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Natural fluctuations in un-magnetized and magnetized plasmas

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There are still open issues within the fluctuation theory of plasmas, in view of the difficulty of formulating adequate theoretical approaches and solving the related equations in particular regimes. A promising alternative approach is direct microphysical modeling based on first principles, as successfully applied to neutral rarefied fluids. Within this approach, the equations of motion of a large ensemble of charged particles are solved numerically while correlations are obtained from statistical analysis of the ensemble at different times. As a first step, in this work we validate the data analysis technique adopted in this numerical scheme for the case of an electron ensemble neglecting Coulomb interactions. The simulation results are compared with the analytical theory of 'natural' fluctuations for both un-magnetized and magnetized plasmas. For the latter, the derivations for arbitrary average distribution functions are presented.

1. Introduction

The Bogoliubov–Klimontovich theory of correlations/fluctuations (Klimontovich 1967; Tsytovich 1995) is one of the most powerful formulations of the kinetic theory of plasmas. The theory of fluctuations has two fundamental constituents: (i) the separation of spatiotemporal scales that allows a decomposition of the distribution functions and electromagnetic fields into average and fluctuating parts, (ii) the separation of the fluctuations into 'natural' and 'induced' parts, the latter induced in a system of charged particles by the fluctuating field generated by the natural fluctuations triggered by particle discreteness and present in any large system of particles. The natural fluctuations are random and not known but the theory is formulated in terms of their correlators, i.e. statistical averages of products of fluctuating quantities, limited to the product of two in the approximation of second moments valid for weakly coupled systems, providing the theoretical expressions for spectral densities of fluctuating plasma quantities.

While the fluctuation theory is rigorous for collisionless fully ionized plasmas (Akhiezer et al. 1975b), in many cases simplifying approximations need to be invoked, for instance when binary particle collisions are not negligible (Sitenko and Gurin 1966), for plasma-molecular systems (Klimontovich et al. 1989) or in the presence of dust grains (Tsytovich and de Angelis 1999; de Angelis et al. 2012). To assess the

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validity domain of theoretical results for the spectral densities of plasma fluctuations in such cases, 'numerical experiments' can be an invaluable tool. Here we propose to utilize molecular dynamics simulations taking advantage of a technique based on the analysis of density correlations that has been successfully applied to neutral atomic gases (Bruno et al. 2006) but not in plasmas.

In this work, we focus on natural fluctuations as an initial step in testing the proposed numerical scheme. In Sec. 2, we discuss the emergence of natural fluctuations within the Klimontovich scheme and their mathematical description within the transition probability approach. The combination of these formalisms allows for the most physically transparent presentation of natural fluctuations. In Sec. 3, we introduce the main statistical quantities of interest and provide basic results for unmagnetized and magnetized plasmas. To our knowledge, despite the fact that the spectral densities of natural density fluctuations are well-studied quantities, rigorous proofs for the space-time correlation function of natural density fluctuations in magnetized plasmas for arbitrary gyrotropic distribution functions do not exist in the literature. Therefore, details of the derivations for the magnetized case are provided in appendices. Finally, in Sec. 4, we present the molecular dynamics method and compare the numerical results with the analytical expressions.

2. The natural fluctuations and the natural correlator

2.1. *The emergence of natural fluctuations in the Klimontovich scheme*

The microscopic phase-space densities or exact distribution functions $f_p^\alpha(\boldsymbol{r},t)$ for Hamiltonian systems consisting of classical point particles can be expressed via the sum of Dirac functions centered at the exact phase-space orbits, $f_p^\alpha(\boldsymbol{r},t) = \sum_{i=1}^{N_\alpha} \delta[\boldsymbol{p}-\boldsymbol{p}_{i\alpha}(t)]\delta[\boldsymbol{r}-\boldsymbol{r}_{i\alpha}(t)]$. In this light, the Klimontovich equation for fully ionized plasmas interacting with the Lorentz force has the general form (Klimontovich 1967)

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}+e_\alpha\left[\boldsymbol{E}(\boldsymbol{r},t)+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}(\boldsymbol{r},t)\right]\cdot\frac{\partial}{\partial \boldsymbol{p}}\right\} f_p^\alpha(\boldsymbol{r},t)=0, \tag{2.1}$$

where $\alpha = \{e,i\}$ denotes the plasma species, $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{B}(\boldsymbol{r},t)$ denote the total fine-grained electromagnetic fields, i.e. the externally applied fields and the fields generated by the plasma particles. The Klimontovich equation can be compactly written as $\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\{f_p^\alpha(\boldsymbol{r},t)\} = 0$ by introducing the electromagnetic phase-space operator $\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)$.

The Klimontovich scheme proceeds with decomposing the exact distribution functions and the electromagnetic phase-space operator in ensemble averaged and fluctuating parts, $f_p^\alpha(\boldsymbol{r},t) = \Phi_p^\alpha(\boldsymbol{r},t) + \delta f_p^\alpha(\boldsymbol{r},t)$ and $\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t) = \widetilde{\mathscr{L}}_\alpha(\boldsymbol{p},\boldsymbol{r},t) + \delta\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)$, where $\langle\delta f_p^\alpha(\boldsymbol{r},t)\rangle = 0$ and $\langle\delta\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\rangle = 0$. Separating temporal-spatial scales, neglecting high-order fluctuations and ensemble averaging, one obtains the Boltzmann and fluctuation equations of the system,

$$\widetilde{\mathscr{L}}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\{\Phi_p^\alpha(\boldsymbol{r},t)\} = -\langle\delta\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\{\delta f_p^\alpha(\boldsymbol{r},t)\}\rangle\,, \tag{2.2}$$

$$\widetilde{\mathscr{L}}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\{\delta f_p^\alpha(\boldsymbol{r},t)\} = -\delta\mathscr{L}_\alpha(\boldsymbol{p},\boldsymbol{r},t)\{\Phi_p^\alpha(\boldsymbol{r},t)\}. \tag{2.3}$$

The general solution of the fluctuation equation has the form $\delta f_p^\alpha(\boldsymbol{r},t) = \delta f_p^{\alpha,(0)}(\boldsymbol{r},t)+\delta f_p^{\alpha,(\mathrm{ind})}(\boldsymbol{r},t)$, where the first adder corresponds to the natural fluctuations and constitutes the solution of the homogeneous equation, whereas the second adder corresponds to the induced fluctuations and constitutes the particular solution.

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Therefore, since the particular part vanishes when $\delta\mathscr{L}_\alpha = 0$, the natural fluctuations are the solutions of the system

$$\widetilde{\mathscr{L}}_\alpha(\boldsymbol{p}, \boldsymbol{r}, t)\{\Phi_{\boldsymbol{p}}^{\alpha}(\boldsymbol{r}, t)\} = 0, \tag{2.4}$$

$$\widetilde{\mathscr{L}}_\alpha(\boldsymbol{p}, \boldsymbol{r}, t)\{\delta f_{\boldsymbol{p}}^{\alpha,(0)}(\boldsymbol{r}, t)\} = 0, \tag{2.5}$$

with the Boltzmann equation collapsing to its Vlasov form. The condition $\delta\mathscr{L}_\alpha = 0$ is equivalent to $\delta\boldsymbol{E} = \delta\boldsymbol{B} = 0$. By the decomposed form of Maxwell's equations the source of fluctuating electromagnetic fields are the plasma particles themselves, hence natural fluctuations are triggered solely by discreteness. This implies that they correspond to the phase-space orbits of a system for which the interactions between plasma particles are switched off. Note that this 'switching off' does not refer to electromagnetic interactions in general, since the particles can still interact with external electromagnetic fields.

Equation (2.5) cannot be solved due to the lack of appropriate initial conditions, meaning that the mathematically necessary initial conditions are not accessible. In fact, natural fluctuations are random variables since their exact value differs in each microscopic realization of the medium. However, their correlations $\langle \delta f_{\boldsymbol{p}}^{\alpha,(0)}(\boldsymbol{r}, t)\delta f_{\boldsymbol{p}'}^{\beta,(0)}(\boldsymbol{r}', t')\rangle$, also known as natural correlator, are a meaningful quantity that can be constructed through single particle phase-space orbits. The existence of correlations between non-interacting particles appears counter-intuitive at first, but, as we shall see, it is a direct consequence of *self-correlations*. The natural correlator is the missing ingredient that, when combining with the decomposed set of Maxwell's equations, provides closure to Klimontovich–Maxwell system (Klimontovich 1967). In particular, in Fourier space, the induced fluctuations and thus the total fluctuations can be found to be proportional to the natural fluctuations, hence the Fourier-transformed natural correlator $\langle \delta f_{\boldsymbol{p},\boldsymbol{k},\omega}^{\alpha,(0)}\delta f_{\boldsymbol{p}',\boldsymbol{k}',\omega'}^{\beta,(0)}\rangle$ allows for a straightforward calculation of both the collision integral and the spectral densities of fluctuations (Tsytovich 1995).

2.2. *The natural correlator in the transition probability approach*

In this subsection, we present the general expressions for the natural correlator in the real space. We shall employ a slightly different and more compact notation for convenience. We introduce the six-dimensional phase-space vector $\boldsymbol{X} = \{\boldsymbol{p}, \boldsymbol{r}\}$, the six-dimensional phase-space orbit vector $\boldsymbol{X}_{i\alpha}(t) = \{\boldsymbol{p}_{i\alpha}(t), \boldsymbol{r}_{i\alpha}(t)\}$ and also use the notation $\Phi_\alpha(\boldsymbol{X}, t)$ for the average distribution function.

Initially, we shall find an expression for the natural correlator at equal time instants, $\langle \delta f_{\boldsymbol{p}}^{\alpha,(0)}(\boldsymbol{r}, t)\delta f_{\boldsymbol{p}'}^{\beta,(0)}(\boldsymbol{r}', t)\rangle$, based on a technique presented in details by Klimontovich (Klimontovich and Silin 1962; Klimontovich 1967). This will serve as an initial condition in the derivation of the natural correlator at different instants, $\langle \delta f_{\boldsymbol{p}}^{\alpha,(0)}(\boldsymbol{r}, t)\delta f_{\boldsymbol{p}'}^{\beta,(0)}(\boldsymbol{r}', t')\rangle$, a derivation which is more intuitive in the transition probability approach (Ichimaru and Yakimenko 1973; Sitenko and Yakimenko 1974; Yakimenko and Zagorodny 1974; Klimontovich et al. 1989; Sosenko and Decyk 1993; Sosenko 1997).

