mathbb{Z}^+}$ for some probability ν on $C^r(M, M)$, we then call $\mathcal{F}$ a *one-sided canonical i.i.d. RDS*; this model corresponds to that introduced and systematically studied by Kifer [**85**] and it can define an iterated function system (IFS) when ν is supported by a finite or countable number of maps. The *two-sided* notions are introduced similarly by taking $\Omega = C^r(M, M)^{\mathbb{Z}}$ and $P = \nu^{\mathbb{Z}}$.

(ii) *RDSs generated by SDEs.* Let $(W, \mathcal{B}(W), P_W)$ be the classical Winner space, i.e. $W = \{w : w(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^d), w(0) = 0\}$ endowed with the usual Polish metric structure and P_W is the Winner measure. It is well known that P_W is invariant and ergodic under $\theta_t : W \to W$ defined by $(\theta_t w)(\cdot) = w(t + \cdot) - w(t)$, $w \in W$ for any $t > 0$. The coordinate process $\{B_t\}_{t\geq 0}$ on $(W, \mathcal{B}(W), P_W)$ describes the classical Brownian motion. Consider the Stratonovich SDE

$$d\xi_t = X_0(\xi_t)\, dt + \sum_{k=1}^{d} X_k(\xi_t) \circ dB_t^k \tag{1.3}$$

where X_k, $0 \leq k \leq d$, are vector fields on M, which are assumed here to be C^∞ to avoid precise smoothness conditions. Its solutions can be realized as a stochastic flow $\{\phi_t : (W, \mathcal{B}(W), P_W) \to \mathrm{Diff}^\infty(M)\}_{t\geq 0}$ (i.e. $\{\phi_t(\cdot)x\}_{t\geq 0}$ is the solution of (1.3) with initial $\xi_0 = x$ for any $x \in M$; see, for example, Kunita [**98**] or Ikeda and Watanabe [**76**]). Putting $\tau = \theta_1$ (or θ_{t_0} for any fixed $t_0 > 0$), define an RDS $\mathcal{F}$ over $(W, \mathcal{B}(W), P_W, \tau)$ by $\mathcal{F} : W \to \mathrm{Diff}^\infty(M)$, $w \mapsto \phi_1(w)$. Then, by Carverhill [**45**, Proposition 1.2.1], $\{f(w, n)\}_{n\geq 0}$ coincides with $\{\phi_n(w)\}_{n\geq 0}$, the time discretization of $\{\phi_t(w)\}_{t\geq 0}$, for P_W-a.e. w. Via this discretization approach, the study of (continuous time) stochastic flows can be reduced in many cases to a discrete time RDS (see, for example, Liu and Qian [**108**, Ch. V]). For a systematic treatment and a historical review on generation of RDSs by stochastic (white noise) as well as random (real noise) differential equations, we refer to Arnold [**6**].

Invariant measures. Defining 'reasonable' invariant measures for an RDS had been a non-trivial task as compared with the deterministic case. For an i.i.d. RDS, probability measures on the phase space which are stationary for the induced Markov processes are of particular interest. Let $\mathcal{F}$ be a C^0 i.i.d. RDS. A Borel probability ρ on M is called a *stationary measure* of $\mathcal{F}$ if $\int P(x, A)\, d\rho(x) = \rho(A)$, or equivalently $\int (f(\omega)\rho)(A)\, dP(\omega) = \rho(A)$, for all Borel $A \subset M$. Such measures always exist and a systematic study of their ergodic properties was presented in Kifer [**85**]. Now we just mention one particular property of them. Consider the skew-product transformation induced naturally by $\mathcal{F}$

$$F : \Omega \times M \to \Omega \times M, \quad (\omega, x) \mapsto (\tau\omega, f(\omega)x), \tag{1.4}$$

which combines together the time evolution of the underlying noise system and the motions of particles in the phase space. We have the following.

PROPOSITION 1.1. (Ohno [**117**]) *Let $\mathcal{F}$ be a one-sided canonical i.i.d. C^r $(0 \leq r \leq \infty)$ RDS. Then:*

(i) *a probability ρ on M is stationary for $\mathcal{F}$ if and only if $P \times \rho$ is F-invariant;*

(ii) *a stationary measure ρ of $\mathcal{F}$ is ergodic from the viewpoint of Markov processes (see, for example, Kifer [**85**]) if and only if $(F, P \times \rho)$ is ergodic.*

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See Carverhill [**45**] for a similar result in the setting of an RDS as introduced in Example (ii).

Due to the Bremen group of Arnold (see Arnold [**6**, Ch. I], Crauel [**49, 50**], Bogenschütz and Gundlach [**35**]), an *invariant measure* of a general stationary (as opposed to the term 'i.i.d.') RDS $\mathcal{F}$ is defined to be a probability measure on $\Omega \times M$ (with the product σ-algebra) which is invariant under the skew-product transformation F and which has marginal P on Ω. Metric and ergodic properties of such measures were systematically studied in the references mentioned just above. In particular, such invariant measures always exist (due to M being compact), and the F-invariance of a probability μ on $\Omega \times M$ which has marginal P on Ω can be characterized by its disintegration $d\mu(\omega, x) = d\mu_\omega(x)\, dP(\omega)$ (Arnold [**6**, Theorem 1.4.5]): μ is F-invariant if and only if for P-a.e. ω one has $E((f(\cdot)\mu_\cdot)(A)|\tau^{-1}\mathcal{A})(\omega) = \mu_{\tau\omega}(A)$ for all Borel $A \subset M$, and, when τ is measurably invertible, the latter fact reads

$$f(\omega)\mu_\omega = \mu_{\tau\omega} \quad \text{for } P\text{-a.e. } \omega. \tag{1.5}$$

Such μ_ω, $\omega \in \Omega$, are called the *sample measures* of μ and they could be regarded as natural 'invariant' measures associated with individual realizations of the system $(\mathcal{F}, \mu)$. An invariant measure μ of $\mathcal{F}$ is said to be *ergodic* if (F, μ) is, and each invariant measure of $\mathcal{F}$ can be decomposed into its ergodic components by *integration* when $(\Omega, \mathcal{A}, P)$ is countably generated and P is ergodic (see Bogenschütz [**30**] and Klünger [**97**]; ergodic decomposition by *fixed partitions*, as done in Rokhlin [**131**] in the deterministic case, and its relation to random invariant sets (see §3.1) are not yet known). Proposition 1.1 shows the relation between stationary and invariant measures for a one-sided canonical i.i.d. RDS. For the two-sided case one has the following.

PROPOSITION 1.2. (Relation between stationary and invariant measures) *Let $\mathcal{F}$ be a two-sided canonical i.i.d. C^r RDS ($0 \le r < \infty$ since Ω is required to be Polish here). Then the set of stationary measures of $\mathcal{F}$ corresponds in a one-to-one way to the set of forward Markov invariant measures of $\mathcal{F}$ (that is, the set of those $\mathcal{F}$-invariant measures ρ^* whose almost every sample measure ρ_ω depends only on the past $(\ldots, \omega_{-2}, \omega_{-1})$ of ω), with the correspondence being given by*

$$\rho \mapsto \rho^* \quad \text{with } \rho_\omega := \lim_{n\to+\infty} f(\tau^{-n}\omega, n)\rho, \ P\text{-a.e. } \omega \quad \text{and} \quad \rho^* \mapsto \rho := \int \rho_\omega\, dP(\omega).$$

Moreover, for a given stationary measure ρ of $\mathcal{F}$, the corresponding ρ^* can be recognized as the unique F-invariant measure whose natural projection on $C^r(M, M)^{\mathbb{Z}^+} \times M$ is $\nu^{\mathbb{Z}^+} \times \rho$, and ρ is ergodic from the viewpoint of Markov processes if and only if (F, ρ^*) is ergodic.

(See Arnold [**6**, Theorems 1.7.2 and 2.1.8] for a proof and for more information.) See §3.1 for a discussion of invariant measures supported by random invariant sets.

2. *Lyapunov exponents and entropy*

Standing hypotheses for this section:

(i) $\mathcal{F}$ is a C^r ($r \ge 0$) RDS on M over a noise system $(\Omega, \mathcal{A}, P, \tau)$;

(ii) F is the skew-product transformation associated with $\mathcal{F}$;

(iii) μ_ω, $\omega \in \Omega$, are the sample measures of an $\mathcal{F}$-invariant measure μ.

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2.1. *Lyapunov exponents and invariant manifolds.*
Lyapunov exponents. Let $\mathcal{F}$ be of class C^1 and μ an $\mathcal{F}$-invariant measure. The following result is a reformulation, by a trivialization argument on $(\Omega \times M, \mu, F)$, of the Oseledec multiplicative ergodic theorem (MET) (Oseledec [**119**] and Ruelle [**139**]).

THEOREM 2.1. (Lyapunov spectrum)
(i) *Assume the integrability condition* $\log^+ |T_x f(\omega)| \in L^1(\Omega \times M, \mu)$ *(where* $\log^+ a := \max\{\log a, 0\}$*). Then there exists a measurable set* $\Delta \subset \Omega \times M$ *such that* $\mu(\Delta) = 1$, $F\Delta \subset \Delta$ *and for each* $(\omega, x) \in \Delta$ *there are numbers*

$$-\infty \leq \lambda^{(1)}(\omega, x) < \lambda^{(2)}(\omega, x) < \cdots < \lambda^{(r(\omega,x))}(\omega, x) < +\infty \tag{2.1}$$

(measurable in (ω, x)*) and an associated filtration by subspaces of* $T_x M$

$$\{0\} = V^{(0)}(\omega, x) \subset V^{(1)}(\omega, x) \subset \cdots \subset V^{(r(\omega,x))}(\omega, x) = T_x M \tag{2.2}$$

(also measurable in (ω, x)*) satisfying*

$$\lim_{n \to +\infty} \frac{1}{n} \log |T_x f(\omega, n)\xi| = \lambda^{(i)}(\omega, x)$$
$$\text{for } \xi \in V^{(i)}(\omega, x) \setminus V^{(i-1)}(\omega, x),\ 1 \leq i \leq r(\omega, x).$$

(ii) (Invertible case) *If* τ *has a measurable inverse,* $f(\omega) \in \mathrm{Diff}^1(M)$ *for* P*-a.e.* ω *and* $\log^+ |T_x f(\omega)|, \log^+ |T_x f(\omega)^{-1}| \in L^1(\Omega \times M, \mu)$*, then there is now a measurable set* $\Delta \subset \Omega \times M$ *such that* $\mu(\Delta) = 1$, $F\Delta = \Delta$ *and for each* $(\omega, x) \in \Delta$ *there are numbers* $\lambda^{(i)}(\omega, x),\ 1 \leq i \leq r(\omega, x)$*, as given in (2.1) (but with* $\lambda^{(1)}(\omega, x) > -\infty$*) and an associated measurable splitting*

$$T_x M = E^{(1)}(\omega, x) \oplus \cdots \oplus E^{(r(\omega,x))}(\omega, x) \tag{2.3}$$

such that

$$\lim_{n \to +\infty} \frac{1}{n} \log |T_x f(\omega, \pm n)\xi| = \pm\lambda^{(i)}(\omega, x)$$
$$\text{for } 0 \neq \xi \in E^{(i)}(\omega, x),\ 1 \leq i \leq r(\omega, x).$$

A point $(\omega, x) \in \Delta$ will be said to be *forward regular* in case (i) and *regular* in case (ii) of the above theorem. Each $\lambda^{(i)}(\omega, x)$ is called a *Lyapunov (characteristic) exponent,* $m^{(i)}(\omega, x) := \dim V^{(i)}(\omega, x) - \dim V^{(i-1)}(\omega, x)$ or $:= \dim E^{(i)}(\omega, x)$ its *multiplicity,* and $\{(\lambda^{(i)}(\omega, x), m^{(i)}(\omega, x)) : 1 \leq i \leq r(\omega, x)\}$ the *Lyapunov spectrum* of $\mathcal{F}$ at (ω, x). If μ is ergodic, i.e. (F, μ) is ergodic, these numbers are constant μ almost everywhere.

'Non-random' exponents and filtrations. If $\mathcal{F}$ is a C^1 canonical i.i.d. RDS and μ is of the form $P \times \rho$ or ρ^* as given in Proposition 1.1 or 1.2 with $\log^+ |T_x f(\omega)| \in L^1(\Omega \times M, \mu)$, the Lyapunov spectra of $(\mathcal{F}, \mu)$ are essentially non-random. This means that μ almost everywhere $\{(\lambda^{(i)}(\omega, x), m^{(i)}(\omega, x)) : 1 \leq i \leq r(\omega, x)\}$ coincides with a non-random spectrum $\{(\lambda^{(i)}(x), m^{(i)}(x)) : 1 \leq i \leq r(x)\}$ (see Proposition 2.10 for a particular interesting result concerning these exponents). This was proved in Kifer [**85**] by employing

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ergodic decomposition of stationary measures. The associated filtration or splitting is, however, random in general (see Kifer [**85**, Example III.2.1]). Based on Furstenberg and Kifer [**63**], Kifer [**84**] proved for $\mathcal{F}$ one-sided and for μ of the form $P \times \rho$ a 'non-random' multiplicative ergodic theorem which allows one to pick out some non-random part $\{0\} = \mathcal{L}^{(0)}(x) \subset \mathcal{L}^{(1)}(x) \subset \cdots \subset \mathcal{L}^{(s(x))}(x) = T_x M$ from the filtration (2.2) such that for all $\xi \in \mathcal{L}^{(i)}(x) \setminus \mathcal{L}^{(i-1)}(x)$ the limit $\lim_{n\to+\infty} n^{-1} \log |T_x f(\omega, n)\xi|$ gives a non-random exponent $\beta^{(i)}(x)$ for P-a.e. ω. The exponents $\beta^{(1)}(x), \ldots, \beta^{(s(x))}(x)$ are usually a proper subset of those given by (2.1), but the largest of them coincide, and these non-random exponents have nice expressions by suitable measures on the projective bundle of M. We refer to Kifer [**85**] for precise statements of these results. Here we remark that for stochastic flows of diffeomorphisms results analogous to those above were developed independently and simultaneously by Carverhill (see the review Carverhill [**46**]).

Stable and unstable manifolds. The following result translates the linear theory of Lyapunov exponents into the nonlinear theory of invariant manifolds.

THEOREM 2.2.

(i) (Stable manifolds) *Assume $\mathcal{F}$ is of class C^2, $f(\omega) \in \mathrm{Diff}^2(M)$ for P-a.e. ω and $\log^+ |f(\omega)|_{C^2} \in L^1(\Omega, P)$ (where $|\cdot|_{C^2}$ denotes the usual C^2-norm). Let μ be an $\mathcal{F}$-invariant measure and put $\Lambda_s = \{(\omega, x)$ forward regular $: -\infty < \lambda^{(1)}(\omega, x) < 0\}$. Then for μ-a.e. $(\omega, x) \in \Lambda_s$, letting $\lambda^{(1)}(\omega, x) < \cdots < \lambda^{(p)}(\omega, x) < 0$ ($p = p(\omega, x)$) be all the negative exponents, the set*

$$W^{s,i}(\omega, x) := \left\{ y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f(\omega, n)x, f(\omega, n)y) \le \lambda^{(i)}(\omega, x) \right\}$$

is the image of $V^{(i)}(\omega, x)$ under a $C^{1,1}$ injective immersion for each $1 \le i \le p$ and it is tangent to $V^{(i)}(\omega, x)$ at point x ($W^{s,i}(\omega, x)$ remains unchanged if one replaces the distance $d(\cdot, \cdot)$ on M with those along $W^{s,i}(F^n(\omega, x))$, $n \ge 0$; the same is true for the unstable manifolds treated just below); in addition, $W^{s,p}(\omega, x)$ coincides with the stable manifold *of $\mathcal{F}$ at (ω, x) defined by $W^s(\omega, x) := \{y \in M : \limsup_{n\to+\infty} n^{-1} \log d(f(\omega, n)x, f(\omega, n)y) < 0\}$.*

(ii) (Unstable manifolds) *Let $(\mathcal{F}, \mu)$ be as in (i). Assume further that τ has a measurable inverse and $\log^+ |f(\omega)^{-1}|_{C^2} \in L^1(\Omega, P)$. Put $\Lambda_u = \{(\omega, x)$ regular $: \lambda^{(i)}(\omega, x) > 0$ for some $i\}$. Then for μ-a.e. $(\omega, x) \in \Lambda_u$, letting $\lambda^{(q)}(\omega, x)$ ($q = q(\omega, x)$) be the smallest positive exponent, the set*

$$W^{u,j}(\omega, x) := \left\{ y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f(\omega, -n)x, f(\omega, -n)y) \le -\lambda^{(j)}(\omega, x) \right\}$$

is the image of $\oplus_{j \le k \le r(\omega,x)} E^{(k)}(\omega, x)$ under a $C^{1,1}$ injective immersion for each $q \le j \le r(\omega, x)$ and it is tangent to this space at x; in addition, $W^{u,q}(\omega, x)$ coincides with the unstable manifold *of $\mathcal{F}$ at (ω, x) defined by $W^u(\omega, x) := \{y \in M : \limsup_{n\to+\infty} n^{-1} \log d(f(\omega, -n)x, f(\omega, -n)y) < 0\}$.*

Based on Ledrappier and Young [**101**] and borrowing some ideas from Brin and Kifer [**42**], the above theorem was proved in Liu and Qian [**108**, Ch. III] along Pesin's line [**122**], which stands at the origin of constructing invariant manifolds associated with

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Lyapunov exponents. A detailed study of local stable and unstable manifolds was also carried out in [**108**]. In particular, taking $W^{s,i}(\omega, x)$ as an example, an open neighborhood of x in $W^{s,i}(\omega, x)$, written $W^{s,i}_{\mathrm{loc}}(\omega, x)$, can be realized as the graph of a $C^{1,1}$ map from an open neighborhood of the origin in $V^{(i)}(\omega, x)$ to $V^{(i)}(\omega, x)^{\perp}$, and on it one can read at one time step the exponential contraction rate $\lambda^{(i)}(\omega, x)$; moreover, when Ω is Polish, Λ_s can be divided into a countable number of pieces such that on each of them $W^{s,i}_{\mathrm{loc}}(\omega, x)$, as an embedded disc, depends C^1 continuously on (ω, x). Hölder continuity of the Lyapunov bundles and the so-called absolute continuity property of these submanifolds were considered there as well. Such properties are crucial to the study of the global behavior of the dynamics, such as the entropy formula (see §2.2), genericity of SRB measures with respect to Lebesgue (see §2.2 and Theorem 3.3), the existence of ergodic attractors and the Bernoulli property etc. (see Pesin [**123**] and Pugh and Shub [**129**] for deterministic results). Here we do not present the precise statements of these properties and refer the reader to [**108**], where the results are formulated for i.i.d. random diffeomorphisms but they hold verbatim in the stationary case.

