J. Plasma Physics (1999), *vol.* 61, *part* 1, *pp.* 135–150. Printed in the United Kingdom
© 1999 Cambridge University Press

Waves in a time-dependent magnetized electron–positron plasma

E. T. ROWE

Special Research Centre for Theoretical Astrophysics, School of Physics, Sydney University, NSW 2006, Australia

(Received 14 July 1998)

Waves in a magnetized electron–positron plasma, supporting a large-amplitude electric field $\mathbf{E}_0$ of superluminal phase speed, are considered. The case of perturbations with the same phase speed as the large-amplitude wave can be treated *exactly*, and we restrict our attention to this case, obtaining *analytical* results. The exact analytical results provide insight into the effect of the magnetic field and the large-amplitude wave on the harmonic structure of the perturbations. Three solutions are found for waves polarized perpendicular to $\mathbf{E}_0$. Waves are amplitude-modulated for weak magnetic fields (relative to the strength of the large-amplitude wave) and frequency-modulated for strong magnetic fields. This suggests that frequency modulation may be relevant to pulsars.

1. Introduction

Large-amplitude nonlinear waves in plasmas have been studied in considerable detail. Akhiezer and Polovin (1956) considered nonlinear wave modes in a cold electron plasma and obtained solutions for the longitudinal and transverse cases, and established some relations for the more general longitudinal–transverse case. Max and Perkins (1971) investigated the case of linearly polarized transverse waves in a homogeneous cold unmagnetized plasma with fixed ions, and also considered the penetration of the waves into a plasma of varying density. That work was extended by Max (1973) to circularly polarized electromagnetic waves, including ion motion. Kennel and Pellat (1976) discussed relativistic nonlinear plasma waves in a magnetic field including transverse circularly polarized waves, longitudinal superluminal waves, transverse superluminal waves (in zero magnetic field) and Alfvén waves. Janicke (1977) considered the superluminal-wave case from a formal point of view, and discussed the general form (and some properties) of solutions in the case of arbitrary particle distribution functions. While no additional explicit solutions were obtained, expansions in the wave strength parameter ($eE_0/m_e\Omega$, where Ω is the wave frequency) were developed.

In order to account for the effect of a large-amplitude wave on the wave modes supported by the medium, a perturbation approach is required. The large-amplitude wave is included explicitly in the description of the equilibrium of the system, which, in general, has temporal (and/or spatial) dependence. Such an approach was adopted in the weakly nonlinear case by, for example, Lakhina and Buti (1981), Weatherall et al. (1981), who gave a numerical treatment of the longitudinal-wave case, and Freund and Papadopoulos (1980), who used an analytical treatment. In

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

these cases the large-amplitude wave was assumed to be *monochromatic*, with at most a relatively low-frequency amplitude modulation. Pustovalov et al. (1978) also considered parametric instabilities in an unmagnetized plasma subject to a laser, modelled either as weakly amplitude-modulated or phase-modulated. The large-amplitude wave considered here contains more harmonic structure than is retained in these calculations.

The fully nonlinear case was considered for a linearly polarized large-amplitude wave in unmagnetized and magnetized plasmas by Luheshi and Stewart (1978a, b) and extended to two-dimensional large-amplitude waves by Luheshi and Stewart (1979). These works involved numerical calculations of Floquet exponents, giving the exponential growth rates for waves in the plasma. However, this approach cannot address the harmonic structure of the waves. An analytical calculation was performed by Asséo et al. (1980) for the case of a large-amplitude triangular wave. A Fourier technique was applied; however, this assumed that the unperturbed particle energy is constant throughout the orbit, which in fact is not the case. It is unclear to what extent constant energy is a good approximation, but clearly information on the harmonic structure of the perturbations is lost.

A motivation for this work, not dealt with here, is a continuing study of linear acceleration emission (LAE) applied to pulsars. LAE has been proposed as one of a myriad of candidates for the radio-pulsar emission mechanism (see e.g. Melrose 1978). In the LAE model it is assumed that a large-amplitude longitudinal wave is present in the pulsar magnetosphere after the pair-creation cascade has taken place. Radio emission is then due to instabilities in this pair plasma. Some form of maser action is required to explain the high brightness temperatures (10^{30} K) of radio pulsar emission, and previous calculations (Rowe 1995) using a single-particle approach, suggest that this is indeed possible. A shortcoming of the single-particle approach is that the refractive index of the medium is undetermined (in the calculations it is assumed to be close to unity) and the large-amplitude wave is assumed not to alter the properties of the plasma. The present approach obviates this assumption. However, we do not specifically seek instabilities here, nor do we treat pulsar radio emission in detail.

In this paper we consider waves in an electron–positron pair plasma supporting a large-amplitude superluminal wave polarized along an ambient magnetic field, in the special case in which the perturbation has the same phase speed as the large-amplitude wave. To the best of the author's knowledge, no *exact analytical* results have previously been presented, and in this case an exact analytical calculation can be performed, giving insight into the full harmonic structure of the waves (which is our main interest here). In general, this structure depends in a non-trivial way on both the ambient magnetic field strength and the strength of the large-amplitude wave, and the analytical results show the full scope of behaviours possible. More generally, these behaviours should also be relevant in a qualitative sense to the $\mathbf{k} \neq 0$ case, which is, however, not amenable to an exact analytical treatment. In Sec. 2 we describe the large-amplitude wave taken as present in our 'equilibrium'. It is identical to the longitudinal case of Akhiezer and Polovin (1956), although we express it here in a more transparent form. In Sec. 3 we develop the appropriate linearized equations, while in Sec. 4 we detail the waves supported by the equilibrium and give analytical solutions. In Sec. 5 we consider the behaviour of these waves in various limits, including the weak- and strong-magnetic-field limits, in which amplitude modulation

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

and frequency modulation respectively are exhibited. We use units with $c = 1$ throughout.

2. The equilibrium

For a cold electron–positron plasma, the relevant MHD equations are the continuity equation and the equation of motion for each species. These are respectively

$$\frac{\partial N_\pm}{\partial t} + \boldsymbol{\nabla} \cdot (N_\pm \mathbf{v}_\pm) = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{p}_\pm}{\partial t} + \mathbf{v}_\pm \cdot \boldsymbol{\nabla} \mathbf{p}_\pm = \pm e(\mathbf{E} + \mathbf{v}_\pm \times \mathbf{B}), \tag{2.2}$$

where a + sign labels positron quantities and a − sign those for the electrons. The plasma is subjected only to the Lorentz force, and we have assumed that there is no net pair creation. Together with Maxwell's equations, these equations completely specify our system.

The assumption of a collisionless plasma means that we can neglect the effects of viscosity, and the cold-plasma assumption implies that no pressure terms appear. In this case the equations of motion for the plasma are just the single-particle equations of motion. The cold-plasma approximation appears to be valid if the plasma is in the presence of a strong electric field (see e.g. Rowe 1992), as is the case discussed here. In essence, the strong electric field minimizes any effect due to disparate particle velocities.