2.2.1. *Expression for the natural correlator at equal time instants.*

The calculations begin with constructing the product $f_\alpha(\boldsymbol{X}, t)f_\beta(\boldsymbol{X}', t)$ by its double series definition and splitting the self-terms, which are the terms that satisfy

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$\{\boldsymbol{X}_{i\alpha}(t) = \boldsymbol{X}_{j\beta}(t),\ \alpha = \beta\}$. This manipulation yields

$$f_\alpha(\boldsymbol{X}, t) f_\beta\left(\boldsymbol{X}', t\right) = \sum_{\substack{\boldsymbol{X}_{i\alpha} \neq \boldsymbol{X}_{j\beta} \\ \text{when } \alpha = \beta}} \sum_{1 \leqslant i \leqslant N_\alpha} \sum_{1 \leqslant j \leqslant N_\beta} \delta\left[\boldsymbol{X} - \boldsymbol{X}_{i\alpha}(t)\right] \delta\left[\boldsymbol{X}' - \boldsymbol{X}_{j\beta}(t)\right] + \delta_{\alpha\beta}\delta\left[\boldsymbol{X} - \boldsymbol{X}'\right] f_\alpha(\boldsymbol{X}, t).$$

We ensemble average this relation, i.e. we average over the Liouville distribution function $N(\boldsymbol{X}_{1\alpha}, \ldots \boldsymbol{X}_{N_\alpha\alpha}, \boldsymbol{X}_{1\beta}, \ldots \boldsymbol{X}_{N_\beta\beta}, t)$ and employ the definitions of the (normalized to unity) one-particle and two-particle average distribution functions, $\bar{\Phi}_\alpha(\boldsymbol{X}_{1\alpha}, t)$ and $\bar{\Phi}_{\alpha\beta}(\boldsymbol{X}_{1\alpha}, \boldsymbol{X}_{1\beta}, t)$ to obtain

$$\begin{aligned}\left\langle f_\alpha(\boldsymbol{X}, t) f_\beta\left(\boldsymbol{X}', t\right)\right\rangle &= \frac{(N_\alpha - 1)N_\beta}{V^2} \bar{\Phi}_{\alpha\beta}\left(\boldsymbol{X}, \boldsymbol{X}', t\right) + \delta_{\alpha\beta} n_\alpha \delta\left[\boldsymbol{X} - \boldsymbol{X}'\right] \bar{\Phi}_\alpha(\boldsymbol{X}, t) \\ &= n_\alpha n_\beta \bar{\Phi}_{\alpha\beta}\left(\boldsymbol{X}, \boldsymbol{X}', t\right) + \delta_{\alpha\beta} n_\alpha \delta\left[\boldsymbol{X} - \boldsymbol{X}'\right] \bar{\Phi}_\alpha(\boldsymbol{X}, t),\end{aligned}$$

where the final result is a consequence of the macroscopic nature of the system, $N_\alpha \simeq N_\alpha - 1$. We use the relation $\delta f_\alpha(\boldsymbol{X}, t) = f_\alpha(\boldsymbol{X}, t) - n_\alpha \bar{\Phi}_\alpha(\boldsymbol{X}, t)$ and the definition of the (normalized to unity) second order correlation function $\bar{g}_{\alpha\beta}(\boldsymbol{X}, \boldsymbol{X}', t)$ via $\bar{\Phi}_{\alpha\beta}(\boldsymbol{X}, \boldsymbol{X}', t) = \bar{\Phi}_\alpha(\boldsymbol{X}, t)\bar{\Phi}_\beta(\boldsymbol{X}', t) + \bar{g}_{\alpha\beta}(\boldsymbol{X}, \boldsymbol{X}', t)$. This leads to a relation for the correlator at equal time instants, $\langle \delta f_\alpha(\boldsymbol{X}, t)\delta f_\beta(\boldsymbol{X}', t)\rangle = n_\alpha n_\beta \bar{g}_{\alpha\beta}(\boldsymbol{X}, \boldsymbol{X}', t) + \delta_{\alpha\beta} n_\alpha \delta[\boldsymbol{X} - \boldsymbol{X}']\bar{\Phi}_\alpha(\boldsymbol{X}, t)$. Obviously, in absence of interactions $\bar{g}_{\alpha\beta}(\boldsymbol{X}, \boldsymbol{X}', t) = 0$ and $\delta f_\alpha^{(0)}(\boldsymbol{X}, t) = \delta f_\alpha(\boldsymbol{X}, t)$. Therefore, recalling that $\Phi_\alpha(\boldsymbol{X}, t) = n_\alpha \bar{\Phi}_\alpha(\boldsymbol{X}, t)$, we are left with the self-terms. The natural correlator at equal time instants has the form

$$\left\langle \delta f_\alpha^{(0)}(\boldsymbol{X}, t)\, \delta f_\beta^{(0)}\left(\boldsymbol{X}', t\right)\right\rangle = \delta_{\alpha\beta}\delta\left[\boldsymbol{X} - \boldsymbol{X}'\right] \Phi_{\boldsymbol{p}}^\alpha(\boldsymbol{r}, t). \tag{2.6}$$

In the language of strongly coupled systems, this equation simply states that the structure factor of an ideal gas is unity, i.e. that non-interacting gases bear no distinctive features (Hansen and McDonald 2006). We note that, since the Fourier normalization choices in fluctuation theory and liquid state theory are opposite, for our case it shall be $(2\pi)^{-3}$.

2.2.2. *Expression for the natural correlator at different time instants.*

With $w_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')$ we denote the retarded Green's function associated with the linear differential operator $\widetilde{\mathscr{L}}_\alpha(\boldsymbol{X}, t)$. By definition, it satisfies the causality condition $w_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t') = 0$ for $t < t'$ and it is the solution of the differential equation $\widetilde{\mathscr{L}}_\alpha(\boldsymbol{X}, t)\{w_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')\} = \delta_{\alpha\beta}\delta(t - t')\delta(\boldsymbol{X} - \boldsymbol{X}')$. Another useful and closely related quantity is the transition probability $P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')$ that expresses the probability density for the transition of a test particle, which at a time instant t' is located at the phase-space point $\boldsymbol{X}'$ and belongs to the plasma species β, to the phase space point $\boldsymbol{X}$ at a time instant t but now belonging to the plasma species α. This function follows the symmetric relation $P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t') = w_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t') + w_{\beta\alpha}(\boldsymbol{X}', t'; \boldsymbol{X}, t)$ and satisfies the homogeneous differential equation $\widetilde{\mathscr{L}}_\alpha(\boldsymbol{X}, t)\{P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')\} = 0$ with an initial condition $P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')|_{t=t'} = \delta_{\alpha\beta}\delta(\boldsymbol{X} - \boldsymbol{X}')$. In analogy with the application of Green's function theory in quantum mechanics and quantum field theory, these quantities are sometimes also called propagators. Since both the average distribution function and the natural fluctuations satisfy the same homogeneous differential equation, see (2.4) and (2.5), for $t > t_0$ we have $\delta f_\alpha^{(0)}(\boldsymbol{X}, t) = \int \sum_\beta P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t_0)\delta f_\alpha^{(0)}(\boldsymbol{X}', t_0) d^6X'$ and $\Phi_\alpha(\boldsymbol{X}, t) = \int \sum_\beta P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t_0)\Phi_\alpha(\boldsymbol{X}', t_0) d^6X'$. Utilizing (2.6), the causality condition and the

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symmetry of the natural correlator with respect to the transformation $(\alpha, \boldsymbol{X}, t) \leftrightarrow (\beta, \boldsymbol{X}', t')$, we end up with an expression valid for any $(t,\ t')$,

$$\langle \delta f_{\alpha}^{(0)}(\boldsymbol{X}, t)\, \delta f_{\beta}^{(0)}\left(\boldsymbol{X}', t'\right)\rangle = w_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t')\Phi_{\beta}\left(\boldsymbol{X}', t'\right) + w_{\beta\alpha}(\boldsymbol{X}', t'; \boldsymbol{X}, t)\Phi_{\alpha}(\boldsymbol{X}, t)\,. \tag{2.7}$$

Assuming that the medium is uniform and stationary, the average distribution function will only depend on the momentum, $\Phi_{\beta}(\boldsymbol{X}', t') = \Phi_{\beta}(\boldsymbol{X}, t) = \Phi_{\boldsymbol{p}}^{\beta}$. Moreover, assuming that the medium is unbounded and $t > t'$, the transition probability will be given by $P_{\alpha\beta}(\boldsymbol{X}, t; \boldsymbol{X}', t') = \delta_{\alpha\beta}\delta[\boldsymbol{X} - \boldsymbol{X}_{\alpha}(t)]_{\boldsymbol{X}'}^{t'}$, where $\boldsymbol{X}_{\alpha}(t)$ corresponds to the phase-space trajectory of a test particle that belongs in the plasma species α with initial conditions $\boldsymbol{X}_{\alpha}(t') = \boldsymbol{X}'$. Combining with (2.7), we obtain a simplified expression for the natural correlator that reads us

$$\langle \delta f_{\alpha}^{(0)}(\boldsymbol{X}, t)\, \delta f_{\beta}^{(0)}\left(\boldsymbol{X}', t'\right)\rangle = \delta_{\alpha\beta}\delta\left[\boldsymbol{X} - \boldsymbol{X}_{\alpha}(t)\right]_{\boldsymbol{X}'}^{t'} \Phi_{\boldsymbol{p}}^{\alpha},$$

or equivalently, reverting to our original notation,

$$\langle \delta f_{\boldsymbol{p},\boldsymbol{r},t}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{r}',t'}^{\beta,(0)}\rangle = \delta_{\alpha\beta}\delta\left[\boldsymbol{p} - \boldsymbol{p}_{\alpha}(t)\right]_{\boldsymbol{p}',\boldsymbol{r}'}^{t'} \delta\left[\boldsymbol{r} - \boldsymbol{r}_{\alpha}(t)\right]_{\boldsymbol{p}',\boldsymbol{r}'}^{t'} \Phi_{\boldsymbol{p}}^{\alpha}. \tag{2.8}$$

This equation reveals that the natural correlator is the sum of self-correlations of all non-interacting particles.