See Ruelle and Wilkinson [**149**] for an interesting application of the set-up and the invariant manifold theory of RDSs in connection with compact group extensions of deterministic maps.

Remark 2.3. ($C^{1,\alpha}$ hypothesis) In the arguments above we made the specific assumption that $\mathcal{F}$ is C^2 (rather than $C^{1,\alpha}$ for some $\alpha \in (0, 1]$), with some integrability conditions being supposed on the C^2-norms. One reason for this is for technical simplicity in some places, and another is that this assumption will be needed for the treatment below of relations between entropy, Lyapunov exponents and dimension, where we often require the map space to be Polish or at least separable so that Lusin's theorem is applicable (see Bahnmüller and Liu [**13**]; it is not clear to the author if this assumption can be reduced to a $C^{1,\alpha}$ one by a suitable trick for these purposes, noting that $C^{1,\alpha}(M, M)$ is in general not separable). However, under analogous assumptions on $C^{1,\alpha}$-norms invariant manifolds can still be constructed along Pesin's line (possibly with a slight loss of smoothness) or by adapting Ruelle's approach (see the next paragraph). Note, however, that a C^1 assumption is usually not sufficient for the invariant manifold theory, especially for the absolute continuity property (see Pugh [**128**] for a counterexample).

Ruelle's approach. Ruelle [**139**] elaborated an abstract invariant manifold theorem via a perturbation result of cocycles and this provides another (as compared with Pesin's) approach of constructing invariant manifolds. This approach is more analytic in nature and has the advantage of easier adaptability to the random case as well as to Hilbert spaces (Ruelle [**140**]). In the random case, stable manifolds were first constructed by Carverhill [**45**] for stochastic flows arising from an SDE by adapting Ruelle's abstract theorem. For a (discrete time) RDS $\mathcal{F}$ of class $C^{k,\alpha}$ ($k \geq 1$ integer, $\alpha \in (0, 1]$) with $\log^+ |f(\omega)|_{C^{k,\alpha}} \in L^1(\Omega, P)$, defining $\Phi_{(\omega,x)} : (\overline{B}(1), 0) \to (\mathbb{R}^m, 0)$ as a local version of $f(\omega)$ obtained by using the exponential map to parametrize neighborhoods of x and $f(\omega)x$ (one perhaps has to use a somewhat 'globalized' exponential map around $f(\omega)x$; Carverhill's trivialization technique of using embedding does not seem easy to be adopted here), one obtains verbatim versions of results in Ruelle [**139**] for local and global stable

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manifolds of $\mathcal{F}$ (which are of the same order of smoothness as $\mathcal{F}$). This approach also works for (non-invertible) endomorphisms (see [**139**, Theorem 6.1], and see also Ruelle [**140**] and Liu [**107**] for related works). For extensions of Ruelle [**139**] to general stochastic flows we refer to Liu and Qian [**108**, Ch. V] for statements and a brief review. An interesting phenomenon in that case, proved by Baxendale [**23**] and Kifer [**87**], is that all the integrability conditions required on the C^r-norms hold true automatically (see Arnold and Imkeller [**8**] for further results in this direction).

Other objects. 'Non-random' stable manifolds corresponding to the 'non-random' multiplicative ergodic theorem were constructed by Brin and Kifer [**42**]. A center manifold theory was developed by Boxler [**38**] for continuous time RDSs in $\mathbb{R}^d$ and generalized invariant manifolds including center and Oseledec manifolds were constructed by Dahlke [**54**] for discrete time RDSs. We refer to Arnold [**6**] for a systematic treatment of these results and applications.

We end this subsection with a very brief remark on *rotation numbers* (see Arnold [**6**, Ch. 6] for a detailed account of related results). In analogy to Lyapunov exponents, such quantities were introduced to describe rotation rates of vectors or planes under the action of the derivative flow of a dynamics (Ruelle [**142**], San Martin [**152**]). Assuming a simple Lyapunov spectrum (i.e. every exponent has multiplicity one), a multiplicative ergodic theorem for rotation numbers was proved by Arnold and San Martin [**9**] for continuous time RDSs; otherwise the situation would be rather different (see Arnold [**6**, Remark 6.5.15]). In dimension two or three, it seems that rotation numbers could be related to some well-known quantities such as curls of a vector field. Some nonlinear behavior associated to such numbers is known, such as the rotation of stable manifolds (Ruffino [**150**]). However, it seems that what one may know about such behavior is far from being as much as in the case of Lyapunov exponents, except for on some low-dimensional or special such as torus spaces.

2.2. *Entropy and its relation to Lyapunov exponents.*

Definition. The (Kolmogorov–Sinai) entropy of a deterministic measure-preserving transformation was introduced to describe the information creation rate of the evolution of a system and also to serve as an isomorphism invariant. For analogous reasons, such a concept was introduced by Kifer [**85**] for i.i.d. RDSs with stationary measures and by Ledrappier and Young [**101**] and Bogenschütz [**29**] for stationary RDSs over an ergodic and measurably invertible noise system. The following is a unified notion, due to Bogenschütz and Crauel [**33**], for general stationary RDSs as defined in §1. Assume $\mathcal{F}$ is of class C^0 and μ is an $\mathcal{F}$-invariant measure. For a finite Borel partition ξ of M, let

$$h_\mu(\mathcal{F}, \xi) = \lim_{n\to+\infty} \frac{1}{n} \int H_{\mu_\omega}\Big(\bigvee_{k=0}^{n-1} f(\omega, k)^{-1}\xi\Big) dP(\omega) \tag{2.4}$$

where $H_\nu(\eta) := -\sum_{C\in\eta} \nu(C) \log \nu(C)$, with the convention $0 \log 0 = 0$, for a finite partition η of M and a probability ν on M (see Walters [**163**]). The above limit can be shown to always exist by combining arguments in Kifer [**85**, Theorem II.1.1] and Bogenschütz [**29**, Theorem 3.1] (if $(\Omega, \mathcal{A}, P, \tau)$ is ergodic and measurably invertible, then

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taking integration in (2.4) is extraneous and the limit exists and is constant for P-a.e. ω by the subadditive ergodic theorem). The *(measure-theoretic) entropy* of $(\mathcal{F}, \mu)$ is then defined by $h_\mu(\mathcal{F}) := \sup h_\mu(\mathcal{F}, \xi)$ with the supremum being taken over the set of all finite Borel partitions of M (the entropy defined for $(\mathcal{F}, P \times \rho)$ given in Proposition 1.1 corresponds to Kifer's notion of entropy for $(\mathcal{F}, \rho)$). $h_\mu(\mathcal{F})$ turns out to coincide with $h_\mu^{\pi^{-1}\mathcal{A}}(F)$, the conditional entropy of $F : (\Omega \times M, \mu) \hookleftarrow$ with respect to $\pi^{-1}\mathcal{A}$ (see Kifer [**85**] for the definition), where $\pi : \Omega \times M \to \Omega$ is the natural projection. Furthermore, it satisfies the Abromov–Rokhlin formula

$$h_\mu(F) = h_P(\tau) + h_\mu(\mathcal{F})$$

(both sides may be $+\infty$) when $(\Omega, \mathcal{A}, P)$ is countably generated (Abromov and Rokhlin [**1**], Bogenschütz and Crauel [**33**]), and this endows $h_\mu(\mathcal{F})$ with the intuitive meaning of being the average gain rate of information created by the combined dynamics $(\Omega \times M, \mu, F)$ minus that of the information created by the 'driving' dynamics (Ω, P, τ).

Using measurable partitions of Lebesgue spaces. The identification of $h_\mu(\mathcal{F})$ and $h_\mu^{\pi^{-1}\mathcal{A}}(F)$ enables us to investigate the entropy of an RDS by means of the classical (conditional) entropy theory. As with a deterministic transformation, just employing finite or countable partitions of the phase space is not enough for the investigation and in many cases one has to appeal to finer partitions (for example, partitions into pieces of stable or unstable manifolds). This was made possible by the conditional entropy theory for measurable partitions (i.e. partitions which can be approximated by finite or countable ones) of Lebesgue spaces (see Rokhlin [**132**]). It is for this reason that we sometimes require $(\Omega, \mathcal{A}, P, \tau)$ to be Polish: $(\Omega \times M, \mu)$, with the completion of the Borel σ-algebra with respect to μ, then defines a Lebesgue space (see, for example, Walters [**163**, Theorem 2.1]). In this case and in the case of τ being measurably invertible, one can obtain the entropy $h_\mu(\mathcal{F})$ by the formula (Liu and Qian [**108**, Theorem 0.5.1])

$$h_\mu^{\pi^{-1}\mathcal{A}}(F) = \sup H_\mu\left(\zeta \middle| \bigvee_{k=1}^{+\infty} F^{-k}\zeta \vee \sigma_0\right)$$

where the supremum is taken over the set of all measurable partitions of $(\Omega \times M, \mu)$ and $\sigma_0 = \{\{\omega\} \times M : \omega \in \Omega\}$. This means that, in the case of $(\Omega, \mathcal{A}, P, \tau)$ being Polish and measurably invertible, if one takes for each $\omega \in \Omega$ a partition ζ_ω of M such that $\zeta := \{\{\omega\} \times \zeta_\omega(x) : \omega \in \Omega, x \in M\}$ constitutes a measurable partition of $(\Omega \times M, \mu)$, then (2.4) can be extended by

$$h_\mu(\mathcal{F}, \zeta) := H_\mu\left(\zeta \middle| \bigvee_{k=1}^{+\infty} F^{-k}\zeta \vee \sigma_0\right) = \int H_{\mu_\omega}\left(\zeta_\omega \middle| \bigvee_{k=1}^{+\infty} f(\omega, k)^{-1}\zeta_{\tau^k\omega}\right) dP(\omega)$$

and one can obtain the entropy $h_\mu(\mathcal{F})$ by taking the supremum of $h_\mu(\mathcal{F}, \zeta)$ over the set of all such ζ's. For more information about conditional (or relative) entropy of measure-preserving transformations of Lebesgue spaces see [**108**, Ch. 0] which is a partial modification of the usual entropy arguments given in Rokhlin [**132**].

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Some other properties. Some properties of the entropy of $(\mathcal{F}, \mu)$ have been known and most of them are analogous to those in the deterministic case, as can be expected. Here we just mention several of them. First, it is an isomorphism invariant (see Bogenschütz [**29**] for a precise statement) and this invariant is complete for Bernoulli random subshifts of finite type (see Gundlach [**65**]; a general random version of Ornstein's well-known theorem in [**118**] is not known). If the noise system $(\Omega, \mathcal{A}, P, \tau)$ satisfies $\tau^{-1}\mathcal{A} = \mathcal{A}$, a version of the Shannon–McMillan–Breiman theorem can be obtained for $(\mathcal{F}, \mu)$ [**29**]; otherwise there have been no results on the extension of this theorem except for a very special situation (i.e. the case of almost all the sample measures being the same) (see Kifer [**85**, Theorem II.1.5]). For random versions of the Kolmogorov–Sinai theorem for calculating the entropy using generators see [**29**] and [**30**]. When $(\Omega, \mathcal{A}, P)$ is countably generated and P is ergodic, the entropy of an invariant measure is equal to the integration of those of its ergodic components; this is a random version of Walters [**163**, Theorem 8.4] and can be proved by analogous arguments. The entropy map of an RDS is affine. It is believed by several authors to be upper semi-continuous under suitable expansivity conditions (in the deterministic case this is a fact which can be proved in a few lines), but, as far as the author knows, a rigorous proof of this (important for the thermodynamic formalism of RDSs) has not yet appeared and it is hoped to come into existence (for the case of a random subshift of finite type a proof was given by [**30**, Proposition 3.2.2], but it does not seem to be acceptable); for this problem note that the set of all $\mathcal{F}$-invariant measures, denoted by $\mathcal{M}(\mathcal{F})$, has the smallest topology which makes $\mu \mapsto \int \phi \, d\mu$ continuous for each ϕ in $L^1_P(\Omega, C(M))$, the space of all measurable $\phi : \Omega \times M \to \mathbb{R}$ such that $\phi(\omega, \cdot) : M \to \mathbb{R}$ is continuous for each $\omega \in \Omega$ and $\int \sup_{x \in M} |\phi(\omega, x)| \, dP(\omega) < +\infty$ (see Arnold [**6**] for more information). For a (continuous time) stochastic flow with a stationary measure the entropy defined for the time-t_0 ($t_0 > 0$) random maps is proportional by t_0 to that for the time-1 random maps (Kifer [**85**, Lemma V.2.2]). In spite of these results, many interesting properties enjoyed by the entropy in the deterministic case have not yet been established rigorously in the random case, taking the Brin–Katok local entropy formula [**41**] as an example.

Ruelle inequality. Roughly speaking, Lyapunov exponents and entropy provide two different ways of measuring the dynamical complexity of an RDS: the Lyapunov exponents measure geometrically how fast nearby orbits diverge (due to the corresponding invariant manifold theory) or how fast volume elements are expanded (by sums of the exponents), and the entropy measures from the viewpoint of information the complexity related to such dynamical behavior. In general, there exists the following relation between these two kinds of quantities.

THEOREM 2.4. (Ruelle inequality) *Let $\mathcal{F}$ be of class C^1 and assume $\log^+ |f(\omega)|_{C^1} \in L^1(\Omega, P)$ where $|f|_{C^1} = \sup_{\xi \in TM, |\xi|=1} |Tf\xi|$. Assume, moreover, that $(\Omega, \mathcal{A}, P, \tau)$ is measurably invertible or is Polish. Then for any $\mathcal{F}$-invariant measure μ one has*

$$h_\mu(\mathcal{F}) \le \int \sum_i \lambda^{(i)}(\omega, x)^+ m^{(i)}(\omega, x) \, d\mu. \tag{2.5}$$

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This theorem was first proved by Ruelle [**137**] (also in an unpublished work by Margulis) for a deterministic C^1 map. An extension to random as well as infinite-dimensional cases, suitable for applications to the Navier–Stokes time evolution, was elaborated later by Ruelle [**141**], where M is allowed to be a Hilbert manifold but Ω is assumed to be a compact metric space. Without knowing this work of Ruelle, Bahnmüller and Bogenschütz [**12**] proved the theorem for the case of $\mathcal{F}$ being over a general but ergodic and measurably invertible noise system (see that article for references to earlier works by Ledrappier, Kifer, Liu and Qian), and the slightly more general version stated here can be proved by minor modification of the arguments in [**12**] or by using the Lebesgue space property of Polish spaces. Hopefully the assumptions on $(\Omega, \mathcal{A}, P, \tau)$ in Theorem 2.4 can be suppressed.

