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TWO THEOREMS ON BALANCED BRACES

WOLFGANG RUMP

Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany (rump@mathematik.uni-stuttgart.de)

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Abstract Two theorems of Gateva-Ivanova [Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups, Adv. Math. 338 (2018), 649–701] on square-free set-theoretic solutions to the Yang–Baxter equation are extended to a wide class of solutions. The square-free hypothesis is almost completely removed. Gateva-Ivanova and Majid's 'cyclic' condition **lri** is shown to be equivalent to balancedness, introduced in Rump [A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), 40–55]. Basic results on balanced solutions are established. For example, it is proved that every finite, not necessarily square-free, balanced brace determines a multipermutation solution.

Keywords: Yang–Baxter equation; cycle set; brace; multipermutation solution

2020 Mathematics subject classifications Primary 16T25; 16W22

1. Introduction

The Yang–Baxter equation

$$(S \otimes 1)(1 \otimes S)(S \otimes 1) = (1 \otimes S)(S \otimes 1)(1 \otimes S)$$

for endomorphisms $S \in \mathrm{End}(X \otimes X)$ in a tensor category is a fundamental relation in the theory of braidings [22]. If the tensor product is replaced by a product of sets, a search for solutions was initiated by Drinfeld [8]. The systematic study of set-theoretic solutions began with the papers of Gateva-Ivanova and Van den Bergh [19], Etingof et al. [9], and Lu et al. [25]. Due to the ubiquity of braidings, numerous connections to other mathematical fields were discovered, including Poisson Lie groups [37], Sklyanin algebras [36], non-commutative regular rings [12, 13], Hopf-Galois extensions [1, 5], right-symmetric Lie algebras [3, 30], and Garside groups [6, 7, 31].

A set-theoretic solution $S(x, y) = ({}^x y, x^y)$ is said to be non-degenerate if S itself and the partial maps $x \mapsto x^y$ and $x \mapsto {}^y x$ are bijective. It is said to be *square-free* if $S(x, x) = (x, x)$ for all $x \in X$. Square-free solutions are non-degenerate if they are *involutive*, that is, $S^2 = 1$. It is this class of solutions which attracted attention in the early nineties of the

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past century because of their conjectured property of providing complete invariants for a class of exceptionally well-behaved Artin–Schelter regular rings of arbitrary dimension [12, 13].

In a recent paper [14], Gateva-Ivanova considers a condition

$$\textbf{lri}\colon ({}^{x}y)^{x} = y = {}^{x}(y^{x}) \quad \text{for all } x, y \in X,$$

introduced in [17], which is slightly more general than 'square-free'. (An equivalent condition is introduced in [28], where its connection to square-freeness is explained.) In [14], she proves two interesting theorems, providing sufficient criteria for *multipermutation solutions* [9] which are much easier to handle than general solutions. Both theorems state that a square-free solution is a multipermutation solution, whenever some ambient solution satisfies **lri**.

We extend and refine these theorems. In particular, we show that 'square-free' can essentially be dropped. As a corollary, simple proofs of Gateva-Ivanova's theorems are obtained. To explain our results, some known facts have to be recalled.

Etingof et al. [9] introduced the *structure group* $(G_X, \circ)$ of a solution $(S; X)$, which allowed a group theoretic analysis of the otherwise notoriously unintuitive solutions. The group G_X is generated by the set X with defining relations

$$x \circ y = {}^{x}y \circ x^{y}.$$

It is easily checked that S is involutive if and only if the equation

$${}^{y}x = y^{x} \cdot x \tag{1}$$

holds in X, where $y \mapsto x \cdot y$ denotes the inverse $\sigma(x)$ of $y \mapsto y^{x}$. Thus, every non-degenerate involutive solution $(S; X)$ can be expressed by a set $(X; \cdot)$ with a single binary operation, which led to the concept of *cycle set* [28].

A striking feature of non-degenerate involutive solutions $(S; X)$ is their unique extendability to a solution $(S; G_X)$ on the structure group $G_X \supset X$. Moreover, this combination of group and solution gives rise to an abelian group structure $a + b := (a \cdot b) \circ a$ on G_X which makes G_X into a *brace* [29], a ring-like structure A_X similar to the Jacobson radical of a ring, with *adjoint group* $A_X^{\circ} := G_X$. The additive group of A_X is a module over the group G_X, and the subset X is invariant under this action, so that the action of any $x \in X \subset G_X$ on X coincides with the left multiplication $\sigma(x)\colon y \mapsto x \cdot y$ in the cycle set.