For the equilibrium, we shall take a homogeneous, cylindrically symmetric system with electrons and positrons moving along a uniform and static ambient magnetic field $B_0\hat{\boldsymbol{z}}$, together with a time-dependent electric field $E_0(t)\hat{\boldsymbol{z}}$. In this case, the ambient magnetic field has no effect on the large-amplitude wave derived below; however, it is relevant to the wave perturbations introduced in Sec. 3. Homogeneity implies, given (2.1) and charge neutrality, that $N_+ = N_- = N_0$ (a constant). The equations of motion (2.2) become

$$\frac{d\gamma_\pm \mathbf{v}_\pm}{dt} = \pm \frac{e\mathbf{E}}{m_e}, \tag{2.3}$$

and Ampère's law can be written in the form

$$\frac{d\mathbf{E}}{dt} = -\frac{m_e \omega_p^2}{2e}(\mathbf{v}_+ - \mathbf{v}_-), \tag{2.4}$$

where $\omega_p^2 = 2N_0 e^2/\epsilon_0 m_e$ is the total plasma frequency. Equation (2.4) indicates that the electric field is entirely determined by the time dependence of the current.

Equations (2.3) and (2.4) furnish a couple of auxiliary equations

$$\frac{d\gamma_\pm}{dt} = \pm \frac{e\mathbf{v}_\pm \cdot \mathbf{E}}{m_e}, \tag{2.5}$$

$$\tfrac{1}{2}\epsilon_0 E^2 + m_e N_0(\gamma_+ + \gamma_-) = \text{const}, \tag{2.6}$$

the latter expressing conservation of energy for the wave–particle system. Special cases involve assuming that either $\mathbf{v}_\pm \cdot \mathbf{E} = 0$ or $\mathbf{v}_\pm \cdot \mathbf{E} = v_\pm E$. In the first case the electric field describes circularly polarized simple harmonic waves, and in the second case plane waves with profiles described by elliptic integrals. The system we consider here fits into the second category.

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

In order to obtain a symmetric waveform (with respect to time inversion or charge conjugation), we shall further assume that the electrons and positrons have the same energy, so that $\gamma_+ = \gamma_- = \gamma$, and then (2.6) implies that

$$E^2 = \frac{2m_e^2\omega_p^2}{e^2}(\gamma_m - \gamma), \tag{2.7}$$

where γ_m is the maximum attained particle Lorentz factor.

In writing out the solution, it is convenient to define the dimensionless quantity

$$n = \left(\frac{\gamma_m - 1}{\gamma_m + 1}\right)^{1/2}, \tag{2.8}$$

and to write γ in terms of a variable ϕ, such that

$$\gamma = \frac{1 + n^2 \sin^2 \phi}{1 - n^2 \sin^2 \phi}. \tag{2.9}$$

Equation (2.7) can now be written in the form

$$E = \frac{2m_e\omega_p}{e}\frac{n}{(1-n^2)^{1/2}}\frac{\cos\phi}{(1 - n^2 \sin^2 \phi)^{1/2}}, \tag{2.10}$$

indicating that the nonlinear equilibrium solution is a periodic function of ϕ. The variable ϕ appears naturally in the integration of (2.5), and we find that it is related to the time t by

$$t = \frac{1}{\omega_p(1-n^2)^{1/2}}\left[2E(\phi, n) - (1-n^2)F(\phi, n) - \frac{n^2 \sin 2\phi}{(1 - n^2 \sin^2 \phi)^{1/2}}\right], \tag{2.11}$$

with $t = 0$ when $\phi = 0$. The functions $F(\phi, n)$ and $E(\phi, n)$ are elliptic integrals of the first and second kinds respectively (see e.g. Byrd and Friedman 1971), and are monotonically increasing functions of ϕ. There is an oscillatory component to t; nevertheless, it is also a monotonically increasing function of ϕ. As a function of time, the waveform resembles a cosine wave for small n and a triangle wave for $n \approx 1$. Thus, as γ_m increases, higher harmonics become important. The solution given here is an alternative form to that given by Akhiezer and Polovin (1956).

The frequency of the wave oscillation is given by

$$\Omega = \frac{\pi\omega_p(1-n^2)^{1/2}}{2[2E(n) - (1-n^2)K(n)]}, \tag{2.12}$$

where $E(n)$ and $K(n)$ are complete elliptic integrals, and is a decreasing function of γ_m (or of wave amplitude), as depicted in Fig. 1. In the small-amplitude limit ($n \to 0$) it approaches ω_p, and in the large-amplitude limit ($n \to 1$) it takes the form

$$\Omega \sim \pi\omega_p/2^{3/2}\gamma_m^{1/2}. \tag{2.13}$$

The solution given here represents a wave with infinite phase speed in this particular reference frame (since E_0 has only time dependence). In any other reference frame the phase speed is finite but greater than that of light (the wave is superluminal), and, depending upon the reference frame, this wave may be transverse, longitudinal or longitudinal–transverse. In the application to pulsars envisaged (but not treated here) a particularly relevant Lorentz transform is one parallel to the

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

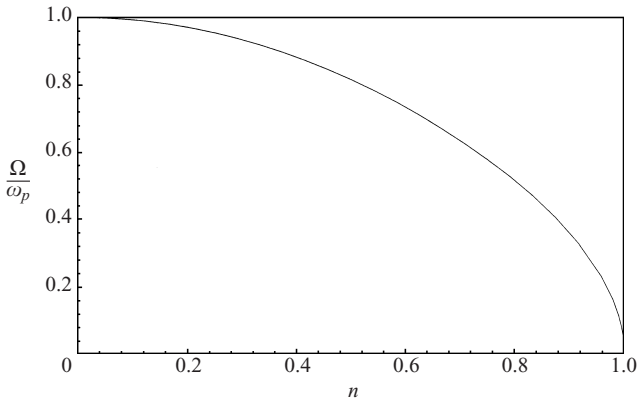


Figure 1. The frequency of a large-amplitude wave, as a function of the parameter n.

magnetic field, such that in the new reference frame there is bulk pair plasma flow along the magnetic field lines (as expected in pulsar magnetospheres). In this case the large-amplitude wave is a longitudinal wave, polarized along the magnetic field.

3. Linearized equations

We now linearize the system and consider the waves supported by the electron–positron plasma in the presence of the large-amplitude wave derived in Sec. 2. We perform all calculations in the same reference frame as in Sec. 2. All equilibrium quantities are henceforth labelled with a subscript 0, while first-order quantities have no subscript. Here we consider only perturbations with $\mathbf{k} = 0$ (homogeneous). While this is restrictive, in this case the allowed waves can be described analytically, allowing for detailed analysis – in particular of their dependence on both the magnetic field and $E_0(t)$. This choice implies that the perturbations are also superluminal. In this particular reference frame the waves have infinite phase speed (or wavelength), but in an arbitrary reference frame they have finite (superluminal) phase speed and finite wavelength. We shall show in Sec. 5.1 how these waves (derived in Sec. 4) are related to the pair-plasma analogues of the magnetoionic modes.