3. The natural correlator in un-magnetized and magnetized plasmas

3.1. *Un-magnetized plasmas*

In absence of external electromagnetic fields, the non-interacting particles are free streaming, which leads to $\boldsymbol{p}_{\alpha}(t) = \boldsymbol{p}'$ and $\boldsymbol{r}_{\alpha}(t) = \boldsymbol{r}' + \boldsymbol{v}(t - t')$. Hence, the natural correlator in the real space becomes

$$\langle \delta f_{\boldsymbol{p},\boldsymbol{r},t}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{r}',t'}^{\beta,(0)}\rangle = \delta_{\alpha\beta}\delta(\boldsymbol{p} - \boldsymbol{p}')\delta[\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{v}(t - t')]\Phi_{\boldsymbol{p}}^{\alpha}. \tag{3.1}$$

Notice that (3.1) possesses a time-space translation symmetry, $\langle \delta f_{\boldsymbol{p},\boldsymbol{r},t}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{r}',t'}^{\beta,(0)}\rangle = \langle \delta f_{\boldsymbol{p},\boldsymbol{r}-\boldsymbol{r}',t-t'}^{\alpha,(0)} \delta f_{\boldsymbol{p}',0,0}^{\beta,(0)}\rangle$. This property is valid in the spatiotemporal scales of ensemble averaged quantities only. For instance, while it holds for the natural correlator, it does not hold for products of fluctuating quantities $\delta f_{\boldsymbol{p},\boldsymbol{r},t}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{r}',t'}^{\beta,(0)}$. Typically, in applications of fluctuation theory concerning calculations of collision integrals or radiation scattering cross-sections, the use of space-time Fourier transforms is more convenient. Using $\int_{-\infty}^{+\infty} \exp\left[-\imath(\omega - \boldsymbol{k}\cdot\boldsymbol{v})t\right]dt = 2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$, it is straightforward to show that the Fourier-transformed natural correlator $\langle \delta f_{\boldsymbol{p},\boldsymbol{k},\omega}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{k}',\omega'}^{\beta,(0)}\rangle$ is given by

$$\langle \delta f_{\boldsymbol{p},\boldsymbol{k},\omega}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{k}',\omega'}^{\beta,(0)}\rangle = \frac{1}{(2\pi)^3}\delta_{\alpha\beta}\delta(\boldsymbol{p} - \boldsymbol{p}')\delta(\omega + \omega')\delta(\boldsymbol{k} + \boldsymbol{k}')\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\Phi_{\boldsymbol{p}}^{\alpha}. \tag{3.2}$$

The physical interpretation of the terms of (3.2) is the following; $\delta_{\alpha\beta}$ is a consequence of the lack of interactions between different plasma species, $\delta(\boldsymbol{p} - \boldsymbol{p}')$ is a result of the absence not only of particle interactions but also of external electromagnetic fields, $\delta(\omega + \omega')$ is a manifestation of the average stationarity of the medium, $\delta(\boldsymbol{k} + \boldsymbol{k}')$ is a manifestation of the average homogeneity of the medium, $\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ stems from the free streaming trajectories followed by all particles, whereas $\Phi_{\boldsymbol{p}}^{\alpha}$ originates from the well-known thermodynamic relation for statistically independent elements $\langle(\delta N)^2\rangle = \langle N\rangle$, with N the total number of particles in a volume V. In (3.2), the numeric constant $(2\pi)^{-3}$ in front of the functional form can differ in papers and textbooks that employ different conventions. In this light, we point out that we followed the

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Fourier analysis convention where the normalization factor $(2\pi)^{-4}$ appears in the direct transform. However, in some textbooks (Sitenko 1967; Aleksandrov et al. 1984) the normalization factor $(2\pi)^{-4}$ appears in the inverse transform, which leads to a numeric constant 2π. We also note that we use the standard normalization $\int \Phi_{\boldsymbol{p}}^{\alpha} d^3p = \int \Phi_{\alpha}(\boldsymbol{v}) d^3v = n_{\alpha}$. In his monographs (Tsytovich 1989, 1995, 2013), Tsytovich uses the Fourier normalization of this work but employs the normalization $\int \Phi_{\boldsymbol{p}}^{\alpha} [d^3p/(2\pi)^3] = \int \Phi_{\alpha}(\boldsymbol{v}) d^3v = n_{\alpha}$, within these conventions he ends up with a Fourier-transformed natural correlator without numeric constants.

In the Fourier domain, the correlations of natural density fluctuations are defined by $\langle \delta n_{\boldsymbol{k},\omega}^{\alpha,(0)} \delta n_{\boldsymbol{k}',\omega'}^{\beta,(0)} \rangle = \int \langle \delta f_{\boldsymbol{p},\boldsymbol{k},\omega}^{\alpha,(0)} \delta f_{\boldsymbol{p}',\boldsymbol{k}',\omega'}^{\beta,(0)} \rangle d^3p d^3p'$. A more useful quantity is the spectral density of natural density fluctuations defined by $S_{\boldsymbol{k},\omega}^{\alpha,(0)} = \langle |\delta n_{\boldsymbol{k},\omega}^{\alpha,(0)}|^2 \rangle$. These quantities are connected with the relation $\langle \delta n_{\boldsymbol{k},\omega}^{\alpha,(0)} \delta n_{\boldsymbol{k}',\omega'}^{\beta,(0)} \rangle = \delta_{\alpha\beta} \delta(\omega + \omega') \delta(\boldsymbol{k} + \boldsymbol{k}') S_{\boldsymbol{k},\omega}^{\alpha,(0)}$. In order to acquire $S_{\boldsymbol{k},\omega}^{\alpha,(0)}$, we need to integrate (3.2) with respect to the momenta $(\boldsymbol{p},\ \boldsymbol{p}')$,

$$S_{\boldsymbol{k},\omega}^{\alpha,(0)} = \frac{1}{(2\pi)^3} \int \delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v}) \Phi_{\alpha}(\boldsymbol{v}) d^3v. \tag{3.3}$$

At this point we remind the reader, that for un-magnetized plasmas the susceptibility of a species α has the general form $\chi_{\boldsymbol{k},\omega}^{\alpha} = (4\pi e_{\alpha}^2)/(m_{\alpha} k^2) \int \frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \imath 0} (\boldsymbol{k} \cdot \frac{\partial \Phi_{\alpha}(\boldsymbol{v})}{\partial \boldsymbol{v}}) d^3v$, application of the Plemelj formula yields an imaginary (dissipative) part $\Im\{\chi_{\boldsymbol{k},\omega}^{\alpha}\} = -(4\pi^2 e_{\alpha}^2)/(m_{\alpha} k^2) \int \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) (\boldsymbol{k} \cdot \frac{\partial \Phi_{\alpha}(\boldsymbol{v})}{\partial \boldsymbol{v}}) d^3v$. The similarities between the latter formula and (3.3) are obvious. In fact, for Maxwellian distributions the two quantities are proportional, which is a clear manifestation of the fluctuation-dissipation theorem (Akhiezer et al. 1975a).

Let us now assume that the average distribution function is isotropic, i.e. that it depends only on the magnitude of the velocity vector, $\Phi_{\alpha}(v)$. Employing spherical coordinates, choosing $\boldsymbol{k} \| \hat{\boldsymbol{z}}$ which leads to $\boldsymbol{k} \cdot \boldsymbol{v} = kv\cos\theta$ and using the integral $\int_{-b}^{+b} \delta(x - |a|) dx = \mathrm{H}(b - |a|)$, with H(.) denoting the Heaviside step function, we end up with

$$S_{\boldsymbol{k},\omega}^{\alpha,(0)} = \frac{1}{(2\pi)^2} \frac{1}{k} \int_{\omega/k}^{+\infty} v \Phi_{\alpha}(v) dv. \tag{3.4}$$

For Maxwellian distribution functions $\Phi_{\alpha}(v) = n_{\alpha} [m_{\alpha}/(2\pi T_{\alpha})]^{3/2} \exp[-(m_{\alpha} v^2)/(2T_{\alpha})]$, with $v_{T\alpha} = \sqrt{T_{\alpha}/m_{\alpha}}$ the thermal velocity, we have

$$S_{\boldsymbol{k},\omega}^{\alpha,(0)} = \frac{1}{(2\pi)^{7/2}} \frac{n_{\alpha}}{k v_{T\alpha}} \exp\left(-\frac{\omega^2}{2k^2 v_{T\alpha}^2} \right). \tag{3.5}$$

Finally, we introduce the two-point space-time correlation function of natural density fluctuations $C_{\boldsymbol{r},\boldsymbol{r}',t,t'}^{\alpha,\beta} = \langle \delta n_{\boldsymbol{r},t}^{\alpha,(0)} \delta n_{\boldsymbol{r}',t'}^{\beta,(0)} \rangle$, which is acquired by integration of the natural correlator, (3.1), with respect to the momenta. We also introduce the space-time correlation function of natural density fluctuations $C_{\boldsymbol{r},t}^{\alpha,(0)}$, which is the inverse Fourier transform of $S_{\boldsymbol{k},\omega}^{\alpha,(0)}$. Due to the spatiotemporal translation symmetry, these quantities are connected with the simple relation $C_{\boldsymbol{r},0,t,0}^{\alpha,\beta} = \delta_{\alpha\beta} C_{\boldsymbol{r},t}^{\alpha,(0)}$. Employing the relation $\delta[a\boldsymbol{x}] = |a|^{-n} \delta(\boldsymbol{x})$, where n is the dimensionality of the arbitrary vector $\boldsymbol{x}$, we obtain

$$C_{\boldsymbol{r},\boldsymbol{r}',t,t'}^{\alpha,\beta} = \delta_{\alpha\beta} \frac{1}{|t - t'|^3} \Phi_{\alpha} \left(\frac{\boldsymbol{r} - \boldsymbol{r}'}{t - t'} \right), \tag{3.6}$$

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or equivalently

$$C^{\alpha,(0)}_{\boldsymbol{r},t} = \frac{1}{t^3}\Phi_\alpha\left(\frac{\boldsymbol{r}}{t}\right). \tag{3.7}$$

We observe that the space-time correlation function of natural density fluctuations reflects the shape of the average distribution function $\Phi_\alpha(\boldsymbol{v})$, a fact exploited in the Thomson scattering technique, which, for wavelengths much smaller than the plasma Debye length, probes natural electron density fluctuations and can thus be used as a diagnostic of the average electron distribution function $\Phi_{\rm e}(\boldsymbol{v})$ (Hutchinson 1987; Froula et al. 2011).