The permutations $\sigma(x)$ of X generate a factor group $G(X)$ of G_X, the *permutation group* of X. Like G_X, the permutation group is also a brace, denoted by $A(X)$, so that the natural group homomorphism $\sigma\colon G_X \twoheadrightarrow G(X)$ becomes a morphism $A_X \twoheadrightarrow A(X)$ of braces. The image σX of $X \subset A_X$ is again a solution, the *retraction* [9] of $(S; X)$. If $X \twoheadrightarrow \sigma X$ is a proper surjection, X is said to be *retractable*. It may happen that the sequence

$$X \twoheadrightarrow \sigma X \twoheadrightarrow \sigma^2 X \twoheadrightarrow \cdots$$

of retractions arrives at a singleton after n steps. Then the solution S is said to be of *multipermutation level* $\operatorname{mpl}(S; X) := n$. The case of multipermutation level 2 was first studied in [16, 18] and extensively pursued in [14]. Gateva-Ivanova's theorems [14] are the following

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Theorem A (Gateva-Ivanova [14, Theorem 8.2]). *For an involutive square-free solution $(S;X)$, the following are equivalent:*

(a) $(S;G_X)$ *satisfies* ***lri****.*

(b) $mpl(S;G_X) \leqslant 2$.

(c) $mpl(S;X) \leqslant 2$.

(d) $(S;X)$ *satisfies* ***lri****, and the equation* ${}^a(b+c) = {}^ab + {}^ac$ *holds for all* $a,b,c \in A_X$.

Theorem B (Gateva-Ivanova [14, Theorem 8.5]). *Let $(S;X)$ be an involutive square-free solution such that $(S;G(X))$ satisfies* ***lri****. Then*

(a) $(S;X)$ *is retractable or a strong twisted union [15] of its G_X-orbits such that each G_X-orbit Z satisfies $mpl(Z) \leqslant 1$.*

(b) If X is finite, $mpl(S;X)$ is finite.

Part (b) of Theorem B is the most striking result. Its proof shows that $\mathrm{mpl}(S;X) \leqslant n$ if X has n adjoint orbits. We prove that Theorem B remains true without the square-free hypothesis (Theorem 2). To achieve this, we note first that for a cycle set X, the condition **lri** is equivalent to the equation $x \cdot y = (y \cdot x) \cdot y$, which states that X is *balanced* [28] (Proposition 1). By recursion, we define products in a cycle set X as $x_0 \cdot \ldots \cdot x_{n+1} = (x_0 \cdot \ldots \cdot x_n) \cdot x_{n+1}$. Then X is of multipermutation level $\leqslant n$ if and only if $x_0 \cdot x_1 \cdot \ldots \cdot x_n$ does not depend on x_0. For square-free cycle sets X, this means that the equation $x_0 \cdot x_1 \cdot \ldots \cdot x_n = x_1 \cdot \ldots \cdot x_n$ holds in X. If this equation is only assumed for x_0, x_n in the same adjoint orbit, we call X *weakly n-reductive*. It turns out that Theorem B holds since square-free cycle sets X for which the brace $A(X)$ is balanced are weakly 1-reductive and weakly 2-reductive. This also explains the close relationship to strong twisted unions. We show that whenever X is weakly n-reductive for two consecutive positive integers n, the same holds for all greater n (Proposition 7). Theorem B immediately follows.

If square-freeness of X is dropped, a deeper analysis will be required. We prove that the retraction of any balanced brace is weakly 2-reductive (Corollary 3 of Proposition 8). To get the inductive argument work, we have to go into the third retraction $B = A/\mathrm{Soc}_3(A)$ of a balanced brace A. It turns out that each adjoint orbit of B generates a right ideal I which is a commutative radical ring with $I^3 = 0$ (Proposition 10). Using this surprising fact, we show that B is weakly 3-reductive, hence weakly n-reductive for all $n \geqslant 2$. In particular, this implies that every non-degenerate cycle set X with finitely many adjoint orbits and $A(X)$ balanced gives rise to a multipermutation solution to the Yang–Baxter equation (Theorem 2).