The linearized equation corresponding to (2.1), together with charge neutrality (from Gauss' law), then implies that $N_+ = N_- = N$ (a constant). Without loss of generality, we may set $N = 0$. Equation (2.2) leads to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \pm \frac{e\mathbf{B}_0}{m_e\gamma_0} \times \right)\boldsymbol{\Gamma}\mathbf{v}_\pm = \pm\frac{e}{m_e}(\mathbf{E} \pm \mathbf{v}_0 \times \mathbf{B}), \tag{3.1a}$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_0 & 0 & 0 \\ 0 & \gamma_0 & 0 \\ 0 & 0 & \gamma_0^3 \end{pmatrix}, \tag{3.1b}$$

where $\mathbf{v}_{+0} = -\mathbf{v}_{-0} = \mathbf{v}_0$. The appearance of γ_0^3 in the last element of $\boldsymbol{\Gamma}$ is a manifestation of the fact that the z component of the equilibrium particle velocity $\mathbf{v}_0$ is non-zero.

The induction equation for the magnetic field implies that the perturbed magnetic field $\mathbf{B}$ is uniform and constant. As such, we are free to incorporate it into the

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

ambient field B_0, and thus we further assume $\mathbf{B} = 0$. The linearized version of Ampère's law takes the same form as (2.4), with $\mathbf{E}$, $\mathbf{v}_+$ and $\mathbf{v}_-$ regarded as first-order quantities.

There are two cases that we can treat separately: $\mathbf{E}$ polarized perpendicular to $\mathbf{B}_0$, and $\mathbf{E}$ polarized along $\mathbf{B}_0$. In the first case we use (3.1) to find

$$\frac{d}{dt}[\gamma_0(\mathbf{v}_+ + \mathbf{v}_-)] + \frac{eB_0}{m_e}\hat{\mathbf{z}} \times (\mathbf{v}_+ - \mathbf{v}_-) = \frac{2e}{m_e} v_0 \hat{\mathbf{z}} \times \mathbf{B} = 0, \tag{3.2a}$$

$$\frac{d}{dt}[\gamma_0(\mathbf{v}_+ - \mathbf{v}_-)] + \frac{eB_0}{m_e}\hat{\mathbf{z}} \times (\mathbf{v}_+ + \mathbf{v}_-) = \frac{2e}{m_e}\mathbf{E}. \tag{3.2b}$$

On multiplying (2.4) by γ_0 and differentiating, we use (3.2) to remove the derivative on the particle velocities. Repeating the procedure gives an equation that we can integrate once immediately. We find

$$\gamma_0 \ddot{\mathbf{E}}_\perp + \dot{\gamma}_0 \dot{\mathbf{E}}_\perp + \left(\omega_p^2 + \frac{\Omega_B^2}{\gamma_0}\right)\mathbf{E}_\perp = \frac{m_e \omega_p^2}{e}\frac{\Omega_B}{\gamma_0}\mathbf{V}, \tag{3.3}$$

where $\mathbf{V}$ is an arbitrary constant vector with units of velocity, $\Omega_B = eB_0/m_e$ is the electron cyclotron frequency, and the dots represent total derivatives in time.

In the case $\mathbf{E} \parallel \mathbf{B}_0$ we rewrite (2.4) as an integro-differential equation for $E_\parallel$, which we then differentiate a sufficient number of times to obtain an ordinary differential equation. We find in this case

$$\gamma_0^3 \ddot{E}_z + 3\gamma_0^2 \dot{\gamma}_0 \dot{E}_z + \omega_p^2 E_z = 0, \tag{3.4}$$

We consider solutions to the $\mathbf{E}_\perp$ case alone in this paper.

4. Solutions of the linearized equations for $\mathbf{E}_\perp$

Equation (3.3) can be separated into its two component scalar equations, and two linearly independent solutions of the homogeneous scalar equations can then be written in terms of ϕ in the form

$$E^+(\phi) = A(\phi)\cos f(\phi), \tag{4.1a}$$

$$E^-(\phi) = A(\phi)\sin f(\phi), \tag{4.1b}$$

with

$$A(\phi) = \left(\frac{1 - \zeta^2 n^2 \sin^2\phi}{1 - n^2 \sin^2\phi}\right)^{1/2}, \tag{4.2a}$$

$$\zeta^2 = \frac{2b + \gamma_m + 1}{2b + \gamma_m - 1}, \tag{4.2b}$$

$$b = \left(\frac{\Omega_B}{\omega_p}\right)^2, \tag{4.2c}$$

$$f(\phi) = \left(\frac{2b}{\gamma_m + 1}\right)^{1/2}\frac{1}{\zeta}\left[F(\phi, n) + \frac{2}{2b + \gamma_m - 1}\Pi(\phi, \zeta^2 n^2, n)\right], \tag{4.2d}$$

and where positive square roots are implied. We shall use $f(\phi)$ and $f(t)$ interchangeably, depending on whether we wish to emphasize the form of f in terms of ϕ or t, and similarly for other functions of ϕ or t. The solution E^- is illustrated in Fig. 2 for various choices of parameters, E^+ being merely a half-period out of phase with

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

E^-. The solutions are both discussed more fully in Sec. 5. A further solution is described by the special solution of (3.3). A closed form cannot be obtained; however a Green-function approach provides a form involving two single quadratures. Denoting this third solution by a superscript s, we have

$$E^s(\phi) = E^-(\phi)\int_0^\phi d\phi'\,\frac{E^+(\phi')}{(1-n^2\sin^2\phi')^{1/2}} + E^+(\phi)\int_\phi^{\pi/2} d\phi'\,\frac{E^-(\phi')}{(1-n^2\sin^2\phi')^{1/2}}. \tag{4.3}$$

The general solution of (3.3) is an arbitrary linear combination of these three solutions, each with independent, arbitrary orientation in the $\perp$ plane:

$$\mathbf{E}_\perp(\phi) = \mathbf{A}_+E^+(\phi) + \mathbf{A}_-E^-(\phi) + \mathbf{A}_sE^s(\phi), \tag{4.4}$$

where $\mathbf{A}_+$, $\mathbf{A}_-$ and $\mathbf{A}_s$ are all arbitary constant vectors in the $\perp$ plane and

$$\mathbf{A}_s = \left(\frac{2}{\gamma_m+1}\right)^{1/2}\frac{m_e\omega_p}{e\zeta}\mathbf{V}. \tag{4.5}$$

In addition, we may solve for the particle velocities using (2.4) and (3.2), introducing a constant of integration. We have

$$\mathbf{v}_\pm = \frac{e\boldsymbol{\Omega}_B}{m_e\omega_p^2\gamma_0}\hat{\mathbf{z}}\times\mathbf{E} \mp \frac{e}{m_e\omega_p^2}\dot{\mathbf{E}} - \frac{\hat{\mathbf{z}}\times\mathbf{V}}{\gamma_0} + \frac{C\hat{\mathbf{z}}}{\gamma_0}, \tag{4.6}$$

where $\mathbf{C}$ is an arbitrary constant. All solutions imply a charge-sign-dependent oscillation along $\mathbf{E}$ together with a sign-independent $\mathbf{E}\times\mathbf{B}$ oscillation and an arbitrary streaming motion along $\mathbf{B}$. The $V \neq 0$ solution has in addition a sign-independent $\mathbf{E}\times\mathbf{B}$ drift term.