3.2. *Magnetized plasmas*

In the case of magnetized plasmas, the mathematical proofs are much more involved and are presented in the appendices. In this subsection, we shall mention the main expressions and briefly discuss them. These expressions are also repeated in the appendices, since we intend for both the main text and the appendices of the manuscript to be self-contained. Let us assume a uniform and static magnetic field $\boldsymbol{B}_{\rm ex} = B_0\hat{\boldsymbol{b}}$ permeating the plasma volume. The non-interacting particles of each species are gyrating with their cyclotron frequency $\omega_{c\alpha} = (e_\alpha B_0)/(m_\alpha c)$. The natural correlator in the real space is given by (see Appendix A for details)

$$\begin{aligned}\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{r}',t'}\rangle = {}& \frac{\delta_{\alpha\beta}}{2\pi p_\perp}\delta\left(p_\perp - p'_\perp\right)\delta\left(p_\parallel - p'_\parallel\right)\delta\left[\boldsymbol{r}-\boldsymbol{r}' + \frac{\boldsymbol{v}-\boldsymbol{v}'}{\omega_{c\alpha}}\times\hat{\boldsymbol{b}} - v_\parallel(t-t')\hat{\boldsymbol{b}}\right]\\ & \times\sum_{l=-\infty}^{+\infty} e^{-\imath l\left(\phi-\phi'\right)}e^{-\imath l\omega_{c\alpha}(t-t')}\Phi^\alpha_{\boldsymbol{p}},\end{aligned} \tag{3.8}$$

and possesses the spatiotemporal translation property. Naturally, the main differences with its un-magnetized counterpart stem from the non-conservation of the polar momentum component (in contrast to the still conserved parallel and perpendicular momentum components) and the periodicity of the motion, which allows the infinite series expansion. With $\mathrm{J}_n(.)$ denoting the Bessel functions of the first kind and nth order and ϕ, ϕ', χ corresponding to the polar angles of the vectors $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{k}$, the space-time Fourier transform is

$$\begin{aligned}\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{k},\omega}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{k}',\omega'}\rangle = {}& \delta(\omega+\omega')\delta(\boldsymbol{k}+\boldsymbol{k}')\frac{\delta_{\alpha\beta}}{(2\pi)^4 p_\perp}\delta(p_\perp - p'_\perp)\delta(p_\parallel - p'_\parallel)\sum_{l,m,n}\mathrm{J}_m\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\\ & \mathrm{J}_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)e^{-\imath(l-m)\phi}e^{-\imath(n-l)\phi'}e^{\imath(n-m)\chi}\delta\left(\omega - k_\parallel v_\parallel - l\omega_{c\alpha}\right)\Phi^\alpha_{\boldsymbol{p}}.\end{aligned} \tag{3.9}$$

We observe the emergence of all cyclotron frequency harmonics whose strength is determined by Bessel functions, as in linear perturbation theory of magnetized plasmas.

For arbitrary average distribution functions $\Phi_\alpha(\boldsymbol{v})$ the spectral density of natural density fluctuations has the general form

$$\begin{aligned}S^{\alpha,(0)}_{\boldsymbol{k},\omega} = {}& \frac{1}{(2\pi)^3}\int\Phi_\alpha(\boldsymbol{v})\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\mathrm{J}_m\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\mathrm{J}_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\\ & e^{\imath(n-m)(\chi-\phi)}\delta\left(\omega - k_\parallel v_\parallel - n\omega_{c\alpha}\right)d^3v.\end{aligned} \tag{3.10}$$

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In the case of gyrotropic distribution functions, $\Phi_\alpha(\boldsymbol{v}) = \Phi_\alpha(v_\perp, v_\parallel)$, we have

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega} = \frac{1}{(2\pi)^3}\int \Phi_\alpha(\boldsymbol{v})\left\{\sum_{n=-\infty}^{+\infty} \mathrm{J}_n^2\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\delta\left(\omega - k_\parallel v_\parallel - n\omega_{c\alpha}\right)\right\} d^3v. \tag{3.11}$$

Finally, in the case of anisotropic Maxwellian distributions (see Appendix C) $\Phi_\alpha(v_\perp, v_\parallel) = n_\alpha[m_\alpha/(2\pi T_{\alpha\perp})][m_\alpha/(2\pi T_{\alpha\parallel})]^{1/2}\exp\{-(m_\alpha/2)[(v_\perp^2/T_{\alpha\perp}) + (v_\parallel^2/T_{\alpha\parallel})]\}$, with $v_{T\alpha\perp}$, $v_{T\alpha\parallel}$ the perpendicular and parallel thermal velocities, we have the simplified form

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega} = \frac{1}{(2\pi)^{7/2}}\frac{n_\alpha}{k_\parallel v_{T\alpha\parallel}}\sum_{n=-\infty}^{+\infty}\exp\left[-\left(\frac{\omega - n\omega_{c\alpha}}{\sqrt{2}k_\parallel v_{T\alpha\parallel}}\right)^2\right] e^{-k_\perp^2\rho_{L\alpha}^2}\mathrm{I}_\mathrm{n}\left(k_\perp^2\rho_{L\alpha}^2\right), \tag{3.12}$$

where $\rho_{L\alpha}$ is the thermal Larmor radius of the species α and with $\mathrm{I}_n(.)$ denoting the modified Bessel function of the first kind and nth order. Remarkably, despite its complex form, (3.11) can be inverted to yield a simple relation for the space-time correlation function of natural density fluctuations for any gyrotropic distribution function (see Appendix B for details),

$$C^{\alpha,(0)}_{\boldsymbol{r},t} = \frac{\omega_{c\alpha}^2}{4t\sin^2\left[\frac{\omega_{c\alpha}t}{2}\right]}\Phi_\alpha\left(\frac{\omega_{c\alpha}}{2\left|\sin\left[\frac{\omega_{c\alpha}t}{2}\right]\right|}\sqrt{x^2+y^2},\ \frac{z}{t}\right). \tag{3.13}$$

This expression possesses the correct un-magnetized limit, since for $\omega_{c\alpha}t \ll 1$ we have $\sin(\omega_{c\alpha}t) \to \omega_{c\alpha}t$.

4. Molecular dynamics calculations of the space-time correlation function of natural density fluctuations

4.1. *Methodology*

The numerical method employed in this work for the calculation of the space-time correlation function of natural density fluctuations is based on molecular dynamics in $(\mathbf{r}, \mathbf{v})$ space using an importance sampling Monte Carlo. Therefore, the method is a particular implementation of molecular dynamics with Monte Carlo initial conditions, similar to those employed in chemical kinetics (Hirst 1990).

Let us consider a single plasma species $\alpha = \beta$. The particles are represented as an ensemble of point particles with phase coordinates $(\mathbf{r}_i, \mathbf{v}_i)$. The coordinates in the configuration space are limited by periodic conditions in a cubic box of size L whose value must be chosen in a manner that optimizes two competing requirements: (i) L must be much larger than the mean distance traveled by a particle in a relevant correlation time scale t, (ii) For a given number density, a fair number of simulated particles must be present in any cell of the numerical mesh (see description below).

At the initial time $t = 0$, the phase coordinates are distributed in the periodic computational domain according to a joint distribution $L^{-3N}\Pi_i\Phi^\alpha_{\mathbf{v}_i}$, where N is the total number of simulated particles. This is obtained by selecting particle positions x_i, y_i, z_i using uniformly distributed random numbers ζ_j in the range $[0, 1]$ as following; $x_i = L\zeta_i$, $y_i = L\zeta_{2i}$, $z_i = L\zeta_{3i}$. The velocity distribution can be arbitrary, here we choose Maxwellian distribution for comparison with the analytical results. Hence, the initial velocity components are selected with the aid of the Box-Muller method, $v_i = (-2T_\alpha \ln\eta_1/m_\alpha)^{1/2}\sin(2\pi\eta_2)$, where the two random numbers η_1 and η_2 are sampled from uniform distributions in the range $[0, 1]$.

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This simulation is mesh-free and there is no discretization in the $(\mathbf{r}, \mathbf{v})$ space. However, a sampling time-space mesh is employed for the calculation of the density correlations, using a correlation time step and spatial steps $\delta x, \delta y, \delta z$ appropriately chosen. This mesh has no effect on the particle motion. The particles are moved individually by time-advancing time steps $\Delta t \leqslant t$. Periodic conditions are implemented in order to remove boundary effects. To this end, particles crossing, e.g. the $x = L$ boundary, are placed back into the box by translation $x_i \to x_i - L$.