Pesin formula. For a deterministic C^2 (or $C^{1,\alpha}$) diffeomorphism f on M and an f-invariant measure ρ, Pesin [**123**] proved that (2.5) will be an equality if ρ is smooth (i.e. absolutely continuous with respect to the Lebesgue measure on M) and this equality is known as the *Pesin (entropy) formula*. The following theorem extends the result of Pesin to RDSs composed of random endomorphisms (maybe non-invertible and with singularities, i.e. points at which the derivative is degenerate).

THEOREM 2.5. (Pesin formula) *Let $\mathcal{F}$ be of class C^2 with $\log^+ |f(\omega)|_{C^2} \in L^1(\Omega, P)$. Assume, moreover, that $(\Omega, \mathcal{A}, P, \tau)$ is Polish and measurably invertible*†. *Let μ be an $\mathcal{F}$-invariant measure. If $\mu_\omega \ll$ Leb for P-a.e. ω, then there holds the equality*

$$h_\mu(\mathcal{F}) = \int \sum_i \lambda^{(i)}(\omega, x)^+ m^{(i)}(\omega, x)\, d\mu. \tag{2.6}$$

This theorem was proved in Liu [**107**], along Pesin's original sketch, for the case of $\mathcal{F}$ being a two-sided canonical C^2 RDS, and the slightly more general version stated above can be easily proved by employing some properties of Polish spaces (see Bahnmüller and Liu [**13**]). Such a result was earlier obtained for expanding RDSs by Bahnmüller [**11**], which extends a proof given by Ledrappier and Young in the deterministic case using the notion of *Jacobian*, introduced by Parry [**121**], of measure-preserving transformations. This notion was then extensively employed in Liu [**107**] to prove Theorem 2.5 for the non-expanding case and it turns out to be a powerful tool to overcome difficulties brought about by non-invertibility of the maps (the inverse limit space method does not seem helpful for the present purpose even in the deterministic case). For a deterministic (maybe non-invertible) $C^{1,\alpha}$ map, Theorem 2.5 was also proved earlier by Thieullen [**154**] through a rather different approach, i.e. by showing that the equality between the fractal (or information) dimension and the Lyapunov dimension of the invariant measure is a sufficient condition to obtain the Pesin formula, even in the infinite-dimensional case. That approach seems adaptable to the present situation by first developing a local entropy theory (including the Brin–Katok formula in [**41**]) for RDSs, but the situation would become different when one considers an i.i.d. RDS with a smooth stationary measure (see Theorem 2.7 below and see Liu [**107**] for more information).

See §3.4 for application of Theorem 2.5 to expanding in average RDSs.

† It was recently proved in [**109**] that the formula (2.6) still holds true without the invertibility assumption on $(\Omega, \mathcal{A}, P, \tau)$, but with the additional condition $\log D(f(\omega)) \in L^1(\Omega, P)$ (see Theorem 2.7). This unifies Theorems 2.5 and 2.7.

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SRB measures of an RDS. In this paragraph we consider RDSs composed of invertible maps on the phase space and of invertible noise systems. Careful observation makes it plausible that the information creation is caused by the expansion of the dynamics and the contracting and neutral directions do not contribute to the entropy. This is made precise by the following theorem, which tells that the entropy is determined by the action of the random maps on the unstable manifolds and the Pesin formula holds true if and only if the invariant measure has conditional measures compatible with the Lebesgue on these submanifolds.

THEOREM 2.6. *Assume $\mathcal{F}$ is of class C^2, $f(\omega) \in \mathrm{Diff}^2(M)$ for P-a.e. ω and $(\Omega, \mathcal{A}, P, \tau)$ is Polish and measurably invertible. Take the integrability conditions* $\log^+ |f(\omega)|_{C^2}$, $\log^+ |f(\omega)^{-1}|_{C^2} \in L^1(\Omega, P)$ *and let μ be an $\mathcal{F}$-invariant measure. Then one can find a measurable partition η, finer than $\{\{\omega\} \times M : \omega \in \Omega\}$, of $(\Omega \times M, \mu)$ which has the following properties.*

(i) *η is* subordinate to W^u-manifolds *of $(\mathcal{F}, \mu)$ in the following sense: for μ-a.e. (ω, x), $\eta_\omega(x) \subset W^u(\omega, x)$ and contains an open neighborhood of x in $W^u(\omega, x)$ (with the neighborhood being taken in the submanifold topology), where η_ω is the partition of M induced by restricting η to $\{\omega\} \times M$, $\eta_\omega(x)$ denotes the element of η_ω containing x, and $W^u(\omega, x) := \{x\}$ if $\lambda^{(i)}(\omega, x) \le 0$ for all i.*

(ii) *η is increasing, i.e. $F^{-1}\eta$ is finer than η; moreover, $\bigvee_{n=0}^{+\infty} F^{-n}\eta$ is equal to the partition into single points, and $\bigwedge_{n=0}^{+\infty} F^n\eta$ generates the σ-algebra consisting of measurable subsets B of $\Omega \times M$ which are unions of some unstable manifolds (i.e. such that $(\omega, x) \in B$ implies $\{\omega\} \times W^u(\omega, x) \subset B$).*

(iii) $h_\mu(\mathcal{F}) = H_\mu(F^{-1}\eta|\eta) = \int H_{\mu_\omega}(f(\omega)^{-1}\eta_{\tau\omega}|\eta_\omega)\, dP(\omega)$.

(iv)
$$H_\mu(F^{-1}\eta|\eta) = \int \log|\det(T_x f(\omega)|_{E^u(\omega,x)})|\, d\mu$$
$(E^u(\omega, x) := \oplus_{\lambda^{(i)}(\omega,x)>0} E^{(i)}(\omega, x))$, or equivalently,
$$h_\mu(\mathcal{F}) = \int \sum_i \lambda^{(i)}(\omega, x)^+ m^{(i)}(\omega, x)\, d\mu$$
if and only if $\mu^\eta_{(\omega,x)} \ll \lambda^u_{(\omega,x)}$ for μ-a.e. (ω, x), where $\mu^\eta_{(\omega,x)}$ is the conditional measure of μ on $\eta(\omega, x)$, the element of η containing (ω, x), which is identified with $\eta_\omega(x)$ by the relation $\eta(\omega, x) = \{\omega\} \times \eta_\omega(x)$, and where $\lambda^u_{(\omega,x)}$ is the Lebesgue measure on $W^u(\omega, x)$ induced by its inherited Riemannian structure ($\lambda^u_{(\omega,x)} = \delta_x$ if $W^u(\omega, x) = \{x\}$). If these conditions are satisfied, the density $\rho = d\mu^\eta_{(\omega,x)}/d\lambda^u_{(\omega,x)}$ is strictly positive and $\log \rho$ is Lipschitz along $W^u(\omega, x)$ for μ-a.e. (ω, x).

An $\mathcal{F}$-invariant measure μ will be called an *SRB measure* if it has smooth conditional measures on the unstable manifolds, as precisely defined in (iv) of the above theorem. Note that this definition is independent of the particular choice of a measurable partition subordinate to the W^u-manifolds, and also note that the partition of $\Omega \times M$ into pieces of the global unstable manifolds is in general not measurable and one needs finer partitions as defined in (i) of Theorem 2.6 for this definition. By transitivity of conditional measures

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(see Rokhlin [**133**]), the SRB property of μ implies that almost every sample measure μ_ω has smooth conditional measures on leaves of the $W^u(\omega, \cdot)$-foliation of M.

A brief history of the above theorem is as follows. First consider a deterministic diffeomorphism f with an invariant measure μ. Observing that the sum of the positive exponents at a point is the exponential expansion rate of the volume element of the unstable manifold at this point and that the information creation is intuitively due to the stretching behavior of f, Ledrappier and Strelcyn [**99**] extended, using Pesin theory, the notion of SRB measures (first defined by Sinai [**153**] for Anosov diffeomorphisms, then by Ruelle [**136**] for Axiom A diffeomorphisms and by Bowen and Ruelle [**40**] for Axiom A flows) to general non-hyperbolic dynamical systems and showed that the SRB property of the invariant measure is sufficient to get the Pesin formula. The existence of a partition with properties (i)–(iii) of Theorem 2.6 in their deterministic versions was proved by Ledrappier and Strelcyn [**99**] for the case of μ being SRB and earlier by Pesin [**123**] for μ being smooth, and the proof in these cases is relatively easy since the hardest part (iii) can be obtained with the help of estimating the entropy via the positive exponents; for μ general the proof is much harder and is due to Ledrappier and Young [**100**, Part I] which, using such a partition, proves that μ being SRB is necessary for the Pesin formula. Theorem 2.6 is a random version of these results and can be proved along the lines of the deterministic case (see Ledrappier and Young [**101**], Liu and Qian [**108**], and Bahnmüller and Liu [**13**]).

Besides being characterized as those measures satisfying the entropy formula, SRB measures have their physical importance: they usually represent the asymptotic behavior of a positive Lebesgue measure set of initial points (see the review Eckmann and Ruelle [**59**] for related results on deterministic systems and see Theorem 3.3 of the present one for information in the random case). The study of the existence of SRB measures, especially in non-uniformly or partially hyperbolic dynamical systems, as well as of their statistical properties such as the decay of correlations and central limit theorems etc., is at present quite active, see the short but excellent reviews of Young [**158**], Viana [**162**] and Hu [**74**] for developments in the deterministic case and see Corollary 2.8 and §3 of the present article for developments in the random case.

Applications to physics of Theorems 2.4 and 2.6 above were made by Ruelle [**148**] where positivity of (statistical mechanics) entropy production in non-equilibrium statistical mechanics was studied in the presence of a thermostat acting by random forces.

Entropy formula for i.i.d. RDSs. An interesting phenomenon due to the presence of noise is that for an i.i.d. RDS and a stationary measure the Pesin formula 'always' holds true and the corresponding sample measures (see Proposition 1.2) are almost surely SRB if the RDS is 'sufficiently' random (Ledrappier and Young [**101**]). To state the results precisely, we first reformulate the definition of the entropy in this particular setting. Assume $\mathcal{F}$ is i.i.d. and ρ is a stationary measure of $\mathcal{F}$. For a finite Borel partition ξ of M, let

$$h_\rho(\mathcal{F}, \xi) = \lim_{n\to+\infty} \frac{1}{n} \int H_\rho\Big(\bigvee_{k=0}^{n-1} f(\omega, k)^{-1}\xi\Big)\, dP(\omega) \tag{2.7}$$

(the limit exists, see Kifer [**85**, Theorem II.1.3] for a similar proof), and define the *entropy*

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of $(\mathcal{F}, \rho)$ as $h_\rho(\mathcal{F}) := \sup h_\rho(\mathcal{F}, \xi)$ with the supremum being taken over the set of all finite Borel partitions of M. Let ν be the probability on $C^r(M, M)$ induced by the map $\Omega \to C^r(M, M)$, $\omega \mapsto f(\omega)$. Denote by $\hat{\mathcal{F}}$ the one-sided canonical i.i.d. RDS with $\hat{P} = \nu^{\mathbb{Z}^+}$, and by $\tilde{\mathcal{F}}$ the two-sided one with $\tilde{P} = \nu^{\mathbb{Z}}$. Then $h_\rho(\mathcal{F})$ can be related to the previous notion of entropy by

$$h_\rho(\mathcal{F}) = h_{\hat{P}\times\rho}(\hat{\mathcal{F}}) = h_{\rho^*}(\tilde{\mathcal{F}}) \tag{2.8}$$

where $\hat{P} \times \rho$ and ρ^* are given by Propositions 1.1 and 1.2, respectively (see Liu [**107**]). If $\mathcal{F}$ is C^1 and $\log^+ |T_x f(\omega)| \in L^1(\Omega \times M, P \times \rho)$, the essentially non-random Lyapunov spectra $\{(\lambda^{(i)}(x), m^{(i)}(x)) : 1 \le i \le r(x)\}$, ρ-a.a. x, of $(\mathcal{F}, \rho)$ can clearly be defined as those of $(\hat{\mathcal{F}}, \hat{P} \times \rho)$ since $\omega \mapsto (f(\omega), f(\tau\omega), \dots)$ is a measure-preserving map from (Ω, P) to $(C^r(M, M)^{\mathbb{Z}^+}, \hat{P})$. By Theorem 2.4, the Ruelle inequality $h_\rho(\mathcal{F}) \le \int \sum_i \lambda^{(i)}(x)^+ m^{(i)}(x)\, d\rho(x)$ holds if $\log^+ |f(\omega)|_{C^1} \in L^1(\Omega, P)$. The following theorem was proved by Ledrappier and Young [**101**] for random diffeomorphisms and then by Liu [**107**] for random non-invertible maps.

THEOREM 2.7. (Pesin formula for i.i.d. RDSs) *Assume $\mathcal{F}$ is of class C^2 and is i.i.d. Assume the integrability conditions* $\log^+ |f(\omega)|_{C^2}$, $\log D(f(\omega)) \in L^1(\Omega, P)$ *where* $D(f(\omega)) := \inf_{x\in M} |\det T_x f(\omega)|$. *Let ρ be a stationary measure of $\mathcal{F}$. If $\rho \ll$ Leb, then one has*

$$h_\rho(\mathcal{F}) = \int \sum_i \lambda^{(i)}(x)^+ m^{(i)}(x)\, d\rho(x). \tag{2.9}$$

Note that, if an i.i.d. RDS is sufficiently random in the sense that its transition probabilities have a density with respect to the Lebesgue, then all its stationary measures are smooth (see, for example, [**108**, §IV.1]). It is particularly the case when $\mathcal{F}$ arises from a stochastic flow of diffeomorphisms solving a non-degenerate SDE (see [**108**, Ch. V] for precise statements concerning the entropy formula in the flow setting). I.i.d. RDSs defined by *a deterministic endomorphism followed by stochastic flows* were introduced by Kifer [**86**] and Baladi and Young [**21**] (see also §3.4 of the present article), and Theorem 2.7 has its application in this model [**107**]. See Tsujii [**155**] for related works on one-dimensional i.i.d. random maps. A brief description of the specific feature of the proof of Theorem 2.7 brought about by the randomness is presented after Proposition 2.10.

From Theorems 2.6 and 2.7 we have the following.

COROLLARY 2.8. (SRB property of sample measures) *Let $\mathcal{F}$ be a two-sided canonical i.i.d. RDS such that $f(\omega) \in \mathrm{Diff}^2(M)$ for P-a.e. ω. Assume the integrability conditions* $\log^+ |f(\omega)|_{C^2}$, $\log^+ |f(\omega)^{-1}|_{C^2} \in L^1(\Omega, P)$. *Let ρ be a stationary measure of $\mathcal{F}$. If $\rho \ll$ Leb, then almost all the sample measures ρ_ω, $\omega \in \Omega$, are SRB or, precisely speaking, the measure ρ^* given by Proposition 1.2 is SRB.*

Remark 2.9. (Non-smoothness of sample measures) Take the assumptions of Corollary 2.8. Then the sample measures ρ_ω, $\omega \in \Omega$, are almost all smooth (i.e. ρ_ω absolutely continuous for P-a.e. ω) if and only if $f(\omega)\rho = \rho$ for P-a.e. ω. Equivalently (and roughly speaking), if the latter condition does not hold, the time evolution on the phase space

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will destroy with full probability (note that P is ergodic now) the smoothness of ρ in some contracting directions but improves it in the stretching directions, ensuring the SRB property of the sample measures (recall the pictures in Proposition 1.2 and Corollary 2.8; see also Theorem 2.11 below). The fact stated above can be proved as follows. Note that, if $\rho_\omega \ll$ Leb for P-a.e. ω, the absolute continuity property of the stable and unstable manifolds (see [**108**, Ch. III]) implies ρ^* is SRB both for $\mathcal{F}$ and its time reversal, yielding $\int \sum_i \lambda^{(i)}(x) m^{(i)}(x)\, d\rho(x) = 0$ by the Pesin formulae. By using the interesting result of Proposition 2.10 below, one proves $f(\omega)\rho = \rho$ for P-a.e.w. The inverse implication is obvious.