5. Discussion

In general, the behaviours of the solutions E^+, E^- and E^s are hidden within the functions $A(\phi)$ and $f(\phi)$. These behaviours depend upon the choice of b and γ_m, as illustrated in Fig. 2, and there are several cases for which further elaboration is possible.

5.1. The limit as $\gamma_m \to 1$

This limit is a useful check on our results, and reproduces standard results for magnetoionic-like waves in a cold magnetized pair plasma, in the absence of the large-amplitude wave. In this case one has $n \to 0$ and

$$\zeta \to \left[1 + \left(\frac{\omega_p}{\boldsymbol{\Omega}_B}\right)^2\right]^{1/2}. \tag{5.1}$$

The elliptic functions $F(\phi, n)$ and $\boldsymbol{\Pi}(\phi, \zeta^2n^2, n)$ then both tend to ϕ (= $\omega_p t$ from (2.11)). Thus

$$E^+(\phi) \to \cos[(\boldsymbol{\Omega}_B^2 + \omega_p^2)^{1/2}t], \tag{5.2a}$$

$$E^-(\phi) \to \sin[(\boldsymbol{\Omega}_B^2 + \omega_p^2)^{1/2}t], \tag{5.2b}$$

as expected for waves in a cold magnetized electron–positron plasma (with $k = 0$). In this particular reference frame $\mathbf{k} = 0$ implies that the waves are at the cutoff frequency of the x- and o-mode equivalents in the pair plasma (i.e they have the

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

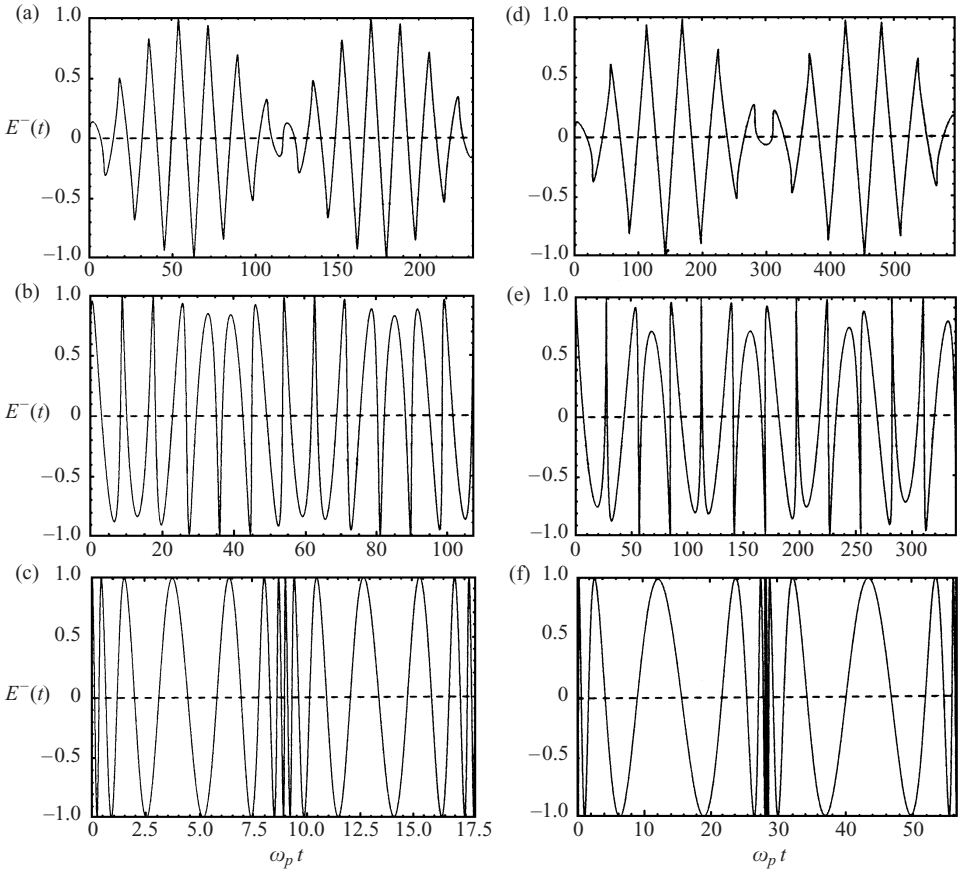


Figure 2. $E^-(t)$ for various choices of γ_m and b. In (a), (b) and (c) $\gamma_m = 10$ and $b = 0.05$, 10 and 500 respectively. In (d), (e) and (f) $\gamma_m = 100$ and $b = 0.2$, 50 and 2000 respectively. In each column the period of the equilibrium waveform is indicated by the width of the lowest panel. Thus in (a) and (d) $E^-(t)$ consists of the equilibrium waveform, amplitude-modulated, with modulation frequency given by the width of the panels. In (c) and (f) the waveform is frequency-modulated with modulation frequency twice the frequency $\mathbf{\Omega}$. (b) and (e) show evidence of both frequency and amplitude modulation.

lowest frequency possible for such waves). Lorentz transforming to another reference frame implies that $k' \neq 0$, and the waves are above cutoff in the new frame.

Equation (4.3) implies that

$$E^s(\phi) \rightarrow \frac{\omega_p}{(\omega_p^2 + \mathbf{\Omega}_B^2)^{1/2}} \left\{ 1 - \cos\left[(\omega_p^2 + \mathbf{\Omega}_B^2)^{1/2} \frac{\pi}{2\omega_p} \right] \times \cos[(\omega_p^2 + \mathbf{\Omega}_B^2)^{1/2} t] \right\}, \quad (5.3)$$

which contains $E^+(\phi)$ and a static term related to the non-zero $\mathbf{E} \times \mathbf{B}$ drift of particles appearing in (4.6).