In the presence of an external magnetic field, the particles follow the Newton–Lorentz equations of motion. Neglecting error terms $O(\Delta t^3)$ (Birdsall and Langdon 1991) this can be replaced by the finite-difference equations

$$\frac{\boldsymbol{r}_{i\alpha}(t+\Delta t)-\boldsymbol{r}_{i\alpha}(t)}{\Delta t} = \boldsymbol{v}_{i\alpha}\,(t+\Delta t/2)\,,$$
$$\frac{\boldsymbol{v}_{i\alpha}\,(t+\Delta t/2)-\boldsymbol{v}_{i\alpha}\,(t-\Delta t/2)}{\Delta t} = \frac{e_\alpha}{2m_\alpha c}\,[\boldsymbol{v}_{i\alpha}\,(t+\Delta t/2)+\boldsymbol{v}_{i\alpha}\,(t-\Delta t/2)]\times \boldsymbol{B}_{\rm ex}.$$

The last equation is a pure rotation from $\boldsymbol{v}_{i\alpha}(t-\Delta t/2)$ to $\boldsymbol{v}_{i\alpha}(t+\Delta t/2)$. Let us consider a uniform static magnetic field along $\hat{\boldsymbol{z}}$. In this case the charged particles follow a circular motion in the $x-y$ plane, characterized by the cyclotron frequency $\omega_{c\alpha} = e_\alpha B_0/m_\alpha c$ and the rotation angle $\theta = \arctan(-\omega_{c\alpha}\Delta t)$. The rotation can be expressed by

$$v_{i\alpha,x}\,(t+\Delta t/2) = v_{i\alpha,x}\,(t-\Delta t/2)\cos\theta + v_{i\alpha,y}\,(t-\Delta t/2)\sin\theta,$$
$$v_{i\alpha,y}\,(t+\Delta t/2) = v_{i\alpha,y}\,(t-\Delta t/2)\cos\theta - v_{i\alpha,x}\,(t-\Delta t/2)\sin\theta.$$

This rotation can be implemented via the Buneman method (Birdsall and Langdon 1991), that is accurate for $\omega_{c\alpha}\Delta t \ll 1$,

$$w = v_{i\alpha,x}\,(t-\Delta t/2) - v_{i\alpha,y}\,(t-\Delta t/2)\tan(\theta/2),$$
$$v_{i\alpha,y}\,(t+\Delta t/2) = v_{i\alpha,y}\,(t-\Delta t/2) + w\sin\theta,$$
$$v_{i\alpha,x}\,(t+\Delta t/2) = w - v_{i\alpha,y}\,(t+\Delta t/2)\tan(\theta/2).$$

In the course of the simulation the aforementioned sampling mesh is used to calculate average values. Namely, the number of particles in any phase cell, $n(i, j)$, is determined based on the simulated particle ensemble, with i corresponding to space and j to time. Then a cycle is run for different values of the spatial scales, i.e. $\delta x = k\Delta x$, where $k = -k_{\max}, \ldots, k_{\max}$. The fluctuations δn are calculated as $\delta n = n - \langle n\rangle$. Finally, the correlator $\langle \delta n(0,0)\,\delta n(\delta x, \delta y, \delta z, t)\rangle$ is estimated as a function of k by averaging over the i, j products of $\delta n(i, j)$ and $\delta n(i+k, j+1)$. Here the periodic boundary conditions are included, e.g. for x-coordinate, $n(N_{\max}+k, j+1) = n(k, j+1)$, $n(-k, j+1) = n(N_{\max}-k, j+1)$ with $N_{\max}$ the number of cells in the x-mesh.

4.2. *Results*

Simulations have been carried out for Maxwellian electron species of a temperature $T_{\rm e}$ with and without the presence of a magnetic field. Here we present results for $T_{\rm e} = 1\,{\rm keV}$ and number density $n_{\rm e} = 10^{20}\,{\rm m}^{-3}$. The box size values have been chosen according to the requirements described in the previous subsection and for the selected correlation times $t = (1,\ 25,\ 250)\,{\rm ps}$, they are $L = (0.35,\ 8,\ 80)\,{\rm mm}$, respectively. Since for the un-magnetized case the trajectories are ballistic, the advancing time step can be the same as the correlation time of interest. For the magnetized case, the time step was set to $\Delta t = 1\,{\rm ps}$ for all t values.

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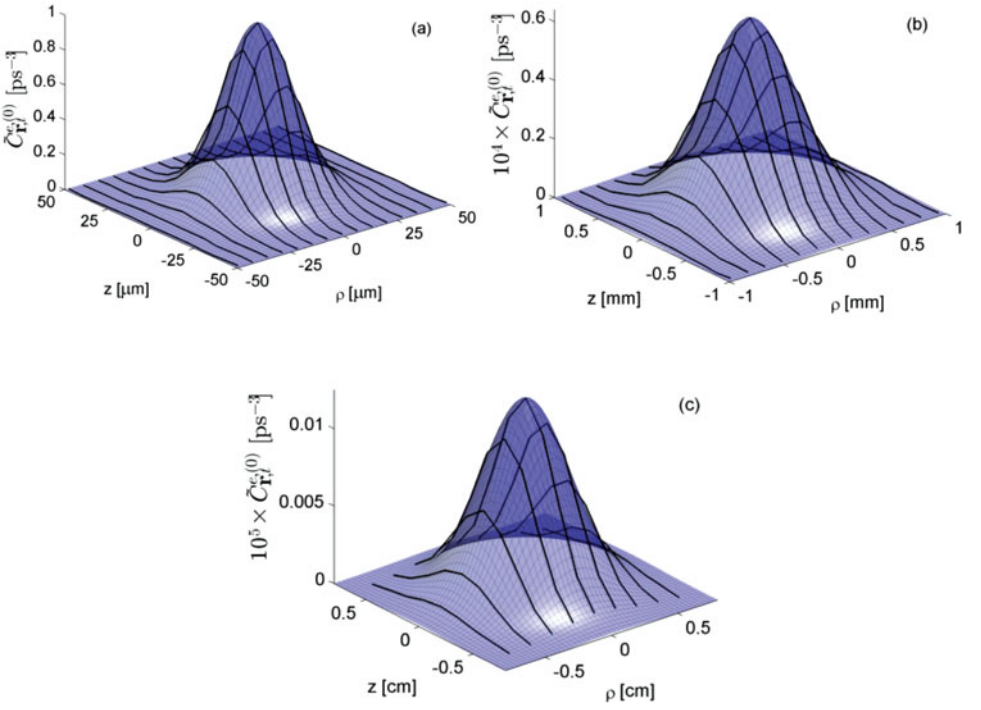


FIGURE 1. The space-time correlation function of natural electron density fluctuations in an un-magnetized plasma, normalized to the Maxwellian distribution exponential pre-factor, as given by (4.1) for correlation times; 1 ps (a), 25 ps (b) and 200 ps (c).

Figure 1 shows the time evolution of the space-time correlation function of natural electron density fluctuations as a function of spatial coordinates ($\rho = \sqrt{(x^2 + y^2)}, z$) in the absence of a magnetic field. The numerical data points are joined by black lines to enable a more clear presentation. Color-coded plots yield values of the space-time correlation function $\tilde{C}^{\mathrm{e},(0)}_{\boldsymbol{r},t}$ as given by (3.7) for the case of Maxwellian distribution and normalized by the pre-exponential factor, i.e.

$$\tilde{C}^{\mathrm{e},(0)}_{\boldsymbol{r},t} = \frac{C^{\mathrm{e},(0)}_{\boldsymbol{r},t}}{n_{\mathrm{e}}[m_{\mathrm{e}}/(2\pi T_{\mathrm{e}})]^{3/2}} = \frac{1}{t^3}\exp\left(-\frac{m_{\mathrm{e}}}{2T_{\mathrm{e}}}\frac{r^2}{t^2}\right). \tag{4.1}$$

As time increases, the peak value of $\tilde{C}^{\mathrm{e},(0)}_{\boldsymbol{r}}$ decreases allowing to observe small but finite correlations on larger spatial scales, from micrometers at $t = 1$ ps (a) to centimeters at $t = 200$ ps (c).

Results for the same parameters and an external magnetic field $B_0 = 1$ T are presented in Fig. 2. Color-coded plots yield values of the space-time correlation function $\tilde{C}^{\mathrm{e},(0)}_{\boldsymbol{r},t}$ as given by (3.13) for the case of Maxwellian distribution and normalized by the pre-exponential factor, i.e.

$$\tilde{C}^{\mathrm{e},(0)}_{\boldsymbol{r},t} = \frac{C^{\mathrm{e},(0)}_{\boldsymbol{r},t}}{n_{\mathrm{e}}[m_{\mathrm{e}}/(2\pi T_{\mathrm{e}})]^{3/2}} = \frac{\omega_{\mathrm{ce}}^2}{4t\sin^2\left[\frac{\omega_{\mathrm{ce}}t}{2}\right]}\exp\left(-\frac{m_{\mathrm{e}}}{2T_{\mathrm{e}}}\left[\frac{\omega_{\mathrm{ce}}^2}{4\sin^2\left[\frac{\omega_{\mathrm{ce}}t}{2}\right]}(x^2+y^2) + \frac{z^2}{t^2}\right]\right). \tag{4.2}$$

For $\omega_{\mathrm{ce}}t \ll 1$, the result collapses to the un-magnetized case as in Fig. 1(a), since the electron trajectories are essentially straight lines. The correlations in the plane perpendicular to the magnetic field decay much more rapidly than those along the

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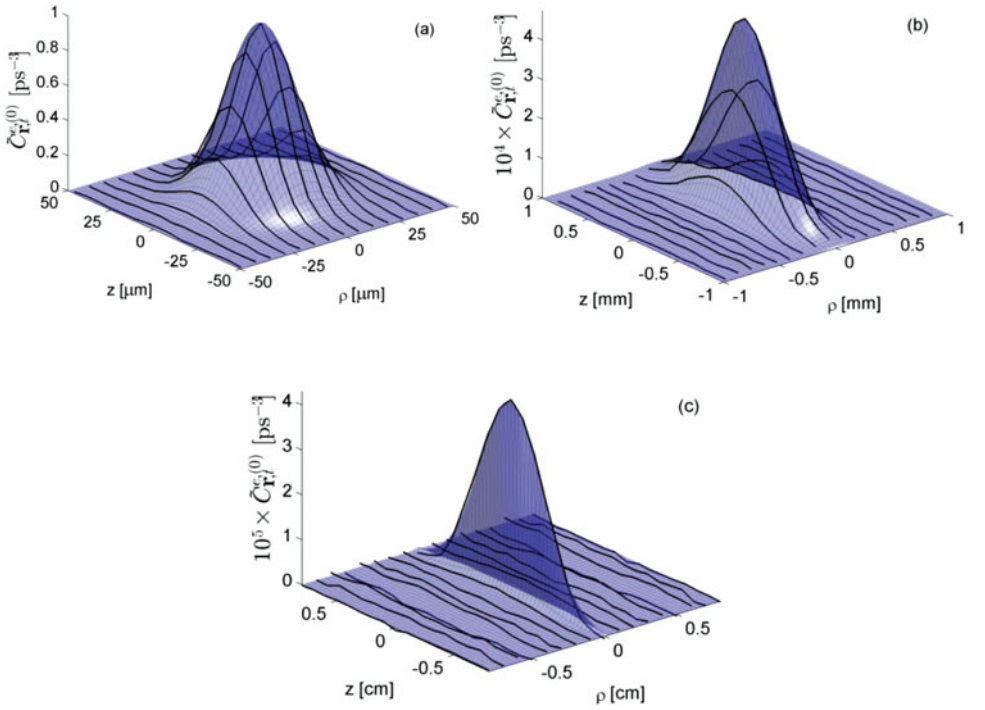


FIGURE 2. The space-time correlation function of natural electron density fluctuations in a magnetized plasma $B_0 = 1$ T, normalized to the Maxwellian distribution exponential pre-factor, as given by (4.2) for correlation times; 1 ps (a), 25 ps (b) and 200 ps (c). The electron cyclotron period is 36 ps.

magnetic field direction (z) and for $t = 200$ ps all correlations in $x - y$ plane are essentially lost as seen from Fig. 2(c).