PROPOSITION 2.10. (Baxendale [**23**], Kifer [**85**]) *Let $\mathcal{F}$ be an i.i.d. RDS with $f(\omega) \in \mathrm{Diff}^1(M)$ for P-a.e. ω and $\log^+ |f(\omega)|_{C^1} \in L^1(\Omega, P)$. Let ρ be a stationary measure of $\mathcal{F}$. If $\rho \ll$ Leb, then*

(i) $\sum_i \lambda^{(i)}(x) m^{(i)}(x) \le 0$, *$\rho$-a.e. x;*

(ii) $\sum_i \lambda^{(i)}(x) m^{(i)}(x) = 0$, *$\rho$-a.e. x if and only if $f(\omega)\rho = \rho$ for P-a.e. ω.*

We now present a very brief description of the random component of the proof of Theorem 2.7, as compared with Pesin's proof in the deterministic case. It is enough to consider $(\hat{\mathcal{F}}, \hat{P} \times \rho)$ and $(\tilde{\mathcal{F}}, \rho^*)$. Even in the diffeomorphisms case, the proof of Theorem 2.7 given by Ledrappier and Young [**101**] consists in analysis along the stable manifolds, generally only for which, in view of Remark 2.9, the smoothness of ρ can be used (as opposed to the unstable manifolds). The smoothness of ρ together with the absolute continuity property of the stable manifolds implies that $\hat{P} \times \rho$ has smooth conditional measures on these submanifolds, and this allows one to measure information creation along the stable manifolds in terms of the Lebesgue measures on them. This statement is seemingly unreasonable since intuitively it has something to do only with the negative exponents, but now the positive and negative exponents do not play a symmetric role in general (in view of Proposition 2.10(i)). We present here an intuitive explanation which is very incomplete and technically inaccurate, but roughly tells us how the use of stable manifolds can lead to the relation of the entropy and the positive exponents. Let ξ be a finite Borel partition of M. One can rewrite the term in (2.7) under the integration signal in the following way,

$$H_\rho\Big(\bigvee_{k=0}^{n-1} f(\omega, k)^{-1}\xi\Big) = H_{f(\omega,n)\rho}\Big(\bigvee_{k=0}^{n-1} f(\tau^k\omega, n-k)\xi\Big).$$

In this way the contraction goes into the picture of information creation. Replacing ξ by a partition into pieces of the stable manifolds and using the conditional entropy theory, the determinant of the derivatives of the associated maps restricted to the stable subspaces (corresponding to the negative exponents) as well as the determinant of the whole derivatives would play a major role, due to the contraction along the stable manifolds, the compatibility of the conditional measures of ρ with the Lebesgue on these manifolds and the total volume change $\rho \to f(\omega, n)\rho$. Finally, the entropy $h_\rho(\mathcal{F})$ turns out to be determined by the latter determinant over the first, the average of whose logarithm gives the integral of the sum of the positive exponents.

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2.3. *Relations between Lyapunov exponents, entropy and dimension.*
A dimension formula for i.i.d. RDSs. In the deterministic case, it was conjectured by Federickson *et al* [**61**] that the Hausdorff dimension of an ergodic SRB measure is 'generically' equal to its Lyapunov dimension; this was proved to be true by Young [**156**] for surface diffeomorphisms but when the phase space has dimension greater than two the conjecture has not been mathematically verified (see Eckmann and Ruelle [**59**] for a detailed review of this topic). However, Ledrappier and Young [**102**] showed that the conjectured scenario is indeed mathematically true in *any* dimension if the dynamical system is subjected to certain types of sufficiently random noise. This means that for an i.i.d. RDS composed of random diffeomorphisms with some conditions to guarantee genuine randomness, the Hausdorff dimension of the sample measures corresponding to an ergodic stationary measure is almost surely equal to the Lyapunov dimension. Below we recall briefly the related notions and formulate this notable result in a more precise way.

Let $\mathcal{F}$ be a two-sided canonical i.i.d. RDS such that $f(\omega) \in \mathrm{Diff}^2(M)$ for P-a.e. ω and $\log^+ |f(\omega)|_{C^2}$, $\log^+ |f(\omega)^{-1}|_{C^2} \in L^1(\Omega, P)$. Let ρ be an ergodic $\mathcal{F}$-stationary measure and ρ^* the corresponding $\mathcal{F}$-invariant measure. Now it is more convenient to write all the Lyapunov exponents of $(\mathcal{F}, \rho)$, or equivalently of $(\mathcal{F}, \rho^*)$, as $+\infty > \lambda_1 > \lambda_2 > \cdots > \lambda_r > -\infty$. Assume $\lambda_1 > 0$ since otherwise the statements below will hold rather trivially. If ρ^* is hyperbolic (i.e. $\lambda_i \neq 0$ for all i) and is SRB, then for P-a.e. ω the pointwise dimension of the sample measure ρ_ω is well defined ρ_ω almost everywhere, that is, the limit

$$d_{\rho_\omega}(x) := \lim_{r \to 0} \frac{\log \rho_\omega(B(x, r))}{\log r} \quad (\text{where } B(x, r) = \{y \in M : d(x, y) < r\})$$

exists for ρ^*-a.e. (ω, x) (see Ledrappier and Young [**101**]; the conclusion is expected to be also true without requiring the SRB property of ρ^* or, more generally, the Eckmann–Ruelle conjecture recently proved by Barreira *et al* [**22**] in the deterministic case is expected to be true for any hyperbolic invariant measure of $\mathcal{F}$, but no confirmation has been made by checking all the details for a rigorous proof). The limit is clearly constant ρ^*-a.e. or, equivalently, for P-a.e. ω the limit is equal ρ_ω-a.e. to a same value, called the *dimension* of ρ_ω and denoted by $\dim(\rho_\omega)$ (which, by a theorem in Young [**156**], coincides with the *Hausdorff dimension* of ρ_ω defined by $\dim_{\mathrm{H}} \rho_\omega := \inf\{\dim_{\mathrm{H}} Z : Z \subset M, \rho_\omega(Z) = 1\}$), and this value is equal P-a.e. to a constant. The *Lyapunov dimension* of the family of sample measures $\{\rho_\omega\}$ is defined as

$$\mathcal{D}(\lambda_1, \ldots, \lambda_r) := \begin{cases} \dim M, & \text{if } \sum_{j=1}^{K} m_j = \dim M \\ \sum_{j=1}^{K} m_j - \dfrac{1}{\lambda_{K+1}} \sum_{j=1}^{K} \lambda_j m_j, & \text{otherwise} \end{cases} \tag{2.10}$$

with K being the largest integer so that $\sum_{j=1}^{K} \lambda_j m_j \geq 0$ and $m_i = \dim E^{(i)}(\omega, x)$. If $\rho \ll \mathrm{Leb}$, then, by Proposition 2.10, one always has $\sum_i \lambda_i m_i \leq 0$ with $\sum_i \lambda_i m_i = 0$ if and only if $\rho_\omega = \rho$, P-a.e. Clearly for the special case $\sum_i \lambda_i m_i = 0$ one has $\dim(\rho_\omega) = \mathcal{D}(\lambda_1, \ldots, \lambda_r) = \dim M$, P-a.e. Under further conditions the coincidence

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of $\dim(\rho_\omega)$ and $\mathcal{D}(\lambda_1, \ldots, \lambda_r)$ is obtained for the general case in the following theorem, where we will employ the 'backward derivative process' naturally induced by $\mathcal{F}$ (with $P = \nu^{\mathbb{Z}}$) on the Grassmannian manifold $\mathrm{Gr}(M)$ whose transition probabilities are given by

$$Q(v, \Gamma) = \nu\{\omega : Tf(\omega)^{-1}v \in \Gamma\}, \quad v \in \mathrm{Gr}(M), \text{ Borel } \Gamma \subset \mathrm{Gr}(M)$$

(recall that $\mathrm{Gr}(M) = \bigcup_{k=1}^{\dim M} \mathrm{Gr}(M, k)$ and $\mathrm{Gr}(M, k)$ is the manifold of k-dimensional subspaces of TM).

THEOREM 2.11. (A dimension formula) (Ledrappier and Young [**102**]) *Let $\mathcal{F}$ be as given above and let ρ be an ergodic $\mathcal{F}$-stationary measure such that $\rho \ll$ Leb and $\lambda_i \neq 0$ for all i. Take the hypothesis that for all $v \in \mathrm{Gr}(M)$ the transition probability $Q(v, \cdot)$ is absolutely continuous with respect to the Lebesgue on $Gr(M)$. Then for P-a.e. ω*

$$\dim(\rho_\omega) = \mathcal{D}(\lambda_1, \ldots, \lambda_r). \tag{2.11}$$

The hypothesis in the theorem is a sufficient randomness condition which appears quite naturally, especially in the setting of stochastic flows arising from SDEs. If $\mathcal{F}$ is generated by the time-one maps of the solution flow of SDE (1.3), then the hypothesis is satisfied if the operator $L = -\tilde{X}_0 + \sum_{k=1}^{d} \tilde{X}_k^2$ on $C^\infty(\mathrm{Gr}(M))$ is hypoelliptic, where $\tilde{X}_k$ is the natural lifting of X_k to $\mathrm{Gr}(M)$, $0 \leq k \leq d$. In particular, if $d \geq \dim M + (\dim M)^2$, then there is an open dense subset in the space of $(d+1)$-tuples of vector fields on M on which the hypothesis is satisfied (see [**102**] for details and for further references). To obtain the desired result (2.11), the above hypothesis can be replaced by the weaker one that, for ρ-a.e. x and $j = K+1$, $K+2$ (K being given in (2.10)), the distribution of $\omega \mapsto \bigoplus_{i \geq j} E^{(i)}(\omega, x)$ is smooth on the Grassmannian manifold of $\sum_{i \geq j} m_i$-dimensional subspaces of $T_x M$, or by a nonlinear version formulated in terms of two-point processes on M. In fact, the latter are more directly related to the key ideas of the proof (which consist of estimating the transverse dimension of ρ_ω with respect to the $W^{s,j}$-manifolds).

Generalized entropy formula. For a deterministic diffeomorphism partial dimensions of an invariant measure were introduced by Ledrappier and Young [**100**, Part II] and a fundamental generalized entropy formula relating entropy, exponents and partial dimensions was given there with a highly complicated proof. Partial dimensions of the sample measures ρ_ω of ρ^* introduced at the beginning of this subsection can be defined similarly (see [**102**]) and the fundamental formula mentioned above is also expected to be true for a general $(\mathcal{F}, \mu)$ as given in Theorem 2.6 (but no rigorous confirmation has appeared). If this formula is indeed true in the random case, the dimension formula (2.11) can then be equivalently formulated in terms of the partial dimensions (see [**100**, Part II] and Eckmann and Ruelle [**59**, §IV.D] for similar arguments in the deterministic case); furthermore, this generalized entropy formula together with the expected random version of the Eckmann–Ruelle conjecture would imply in the two-dimensional case an extension of Young's notable formula in [**156**].

3. *Thermodynamic formalism*

The thermodynamic formalism in terms of deterministic dynamical systems (often with time set $\mathbb{Z}^d$, $\mathbb{R}^d$, $d \geq 1$ or another group G) has been widely developed and for this theory

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we refer to the marvellous (though austere) book by Ruelle [**138**]. In this section we review some relevant progress in the setting of RDSs. In what follows $(\Omega, \mathcal{A}, P, \tau)$ continues to be a measure-preserving system which serves as the underlying noise system of our RDSs.

3.1. *The notion of equilibrium states of bundle RDSs.*

Bundle RDSs. In this section we are very often concerned with an RDS lying only on some subsets which are 'invariant' under the random maps instead of on the entire phase space, and this leads to the notion of a bundle RDS (Bogenschütz [**30**]). Here we go a bit beyond the framework considered in the previous sections and introduce this notion in its general form. Let X be a compact metric space. Let $\{E_\omega\}_{\omega\in\Omega}$ be a family of compact subsets of X such that $\omega \mapsto E_\omega$ is measurable (with respect to the Hausdorff topology), implying that $E := \cup\{\omega\} \times E_\omega$ constitutes a measurable subset of $\Omega \times X$ (the inverse is true if $\mathcal{A}$ is P-complete, see Arnold [**6**, Proposition 1.6.2]; the reader is referred to Crauel [**50**] for a random set theory). If $\{f(\omega) : E_\omega \to E_{\tau\omega}\}_{\omega\in\Omega}$ is a family of continuous maps such that $(\omega, x) \mapsto f(\omega)x$ is a measurable map from E to X, we then call $\mathcal{F} := \{f(\omega, n) : E_\omega \to E_{\tau^n\omega} \mid n \in \mathbb{Z}^+, \omega \in \Omega\}$ a (continuous) *bundle RDS*, referring to E as a compact bundle over Ω and to the E_ω's as its fibers, where $f(\omega, n)$ are defined by (1.1). *In this subsection we will always assume that $\mathcal{F}$ is such a bundle RDS*. Typical examples of such RDSs include random subshifts of finite type and small random perturbations of hyperbolic sets, which will be discussed in §§3.2–3.4. More interesting is the case where X is taken to be a Polish space and $\{E_\omega\}_{\omega\in\Omega}$ is a random (compact) attractor of an RDS on X. Such attractors have been shown to exist for some stochastic partial differential equations such as Navier–Stokes and reaction–diffusion equations with additive noise, and their study now constitutes an important research line whose spirit coincides with that of Ruelle [**141**]. In this paper we confine ourselves to the compact phase space case and rather refer to Crauel [**50**] for a nice measure (and ergodic) theory on random closed sets in Polish spaces and to Arnold [**6**], Crauel [**51**], Crauel and Flandoli [**52, 53**], Keller and Schmalfuss [**80**] and the further references therein for a theory of random attractors. In the light of Ruelle [**141**], relations between exponents, entropy and dimension as well as their estimation might be a topic in this theory that deserves further investigation. For example, the Ruelle inequality remains a problem when the compact sets $E_{\tau^n\omega}$ leave with positive probability any bounded domain as n increases (this may happen (see [**52**]) and the set of invariant measures supported by E is always non-empty (see the next paragraph)).

We will denote by $\mathcal{M}(E, \mathcal{F})$ the collection of $\mathcal{F}$-invariant measures, i.e. those probability measures μ on $\Omega\times X$ such that μ has marginal P on Ω, $\mu(E) = 1$ and $F\mu = \mu$, where $F : E \to E$ is the skew-product transformation associated with $\mathcal{F}$ defined in the same way as (1.4). This set is always non-empty (even if X is allowed to be a Polish space, see [**50**, Corollary 6.13]). The entropy $h_\mu(\mathcal{F})$ of $(\mathcal{F}, \mu)$ for $\mu \in \mathcal{M}(E, \mathcal{F})$ is defined in the same way as in §2.2. Properties of invariant measures and of their entropies described respectively in §1 and in §2.2 for the trivial bundle $\Omega \times X$ case carry over to the present non-trivial bundle situation (see [**6**, Ch. 1], [**50, 97, 30**]).

Variational principle. We first give a review of the notion of pressure for bundle RDSs, which was first introduced by Ruelle [**135**] for a deterministic system and was adapted to

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the random case by Bogenschütz [**29**] (for the trivial bundle $\Omega \times X$ case) and Kifer [**96**] (for the non-trivial bundle case). We will use a measurable family of continuous functions on the fibers E_ω as an observable or a potential. More precisely, the space of observables or potentials will be taken as $L_E^1(\Omega, C(X))$ which consists of all families $\phi = \{\phi_\omega \in C(E_\omega)\}$ such that $(\omega, x) \mapsto \phi_\omega(x)$ is measurable on E and $\|\phi\| := \int |\phi_\omega|_\infty \, dP(\omega) < +\infty$, where $|\cdot|_\infty$ is the supremum norm (see Bogenschütz and Gundlach [**35**] for a discussion on structural properties of this space). For $\phi \in L_E^1(\Omega, C(X))$, $\varepsilon > 0$ and $n \geq 1$ put

$$\pi_{\mathcal{F}}(\phi)(\omega, n, \varepsilon) = \sup\left\{ \sum_{x \in G} \exp\left[\sum_{k=0}^{n-1} \phi_{\tau^k \omega}(f(\omega, k)x) \right] : G \text{ an } (\omega, n, \varepsilon)\text{-separated subset of } E_\omega \right\},$$

with the (ω, n, ε)-separated set being defined in an analogous way to the deterministic case (see Walters [**163**]). It is measurable in ω with respect to the P-completion of $\mathcal{A}$ and we can define

$$\pi_{\mathcal{F}}(\phi) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \int \log \pi_{\mathcal{F}}(\phi)(\omega, n, \varepsilon) \, dP(\omega). \tag{3.1}$$

The map $\pi_{\mathcal{F}} : L_E^1(\Omega, C(X)) \to \mathbb{R} \cup \{\infty\}$, $\phi \mapsto \pi_{\mathcal{F}}(\phi)$ is then called the *(topological) pressure* of $\mathcal{F}$. It can be shown that for any $\phi \in L_E^1(\Omega, C(X))$

$$\pi_{\mathcal{F}}(\phi) = \int \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \pi_{\mathcal{F}}(\phi)(\omega, n, \varepsilon) \, dP(\omega) \tag{3.2}$$

(if $(\Omega, \mathcal{A}, P, \tau)$ is ergodic then (3.2) holds true P-a.e. without taking the integral) and replacing lim sup with lim inf will result in the same pressure value (Kifer [**96**]). In some cases it is more natural to use a random variable $\varepsilon : \Omega \to (0, +\infty)$ instead of a constant one to define an (ω, n, ε)-separated set, and this will lead to the same pressure provided that the random ε belongs to a natural class of positive random variables (see Bogenschütz and Ochs [**36**] and Kifer [**96**] for details). It is shown in Gundlach [**68**] that the pressure can be equivalently defined by using (ω, n, ε)-spanning sets. See Gundlach [**67**] (this preprint contains a number of confusing points and a clear version was presented in Gundlach [**68**]) for a detailed study of the pressure (including its analyticity) of random subshifts of finite type. The following theorem is called the *variational principle* for bundle RDSs. It was first proved by Ledrappier and Walters [**103**] for the case of Ω being a compact metric space, τ being continuous and ϕ being a continuous function on E, and then by Bogenschütz [**29**] for the trivial bundle $\Omega \times X$ case under some invertibility assumptions. The present version is due to Kifer [**96**].