5.2. The limit $b \ll \gamma_m - 1$

The plots of $E^-(\phi)$ in this limit show that the waveform consists of a relatively high-frequency wave, with amplitude modulation at a lower frequency (see Fig. 2). Generally, we may write

$$f(t) = \mathbf{\Omega} t + P(t) + \Delta\mathbf{\Omega} t, \quad (5.4)$$

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

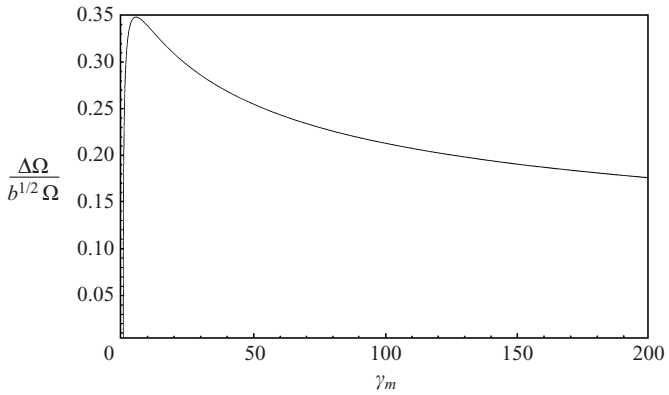


Figure 3. The modulation frequency $\Delta\Omega$ in units of $b^{1/2}\Omega$ for the case $b \ll \gamma_m - 1$ (see (5.8)), as a function of γ_m. For $\gamma_m \sim 1$, one has $\Delta\Omega/b^{1/2}\Omega \sim (\gamma_m - 1)^{1/2}/8^{1/2}$, while for $\gamma_m \gg 1$, the function decreases as $\Delta\Omega/b^{1/2}\Omega \sim 2^{1/2} \ln \gamma_m/\pi\gamma_m^{1/2}$.

where Ω is given by (2.12), $\Delta\Omega$ is given by

$$\Delta\Omega = \Omega\left\{\frac{2}{\pi}\left(\frac{2b}{\gamma_m + 1}\right)^{1/2}\frac{1}{\zeta}\left[K(n) + \frac{2}{2b + \gamma_m - 1}\Pi(\zeta^2 n^2, n)\right] - 1\right\}, \tag{5.5}$$

and $P(t)$ is a purely periodic function with period π/Ω, thus explicitly separating out the secular part of $f(t)$. Hence

$$E^+(\phi) = A(\phi)\left\{\cos[\Omega t + P(t)]\cos\Delta\Omega t - \sin[\Omega t + P(t)]\sin\Delta\Omega t\right\}, \tag{5.6a}$$

$$E^-(\phi) = A(\phi)\left\{\sin[\Omega t + P(t)]\cos\Delta\Omega t + \cos[\Omega t + P(t)]\sin\Delta\Omega t\right\}. \tag{5.6b}$$

In the Appendix it is shown that in the limit $b \ll \gamma_m - 1$

$$E^+(\phi) \to \{E_0^* + O(b^{1/2})\}\cos\Delta\Omega t + O(b^{1/2})\sin\Delta\Omega t, \tag{5.7a}$$

$$E^-(\phi) \to \{E_0^* + O(b^{1/2})\}\sin\Delta\Omega t + O(b^{1/2})\cos\Delta\Omega t, \tag{5.7b}$$

where E_0^* has the form of (2.10) but normalized to unit amplitude. The waves thus have the same form as the equilibrium waveform (2.10), with frequency Ω but modulated by a low-frequency wave of frequency $\Delta\Omega$.

In our limit of interest (see the Appendix)

$$\frac{\Delta\Omega}{\Omega} \sim \left[\frac{8b}{\pi^2(\gamma_m - 1)}\right]^{1/2}[K(n) - E(n)], \tag{5.8}$$

gives the envelope frequency, as depicted in Fig. 3. We expect the above approximations to hold as long as $\Delta\Omega/\Omega < 1$, so that the modulations occur over many periods of the high-frequency component.

If $b \to 0$, our results must satisfy the second-order ordinary differential equation (3.3) with $\Omega_B = 0$. In this case E^s is invalid, since it specifically requires a non-zero constant on the right-hand side of (3.3), and $E^+(\phi) \to E_0^*$ as $\Delta\Omega \to 0$. In the case of $E^-(\phi)$ we need to retain terms to order $b^{1/2}$. Specifically

$$E^-(\phi) \sim \left(\frac{2b}{\gamma_m - 1}\right)^{1/2}\left\{E_0^*[F(\phi, n) - E(\phi, n)] + \sin\phi\right\}, \tag{5.9}$$

as shown in the Appendix. These solutions are indeed solutions of (3.3) with $\Omega_B = 0$.

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

The last result has a term that grows secularly with time. Although the waves are clearly stable in the magnetized case, we cannot be certain that this mode in the unmagnetized case is unstable. Secular terms probably indicate the failure of the perturbation analysis to aptly describe the nature of the wave modes in this case.

5.3. The limit $b \gg \gamma_m - 1$

In this limit Fig. 2 indicates that the waveforms are frequency-modulated, rather than amplitude modulated. We write

$$f(t) = \int_0^t \omega(t')\, dt', \tag{5.10}$$

where $\omega(t') = f'(t')$ generalizes the notion of frequency, allowing for time dependence. We find generally that

$$\frac{\omega(\phi)}{\Omega_B} = \zeta \frac{(1 - n^2 \sin^2 \phi)^2}{(1 + n^2 \sin^2 \phi)(1 - n^2 \zeta^2 \sin^2 \phi)}. \tag{5.11}$$

A maximum and a minimum occur at $\phi = 0$ and $\phi = \frac{1}{2}\pi$ respectively and the function has period $t = \pi/\Omega$. We find that

$$\frac{\omega(0)}{\Omega_B} = \zeta, \tag{5.12a}$$

$$\frac{\omega(\frac{1}{2}\pi)}{\Omega_B} = \frac{\omega_p^2}{2\gamma_m \Omega_B^2} \left\{ \left[2\left(\frac{\Omega_B}{\omega_p}\right)^2 + \gamma_m \right]^2 - 1 \right\}^{1/2}, \tag{5.12b}$$

and in our limit of interest $\omega(0) \to \Omega_B$ and $\omega(\frac{1}{2}\pi) \to \Omega_B/\gamma_m$. Waveforms have relatively high frequency near maxima of $\omega(t)$ and relatively low frequency near minima, and the frequency modulation has period π/Ω. For $b \gg \gamma_m - 1$,

$$\frac{\omega(\phi)}{\Omega_B} \approx \frac{1 - n^2 \sin^2 \phi}{1 + n^2 \sin^2 \phi}, \tag{5.13}$$

which tends to a square-well profile as γ_m increases, as shown in Fig. 4. This implies that the high-frequency phase is short-lived compared with the low-frequency phase, as is also indicated by Figs 2(c,f).

We expect this limit to give a valid description of the waveform as long as the lower frequency is higher than that of the equilibrium waveform (and of the underlying periodicity), $\Omega_B/\gamma_m \gg \Omega$. This implies that

$$\frac{b}{\gamma_m} \gg \left[\left(\frac{\Omega}{\omega_p}\right)^2 \gamma_m \right]_{\max} = \tfrac{1}{8}\pi^2, \tag{5.14}$$

is required.