In conclusion, simulations for non-interacting charged particles have reproduced analytical results for the correlator of natural density fluctuations in both un-magnetized and magnetized plasmas. The numerical approach employed enables one to include electrostatic interactions between particles, and hence also provides a possibility to check theoretical predictions for the spectral densities of plasma fluctuations not only in fully-ionized plasmas but also in the presence of neutrals as well as dust grains. This is particularly important for realistic plasma environments, where modifications in the spectra due to the presence of solid particulates have been proposed as a diagnostic of dust in laboratory and space plasmas (Ratynskaia et al. 2010; Tolias et al. 2012; Ratynskaia et al. 2014).

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Appendix A. The natural correlator and the spectral density of natural density fluctuations in magnetized plasmas

A.1. *The natural correlator in the real space*

Let us assume a static uniform magnetic field $\boldsymbol{B}_{ex} = B_0\hat{\boldsymbol{b}}$ and a cartesian coordinate system such that $\hat{\boldsymbol{b}} \parallel \hat{\boldsymbol{z}}$. In order to calculate the transition probability, we need to solve

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Newton's second law $d\boldsymbol{v}_\alpha/dt = \omega_{c\alpha}(\boldsymbol{v}_\alpha \times \hat{\boldsymbol{b}})$ with the initial conditions $\boldsymbol{v}_\alpha(t') = \boldsymbol{v}'$ and $\boldsymbol{r}_\alpha(t') = \boldsymbol{r}'$. In cylindrical coordinates, we have $\boldsymbol{p}_\alpha(t) = \{p'_\perp, \phi' - \omega_{c\alpha}(t-t') + 2\pi l, p'_\parallel\}$, since the parallel and perpendicular momenta are constants of motion and there is a 2π periodicity in the polar angle. As a result, the momentum part of the transition probability becomes $\delta[\boldsymbol{p} - \boldsymbol{p}_\alpha(t)]^{t'}_{\boldsymbol{p}',\boldsymbol{r}'} = (1/p_\perp)\delta(p_\perp - p'_\perp)\delta(p_\parallel - p'_\parallel)\sum_{l=-\infty}^{+\infty}\delta[\phi - \phi' + \omega_{c\alpha}(t-t') - 2\pi l]$. The infinite sum that emerges has the well-studied Dirac comb form. It is a periodic function with $T = 2\pi$ and can be expanded as a complex Fourier series yielding $\sum_{l=-\infty}^{+\infty}\delta(x - lT) = (1/T)\sum_{l=-\infty}^{+\infty}\exp[\pm\imath(2\pi lx)/(T)]$. Therefore, we acquire

$$\delta\left[\boldsymbol{p} - \boldsymbol{p}_\alpha(t)\right]^{t'}_{\boldsymbol{p}',\boldsymbol{r}'} = \frac{1}{2\pi p_\perp}\delta\left(p_\perp - p'_\perp\right)\delta\left(p_\parallel - p'_\parallel\right)\sum_{l=-\infty}^{+\infty} e^{-\imath l\left(\phi-\phi'\right)}e^{-\imath l\omega_{c\alpha}(t-t')}. \tag{A 1}$$

To acquire $\boldsymbol{r}_\alpha(t)$, we simply integrate $\boldsymbol{v}_\alpha(t)$ in the time interval (t', t). In this case a vector form is more convenient, yielding $\boldsymbol{r}_\alpha(t) = \boldsymbol{r}' - (1/\omega_{c\alpha})[\boldsymbol{v}_\alpha(t) - \boldsymbol{v}'] \times \hat{\boldsymbol{b}} + [\boldsymbol{v}_\alpha(t) \cdot \hat{\boldsymbol{b}}](t-t')\hat{\boldsymbol{b}}$. Consequently, the position part of the transition probability becomes

$$\delta\left[\boldsymbol{r} - \boldsymbol{r}_\alpha(t)\right]^{t'}_{\boldsymbol{p}',\boldsymbol{r}'} = \delta\left[\boldsymbol{r} - \boldsymbol{r}' + \frac{\boldsymbol{v} - \boldsymbol{v}'}{\omega_{c\alpha}} \times \hat{\boldsymbol{z}} - v_\parallel(t-t')\hat{\boldsymbol{z}}\right]. \tag{A 2}$$

Combining (A 1) and (A 2) with (2.8), we obtain a general relation for the natural correlator

$$\left\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{r}',t'}\right\rangle = \frac{\delta_{\alpha\beta}}{2\pi p_\perp}\delta\left(p_\perp - p'_\perp\right)\delta\left(p_\parallel - p'_\parallel\right)\delta\left[\boldsymbol{r} - \boldsymbol{r}' + \frac{\boldsymbol{v} - \boldsymbol{v}'}{\omega_{c\alpha}} \times \hat{\boldsymbol{z}} - v_\parallel(t-t')\hat{\boldsymbol{z}}\right] \times \sum_{l=-\infty}^{+\infty} e^{-\imath l\left(\phi-\phi'\right)}e^{-\imath l\omega_{c\alpha}(t-t')}\Phi^\alpha_{\boldsymbol{p}}.$$

Clearly, the natural correlator satisfies $\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{r}',t'}\rangle = \langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r}-\boldsymbol{r}',t-t'}\delta f^{\beta,(0)}_{\boldsymbol{p}',0,0}\rangle$, which is a manifestation of the stationarity and homogeneity of the medium. Thus, the correlations of natural fluctuations between the phase space point $(\boldsymbol{p}, \boldsymbol{r})$ at t and the phase-space point $(\boldsymbol{p}', \boldsymbol{r}')$ at t', can be viewed as a function of $(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{r} - \boldsymbol{r}', t - t')$ only. We use the notation $\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',0,0}\rangle = \langle \delta f^{\alpha,(0)}_{\boldsymbol{p}}\delta f^{\beta,(0)}_{\boldsymbol{p}'}\rangle(\boldsymbol{r}, t)$ with

$$\langle \delta f^{\alpha,(0)}_{\boldsymbol{p}}\delta f^{\beta,(0)}_{\boldsymbol{p}'}\rangle(\boldsymbol{r}, t) = \frac{\delta_{\alpha\beta}}{2\pi p_\perp}\delta\left(p_\perp - p'_\perp\right)\delta(p_\parallel - p'_\parallel)\delta\left[\boldsymbol{r} + \frac{\boldsymbol{v} - \boldsymbol{v}'}{\omega_{c\alpha}} \times \hat{\boldsymbol{z}} - v_\parallel t\hat{\boldsymbol{z}}\right] \times \sum_{l=-\infty}^{+\infty} e^{-\imath l\left(\phi-\phi'\right)}e^{-\imath l\omega_{c\alpha}t}\Phi^\alpha_{\boldsymbol{p}}. \tag{A 3}$$

A.2. *The natural correlator in the Fourier domain*

The time and space translation property of the natural correlator is particularly useful in the Fourier domain. Fourier transforming in $(\boldsymbol{r}, \boldsymbol{r}', t, t')$ we have

$$\left\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{k},\omega}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{k}',\omega'}\right\rangle = \frac{1}{(2\pi)^4}\delta(\omega + \omega')\delta(\boldsymbol{k} + \boldsymbol{k}')\int \langle \delta f^{\alpha,(0)}_{\boldsymbol{p}}\delta f^{\beta,(0)}_{\boldsymbol{p}'}\rangle(\boldsymbol{r}, t)e^{-\imath\boldsymbol{k}\cdot\boldsymbol{r}}e^{\imath\omega t}d^3r dt. \tag{A 4}$$

Proceeding with the calculations, it is preferable to employ the cylindrical coordinate representations $(v_\perp, \phi, v_\parallel)$, $(v'_\perp, \phi', v'_\parallel)$ and $(k_\perp, \chi, k_\parallel)$ for the vectors $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\boldsymbol{k}$. On the contrary, the scalar triple product is more easily computed in cartesian unit vectors. Using the above conventions, together with the aid of the δ-functions and some

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trigonometric identities, we have $\boldsymbol{k}\cdot[(\boldsymbol{v}-\boldsymbol{v}')\times\hat{\boldsymbol{z}}]=-k_\perp v_\perp[\sin(\chi-\phi)+\sin(\phi'-\chi)]$. Moreover, by the generating function of the Bessel functions of the first kind, we have $e^{-\imath z\sin\psi}=\sum_{n=-\infty}^{+\infty}\mathrm{J}_n(z)e^{-\imath n\psi}$, which we apply for both $\sin(\chi-\phi)$ and $\sin(\phi'-\chi)$. Combining (A 3) and (A 4) with the above, the d^3rdt integrations are trivial and we end up with

$$\begin{aligned}\langle\delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{k},\omega}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{k}',\omega'}\rangle=\delta_{\alpha\beta}\delta(\omega+\omega')\delta(\boldsymbol{k}+\boldsymbol{k}')\frac{1}{(2\pi)^4p_\perp}\delta(p_\perp-p'_\perp)\delta(p_\parallel-p'_\parallel)\\ \times\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\mathrm{J}_m\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\mathrm{J}_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\\ \times e^{-\imath(l-m)\phi}e^{-\imath(n-l)\phi'}e^{\imath(n-m)\chi}\delta\left(\omega-k_\parallel v_\parallel-l\omega_{c\alpha}\right)\Phi^\alpha_{\boldsymbol{p}}.\end{aligned}\tag{A 5}$$