THEOREM 3.1. (Variational principle) *Assume that $(\Omega, \mathcal{A}, P)$ is a Lebesgue space. Then for any $\phi \in L_E^1(\Omega, C(X))$ the following holds:*

$$\pi_{\mathcal{F}}(\phi) = \sup\left\{ h_\mu(\mathcal{F}) + \int \phi \, d\mu : \mu \in \mathcal{M}(E, \mathcal{F}) \right\}. \tag{3.3}$$

Since a Lebesgue space is countably generated, by ergodic decomposition, the supremum in (3.3) can be taken over the set of ergodic measures of $\mathcal{F}$ when P is

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ergodic. As indicated in Kifer [**96**], it would be interesting to know if the Lebesgue space assumption on $(\Omega, \mathcal{A}, P)$ can be suppressed for the conclusion of Theorem 3.1. Note that, if $\mathcal{F}$ is a one-sided canonical i.i.d. RDS on a compact manifold M corresponding to $\Omega = C^r(M, M)^{\mathbb{Z}^+}$ $(0 \le r < +\infty$ integer) and $P = \nu^{\mathbb{Z}^+}$ as described in Example (i) of §1, the variational equality (3.3) holds true but in general one only has

$$\pi_{\mathcal{F}}(\phi) \ge \sup \left\{ h_{P\times\rho}(\mathcal{F}) + \int \phi \, dP \times \rho : \rho \text{ an } \mathcal{F}\text{-stationary measure} \right\} \tag{3.4}$$

and (3.4) may be a strict inequality (see Kifer [**89**] for an example and see §3.4 for a special but important case where (3.4) holds as an equality).

Any member μ of $\mathcal{M}(E, \mathcal{F})$ for which the supremum in (3.3) is attained will be called an *equilibrium state* of $\mathcal{F}$ for ϕ. As in the deterministic case, existence and uniqueness of such states as well as their statistical properties (especially their Gibbs state property) constitute a significant topic of research, and usually the uniqueness problem (concerning conditions both on $\mathcal{F}$ and ϕ) is much more subtle than the former one (the existence can be simply guaranteed for any $\phi \in L^1_E(\Omega, C(X))$ if the entropy map is upper semi-continuous). Some results have been obtained in this direction for some specific bundle RDSs which will be discussed in the subsequent subsections. Adopting ideas of Haydn and Ruelle [**72**] and of Ruelle [**145**], Gundlach and Kifer [**71**] obtained the existence and uniqueness of equilibrium states and their coincidence with Gibbs states for a natural class of potentials in the case of general bundle RDSs possessing appropriate expansiveness and specification properties.

3.2. *Random subshifts of finite type.*

Notion. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ be the one-point compactification of $\mathbb{N} = \{1, 2, \dots\}$, put $X = \prod_{i=0}^{+\infty} \bar{\mathbb{N}}$ and consider the shift operator $\sigma : X \to X$ defined by $(\sigma x)_i = x_{i+1}$ for $x = (x_i) \in X$. X is compact metrizable, for example, by the metric $d(x, y) := \sum_{i=0}^{+\infty} 2^{-i} |x_i^{-1} - y_i^{-1}|$, $x = (x_i)$, $y = (y_i) \in X$, with the convention $\infty^{-1} = 0$. On the subset $\{x \in X : x_i < +\infty \text{ for all } i \in \mathbb{Z}^+\}$ one can fix arbitrarily $0 < \theta < 1$ and introduce another metric $\hat{d}_\theta(x, y) := \inf\{\theta^n : n \in \mathbb{Z}^+, x \overset{n}{\sim} y\}$ (where $x \overset{0}{\sim} y :\Leftrightarrow x_0 \ne y_0$ and $x \overset{n}{\sim} y :\Leftrightarrow x_i = y_i$, $0 \le i \le n-1$, for $n \ge 1$) which is topologically equivalent to $d(\cdot, \cdot)$ but with respect to which the shift σ is locally expanding. Let $k : \Omega \to \mathbb{N}$ be a random variable with $\log k \in L^1(\Omega, P)$ and let $A = \{A(\omega) = (a_{ij}(\omega))\}_{\omega\in\Omega}$ be a random transition matrix, i.e. a family of $k(\omega) \times k(\tau\omega)$ matrices $A(\omega)$ with measurable (in ω) entries 0 or 1 and with at least one non-zero entry in each row and each column. Put

$$\Sigma_A(\omega) = \{x \in X : x_i \le k(\tau^i\omega) \text{ and } a_{x_i x_{i+1}}(\tau^i\omega) = 1 \text{ for all } i \in \mathbb{Z}^+\}.$$

Then the bundle RDS determined by $\{\sigma : \Sigma_A(\omega) \to \Sigma_A(\tau\omega)\}$ is called a *(one-sided) random subshift of finite type* (or a random topological Markov chain) and will be referred to as σ_A, with $\Sigma_A := \cup\{\omega\} \times \Sigma_A(\omega)$ being the corresponding bundle. Assuming the invertibility of $(\Omega, \mathcal{A}, P, \tau)$, the two-sided version can be defined analogously. When k and A are constant P-a.e., one gets the deterministic shift systems, which have been analyzed since, several decades ago both for their intrinsic topological and ergodic-theoretical properties and for applications to the thermodynamic formalism of statistical

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mechanics, to uniformly hyperbolic and other dynamical systems, or to other fields (the reader is referred to the monograph Baladi [**16**] for a comprehensive treatment of this topic). As will be seen, random shift systems exhibit some analogous pictures. The following mixing condition on σ_A will often be assumed: A is called *aperiodic* if there exists a random variable $M : \Omega \to \mathbb{N}$ such that $A(\omega)\cdots A(\tau^{M(\omega)}\omega)$ has no zero entries P-a.e.

Gibbs states. Gibbs distributions (measures) on a symbolic space are of great interest from the viewpoint of statistical mechanics, but they may not be invariant under the shift operator. The Ruelle transfer operator provides a powerful tool to find shift-invariant measures which are absolutely continuous with respect to a Gibbs measure (for a potential) and to investigate their statistical properties such as the decay of correlations etc. (for extensions to other settings of the Ruelle transfer operator see §3.4). Under some mixing conditions on the shift and under some Hölder (i.e. exponential decay) or summability conditions on the potential, the Gibbs measure and the absolutely continuous shift-invariant measure often uniquely exist and the latter also has a kind of Gibbs property and is the unique equilibrium state for the potential (we refer to Sinai [**153**] and to the excellent monographs Ruelle [**138**], Bowen [**39**] and Baladi [**16**] for precise statements and for historical background in the case of a deterministic shift system). In what follows we review some analogous results in the random setting, mainly due to Bogenschütz and Gundlach [**35**], Gundlach [**65, 68**] and Khanin and Kifer [**81**] (see these papers for further references).

We consider below the one-sided random shifts and, for many purposes, problems in the two-sided case can be reduced to the former one via a cohomology approach [**68**] (this is useful for applying the Ruelle transfer operator method since the former is locally expanding; here we note that the transfer operator method can also be used for hyperbolic systems (where expansion and contraction coexist) by 'quotienting out' the non-expanding directions, see Liverani [**111**], Viana [**162**, Ch. 4] and see Rugh [**151**] etc. for related strategy). Let σ_A be given. For any given $\alpha \in (0,1)$ denote by $H_A^1(\alpha)$ the set of those families $\phi \in L^1_{\Sigma_A}(\Omega, C(X))$ (see §3.1 for the definition) such that for some constant $c > 0$ one has P-a.e. $|\phi_\omega(x) - \phi_\omega(y)| \le c\alpha^n$ whenever $x \overset{n}{\sim} y$ (such a ϕ is also called *equi-Hölder continuous* (with respect to $\hat{d}_\theta(\cdot,\cdot)$), α may be referred to as the decay rate of ϕ). A probability ν on Σ_A with marginal P on Ω is called a *Gibbs measure* (with the empty boundary condition on the left) for $\phi \in H_A^1(\alpha)$ if for P-a.e. ω the conditional probability $\nu_{\omega,n}(\cdot|y)$ of ν_ω on $\{x \in \Sigma_A(\omega) : \sigma^n x = y\}$ is given by

$$\nu_{\omega,n}(x|y) = P_{\omega,n}(x|y) := \exp\left[\sum_{k=0}^{n-1}\phi_{\tau^k\omega}(\sigma^k x)\right] \Big/ \sum_{z\in\Sigma_A(\omega):\sigma^n z=y} \exp\left[\sum_{k=0}^{n-1}\phi_{\tau^k\omega}(\sigma^k z)\right] \tag{3.5}$$

for $\sigma^n\nu_\omega$-a.e. $y \in \Sigma_A(\tau^n\omega)$ and for all $n \in \mathbb{N}$. A measure μ on Σ_A is called a *ϕ-Gibbs state* if $\mu \in \mathcal{M}(\Sigma_A, \sigma_A)$ (i.e. μ is an invariant measure of the bundle RDS σ_A) and $\mu_\omega \ll \nu_\omega$ for P-a.e. ω, where ν is a Gibbs measure for $\phi \in H_A^1(\alpha)$. To construct such measures we now introduce the random Ruelle transfer operators $\mathcal{L}_\phi(\omega) : C(\Sigma_A(\omega)) \to C(\Sigma_A(\tau\omega))$

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associated with $\phi \in H_A^1(\alpha)$ which are defined by

$$(\mathcal{L}_\phi(\omega)g)(x) = \sum_{y \in \Sigma_A(\omega):\sigma y = x} e^{\phi_\omega(y)} g(y) \quad \text{for all } g \in C(\Sigma_A(\omega)), \tag{3.6}$$

and we will write $\mathcal{L}_\phi(\omega, n) = \mathcal{L}_\phi(\tau^{n-1}\omega) \circ \cdots \circ \mathcal{L}_\phi(\omega)$ and denote by $\mathcal{L}_\phi^*(\omega)$ the dual of $\mathcal{L}_\phi(\omega)$ acting on finite signed measures m on $\Sigma_A(\tau\omega)$ by $\int g\, d\mathcal{L}_\phi^*(\omega)m := \int \mathcal{L}_\phi(\omega)g\, dm$ for all $g \in C(\Sigma_A(\omega))$ (see, for example, Przytycki and Urbański [**127**, Ch. 3] for a description in the deterministic case on why it is natural to consider these kinds of operators; the random ones were first employed by Ferrero and Schmitt [**60**] and Kifer [**89**] in some simple or limited situations). Following the pioneering work of Ruelle [**134**] but using projective metric techniques (instead of Ruelle's Arzela–Ascoli-type argument), the theorem that follows was proved by [**35**] for the case where A is uniformly aperiodic (i.e. $M(\cdot) = \text{Const.}$ P-a.e.) and in the present form by Gundlach [**65, 68**].

THEOREM 3.2. (The Random Ruelle–Perron–Frobenius theorem) *Assume that* $(\Omega, \mathcal{A}, P, \tau)$ *is measurably invertible and ergodic, with* $\mathcal{A}$ *being countably generated. Assume that* A *is aperiodic. Let* $\phi \in H_A^1(\alpha)$. *If*

$$|\log \mathcal{L}_\phi(\omega, M(\omega))\mathbf{1}|_\infty \in L^1(\Omega, P) \tag{3.7}$$

(recall that $|\cdot|_\infty$ *is the supremum norm; (3.7) holds true, for example, when* $M(\cdot) = \text{Const.}$ P*-a.e. since* $\log k \in L^1(\Omega, P)$*), then there exist a positive random variable* λ *with* $\log\lambda \in L^1(\Omega, P)$*, a family of positive functions* $h = \{h_\omega \in C(\Sigma_A(\omega))\}$ *with* $\log h := \{\log h_\omega\} \in H_A^1(\alpha)$*, and probabilities* ν, μ *on* Σ_A *with marginal* P *on* Ω *such that the following hold true for* P*-a.e.* ω.

(i) $\mathcal{L}_\phi(\omega)h_\omega = \lambda(\omega)h_{\tau\omega}$, $\mathcal{L}_\phi^*(\omega)\nu_{\tau\omega} = \lambda(\omega)\nu_\omega$, $\int h_\omega\, d\nu_\omega = 1$ *and*

$$\lim_{n\to+\infty} \left| \lambda(\omega, n)^{-1}\mathcal{L}_\phi(\omega, n)g - h_{\tau^n\omega} \int g\, d\nu_\omega \right|_\infty = 0 \quad \textit{for all } g \in C(\Sigma_A(\omega)) \tag{3.8}$$

(where $\lambda(\omega, n) := \lambda(\tau^{n-1}\omega)\cdots\lambda(\omega)$*) with exponential speed of convergence for* g *in a dense subset* $\mathcal{V}(\omega)$ *of* $C(\Sigma_A(\omega))$ *(see [**35, 68**] for the definition of* $\mathcal{V}(\omega)$ *which depends on the Hölder constant* c *and the decay rate* α *of* ϕ *and see [**35**, Proof of Proposition 3.14] and [**68**, Proof of Proposition 4.4.2] for an explicit estimate of the exponential speed;* $\mathcal{V}(\omega)$ *contains Hölder continuous functions* g *with decay rate* α *and one can check this by writing* $g = (g^+ + C) - (g^- + C)$ *for a sufficiently large* C*).*

(ii) *Define* $\Phi = \{\Phi_\omega\}$ *by* $\Phi_\omega(x) = \phi_\omega(x) + \log h_\omega(x) - \log h_{\tau\omega}(\sigma x) - \log\lambda(\omega)$. *Then* $\mathcal{L}_\Phi(\omega)\mathbf{1} = \mathbf{1}$, $\mathcal{L}_\Phi^*(\omega)\mu_{\tau\omega} = \mu_\omega$, $d\mu_\omega = h_\omega\, d\nu_\omega$ *and*

$$\lim_{n\to+\infty} \left| \mathcal{L}_\Phi(\omega, n)g - \int g\, d\mu_\omega \right|_\infty = 0 \quad \textit{for all } g \in C(\Sigma_A(\omega)) \tag{3.9}$$

with the same exponential convergence property as (3.8).

All the objects are P *almost all uniquely determined by these properties.*

This theorem has a lot of corollaries (see [**35, 65, 68**] for proofs). First, ν is a ϕ-Gibbs measure and can be given by

$$\int g\, d\nu_\omega = \lim_{n\to+\infty} \frac{\mathcal{L}_\phi(\omega, n)g}{\mathcal{L}_\phi(\omega, n)\mathbf{1}}, \quad \forall g \in C(\Sigma_A(\omega));$$

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μ is a ϕ-Gibbs state and a Φ-Gibbs measure, which has the following kind of Gibbs property: there exist $c_1, c_2 > 0$ such that P-a.e.

$$c_1 \le \frac{\mu_\omega\{y \in \Sigma_A(\omega) : y_i = x_i \text{ for } i = 0, 1, \ldots, m-1\}}{\exp\left[\sum_{k=0}^{m-1} \Phi_{\tau^k\omega}(\sigma^k x)\right]} \le c_2 \tag{3.10}$$

for all $m \in \mathbb{N}$ and all $x \in \Sigma_A(\omega)$. Furthermore, $\pi_{\sigma_A}(\phi) = \int \log \lambda \, dP$, the variational principle holds for (σ_A, ϕ) (without requiring $(\Omega, \mathcal{A}, P)$ to be Lebesgue) and μ is the unique equilibrium state of σ_A for ϕ. (The uniqueness of the ϕ-Gibbs measure and state was proved in [**81**], see below. See [**35, 68**] for the notion, existence, uniqueness and the equivalence with the equilibrium state of an invariant *quasi-Gibbs measure*, μ is in fact such a measure for ϕ.) (3.9) implies that (σ_A, μ) is *randomly mixing* in the sense that P-a.e.

$$\lim_{n\to+\infty} \left| \int (\psi_{\tau^n\omega} \circ \sigma^n) g \, d\mu_\omega - \int \psi_{\tau^n\omega} \, d\mu_{\tau^n\omega} \int g \, d\mu_\omega \right| = 0 \tag{3.11}$$

for any $\phi \in L^1_{\Sigma_A}(\Omega, C(X))$ and any $g \in C(\Sigma_A(\omega))$ (see [**68**] for refined mixing properties), with exponential speed of convergence for $g \in \mathcal{V}(\omega)$. This type of fast correlation decay property can lead to a ω-wise central limit theorem for random Hölder continuous observables under some further integrability conditions, see Kifer [**94**] (or see Kifer [**95**] for a short review). It was shown in [**68**] that $\hat{\mathcal{L}}_\Phi : H^1_A(\alpha) \to H^1_A(\alpha)$ defined by $(\hat{\mathcal{L}}_\Phi \psi)_\omega = \mathcal{L}_\Phi(\tau^{-1}\omega)\psi_{\tau^{-1}\omega}$ is well defined, it has 1 as a simple isolated eigenvalue and the rest of the spectrum is contained in a disc of radius strictly smaller than 1; however, as compared with the deterministic case (see, for example, Baladi [**16**]), much less has been known about the spectral properties of the operators $\mathcal{L}_\phi$ and $\mathcal{L}_\Phi$ and about their relation to the convergence speed of (3.11) (see Bogenschütz [**32**] and Baladi *et al* [**17**] for related works in other settings of RDSs).