Approximate forms for $E^+(\phi)$, $E^-(\phi)$ and $E^s(\phi)$ can be derived in this limit. We have $\zeta \to 1$, and since there is no singularity in $\Pi(\phi, \zeta^2 n^2, n)$ for $\zeta = 1$, we find

$$f(\phi) \to \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} F(\phi, n), \tag{5.15}$$

and $A(\phi) \to 1$, the forms for $E^+(\phi)$ and $E^-(\phi)$ following from (4.1). The function $f(\phi)$ may be split into a periodic part and a secular part in ϕ. We write this

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

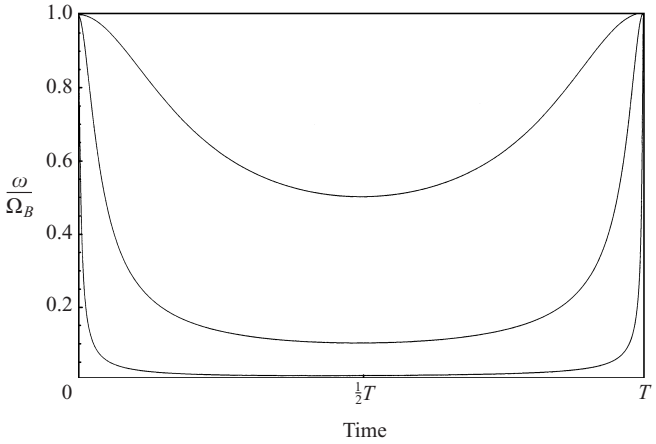


Figure 4. ω/Ω_B as a function of time for $\gamma_m = 2$ (upper curve), 10 (middle curve) and 100 (bottom curve). The period T of the frequency modulation is different in each case, and for $\gamma_m = 10$ and $\gamma_m = 100$ may be obtained from Figs 2(c,f).

explicitly as

$$f(\phi) \to \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} \left\{ \frac{2K(n)}{\pi}\phi + \left[F(\phi, n) - \frac{2K(n)}{\pi}\phi\right]\right\}. \tag{5.16}$$

Similarly, separating ϕ explicitly into a secular term in t and a periodic term $\phi(t) = \Omega t + [\phi(t) - \Omega t]$, we find

$$f(\phi) \to \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} \left\{ \frac{2K(n)}{\pi}\Omega t + \left[F(\phi, n) - \frac{2K(n)}{\pi}\Omega t\right]\right\}. \tag{5.17}$$

Hence the waves consist of a high-frequency part (relative to Ω) with frequency

$$\omega = \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} \frac{2K(n)}{\pi}\Omega, \tag{5.18}$$

and a periodically varying phase shift

$$\Phi = \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} \left[F(\phi(t), n) - \frac{2K(n)}{\pi}\Omega t\right]. \tag{5.19}$$

The above furnishes an alternative description (though less convenient) for the frequency modulation appearing in this limit.

We may also derive an approximate form for $E^s(\phi)$. The integrals involved in (4.3) become

$$\int_0^{\phi} d\phi' \frac{\cos[\alpha F(\phi', n)]}{(1 - n^2 \sin^2\phi')^{1/2}} = \frac{1}{\alpha}\sin[\alpha F(\phi, n)], \tag{5.20a}$$

$$\int_{\phi}^{\frac{\pi}{2}} d\phi' \frac{\sin[\alpha F(\phi', n)]}{(1 - n^2 \sin^2\phi')^{1/2}} = \frac{1}{\alpha}\left\{\cos[\alpha F(\phi, n)] - \cos[\alpha K(n)]\right\}, \tag{5.20b}$$

with $\alpha = \{2b/(\gamma_m + 1)\}^{1/2}$. Thus

$$E^s(\phi) \to \left(\frac{\gamma_m + 1}{2b}\right)^{1/2} \{1 - \cos[\alpha K(n)]\cos[\alpha F(\phi, n)]\}, \tag{5.21}$$

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

with decreasing amplitude as b increases. As in the $\gamma_m \to 1$ limit, this third mode involves a DC component.

6. Conclusions

We have considered a magnetized electron–positron plasma subject to a large-amplitude time-dependent electric field, and have sought analytical results for the waves that may propagate in such a system. We have used a perturbation approach and restricted consideration to the case $\mathbf{k} = 0$ (superluminal waves). While this is restrictive, we have been able to investigate *analytically* the effects of both a magnetic field and a large amplitude electric oscillation on the pair plasma. In the limit of no large-amplitude wave, the perturbations reduce to waves in the pair-plasma analogues of the magnetoionic x and o modes. Most importantly, in the presence of a large-amplitude wave we find evidence of both amplitude-modulated waves and frequency-modulated waves. These are limiting cases of the same wave modes, depending on the relative strength of the magnetic field and the large-amplitude wave, and qualitatively this result is expected to be relevant to the more general $\mathbf{k} \neq 0$ case. This more general case appears not to be amenable to an exact analytical treatment as is used here, but it may be possible, for example, to assume a frequency-modulated perturbation (for the strong-magnetic-field case) and obtain approximate results.

We have not considered applications here. However, a possible application to consider is to the pulsar magnetosphere, in which strong magnetic fields and electric fields or waves are expected. The value of b near the star would be of order 10^{18} for a magnetic field of 10^8 T and a particle number density of the order of the Goldreich–Julian density, which is $n_{GJ} \approx 10^{17}$ m^{-3} for the same magnetic field and a rotation period of the order of 1 s. Thus, for conservative values of γ_m, the frequency-modulated case may be of interest.

Acknowledgements

The author wishes to thank G. W. Rowe for helpful comments on the manuscript. The idea for the work presented here was partly conceived while the author was at the Institute of Astronomy in Utrecht, The Netherlands. The author wishes to acknowledge their support and that of the NFRA and NWO during his time there.

Appendix. Derivation of the results in the limit $b \ll \gamma_m - 1$

In this appendix we derive (5.7), (5.8) and (5.9). In the limit $b \ll \gamma_m - 1$ the quantity $\zeta \to 1/n$. We write

$$\zeta^2 \sim \frac{1}{n^2} - \delta, \tag{A 1a}$$

$$\delta = \frac{4b}{(\gamma_m - 1)^2}, \tag{A 1b}$$

which is obtained from (4.2), for small b. We now find

$$f(\phi) \sim \left(\frac{2b}{\gamma_m + 1}\right)^{1/2} n\left[F(\phi, n) + \frac{2}{\gamma_m - 1}\left(1 - \frac{2b}{\gamma_m - 1}\right)\Pi(\phi, 1 - n^2\delta, n)\right], \tag{A 2}$$

correct to $O(b^{1/2})$.