A.3. *The spectral density of natural density fluctuations*

Natural plasma density fluctuations are given by $\delta n^{\alpha,(0)}_{\boldsymbol{k},\omega}=\int\delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{k},\omega}d^3p$, whereas the correlator of natural plasma density fluctuations is given by $\langle\delta n^{\alpha,(0)}_{\boldsymbol{k},\omega}\delta n^{\beta,(0)}_{\boldsymbol{k}',\omega'}\rangle=\int\langle\delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{k},\omega}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{k}',\omega'}\rangle d^3pd^3p'$ and the spectral density of natural plasma density fluctuations is defined by $S^{\alpha,(0)}_{\boldsymbol{k},\omega}=\langle|\delta n^{\alpha,(0)}_{\boldsymbol{k},\omega}|^2\rangle$. Owing to the reality property of the Fourier transform $\delta n^{\alpha,(0)}_{-\boldsymbol{k},-\omega}=\delta n^{\alpha*,(0)}_{\boldsymbol{k},\omega}$, it is straightforward that $\langle\delta n^{\alpha,(0)}_{\boldsymbol{k},\omega}\delta n^{\beta,(0)}_{\boldsymbol{k}',\omega'}\rangle=\delta_{\alpha\beta}\delta(\omega+\omega')\delta(\boldsymbol{k}+\boldsymbol{k}')S^{\alpha,(0)}_{\boldsymbol{k},\omega}$ which leads to

$$\begin{aligned}S^{\alpha,(0)}_{\boldsymbol{k},\omega}=\int\frac{1}{(2\pi)^4p_\perp}\delta(p_\perp-p'_\perp)\delta(p_\parallel-p'_\parallel)\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\mathrm{J}_m\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\mathrm{J}_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\\ e^{-\imath(l-m)\phi}e^{-\imath(n-l)\phi'}e^{\imath(n-m)\chi}\delta\left(\omega-k_\parallel v_\parallel-l\omega_{c\alpha}\right)\Phi^\alpha_{\boldsymbol{p}}d^3pd^3p'.\end{aligned}\tag{A 6}$$

The d^3p' integration can be carried out by using cylindrical coordinates, the relation $(1/p_\perp)\delta(p_\perp-p'_\perp)=(1/p'_\perp)\delta(p_\perp-p'_\perp)$ and the orthogonality property of complex exponentials $\int_0^{2\pi}e^{-\imath(n-l)\phi'}d\phi'=2\pi\delta_{nl}$, with the Kronecker-delta automatic canceling all terms of the l-series apart from $n=l$. Consequently, the general relation for the spectral density of natural plasma density fluctuations for arbitrary average distribution functions has the form

$$\begin{aligned}S^{\alpha,(0)}_{\boldsymbol{k},\omega}=\frac{1}{(2\pi)^3}\int\Phi_\alpha(\boldsymbol{v})\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\mathrm{J}_m\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\mathrm{J}_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\\ e^{\imath(n-m)(\chi-\phi)}\delta\left(\omega-k_\parallel v_\parallel-n\omega_{c\alpha}\right)d^3v.\end{aligned}\tag{A 7}$$

In the special case of gyrotropic average distribution functions, due to the lack of dependence on the polar velocity angle, the $d\phi$ integration can also be carried out with the orthogonality property, i.e. $\int_0^{2\pi}e^{\imath(n-m)(\chi-\phi)}d\phi=2\pi\delta_{nm}$. Therefore, in the gyrotropic case, the spectral density of natural plasma density fluctuations has the simplified form

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega}=\frac{1}{(2\pi)^3}\int\Phi_\alpha(\boldsymbol{v})\left\{\sum_{n=-\infty}^{+\infty}\mathrm{J}_n^2\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\delta\left(\omega-k_\parallel v_\parallel-n\omega_{c\alpha}\right)\right\}d^3v.\tag{A 8}$$

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Appendix B. The space-time correlation function of natural density fluctuations in magnetized plasmas for arbitrary gyrotropic distribution functions

B.1. *Fourier inversion method*

The space-time correlation function of natural density fluctuations can be found by Fourier inversion of the spectral density of natural density fluctuations, i.e.

$$C^{\alpha,(0)}_{\boldsymbol{r},t} = \int S^{\alpha,(0)}_{\boldsymbol{k},\omega} e^{-\imath\omega t+\imath\boldsymbol{k}\cdot\boldsymbol{r}} d^3k d\omega. \tag{B 1}$$

In order to carry out the Fourier inversion it is necessary to employ a number of identities from the theory of Bessel functions (Watson 1966). Let us start with Parseval's integral, a special case of Bessel's integrals, that reads us $\int_0^{2\pi} \exp(\imath a \sin\phi) d\phi = 2\pi \mathrm{J}_0(a)$. Using Parseval's integral, the representation $x\cos\phi + y\sin\phi = d\sin(\phi+\theta)$ with $d = \sqrt{x^2+y^2}$ and $\theta = \arctan(x/y)$ together with the periodicity of the integrand, it is straightforward to prove that

$$\int_0^{2\pi} e^{\imath a(x\cos\phi + y\sin\phi)} d\phi = 2\pi \mathrm{J}_0\left(a\sqrt{x^2+y^2}\right). \tag{B 2}$$

Moreover, Neumann's addition formula reads us $\mathrm{J}_0(w) = \sum_{k=-\infty}^{+\infty} \mathrm{J}_k(Z)\mathrm{J}_k(z) e^{\imath k\phi}$ with $w = \sqrt{Z^2+z^2-2Zz\cos\phi}$. We set $Z = z$ and $k = -n$, we also use the relations $\mathrm{J}_{-n}(z) = (-1)^n \mathrm{J}_n(z)$ and $\sqrt{2(1-\cos\phi)} = 2|\sin[\phi/2]|$, for z a non-negative real number we obtain

$$\sum_{n=-\infty}^{+\infty} \mathrm{J}_n^2(z) e^{-\imath n\phi} = \mathrm{J}_0\left(2z\left|\sin\left[\frac{\phi}{2}\right]\right|\right). \tag{B 3}$$

Furthermore, the Bessel closure equation, one of the orthogonality relations of the Bessel functions, reads us

$$\int_0^{\infty} x\mathrm{J}_n(ax)\mathrm{J}_n(bx) dx = \frac{1}{a}\delta(a-b). \tag{B 4}$$

Finally, it is necessary to employ some basic identities from the theory of Dirac functions. These are the scaling and Fourier properties, that respectively read us $\delta(ax) = (1/|a|)\delta(x)$ and

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm\imath(x-x_0)t} dt. \tag{B 5}$$

The derivation begins with the choice of appropriate coordinates for the wave-vector $\boldsymbol{k}$, the position $\boldsymbol{r}$ and the velocity $\boldsymbol{v}$. We choose cylindrical coordinates for $\boldsymbol{k}$, $\boldsymbol{v}$ and cartesian coordinates for $\boldsymbol{r}$. Therefore, we have $\boldsymbol{k} = k_\perp\cos\chi\hat{\boldsymbol{x}} + k_\perp\sin\chi\hat{\boldsymbol{y}} + k_\parallel\hat{\boldsymbol{z}}$, $\boldsymbol{v} = v_\perp\cos\phi\hat{\boldsymbol{x}} + v_\perp\sin\phi\hat{\boldsymbol{y}} + v_\parallel\hat{\boldsymbol{z}}$, $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ and $d^3k = k_\perp dk_\perp dk_\parallel d\chi$, $d^3v = v_\perp dv_\perp dv_\parallel d\phi$. The order of the calculations is the following: integration over the frequency $d\omega$ with the δ—function definition, integration over the wave-vector angle $d\chi$ with (B 2), integration over the parallel wave-vector $dk_\parallel$ with (B 5), summation over the index n with (B 3) and integration over the perpendicular wave-vector $dk_\perp$ with (B 4). Applying the scaling property, we have

$$C^{\alpha,(0)}_{\boldsymbol{r},t} = \frac{1}{2\pi}\int \frac{1}{v_\perp}\frac{\omega_{c\alpha}^2}{4t\sin^2\left[\frac{\omega_{c\alpha}t}{2}\right]}\delta\left(v_\parallel - \frac{z}{t}\right)\delta\left(v_\perp - \frac{\omega_{c\alpha}}{2\left|\sin\left[\frac{\omega_{c\alpha}t}{2}\right]\right|}\sqrt{x^2+y^2}\right)\Phi_\alpha(\boldsymbol{v}) d^3v.$$

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For a gyrotropic distribution function we have $\Phi_\alpha(\boldsymbol{v}) = \Phi_\alpha(v_\perp, v_\parallel)$. Therefore, the polar integration $d\phi$ is trivial and the integration over the parallel and perpendicular velocity components $dv_\perp$, $dv_\parallel$ can be carried out with the aid of the $\delta-$functions. Reverting to cylindrical co-ordinates for the position with $\rho = \sqrt{x^2+y^2}$, we end up with the relation

$$C^{\alpha,(0)}_{\boldsymbol{r},t} = \frac{\omega_{c\alpha}^2}{4t\sin^2\left[\frac{\omega_{c\alpha}t}{2}\right]}\Phi_\alpha\left(\frac{\omega_{c\alpha}}{2\left|\sin\left[\frac{\omega_{c\alpha}t}{2}\right]\right|}\rho,\ \frac{z}{t}\right). \tag{B 6}$$

B.2. *Transition probability method.*

Alternatively, the two-point space-time correlation function of natural density fluctuations can be found by integrating the transition probability relation for $\langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{r}',t'}\rangle$ over the momenta. In this expression, the stationarity and homogeneity properties of the medium have not been applied yet, meaning that the space-time correlation function will be a function of $(\boldsymbol{r}, \boldsymbol{r}', t, t')$. As we shall see, the spatial and temporal translation properties will arise naturally in the final formula,

$$C^{\alpha,\beta}_{\boldsymbol{r},\boldsymbol{r}',t,t'} = \int \langle \delta f^{\alpha,(0)}_{\boldsymbol{p},\boldsymbol{r},t}\delta f^{\beta,(0)}_{\boldsymbol{p}',\boldsymbol{r}',t'}\rangle d^3p d^3p'. \tag{B 7}$$