We remark that for two-sided random subshifts the construction of Gibbs states (see [**81**] for the definition which concerns boundary conditions) and of equilibrium states can be reduced to the one-sided case via the cohomology approach (due originally to Sinai [**153**]) with the properties (3.10) and (3.11) etc. being preserved (up to minor changes) under the pull-back (the two-sided system is in fact the natural extension of the one-sided one; see [**68**] for details). This can also be done by including the two-sided shifts in the general set-up of RDSs in Gundlach and Kifer [**71**] satisfying some expansiveness and specification conditions.

The assumption on the equi-Hölder continuity of the potentials ϕ seems too strong and a more natural one is to allow the Hölder constant c of ϕ_ω's to be a random variable $c : \Omega \to [1, +\infty)$ with $\log c \in L^1(\Omega, P)$. Under this and other more natural integrability conditions the existence and uniqueness of Gibbs measures and states as well as the unique equilibrium state and ergodicity properties of the latter were proved by Khanin and Kifer [**81**] both for the one-sided and the two-sided cases (the Gibbs measure and state coincide in the latter case), both using the Ruelle operator and direct statistical mechanics methods. See Maume Deschamps [**114**] for extensions to non-Hölderian potentials.

Other topics and applications. See Kifer [**95**] for a short review on large deviation aspects of random subshifts. See Kifer [**92**] for multi-dimensional extensions, and see Gundlach

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and Kifer [**69, 71**] for random sofic shifts. As reviewed in [**95**], such set-ups have applications when studying randomly evolving graphs or random networks. Interesting applications of the Gibbs state theory of random subshifts were made by Demetrius and Gundlach [**55**] in their investigations of evolutionary population dynamics. See the next subsection for applications to random hyperbolic systems, and see Kifer [**91**] for applications in dimension theory of random iterated function systems.

3.3. *Random hyperbolic diffeomorphisms.* This subsection mainly concerns *hyperbolic* dynamics and *expanding* ones will be discussed in §3.4.

Random perturbations of Axiom A basic sets. We first consider small random diffeomorphism-type perturbations of hyperbolic sets of deterministic diffeomorphisms. This was first studied by Young [**157**] and we will see that this provides the strongest possible set-up of hyperbolic bundle RDSs (Liu [**106**]). In this subsection let M be a Riemannian manifold without boundary and O an open subset of M with compact closure. Let $f : O \to M$ be a C^r ($r \geq 1$ integer) diffeomorphism to the image $f(0)$. For convenience of the reader we recall some related concepts from deterministic dynamical systems. A compact set $\Lambda_0 \subset O$ is called a *hyperbolic set* of f if it is f-invariant (i.e. $f\Lambda_0 = \Lambda_0$) and there is a Tf-invariant splitting $T_{\Lambda_0}M = E^s \oplus E^u$ together with $0 < \lambda_0 < 1$ such that for some constant $C_0 > 0$ and all $n \geq 0$

$$|Tf^n\xi| \leq C_0\lambda_0^n|\xi|, \ \forall \xi \in E^s; \quad |Tf^n\eta| \geq C_0^{-1}\lambda_0^{-n}|\eta|, \ \forall \eta \in E^u.$$

A hyperbolic set Λ_0 of f is called an *Axiom A basic set* if it is locally maximal (i.e. $\bigcap_{n=-\infty}^{+\infty} f^n U = \Lambda_0$ for some neighborhood U of Λ_0) and f is topologically transitive on it. Now let Λ_0 be a hyperbolic set of f and consider the following random perturbation model of (f, Λ_0).

Assume in this subsection that $(\Omega, \mathcal{A}, P, \tau)$ is measurably invertible and ergodic with $\mathcal{A}$ being countably generated. Let

$$\mathcal{F} : (\Omega, \mathcal{A}, P) \to \mathcal{U}(f)$$

be a random variable taking values in an open neighborhood $\mathcal{U}(f)$ of f in $C^r(O, M)$, with $f(\omega, n)$, $\omega \in \Omega$, $n \in \mathbb{Z}$ being defined by (1.1), but being defined wherever they make sense (recall that $f(\omega) := \mathcal{F}(\omega)$). Standard structural stability arguments yield that, if $\mathcal{U}(f)$ is given sufficiently small, then for some constant $\varepsilon_0 > 0$ there exists a unique measurable (in ω) family of compact sets $\{\Lambda_\omega \subset O\}_{\omega\in\Omega}$ such that $f(\omega)\Lambda_\omega = \Lambda_{\tau\omega}$ for all $\omega \in \Omega$ and that there is a measurable family of homeomorphisms $\{h_\omega : \Lambda_0 \to \Lambda_\omega\}_{\omega\in\Omega}$ which satisfies $d(h_\omega, \mathrm{id}) < \varepsilon_0$, $\forall \omega \in \Omega$, and makes the following diagram commutative for each ω,

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{f} & \Lambda_0 & \xrightarrow{f} & \Lambda_0 & \xrightarrow{f} & \cdots \\
 & & \downarrow{\scriptstyle h_\omega} & & \downarrow{\scriptstyle h_{\tau\omega}} & & \\
\cdots & \xrightarrow{f(\tau^{-1}\omega)} & \Lambda_\omega & \xrightarrow{f(\omega)} & \Lambda_{\tau\omega} & \xrightarrow{f(\tau\omega)} & \cdots
\end{array}$$

(see Ruelle [**147**], Liu [**106**]). This picture results in a bundle RDS $\{f(\omega, n) : \Lambda_\omega \to \Lambda_{\tau^n\omega}\}_{n\in\mathbb{Z},\omega\in\Omega}$ which will be denoted by $\mathcal{F}_\Lambda$, with $\Lambda = \bigcup_{\omega\in\Omega}\{\omega\} \times \Lambda_\omega$ denoting the

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corresponding bundle. For small $\mathcal{U}(f)$, $\mathcal{F}_\Lambda$ is *hyperbolic* in the following sense: there exist measurable (in (ω, x)) splittings $T_xM = E^s_{(\omega,x)} \oplus E^u_{(\omega,x)}$, $(\omega, x) \in \Lambda$, such that for some constants $C > 0$, $\lambda_0 \le \lambda < 1$ and for all $n \ge 0$

$$|T_x f(\omega, n)\xi| \le C\lambda^n|\xi|, \ \forall \xi \in E^s_{(\omega,x)}; \quad |T_x f(\omega, n)\eta| \ge C^{-1}\lambda^{-n}|\eta|, \ \forall \eta \in E^u_{(\omega,x)}.$$

(See Liu [**106**] for some further structural stability and stable manifolds statements.)

If Λ_0 is an Axiom A basic set, then it can be modeled by a (deterministic) subshift of finite type via a Markov partition. More precisely, there exists a two-sided topologically transitive deterministic subshift $(\bar{\Sigma}_A, \bar{\sigma})$ and a Lipschitz (with respect to $\hat{d}_\theta(\cdot,\cdot)$ for a suitable $0 < \theta < 1$) surjective map $\pi : \bar{\Sigma}_A \to \Lambda_0$ such that $f \circ \pi = \pi \circ \bar{\sigma}$, with π being one-to-one over the set of points in Λ_0 whose orbits do not hit the boundaries of the rectangles of the Markov partition. Via this machinery, thermodynamic formalism of statistical mechanics was applied to Axiom A diffeomorphisms by Sinai [**153**] (for Anosov diffeomorphisms) and by Ruelle [**136**] (for Axiom A attractors and for the formalism of equilibrium states), leading to a successful ergodic theory of the latter (see Bowen [**39**] for an excellent systematic treatment). For the case of the bundle RDS $\mathcal{F}_\Lambda$, the deterministic shift $(\bar{\Sigma}_A, \bar{\sigma})$ can also be used as a symbolic representation of $\mathcal{F}_\Lambda$ via the family of maps $\{\pi_\omega := h_\omega \circ \pi\}$ which is uniformly Lipschitz (with respect to $\hat{d}_{\theta'}(\cdot,\cdot)$ for a suitable $0 < \theta' < 1$) due to the uniform expansivity of the random maps in $\mathcal{F}_\Lambda$. This allows one to apply the thermodynamic formalism of random subshifts reviewed in §3.2 to $\mathcal{F}_\Lambda$ (although the shift $(\bar{\Sigma}_A, \bar{\sigma})$ is deterministic, the potentials and the Gibbs states are random now), obtaining an ergodic theory of $\mathcal{F}_\Lambda$ parallel to Bowen [**39**]. In particular, we have the following result [**106, 110**], where a family of functions $\phi = \{\phi_\omega \in C(\Lambda_\omega)\} \in L^1_\Lambda(\Omega, C(\overline{O}))$ is called *equi-Hölder continuous* if there are constants $c > 0$ and $\alpha > 0$ such that $|\phi_\omega(x) - \phi_\omega(y)| \le cd(x, y)^\alpha$ for any $x, y \in \Lambda_\omega$ and for P-a.e. ω.

THEOREM 3.3. *Let Λ_0 be an Axiom A basic set of f.*

(i) (Equilibrium states) *Let $r = 1$ and let $\mathcal{U}(f)$ be a sufficiently small open neighborhood of f in $C^1(O, M)$. Then for any equi-Hölder continuous $\phi \in L^1_\Lambda(\Omega, C(\overline{O}))$ there exists a unique equilibrium state, written μ_ϕ, of $\mathcal{F}_\Lambda$ for ϕ. μ_ϕ is ergodic and, if (f, Λ_0) is topologically mixing, $(\mathcal{F}_\Lambda, \mu_\phi)$ is further randomly mixing in a sense analogous to (3.11) and enjoys the ω-wise CLT (central limit theorem) for any Hölder continuous observable $\psi : \overline{O} \to \mathbb{R}$ (see below for the precise meaning).*

(ii) (Generalized SRB measure) *Let $r = 2$ and let $\mathcal{U}(f)$ be a sufficiently small open neighborhood of f in $C^2(O, M)$. Then $J^{(u)} = \{J^{(u)}_\omega \in C(\Lambda_\omega)\}$ defined by $J^{(u)}_\omega(x) = -\log|\det(T_x f(\omega)|_{E^u_{(\omega,x)}})|$ is an equi-Hölder continuous element of $L^1_\Lambda(\Omega, C(\overline{O}))$ and the unique equilibrium state of $\mathcal{F}_\Lambda$ for $J^{(u)}$, written $\mu^{(u)}$ (and called the* generalized SRB measure *of $\mathcal{F}_\Lambda$), satisfies*

$$\pi_{\mathcal{F}_\Lambda}(J^{(u)}) = h_{\mu^{(u)}}(\mathcal{F}_\Lambda) - \sum_{\lambda^{(i)}>0} \lambda^{(i)} m^{(i)}$$

where $\{(\lambda^{(i)}, m^{(i)}) : 1 \le i \le r\}$ is the Lyapunov spectrum of $(\mathcal{F}_\Lambda, \mu^{(u)})$.
If Λ_0 is an Axiom A attractor, then $\pi_{\mathcal{F}_\Lambda}(J^{(u)}) = 0$ and hence $\mu^{(u)}$ satisfies the Pesin formula $h_{\mu^{(u)}}(\mathcal{F}_\Lambda) = \sum_{\lambda^{(i)}>0} \lambda^{(i)} m^{(i)}$ or, equivalently, $\mu^{(u)}$ is an SRB measure of

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*$\mathcal{F}_\Lambda$. $\mu^{(u)}$ has some kind of genericity with respect to the Lebesgue measure (see Liu [**106**] for details and see Ruelle [**147**] for a time-dependent but non-probabilistic treatment which is perhaps more suitable for applications in physics).*

(iii) (Stochastic stability) *Let $r = 2$. When the neighborhood $\mathcal{U}(f)$ of f in $C^2(O, M)$ shrinks to $\{f\}$, the generalized SRB measure $\mu^{(u)}$ of $\mathcal{F}_\Lambda$ converges weakly to $P \times \mu_0^{(u)}$, where $\mu_0^{(u)}$ is the generalized SRB measure of f on Λ_0, and, in particular, the marginal of $\mu^{(u)}$ on $\overline{O}$ converges weakly to $\mu_0^{(u)}$ (see [**110**] for precise statements).*

The CLT statement in (i) of the above theorem means that for any Hölder continuous observable $\psi : \overline{O} \to \mathbb{R}$ it holds P-a.e. that for any $a \in \mathbb{R}$, with μ denoting μ_ϕ for short,

$$\lim_{n\to+\infty} \mu_\omega \left\{ x \in \Lambda_\omega : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[\psi(f(\omega, k)x) - \int \psi \, d\mu_{\tau^k\omega} \right] \le a \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-t^2/2\sigma^2} \, dt \tag{3.12}$$

where $\sigma \ge 0$ is given by

$$\sigma^2 := \lim_{n\to+\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} \left[\psi(f(\omega, k)x) - \int \psi \, d\mu_{\tau^k\omega} \right] \right)^2 d\mu_\omega$$

(being constant P-a.e.), and in the case when $\sigma = 0$ the right-hand side of (3.12) is understood to be the Dirac measure at zero. To prove this (and possibly a stronger) result, note that μ_ϕ is in fact the projection of the Gibbs state of the shift $(\bar{\Sigma}_A, \bar{\sigma})$ for the potential $\phi^* = \{\phi_\omega \circ h_\omega \circ \pi\}$, and it is then enough (and easy) to check that the CLT for random subshifts proved in Kifer [**94**] applies in the present setting. In view of the volume lemma (see Liu [**106**]), it seems that some large deviation results could be proved for $\mathcal{F}_\Lambda$ as done by Kifer [**88, 89**] for random expanding maps. Stochastic stability of the SRB measure of an Axiom A attractor with respect to i.i.d. random diffeomorphism-type perturbations was earlier proved by Young [**157**] via a different approach (i.e. by the method of pushing forward the Lebesgue measure on a piece of unstable manifold), and this is a weaker form of (iii) of Theorem 3.3. A similar result was also obtained by Kifer [**90**] via a variational argument and see Tsujii [**155**] for a similar treatment. Stochastic stability of many other statistical properties, for example, the decay rate of correlations, and geometrical such as dimensional properties of a deterministic hyperbolic system could be expected (for this see the comprehensive book Pesin [**125**]); see Blank *et al* [**28**] for some results on the former aspect of two-dimensional hyperbolic maps.

For perturbations of continuous time hyperbolic systems by flows of random diffeomorphisms arising from random or stochastic differential equations, the problem is more difficult and no successful ergodic theory (such as that given by Bowen and Ruelle [**40**] for Axiom A flows) has been elaborated (see Gundlach and Kifer [**70**] for a first attack on this problem). For perturbations of such systems by continuous time Markov processes we refer to Kifer [**83, 86**].

General random hyperbolic sets. The set-up discussed above is too strong because all the constants related to the hyperbolicity of the random maps are non-random and this does

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not even include the simple RDS defined by $f(\omega) := f^{n(\omega)}$, where f is a deterministic hyperbolic map and $n : \Omega \to \mathbb{Z}^+$ is a random variable. More general random hyperbolic sets allowing random contraction and expansion rates were introduced and studied by Gundlach and Kifer [**70**] and further examples of such systems include random homoclinic points (Gundlach [**66**]) and random torus automorphisms (see [**70**], they are essentially due to a possibly unpublished work Arnoux and Fisher [**10**]) (to repeat, the expanding case will be discussed in §3.4). An ergodic theory of such random hyperbolic sets was developed by [**70**] via constructing random Markov partitions, but difficulties occur when transferring symbolic results back to the original system and some results can perhaps only be formulated at the symbolic level. Methods without involving Markov partitions could be expected, as developed in a number of works in the deterministic case, for example, Young [**159**], Blank *et al* [**28**] and Leplaideur [**104**], and for relevant works in the random case we mention, Young [**157**], Viana [**161**] and also [**28**].