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

The function $\Pi(\phi, 1-n^2\delta, n)$ has a singularity at $\phi = \frac{1}{2}\pi$ if $\delta = 0$. Using the addition rule (see e.g. Byrd and Friedman 1971)

$$\Pi(\phi,\alpha^2,k)+\Pi(\phi,k^2/\alpha^2,k)$$
$$= F(\phi,k)+\left\{\frac{\alpha^2}{(1-\alpha^2)(\alpha^2-k^2)}\right\}^{1/2}\tan^{-1}\left\{\left[\frac{(1-\alpha^2)(\alpha^2-k^2)}{\alpha^2(1-k^2\sin^2\phi)}\right]^{1/2}\tan\phi\right\}, \quad (A\,3)$$

with k and α replaced by n and ζ^{-1} respectively, we obtain

$$\Pi(\phi,1-n^2\delta,n)\sim F(\phi,n)-\Pi(\phi,n^2,n)$$
$$+\frac{1}{n(1-n^2)^{1/2}\delta^{1/2}}\tan^{-1}\left\{\left[\frac{n^2(1-n^2)\delta}{1-n^2\sin^2\phi}\right]^{1/2}\tan\phi\right\}, \quad (A\,4)$$

where we have retained terms to $O(b^0)$ only, except inside the $\tan^{-1}$ function, for which the lowest non-zero order is $b^{1/2}$.

At $\phi = \frac{1}{2}\pi$, the $\tan^{-1}$ function evaluates to $\frac{1}{2}\pi$, and the singularity for $\delta = 0$ is explicit and $O(b^{-1/2})$. For $\phi \neq \frac{1}{2}\pi$, the $\tan^{-1}$ function may be expanded for small δ, and there is no singularity. The definition of the $\tan^{-1}$ function used here is that it is the branch of the multivalued function that takes values in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The addition formula is thus valid for $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. The elliptic integrals are defined for values outside this range, and so we need to extend the range of validity of the addition formula. This is achieved by taking a different branch of the multivalued $\tan^{-1}$ for each interval $[-\frac{1}{2}\pi + n\pi, \frac{1}{2}\pi + n\pi]$, where n is an integer. We make this explicit by regarding $\tan^{-1}$ as the usual single-valued function and making the following replacement for $\phi > 0$:

$$\tan^{-1}(\beta\tan\phi)\rightarrow\begin{cases}\tan^{-1}(\beta\tan\phi)+\pi\sum_{n=0}^{\infty}H(n,\phi) & (\phi\neq\frac{1}{2}\pi+n\pi),\\ \frac{1}{2}\pi(n+1) & (\phi=\frac{1}{2}\pi+n\pi),\end{cases} \quad (A\,5)$$

where $H(n,\phi) = H(\phi - \frac{1}{2}(2n+1)\pi)$, H is the Heaviside step function and β is an arbitrary positive-definite quantity. Alternatively, we may notionally absorb the values at $\phi = \frac{1}{2}\pi + n\pi$ into the step functions (effectively defining $H(0) = \frac{1}{2}$) and consider $\tan^{-1}(\pm\infty)$ to be undefined or zero. We shall adopt this convention.

We now use (see e.g. Byrd and Friedman 1971)

$$\Pi(\phi,n^2,n)=\frac{1}{1-n^2}\left[E(\phi,n)-\frac{n^2\sin\phi\cos\phi}{(1-n^2\sin^2\phi)^{1/2}}\right], \quad (A\,6)$$

and find

$$f(\phi)\sim\left(\frac{2b}{\gamma_m-1}\right)^{1/2}\left[F(\phi,n)-E(\phi,n)+\frac{n^2\sin\phi\cos\phi}{(1-n^2\sin^2\phi)^{1/2}}\right]$$
$$+\tan^{-1}\left\{\left[\frac{8b}{(\gamma_m-1)(\gamma_m+1)^2}\right]^{1/2}\frac{\tan\phi}{(1-n^2\sin^2\phi)^{1/2}}\right\}$$
$$+\pi\sum_{n=0}^{\infty}H(n,\phi). \quad (A\,7)$$

The singularity found earlier is removed by the premultiplying factor $O(b^{1/2})$ appearing in (A 2). A similar procedure to the above is used to derive (5.8) from (5.5). In this case the relevant form for the function $\Pi(\zeta^2 n^2, 1)$ appearing in (5.5) is ob-

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

tained directly from (A 4) evaluated at $\phi = \frac{1}{2}\pi$, and, together with (A 6), we finally obtain (5.8).

The quantity $\Omega t + P(t) = f(\phi) - \Delta\Omega t$ appearing in (5.6) can now be written out in full:

$$\begin{aligned}\Omega t + P(t) &= \pi \sum_{n=0}^{\infty} H(n,\phi) + \tan^{-1}\left\{\left[\frac{8b}{(\gamma_m - 1)(\gamma_m + 1)^2}\right]^{1/2} \frac{\tan\phi}{(1 - n^2\sin^2\phi)^{1/2}}\right\} \\ &+ \left(\frac{2b}{\gamma_m - 1}\right)^{1/2}\left\{F(\phi,n) - E(\phi,n) + \frac{n^2\sin\phi\cos\phi}{(1 - n^2\sin^2\phi)^{1/2}} - \frac{2\Omega}{\pi}[K(n) - E(n)]t\right\}, \\ &= \pi \sum_{n=0}^{\infty} H(n,\phi) + D(\phi), \end{aligned} \tag{A 8}$$

where $D(\phi) = O(b^{1/2})$.

Here, to lowest order, the function is a 'staircase' of step functions, with step height and width π. The higher-order terms represent a deviation from perfectly 'square steps'. We thus write

$$\cos[\Omega t + P(t)] = \cos\left[\pi \sum_{n=0}^{\infty} H(n,\phi)\right] \cos D(\phi) - \sin\left[\pi \sum_{n=0}^{\infty} H(n,\phi)\right] \sin D(\phi), \tag{A 9a}$$

$$\sin[\Omega t + P(t)] = \sin\left[\pi \sum_{n=0}^{\infty} H(n,\phi)\right] \cos D(\phi) + \cos\left[\pi \sum_{n=0}^{\infty} H(n,\phi)\right] \sin D(\phi). \tag{A 9b}$$

Moreover, we have

$$\cos[\pi H(n,\phi)] = \begin{cases} 1 & (\phi < \frac{1}{2}(2n+1)\pi), \\ 0 & (\phi = \frac{1}{2}(2n+1)\pi), \\ -1 & (\phi > \frac{1}{2}(2n+1)\pi), \end{cases} \tag{A 10a}$$

$$\sin[\pi H(n,\phi)] = \begin{cases} 0 & (\phi \neq \frac{1}{2}(2n+1)\pi), \\ 1 & (\phi = \frac{1}{2}(2n+1)\pi), \end{cases} \tag{A 10b}$$

keeping in mind that we have defined $H(0) = \frac{1}{2}$, and we can prove by induction that

$$\cos\left[\pi \sum_{n=0}^{N} H(n,\phi)\right] = \prod_{n=0}^{N} \cos[\pi H(n,\phi)] \tag{A 11}$$

for any positive integer N. This result gives a representation of a finite train of a square wave, with the step functions implying a sign flip at $\phi = \frac{1}{2}(2n+1)\pi$ for each n. Similarly, we find by induction that

$$\sin\left[\pi \sum_{n=0}^{N} H(n,\phi)\right] = \sum_{n=0}^{N} (-1)^n \sin[\pi H(n,\phi)]\,, \tag{A 12}$$

representing a finite segment of an alternating Dirac comb with teeth of unit height. The above results also hold for $N \to \infty$.