In these calculations, it is not convenient to use the position part of the transition probability as described in (A 2). An alternative representation needs to be sought. Let us first define the vectors $\boldsymbol{R}_\alpha = \boldsymbol{r}_\alpha(t) - \boldsymbol{r}'$ and $\boldsymbol{R}_\mathrm{d} = \boldsymbol{r} - \boldsymbol{r}'$. We express them in cylindrical coordinates of the form $\boldsymbol{R}_\alpha = (R_\alpha,\ \Theta_\alpha,\ Z_\alpha)$ and $\boldsymbol{R}_\mathrm{d} = (R_\mathrm{d},\ \Theta_\mathrm{d},\ Z_\mathrm{d})$. Using the equation of motion, it is straightforward to show that

$$\boldsymbol{R}_\alpha = \left\{2\frac{v_\perp}{\omega_{c\alpha}}\left|\sin\left[\frac{\omega_{c\alpha}(t-t')}{2}\right]\right|,\ \phi' - \frac{\omega_{c\alpha}(t-t')}{2},\ v_\parallel(t-t')\right\},$$

$$\boldsymbol{R}_\mathrm{d} = \left\{\sqrt{(x-x')^2+(y-y')^2},\ \arctan\left(\frac{y-y'}{x-x'}\right),\ z-z'\right\}.$$

In this notation, the r-part of the transition probability has the form $\delta[\boldsymbol{r}-\boldsymbol{r}_\alpha(t)]^{t'}_{\boldsymbol{p}',\boldsymbol{r}'} = \delta(\boldsymbol{R}_\mathrm{d} - \boldsymbol{R}_\alpha)$. Considering the $\delta-$function in cylindrical coordinates we have

$$\delta\left[\boldsymbol{r} - \boldsymbol{r}_\alpha(t)\right]^{t'}_{\boldsymbol{p}',\boldsymbol{r}'} = (1/R_\alpha)\,\delta\left(R_\mathrm{d} - R_\alpha\right)\delta\left(\Theta_\mathrm{d} - \Theta_\alpha\right)\delta\left(Z_\mathrm{d} - Z_\alpha\right). \tag{B 8}$$

Combining (A 1), (B 7) and (B 8) we have

$$C^{\alpha,\beta}_{\boldsymbol{r},\boldsymbol{r}',t,t'} = \delta_{\alpha\beta}\int \frac{(2\pi)^{-1}}{R_\alpha p'_\perp}\delta(p_\perp - p'_\perp)\delta(p_\parallel - p'_\parallel)\delta(R_\mathrm{d} - R_\alpha)\delta(\Theta_\mathrm{d} - \Theta_\alpha)\delta(Z_\mathrm{d} - Z_\alpha)$$
$$\sum_{l=-\infty}^{+\infty} e^{-\imath l\left[\phi - \phi' + \omega_{c\alpha}(t-t')\right]}\Phi^\alpha_{\boldsymbol{p}} d^3p'd^3p.$$

The $dp'_\parallel$, $dp'_\perp$ and $d\phi'$ integrations are carried out with the aid of δ-functions $\delta(p_\parallel - p'_\parallel)$, $\delta(p_\perp - p'_\perp)$ and $\delta(\Theta_\mathrm{d} - \Theta_\alpha)$, respectively. For gyrotropic average distribution functions, the $d\phi$ integration is carried out with the orthogonality property of complex exponentials, which also disposes of the l-series since only the $l = 0$ term is non-zero. Therefore, we get

$$C^{\alpha,\beta}_{\boldsymbol{r},\boldsymbol{r}',t,t'} = \delta_{\alpha\beta}\int_{-\infty}^{\infty}\int_0^\infty \frac{1}{R_\alpha}\delta(R_\mathrm{d} - R_\alpha)\delta(Z_\mathrm{d} - Z_\alpha)\Phi_\alpha(v_\perp, v_\parallel)v_\perp dv_\perp dv_\parallel.$$

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Finally, after employing the scaling property of the $\delta-$functions we can perform the $dv_\perp$, $dv_\parallel$ integrations to obtain

$$C^{\alpha,\beta}_{\boldsymbol{r},\boldsymbol{r}',t,t'} = \frac{\delta_{\alpha\beta}\omega_{c\alpha}^2}{4|t-t'|\sin^2\left[\frac{\omega_{c\alpha}(t-t')}{2}\right]}\Phi_\alpha\left[\frac{\omega_{c\alpha}/2}{\left|\sin\left[\frac{\omega_{c\alpha}(t-t')}{2}\right]\right|}\sqrt{(x-x')^2+(y-y')^2}, \frac{z-z'}{t-t'}\right]. \tag{B 9}$$

The relation between the one-point and the two-time space-time correlation function of natural density fluctuations is $C^{\alpha,\beta}_{\boldsymbol{r},0,t,0} = \delta_{\alpha\beta}C^{\alpha,(0)}_{\boldsymbol{r},t}$, which implies that (B 6), (B 9) are equivalent.

Appendix C. The spectral density of natural plasma density fluctuations in magnetized plasmas for anisotropic Maxwellian distribution functions

Let us assume that the average distribution function $\Phi_\alpha(\boldsymbol{v})$ has a bi-Maxwellian character, i.e. that the parallel and perpendicular to the magnetic field motions are characterized by different temperatures $T_{\alpha\parallel}$ and $T_{\alpha\perp}$, respectively. The general form of the bi-Maxwellian, normalized to the species density, is

$$\Phi_\alpha(v_\perp, v_\parallel) = n_\alpha \frac{m_\alpha}{2\pi T_{\alpha\perp}}\sqrt{\frac{m_\alpha}{2\pi T_{\alpha\parallel}}}\exp\left[-\frac{m_\alpha}{2}\left(\frac{v_\perp^2}{T_{\alpha\perp}}+\frac{v_\parallel^2}{T_{\alpha\parallel}}\right)\right].$$

In analogy with the isotropic Maxwellian distribution, we can define the parallel and perpendicular thermal velocities $v_{T\alpha\parallel} = \sqrt{T_{\alpha\parallel}/m_\alpha}$ and $v_{T\alpha\perp} = \sqrt{T_{\alpha\perp}/m_\alpha}$, but also the thermal Larmor radius $\rho_{L\alpha} = v_{T\alpha\perp}/\omega_{c\alpha}$. The bi-Maxwellian is a gyrotropic distribution function, hence the spectral density can be calculated by applying (A 8). The computations proceed as usual with selecting cylindrical coordinates, the polar integration is trivial, the $dv_\parallel$ integration is carried out with the aid of the $\delta-$function, while the $dv_\perp$ integration is carried out with the aid of Weber's second exponential integral (Watson 1966), $\int_0^\infty \exp(-p^2t^2)\mathrm{J}_n(at)\mathrm{J}_n(bt)tdt = (1/2p^2)\exp[-(a^2+b^2)/4p^2]\mathrm{I}_n(ab/2p^2)$ where $\mathrm{I}_n(.)$ denotes the modified Bessel function of the first kind and nth order. In particular, the limiting case $a = b$ of the integral is needed. Overall, we end up with

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega} = \frac{1}{(2\pi)^{7/2}}\frac{n_\alpha}{k_\parallel v_{T\alpha\parallel}}\sum_{n=-\infty}^{+\infty}\exp\left[-\left(\frac{\omega-n\omega_{c\alpha}}{\sqrt{2}k_\parallel v_{T\alpha\parallel}}\right)^2\right]e^{-k_\perp^2\rho_{L\alpha}^2}\mathrm{I}_n\left(k_\perp^2\rho_{L\alpha}^2\right). \tag{C 1}$$

Of particular interest are the limiting cases. In the strongly-magnetized limit we have $B_0 \to \infty$, $\omega_{c\alpha} \to \infty$ and $\rho_{L\alpha} \to 0$. Singling out the $n = 0$ term of the infinite series and recalling that $\mathrm{I}_0(0) = 1$, $\mathrm{I}_n(0) = 1$ for $n \neq 0$, we have

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega} = \frac{1}{(2\pi)^{7/2}}\frac{n_\alpha}{k_\parallel v_{T\alpha\parallel}}\exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{T\alpha\parallel}^2}\right). \tag{C 2}$$

Recalling the well-known relation in the case of un-magnetized plasmas, we notice that the only difference is the presence of $k_\parallel$ instead of k, since the motion of the particles is limited solely to the dimension along the magnetic field. Strictly speaking, this is not a pure one-dimensional result, because the applied Fourier transform is three dimensional.

In the un-magnetized limit we have $B_0 \to 0$. The proof is straightforward but it involves some subtleties in the treatment of the limits. The single limit ($B_0 \to 0$) should

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be decomposed in the double limit ($\omega_{c\alpha} \to 0$, $\rho_{L\alpha} \to \infty$), and then the consecutive single limits taken with ($\omega_{c\alpha} \to 0$) first. A different order will lead to indeterminate $0 \times \infty$ forms. We shall also decompose the series for $n > 0$, $n < 0$ and $n = 0$, use the even parity property of the modified Bessel function of the first kind with respect to the order, i.e. $\mathrm{I}_{-n}(x) = \mathrm{I}_n(x)$, use the sum rule $\sum_{n=1}^{+\infty} \mathrm{I}_n(x) = \frac{1}{2}(e^x - \mathrm{I}_0(x))$ and the fact that since in the limit $B_0 \to 0$ $\boldsymbol{b}$ is a null vector and all vectors are both parallel and perpendicular to it, we can consider $\boldsymbol{k} \| \boldsymbol{b}$ and hence $k_\| = k$. Finally, in this limit the temperature anisotropy has no physical basis. As expected, the un-magnetized result reads us

$$S^{\alpha,(0)}_{\boldsymbol{k},\omega} = \frac{1}{(2\pi)^{7/2}} \frac{n_\alpha}{k v_{T\alpha}} \exp\left(-\frac{\omega^2}{2k^2 v^2_{T\alpha}}\right). \tag{C 3}$$

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