Stochastic stability of non-uniformly hyperbolic systems. Many interesting results have been obtained, as mentioned in §2.2, on the existence of SRB measures as well as on their statistical properties in non-uniformly or partially hyperbolic deterministic systems. Whether the scenarios of these results are stable with respect to random perturbations constitutes a very interesting topic, and this subject is closely related to the important conjecture of Palis [**120**] (see Araújo [**5**] for important contributions in the direction of proving the stochastic stability part of the conjecture). In addition to considering random map perturbations, one can also consider another type of random perturbations, i.e. perturbations by Markov chains, which seems more natural (but less rich from the viewpoint of dynamical systems) than the former. More precisely, let $f : M \to M$ be a dynamical system preserving a probability μ_0. A random perturbation $\mathcal{F}$ can be taken as a Markov chain with transition probabilities $P(x, \cdot)$, $x \in M$ (one could think of $\mathcal{F}$ as 'being sufficiently random' if $P(x, \cdot)$, $x \in M$, has a density with respect to the Lebesgue measure on M). The system (f, μ_0) is said to be *stochastically stable with respect to Markov chain perturbations* if, for any one-parameter family of random perturbations $\{\mathcal{F}_\varepsilon : \varepsilon > 0\}$ which satisfies $P_\varepsilon(x, \cdot) \to \delta_{fx}$ weakly as $\varepsilon \to 0$ (one could also consider some special perturbations), the stationary measures μ_ε of $\mathcal{F}_\varepsilon$ converge weakly to μ_0 as $\varepsilon \to 0$. Following Sinai's work [**153**] on Anosov systems, SRB measures of Axiom A attractors have also been proved to be stochastically stable in this sense by Kifer; we refer to Kifer [**86**] for a systematic treatment of the topic and for relevant results on some other one-dimensional dynamical systems, and we refer to Metzger [**115**] for stochastic stability of this kind for Lorenz maps and flows. However, few results of stochastic stability (considered maybe in both senses) have been known for multi-dimensional non-uniformly or partially hyperbolic maps, and any positive or negative answer in this direction would be desired. Non-uniformly hyperbolic systems considered in Hu [**74, 75**], Jacobson and Newhouse [**77**], Pesin [**124**] etc. and partially hyperbolic systems considered in Pesin and Sinai [**126**] (including time-1 maps of Axiom A flows) as well as those reviewed in Palis [**120**] and Viana [**162**] might be good candidates to be studied. It was announced in Viana [**161**, Ch. 6] (see also Palis [**120**]) that the unique SRB measure supported by the so-called Hénon-like attractor of the map

$$f = f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (1 - ax^2 + y, bx),$$

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where (a, b) is from the 'good' parameter set $E \subset \mathbb{R}^2$ which is of positive Lebesgue measure (see Benedicks and Carleson [**25**], Benedicks and Young [**27**]), has been proved by Benedicks and Viana to be stochastically stable with respect to Markov chain perturbations $\mathcal{F}_\varepsilon$, $\varepsilon > 0$ whose transition probabilities $P_\varepsilon((x, y), \cdot)$, $(x, y) \in \mathbb{R}^2$, are given by

$$P_\varepsilon((x, y), A) = \nu_\varepsilon\{\omega_0 : f_{\omega_0}(x, y) := f(x, y) + \omega_0 \in A\} \quad \text{for Borel } A \subset \mathbb{R}^2,$$

where ν_ε is a probability on the ε-neighborhood of $0 \in \mathbb{R}^2$ with $\nu_\varepsilon \ll$ Leb and with $d\nu_\varepsilon/d$ Leb being uniformly bounded away from zero and infinity. It would be nice to know if the interesting random correlation decay results obtained by Baladi and Viana [**20**] and Baladi *et al* [**17**] for unimodal maps could be extended to this two-dimensional case.

It is very interesting to see that, based on random perturbations, Chernov [**47**] and Liverani *et al* [**112, 113**] developed a new approach which is quite efficient to investigate statistical properties of non-uniformly hyperbolic deterministic maps.

Finally, we use this place to mention that Freidlin and Wentzell [**62**] elaborated a theory of random perturbations of dynamical systems of a different (and more realistic) nature, which mainly concerns stochastic stability of solutions of differential equations on a finite time interval.

3.4. *Random expanding maps.* In this subsection we mostly consider canonical RDSs as introduced in §1. For general RDSs some of the results presented below could be true at least when $(\Omega, \mathcal{A}, P)$ is Polish or is Lebesgue.

Random perturbations of expanding maps. Let M be as given in §1 and let $f : M \to M$ be a $C^{r,\alpha}$ ($1 \le r < +\infty$ integer, $\alpha \in (0, 1]$) map which is *expanding*, i.e. there exists $\lambda > 1$ such that $|Tf\xi| \ge \lambda|\xi|$ for all $\xi \in TM$. It is well known that f admits a unique absolutely continuous invariant measure $d\mu_0 = h_0 d$ Leb and (f, μ_0) is exact. The density h_0 can be obtained by showing that the Ruelle transfer operator $\mathcal{L}_f : H \to H$, where H is a suitable Banach space of functions on M ($H = C^{r'}(M)$ or $C^{r',\alpha'}(M)$ ($0 \le r' < r$ integer, $0 < \alpha' \le \alpha$), for example) and $\mathcal{L}_f$ is defined by

$$(\mathcal{L}_f h)(x) = \sum_{y: fy = x} \frac{h(y)}{|\det T_y f|} \quad \text{for } h \in H, \tag{3.13}$$

has 1 as a simple eigenvalue and by taking h_0 as its normalized eigenfunction. Moreover, the spectral properties of $\mathcal{L}_f$ are closely related to statistical properties of (f, μ_0), especially to the exponential mixing rate, written as $\tau_{0,H}$ (which depends subtly on H), of (f, μ_0) for observables in H, which is defined as the smallest number such that for each $\tau > \tau_{0,H}$ and each pair $\phi, \psi \in H$,

$$\left| \int (\phi \circ f^n)\psi \, d\mu_0 - \int \phi \, d\mu_0 \int \psi \, d\mu_0 \right| \le C\tau^n, \quad \forall n \ge 1 \tag{3.14}$$

for some constant $C = C(\tau, \phi, \psi)$, and to other dynamical properties of f. We refer to the celebrated paper Ruelle [**143**] and survey papers Ruelle [**146**] and Baladi [**15**] for an abundance of results on spectral theory of more generally weighted transfer operators of f and on its relation to analytic properties of dynamical zeta functions or Fredholm determinants.

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Stochastic stability of (f, μ_0) and, more generally, of spectral properties of $\mathcal{L}_f$ has been studied by several authors. Baladi and Young [**21**] considered i.i.d. random perturbations by *time-ε-maps of stochastic flows* of a C^r ($2 \le r < +\infty$ integer) expanding map f with $\mathcal{L}_f$ acting on $H = C^{r-1}(M)$ (in this case $\mathcal{L}_f$ has 1 as a maximal and simple eigenvalue, it is quasi-compact with essential spectral radius bounded from above by $\lambda^{-(r-1)}$ and $\tau_{0,H} = |\sigma_0| := \sup\{|z| : z \in \sigma_0\}$ where $\sigma_0 \cup \{1\}$ is the spectrum of $\mathcal{L}_f$, see Ruelle [**143**]). More precisely, letting $\{\phi_t : (W, \mathcal{B}(W), P_W) \to \mathrm{Diff}^\infty(M)\}_{t\ge 0}$ be the stochastic flow arising from SDE (1.3) and letting $\varepsilon > 0$, the perturbation, written $\mathcal{F}_\varepsilon$, was taken as the one-sided canonical i.i.d. RDS corresponding to $\Omega = C^r(M, M)^{\mathbb{Z}^+}$ and $P_\varepsilon = \nu_\varepsilon^{\mathbb{Z}^+}$ where ν_ε is the distribution of $w \mapsto \phi_\varepsilon(w) \circ f$. From another point of view this is a perturbation of (f, μ_0) by Markov chains (in fact diffusions) with transition probabilities $Q_\varepsilon(x, \cdot)$, $x \in M$, given by

$$Q_\varepsilon(x, A) := P_W\{w : \phi_\varepsilon(w)(fx) \in A\} \quad \text{for Borel } A \subset M.$$

When ε is small, $\mathcal{F}_\varepsilon$ admits a unique absolutely continuous stationary measure $d\rho_\varepsilon = h_\varepsilon d\,\mathrm{Leb}$ ($\mathcal{F}_\varepsilon$ is in fact *expanding in average*, see below) and h_ε can be obtained as the eigenfunction with $\int h_\varepsilon d\,\mathrm{Leb} = 1$ of the simple eigenvalue 1 of the *integrated* transfer operator $\mathcal{L}_{\mathcal{F}_\varepsilon} : C^{r-1}(M) \to C^{r-1}(M)$ defined by

$$(\mathcal{L}_{\mathcal{F}_\varepsilon} h)(x) = \int (\mathcal{L}_g h)(x)\, d\nu_\varepsilon(g) \quad \text{for } h \in C^{r-1}(M) \tag{3.15}$$

where $\mathcal{L}_g$ is defined analogously to (3.13) (see Ruelle [**144**]; note that $d\rho = h d\,\mathrm{Leb}$ with $h \in L^1(M, \mathrm{Leb})$ is $\mathcal{F}_\varepsilon$-stationary if and only if $\mathcal{L}_{\mathcal{F}_\varepsilon} h = h$). By first proving some perturbative lemmas for spectra of abstract operators, it was shown by Baladi and Young [**21**] that, as $\varepsilon \to 0$, the C^{r-1}-norm $|h_\varepsilon - h_0|_{C^{r-1}}$ of $h_\varepsilon - h_0$ tends to zero (much stronger than $\rho_\varepsilon \to \mu_0$!) and, for each $\delta > 0$, the spectrum of $\mathcal{L}_{\mathcal{F}_\varepsilon}$ restricted to $\{z : |z| > \lambda^{-(r-1)} + \delta\}$ converges to that of $\mathcal{L}_f$ (restricted to the same domain). In particular, if $\tau_{0,r-1} > \lambda^{-(r-1)}$ (here $\tau_{0,r-1} := \tau_{0,H}$ for $H = C^{r-1}(M)$), then $\tau_{\varepsilon,r-1} \to \tau_{0,r-1}$ (it is not known if a result similar to the second limit of (3.16) below holds without the above assumption), where $\tau_{\varepsilon,r-1}$ is the exponential mixing rate of $(\mathcal{F}_\varepsilon, \rho_\varepsilon)$ for observables in $C^{r-1}(M)$ defined (in an equivalent way to (3.14)) by

$$\tau_{\varepsilon,r-1} := \sup_{\phi,\psi \in C^{r-1}(M)} \limsup_{n\to+\infty} \frac{1}{n} \log \left| \int [(Q_\varepsilon^*)^n \phi]\psi \, d\rho_\varepsilon - \int \phi \, d\rho_\varepsilon \int \psi \, d\rho_\varepsilon \right|$$

with $(Q_\varepsilon^*\phi)(x) := \int \phi(y) Q_\varepsilon(x, dy)$. (See [**21**] for more information and for conjectures.) For small ε, the entropy formula

$$h_{\rho_\varepsilon}(\mathcal{F}_\varepsilon) = \iint \log |\det T_x[\phi_\varepsilon(w) \circ f]| \, d\rho_\varepsilon(x)\, dP_W(w)$$

holds and it, together with $|h_\varepsilon - h_0|_{C^{r-1}} \to 0$, imply $h_{\rho_\varepsilon}(\mathcal{F}_\varepsilon) \to h_{\mu_0}(f)$ as $\varepsilon \to 0$ (Liu [**107**]). Stochastic stability of (f, μ_0) with respect to more general Markov chain perturbations was earlier proved by Kifer [**82**] (see also [**86**]).

If one considers perturbations of f by non-i.i.d. random maps which lie in a small neighborhood of f in a map space, stability results analogous to those mentioned just

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above were obtained by Baladi *et al* [**19**] for the two-sided case by considering the random transfer operators $\mathcal{L}_{f(\omega)}$ instead of the integrated one (3.15) and by employing Birkhoff cone techniques. For simplicity, we assume that f is C^r with $2 \le r < +\infty$ being integer (see [**19**] for the Hölder case). For $\varepsilon > 0$ let $\mathcal{F}_\varepsilon$ be the canonical RDS corresponding to $\Omega_\varepsilon = B_\varepsilon(f)^{\mathbb{Z}}$ and a shift-invariant probability P_ε on Ω_ε, where $B_\varepsilon(f)$ is the ε-neighborhood of f in $C^r(M, M)$ (endowed with the C^r-metric). Let $\varepsilon > 0$ be small enough. Then $\mathcal{F}_\varepsilon$ admits a unique invariant measure μ_ε whose sample measures $\mu_{\varepsilon,\omega}$ satisfy $\mu_{\varepsilon,\omega} \ll \text{Leb}$ P_ε-a.e. (see the next topic for the result in the more general framework of expanding in average RDSs). It was shown by [**19**] that for *each* $\omega \in \Omega_\varepsilon$ the density $h_{\varepsilon,\omega} = d\mu_{\varepsilon,\omega}/d\,\text{Leb}$ can be obtained as the unique C^{r-1} solution of $\mathcal{L}_{f(\omega)} h_{\varepsilon,\omega} = h_{\varepsilon,\tau\omega}$ with $\int h_{\varepsilon,\omega} d\,\text{Leb} = 1$ and that

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} |h_{\varepsilon,\omega} - h_0|_{C^{r-1}} = 0, \quad \limsup_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} \tau_{\varepsilon,\omega,r-1} \le \max\{\tau_{0,r-1}, \theta_{r-1}\} \tag{3.16}$$

where $\theta_{r-1} < 1$ is a number (see [**19**] for the definition) which was conjectured to be equal to the essential spectral radius of $\mathcal{L}_f : C^{r-1}(M) \to C^{r-1}(M)$ (see [**19**] for further conjectures) and $\tau_{\varepsilon,\omega,r-1}$ is the random exponential mixing rate of $\mathcal{F}_\varepsilon$ for observables in $C^{r-1}(M)$ defined by

$$\tau_{\varepsilon,\omega,r-1} := \sup_{\phi,\psi\in C^{r-1}(M)} \limsup_{n\to+\infty} \frac{1}{n} \log \times \left| \int [\phi \circ f(\omega, n)]\psi \, d\mu_{\varepsilon,\omega} - \int \phi \, d\mu_{\varepsilon,\tau^n\omega} \int \psi \, d\mu_{\varepsilon,\omega} \right|. \tag{3.17}$$

Similar stability results were obtained by Bogenschütz [**32**] for more general equilibrium states via a different approach, i.e. by proving the stability under a certain type of random perturbation of the Lyapunov spectrum and the corresponding Oseledec spaces of a quasi-compact linear cocycle (random stability of the equilibrium states in the sense of weak convergence of measures was earlier proved by Bogenschütz [**31**]). The one-sided case will be discussed below.

If one is concerned with a general expanding invariant set of a map, structural stability arguments similar to those in the hyperbolic case reviewed in §3.3 hold (but now both for one-sided and two-sided cases). Using this approach and the Bowen–Ruelle formula, the Hausdorff dimension of conformal repellers was proved by Bogenschütz and Ochs [**36**] to be stable under random map perturbations. See Kifer [**93**] or the ICM report Kifer [**95**] for computation of the Hausdorff dimension of random Cantor sets by realizing them as 'invariant sets' of random expanding maps on tori via the base expansion and then using the Bowen–Ruelle formula.

There have been a number of results on stochastic stability of absolutely continuous invariant measures of *piecewise expanding or non-uniformly expanding maps* on intervals and on higher-dimensional manifolds. Here we do not get into a detailed review of these results and rather just recall some of the main literature. Unlike the smooth expanding maps treated just above, such maps are in general not structurally or measure-theoretically stable with respect to *deterministic* perturbations. For example, the quadratic map $f_a : [-1, 1] \to [-1, 1]$ defined by $f_a(x) = 1 - ax^2$, where $a \in [0, 2]$ is a parameter, has an absolutely

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continuous invariant measure for a in a positive Lebesgue measure set $E \subset [0, 2]$ but has a periodic orbit which attracts the orbit of Lebesgue-a.e. point for a in an open and dense subset of $[0, 2]$ (see Young [**158**] for an overview of these works due to Jacobson, Benedicks, Carleson and others). It was also found that *not* all piecewise expanding maps are stochastically stable (see Baladi and Young [**21**] for an example which is due essentially to Keller [**79**]). However, under rather mild 'generic' conditions stochastic stability does hold for these maps (and in some cases also for their decay of correlations). For stochastic stability results on piecewise expanding maps see Keller [**79**], Kifer [**86**, Ch. 4], Baladi and Young [**21**] (all these three references deal with Markov chain perturbations) and Viana [**161**, Ch. 3] (which treats random map perturbations). For such results on unimodal maps $f(x) = 1 - ax^2$ (for 'good' a's in a positive Lebesgue measure subset of $[0, 2]$), see Katok and Kifer [**78**] (see also Kifer [**86**, Ch. 4], which contains in addition a discussion on random perturbations of Lorenz-type dynamical systems), Benedicks and Young [**26**] (which is independent of but contains a similar result to an unpublished work of Collet) and Baladi and Viana [**20**]. We remark that [**26**] and [**20**] deal with Markov chain perturbations of a special class whose transition probabilities are deduced from the random 'additive' maps

$$([-\varepsilon, \varepsilon], \nu_\varepsilon) \ni t \mapsto f_t(x) := 1 - ax^2 + t \tag{3.18}$$

where ν_ε is a probability on $[-\varepsilon, \varepsilon]$ with $\nu_\varepsilon \ll$ Leb and with $d\nu_\varepsilon/d\,\mathrm{Leb}$ satisfying some uniformity conditions, and [**78**] considers more general perturbations but under a Misiurewicz assumption. Some stochastic stability results of higher-dimensional non-uniformly expanding maps were obtained in the recent work Alves and Araújo [**2**].