We next consider $A(\phi)$. Using (A 1), we have

$$A(\phi) = \left(\frac{\cos^2\phi}{1 - n^2\sin^2\phi}\right)^{1/2} + O(b), \tag{A 13}$$

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

where the square root is taken to be strictly positive. This is a natural choice, since then $0 < A(\phi) < 1$ in (4.2). We now have

$$E^+(\phi) = \left\{ \left(\frac{\cos^2 \phi}{1 - n^2 \sin^2 \phi} \right)^{1/2} \prod_{n=0}^{\infty} \cos[\pi H(n, \phi)] + O(b^{1/2}) \right\} \cos \Delta\Omega t$$
$$- \left\{ \left(\frac{\cos^2 \phi}{1 - n^2 \sin^2 \phi} \right)^{1/2} \sum_{n=0}^{\infty} (-1)^n \sin[\pi H(n, \phi)] + O(b^{1/2}) \right\} \sin \Delta\Omega t. \quad \text{(A 14)}$$

As stated above, the product of cosine functions is a square wave with sign flips at $\phi = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$, at which points $\cos\phi$ vanishes. Hence we have

$$(\cos^2 \phi)^{1/2} \prod_{n=0}^{\infty} \cos[\pi H(n, \phi)] = \cos \phi. \quad \text{(A 15)}$$

The effect of the sum over sine terms is to select out values at $\phi = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$ only. Thus

$$(\cos^2 \phi)^{1/2} \sum_{n=0}^{\infty} (-1)^n \sin[\pi H(n, \phi)] = 0. \quad \text{(A 16)}$$

Hence, finally,

$$E^+(\phi) = \left[\frac{\cos \phi}{(1 - n^2 \sin^2 \phi)^{1/2}} + O(b^{1/2}) \right] \cos \Delta\Omega t + O(b^{1/2}) \sin \Delta\Omega t,$$
$$= \left[E_0^* + O(b^{1/2}) \right] \cos \Delta\Omega t + O(b^{1/2}) \sin \Delta\Omega t, \quad \text{(A 17)}$$

as required. We derive the second of (5.7) in a similar fashion.

A.1. The limit as $b \to 0$

In the limit as $b \to 0$, $E^-(\phi)$ as given by (5.7) vanishes. However, the ordinary differential equation from which $E^+(\phi)$, $E^-(\phi)$ and $E^s(\phi)$ are derived, (3.3), becomes

$$\gamma_0 \ddot{\mathbf{E}}_\perp + \dot{\gamma}_0 \dot{\mathbf{E}}_\perp + \omega_p^2 \mathbf{E}_\perp = 0, \quad \text{(A 18)}$$

and has two solutions. One solution is $E_0^*(\phi)$, which is readily derived from $E^+(\phi)$ in this limit. To derive the other solution from (5.7), we need to retain all terms of $O(b^{1/2})$. A constant factor of $O(b^{1/2})$ can then be regarded as part of an arbitrary constant of integration.

The first $O(b^{1/2})$ term in (5.7) falls away, since $\sin \Delta\Omega t$ is itself of $O(b^{1/2})$. The second $O(b^{1/2})$ term, premultiplying $\cos \Delta\Omega t$, is to be retained. Thus we use (A 9) and write, to $O(b^{1/2})$

$$\sin[\Omega t + P(t)] = \sin\left[\pi \sum_{n=0}^{\infty} H(n, \phi) \right] + D(\phi) \cos\left[\pi \sum_{n=0}^{\infty} H(n, \phi) \right]. \quad \text{(A 19)}$$

Equation (5.6*b*) to $O(b^{1/2})$ is then

$$E^-(\phi) \to E_0^*[\Delta\Omega t + D(\phi)]. \quad \text{(A 20)}$$

Expanding the $\tan^{-1}$ in $D(\phi)$ to $O(b^{1/2})$ and simplifying, we finally obtain

$$E^-(\phi) \to \left(\frac{2b}{\gamma_m - 1} \right)^{1/2} \{ E_0^*[F(\phi, n) - E(\phi, n)] + \sin \phi \}, \quad \text{(A 21)}$$

which is a solution of (A 18).

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press

References

Akhiezer, A. I. and Polovin, R. V. 1956 Theory of wave motion of an electron plasma. *Soviet Phys. JETP* **3**, 696–705.

Asséo, E., Llobet, X. and Schmidt, G. 1980 Instability of large-amplitude electromagnetic waves in plasmas. *Phys. Rev.* **A22**, 1293–1294.

Byrd, P. F. and Friedman, M. D. 1971 *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer-Verlag, New York.

Freund, H. P. and Papadopoulos, K. 1980 Oscillating two-stream and parametric decay instabilities in a weakly magnetised plasma. *Phys. Fluids* **23**, 139–146.

Janicke, L. 1977 Non-linear electromagnetic waves in a relativistic plasma. *Plasma Phys.* **19**, 209–228.

Kennel, C. F. and Pellat, R. 1976 Relativistic nonlinear plasma waves in a magnetic field. *J. Plasma Phys.* **15**, 335–355.

Lakhina, G. S. and Buti, B. 1981 Generation of a d.c. field by non-linear electromagnetic waves in relativistic plasmas. *Astrophys. Space Sci.* **79**, 25–36.

Luheshi, M. and Stewart, P. 1978a Parametric instabilities III. *Astron. Astrophys.* **64**, 259–265.

Luheshi, M. and Stewart, P. 1978b Magneto-parametric instabilities III. *Astron. Astrophys.* **70**, 79–85.

Luheshi, M. and Stewart, P. 1979 Parametric instabilities of commensurable and incommensurable nonlinear plasma waves. *Astron. Astrophys.* **75**, 185–192.

Max, C. 1973 Steady-state solutions for relativistically strong electromagnetic waves in plasmas. *Phys. Fluids* **16**, 1277–1288.

Max, C. and Perkins, F. 1971 Strong electromagnetic waves in overdense plasmas. *Phys. Rev. Lett.* **27**, 1342–1345.

Melrose, D. B. 1978 Amplified linear acceleration emission applied to pulsars. *Astrophys. J.* **225**, 557–573.

Pustovalov, V. V., Silin, V. P. and Chernikov, A. A. 1978 Time-dependent theory of parametric plasma instabilities. *Soviet J. Quantum Electron.* **8**, 1097–1101.

Rowe, E. T. 1992 Linear acceleration emission. PhD thesis, University of Sydney.

Rowe, E. T. 1995 A self-consistent treatment of linear acceleration emission. *Astron. Astrophys.* **296**, 275–288.

Weatherall, J. C., Goldman, M. V. and Nicholson, D. R. 1981 Parametric instabilities in weakly magnetised plasma. *Astrophys. J.* **246**, 306–313.

https://doi.org/10.1017/S0022377898007387 Published online by Cambridge University Press