Two-sided expanding in average RDSs. Let $\mathcal{F}$ be a *two-sided* canonical C^1 RDS on M. Assume that for P-a.e. ω the map $f(\omega)$ has no singularities (i.e. to repeat, points at which $Tf(\omega)$ is degenerate). Define $|f|_{C^1}^- = \inf_{\xi \in TM, |\xi|=1} |Tf\xi|$ for $f \in C^1(M, M)$. If $\log |f(\omega)|_{C^1}^- \in L^1(\Omega, P)$ and one has P-a.e.

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f(\tau^k \omega)|_{C^1}^- =: a(\omega) > 0 \tag{3.19}$$

(the limit exists for P-a.e. ω by Birkhoff theorem), $\mathcal{F}$ is then said to be *expanding in average*. This model was introduced in Khanin and Kifer [**81**] and the thermodynamic formalism for this model was developed there, generalizing previous results in Kifer [**89**]. To prevent a lengthy presentation we now only recall the most salient results under slightly more restrictive integrability conditions on the potentials. For $\beta > 0$ let $H^1(\beta)$ be the space of families $\phi = \{\phi_\omega \in C(M)\}$ which are such that $(\omega, x) \to \phi_\omega(x)$ is measurable on $\Omega \times M$, $|\phi_\omega|_\infty \in L^1(\Omega, P)$ ([**81**] uses the weaker condition $\sup_{x,y \in M} |\phi_\omega(x) - \phi_\omega(y)| \in L^1(\Omega, P)$) and P-a.e.

$$|\phi_\omega(x) - \phi_\omega(y)| \le C_\phi(\omega) d(x, y)^\beta, \quad \forall x, y \in M$$

for some random variable C_ϕ with $\log C_\phi \in L^1(\Omega, P)$.

THEOREM 3.4. [**81**] *Let $\mathcal{F}$ be an expanding in average RDS as defined above, with $\log^+ |f(\omega)|_{C^1} \in L^1(\Omega, P)$ (which implies, in particular, that $\log \deg(f(\omega)) \in L^1(\Omega, P)$,*

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*see [**107**]). Then for any $\phi \in H^1(\beta)$ there exists a unique ϕ-Gibbs measure ν_ϕ and a unique ϕ-Gibbs state μ_ϕ of $\mathcal{F}$. μ_ϕ is the unique equilibrium state of $\mathcal{F}$ for ϕ. When $(\Omega, \mathcal{A}, P, \tau)$ is ergodic (respectively mixing), the skew product system $(\Omega \times M, F, \mu_\phi)$ is ergodic (respectively mixing).*

The definitions of the ϕ-Gibbs measure and state here are analogous to those in the case of random subshifts, namely, they are given by replacing in (3.5) Σ_A and σ^n by $\Omega \times M$ and $f(\omega, n)$, respectively. Construction of these measures can also be done by two methods. One is a direct statistical mechanics method: obtain $(\nu_\phi)_\omega$ by proving that with probability one $P_{\omega,n}(\cdot|y) \to (\nu_\phi)_\omega$ weakly as $n \to +\infty$ for any $y \in M$ and take $(\mu_\phi)_\omega$ as the (unique) limit point of $f(\tau^{-m}\omega, m)(\nu_\phi)_{\tau^{-m}\omega}$ as $m \to +\infty$. The other employs random Ruelle transfer operators $\mathcal{L}_\phi(\omega) : C(M) \to C(M)$ defined analogously to (3.6), leading to a random Ruelle–Perron–Frobenius (RPF) theorem [**81**, Theorem 3.1] similar to Theorem 3.2 and leading to similar corollaries (for a version of the important Gibbs property (3.10) one should replace the cylinder set by $B_{\omega,m}(x, \delta) := \{y \in M : d(f(\omega, i)x, f(\omega, i)y) \le \delta \text{ for } i = 0, 1, \ldots, m-1\}$ for small $\delta > 0$, but this needs subtle modifications; see [**81**] for details of these results). The random mixing rate of $(\mathcal{F}, \mu_\phi)$ defined analogously to (3.17) has not been clearly studied, but an ω-wise CLT was proved in Kifer [**94**] for $(\mathcal{F}, \mu_\phi)$ and for observables $\psi \in H^1(\beta)$ satisfying some further integrability conditions. Markov partitions were constructed by Bogenschütz and Gundlach [**34**] for $\mathcal{F}$ when $f(\omega)$ belongs P-a.e. to a compact set of expanding maps and in such cases the thermodynamic formalism for $\mathcal{F}$ can also be studied by passing to the symbolic coding and using results from §3.2.

Let $\mathcal{F}$ be as given in Theorem 3.4. If it is C^2 and satisfies $\log^+ |f(\omega)|_{C^2} \in L^1(\Omega, P)$, then $J = \{J_\omega : J_\omega(x) = -\log|\det T_x f(\omega)|\} \in H^1(1)$, the Gibbs measure ν_J is equal to $P \times m$ (where m is the normalized Lebesgue measure) and the Gibbs state μ_J is given by

$$d(\mu_J)_\omega = h_\omega \, dm = \lim_{n \to +\infty} f(\tau^{-n}\omega, n)m \tag{3.20}$$

(where $h_\omega > 0$ is from the random RPF theorem). In this context it is natural to consider both the *future* and the *past* rate of convergence to equilibrium, that is, the rates of convergence to zero of the quantities

$$R^{(\mathrm{f})}_{\omega,\phi}(n) := |f(\omega, n)(\phi m) - (\mu_J)_{\tau^n\omega}| \quad \text{and} \quad R^{(\mathrm{p})}_{\omega,\phi}(n) := |f(\tau^{-n}\omega, n)(\phi m) - (\mu_J)_\omega| \tag{3.21}$$

for suitable $\phi \in L^1(M, m)$, where $|\cdot|$ denotes the total variation of a signed measure, and to consider the *future* and the *past* decay rate of *random operational correlations*:

$$C^{(\mathrm{f})}_{\omega,\phi,\psi}(n) := \left| \int [\phi \circ f(\omega, n)]\psi \, dm - \int \phi \, d(\mu_J)_{\tau^n\omega} \int \psi \, dm \right|, \tag{3.22}$$

$$C^{(\mathrm{p})}_{\omega,\phi,\psi}(n) := \left| \int [\phi \circ f(\tau^{-n}\omega, n)]\psi \, dm - \int \phi \, d(\mu_J)_\omega \int \psi \, dm \right| \tag{3.23}$$

for ϕ, ψ in a suitable space of observables (see Baladi *et al* [**19**] for the operational significance of, and experimental access to, these rates). A considerable study of these rates has been recently elaborated by Baladi *et al* [**17**] for i.i.d. unimodal maps (see [**19, 43, 44**] for historically earlier studies under much stronger assumptions), but such a study is still

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lacking in the present general context (see [**19**] for some partial results, see below for related works by Buzzi on random piecewise monotone maps). The Pesin formula holds for $(\mathcal{F}, \mu_J)$ by Theorem 2.5, and, since condition (3.19) easily implies the positiveness of all the Lyapunov exponents, one has $h_{\mu_J}(\mathcal{F}) = \int \log|\det T_x f(\omega)|\, d\mu_J$. This yields that the pressure $\pi_{\mathcal{F}}(J) = 0$ and, by Theorems 2.5 and 3.4, μ_J is the unique $\mathcal{F}$-invariant measure with absolutely continuous sample measures. (When P is ergodic, large deviation bounds for the limit $n^{-1}\sum_{k=0}^{n-1}\delta_{F^k(\omega,x)} \to \mu_J$, $P \times m$-a.e. (ω, x) were given by [**89**] and [**81**, Theorem B].) If $\mathcal{F}$ is further i.i.d., $\rho := \int (\mu_J)_\omega\, dP(\omega)$ (which clearly satisfies $\rho \ll$ Leb) is $\mathcal{F}$-stationary since by (3.20) $(\mu_J)_\omega$ depends only on the past of ω (see Proposition 1.2, see also [**89**]). By results in §2 and Theorem 3.4, $P \times \rho$ is (see the last paragraph for the uniqueness problem) an equilibrium state of the one-sided version $\mathcal{F}^+$ of $\mathcal{F}$ for the analogous potential J ((3.4) is thus an equality for $\mathcal{F}^+$ and J) and ρ is the unique absolutely continuous stationary measure of $\mathcal{F}$ or $\mathcal{F}^+$.

Buzzi [**43, 44**] considered (two-sided) expanding in average piecewise monotone interval (and also higher-dimensional) maps. The existence of invariant measures with absolutely continuous sample measures and both future and past exponential decays of the random correlations were proved there under some further assumptions to control the distortion of the maps and to assure a strong mixing property (see also [**44**] for a discussion on relations of the various mixing properties: random mixing, integrated mixing and mixing of the skew-product system). Baladi *et al* [**17**] considered two-sided compositions of i.i.d. random unimodal maps given by (3.18) and, as alluded to above, a considerable study of the decay rates of (3.21)–(3.23) was elaborated there. However, no stochastic stability results were proved in these papers.

One-sided expanding RDSs. Since the lack of invertibility of $(\Omega, \mathcal{A}, P, \tau)$ may cause essential difficulties in the equilibrium states theory of RDSs, it would be interesting to know if the results (Theorem 3.4 for example) obtained above in the two-sided setting can be extended to the one-sided case. The answer has not been made clear in its fairly full generality, but many results were elaborated by Baladi [**14**] on equilibrium states (including their correlation spectrum) theory of one-sided i.i.d. RDSs under some specific and somewhat restrictive assumptions (a description was given by [**14**] in an appendix on how most of the results could be extended to the non-i.i.d. case, as indicated by Ruelle, and it is also expected that the assumptions made there could be relaxed to various more realistic ones). In particular, new types of equilibrium states, named *annealed* equilibrium states, was introduced there, whereas the previous ones defined by the variational principle Theorem 3.1 were renamed *quenched* equilibrium states. We now only recall some of the main points of [**14**]. Set $(\Omega, P) = (E^{\mathbb{Z}^+}, \nu^{\mathbb{Z}^+})$, where E is a compact subset of $C^r(M, M) \times C^r(M, \mathbb{R}^+_*)$ ($r \geq 1$ integer, $\mathbb{R}^+_* = (0, +\infty)$) such that the first factor of each of its elements is an expanding map and ν is a Borel probability on E. Write $\xi = (f_\xi, g_\xi)$ for $\xi \in E$. Then define an RDS $\mathcal{F}$ by letting the random map $f(\omega)$ be f_{ω_0} for each $\omega = (\omega_0, \omega_1, \dots) \in \Omega$, and supported by the random function $g = \{g_\omega := g_{\omega_0}\}$ (negative or complex functions were also considered in [**14**]). Associated to $(\mathcal{F}, g)$, one considers the *integrated annealed transfer operator* $\hat{\mathcal{L}}$ which acts on measurable functions

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$h : \Omega \times M \to \mathbb{R}$ (we write sometimes $h(\omega, x) = h_\omega(x)$) by

$$(\hat{\mathcal{L}}h)(\omega, x) = \int (\mathcal{L}_\xi h_{\xi\wedge\omega})(x)\, d\nu(\xi) := \int \sum_{y: f_\xi y = x} g_\xi(y) h_{\xi\wedge\omega}(y)\, d\nu(\xi)$$

where $\xi \wedge \omega := (\xi, \omega_0, \omega_1, \dots) \in \Omega$. If $r \geq 2$ and $g_\omega(x) = 1/|\det T_x f(\omega)|$, as can be partly seen from the previous argument of this subsection, $\hat{\mathcal{L}}$ acting on $C^{r-1}(M)$ or on a larger space $\mathcal{B}$ consisting of suitable random functions gives rise (via its maximal eigenfunction) to the quenched equilibrium state $P \times \rho$ of $\mathcal{F}$ for $\log g$ (see [**14**, §3.1] for the definition of $\mathcal{B}$). However, it was found by Baladi [**14**] that for general g the operator $\hat{\mathcal{L}}$ acting on $\mathcal{B}$ no longer always gives rise to a quenched equilibrium state $\mu^{(\mathrm{q})}$, but to an *annealed* one $\mu^{(\mathrm{a})}$ of $\mathcal{F}$ for $\log g$, that is, $\mu^{(\mathrm{a})}$ is a probability on $\Omega \times M$ (whose projection on Ω may not be $P = \nu^{\mathbb{Z}^+}$) which is F-invariant (F the skew-product transformation) and realizes the following supremum taken over the set of all F-invariant probabilities μ on $\Omega \times M$

$$\pi^{(\mathrm{a})}_{\mathcal{F}}(\log g) := \sup \left\{ h_\mu^{\pi^{-1}\mathcal{A}}(F) + h^\nu(\pi\mu) + \int \log g(\omega, x)\, d\mu(\omega, x) \right\}$$

where $\pi : \Omega \times M \to \Omega$ is the natural projection,

$$h^\nu(\pi\mu) = -\int \log \frac{d(\pi\mu)}{d(\nu \times \pi\mu)}\, d(\pi\mu)$$

if $\pi\mu \ll \nu \times \pi\mu$ and $h^\nu(\pi\mu) = -\infty$ otherwise (i.e. $h^\nu(\pi\mu)$ is minus the relative entropy of $\pi\mu$ with respect to $\nu \times \pi\mu$) and $\pi^{(\mathrm{a})}_{\mathcal{F}}(\log g)$ is called the *annealed topological pressure* of $\mathcal{F}$ for $\log g$. In analogy to the quenched case, *annealed Gibbs measures and states* can be introduced. Via a detailed study of the spectral properties of $\hat{\mathcal{L}}$ acting on $\mathcal{B}$ (in particular, $R = \exp \pi^{(\mathrm{a})}_{\mathcal{F}}(\log g)$ is a simple maximal eigenvalue of $\hat{\mathcal{L}} : \mathcal{B} \to \mathcal{B}$) and based on Ruelle [**144**], the existence and uniqueness as well as their coincidence of the annealed equilibrium and Gibbs states were proved for $(\mathcal{F}, \log g)$ by Baladi [**14**] (these results solve a conjecture of Ruelle [**146**, §7.4]) and, moreover, the discrete spectrum of $\hat{\mathcal{L}}$ was shown there to have exact relations to the correlation spectrum of this state μ_g (yielding exponential decay of correlations for (F, μ_g) and for suitable observables on $\Omega \times M$ (note that $\mathcal{F}$ is assumed to be i.i.d. now)), to the poles of an annealed zeta function and to the zeros of an annealed Fredholm determinant. Stability results were also obtained on these objects by applying perturbative results of Baladi and Young [**21**]. Similar results were also obtained there for the quenched case by considering the *integrated quenched transfer operator* $\hat{\mathcal{M}}$ acting on $\mathcal{B}$ (see [**14**] for the definition) which gives rise to the quenched equilibrium (and Gibbs) state. In the case of $r \geq 2$ and $g_\omega(x) = 1/|\det T_x f(\omega)|$, the quenched and annealed states coincide; one-dimensional i.i.d. random Ising models give examples where they differ (see [**14**, §2.4]).

In addition to Baladi [**14**], we mention a related work Denker and Gordin [**56**] which deals with fibred systems (Y, T, X, S, π) where Y, X are compact metric spaces and $T : Y \to Y$, $S : X \to X$, $\pi : Y \to X$ are continuous maps such that $\pi \circ T = S \circ \pi$ and where the invertibility of S is not assumed. Under the assumption that (Y, T, X, S, π) is fibrewise expanding and under some other conditions, unique Gibbs measure was proved in [**56**], using the Ruelle transfer operator method, to exist for every Hölder continuous

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potential $\phi : Y \to \mathbb{R}$ (see that paper for results on properties of the measure), but, when there is neither the invertibility nor expandingness assumption on $S : X \to X$, no equilibrium states theory has been elaborated in this set-up.

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