Theorem A, and a similar result which is used for its proof, will also be proved, up to a trivial exception, without the square-free hypothesis (Theorem 1). The exception consists in the fact that 'trivial' solutions in the sense that all permutations $\sigma(x)$ are equal, need not be trivial in the sense that all $\sigma(x)$ are identity maps. Thus, only the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) and the implication (b) $\Rightarrow$ (c) hold without the condition of square-freeness. Apart from this exception, the conditions of Theorem A describe exactly the involutive non-degenerate solutions which determine a rack [10, 23].

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In proving our theorems, we found a simple but useful relationship between balancedness and the square map $x \mapsto x \cdot x$, the 'T-structure' [9] of a non-degenerate cycle set. Namely, a cycle set is balanced if and only if the adjoint group respects the square map (Proposition 2), while a brace A is balanced if and only if the inversion in A° is invariant under the adjoint action (Proposition 6). It seems that balanced braces and their corresponding solutions to the Yang–Baxter equation form a nice class which should be further investigated. At present, it is far from obvious which of the known braces and cycle sets are of that type.

2. Cycle sets and braces

In this section, we explain our terminology and reformulate the Theorems B and B. Recall that a set-theoretic solution $S(x, y) = ({}^x y, x^y)$ to the Yang–Baxter equation and its associated 'R-matrix' $R(x, y) = (x^y, {}^x y)$ is said to be *left non-degenerate* [28] if the partial map $x \mapsto x^y$ is bijective for all $y \in X$. As above, we write $y \mapsto x \cdot y$ for the inverse of $y \mapsto y^x$. If S and its component maps $y \mapsto y^x$ and $y \mapsto {}^x y$ are bijective, S is said to be *non-degenerate* [9]. There are two important special cases of left non-degenerate set-theoretic solutions, namely, the *involutive* ones with $S^2 = 1$, and the *rack solutions* where the first component of S is trivial, that is, ${}^x y = y$. The involutive property is equivalent to Equation (1). Thus, in both cases, the solution is determined by the map $y \mapsto x \cdot y$. By [28, Proposition 1], involutive left non-degenerate solutions are characterized by the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z). \tag{2}$$

A structure $(X; \cdot)$ satisfying Equation (2) is said to be a *cycle set* [28]. On the other hand, a *rack* [10, 23] is given by the self-distributivity equation

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z). \tag{3}$$

For cycle sets and racks, left non-degeneracy (i.e. bijectivity of the maps $y \mapsto x \cdot y$) is assumed. Replacing the operation $x \cdot y$ by its inverse $y \rhd x := y^x$, left distributivity (3) turns into right distributivity

$$(z \rhd y) \rhd x = (z \rhd x) \rhd (y \rhd x), \tag{4}$$

which gives a right-hand version of a rack.

By [34, Corollary of Proposition 2], every non-degenerate solution $S\colon X \times X \to X \times X$ has an associated rack solution (4), also called the *derived solution* [8, 25], given by

$$x \rhd y := ({}^{x \cdot y} x)^y.$$

The map $x \mapsto x \rhd y$ is the identity map if and only if Equation (1) holds in X. This happens if and only if the solution S is involutive, i.e. given by a cycle set. In other words, the distance between arbitrary (left non-degenerate) solutions S and rack solutions is measured by a cycle set. Informally, this can be expressed by a 'short exact sequence'

$$\boxed{\text{Cycle set}} \rightarrowtail \boxed{\text{Solution}} \twoheadrightarrow \boxed{\text{Rack}}$$

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The concept of *cycle set* is inspired by a mysterious cyclicity which occurred in connection with a class of skew-polynomial rings [12]. It turned out that these non-commutative regular rings [11] are equivalent to *square-free* cycle sets [28], that is, cycle sets satisfying the equation $x \cdot x = x$. For any pair of distinct elements x, y of a cycle set X, the powers $\sigma(x)^n(y)$ with $n \in \mathbb{Z}$, where $\sigma(x)(y) := x \cdot y$, form a (possibly infinite) cycle $C_x(y)$. If X is square-free, the cycles $C_x(y)$ and $C_y(x)$ are disjoint, all elements of $C_y(x)$ act in the same way on $C_x(y)$, and vice versa. More generally, this cyclic behaviour occurs if X is *balanced* [28], that is,

$$ {}^{x}y = x \cdot y \tag{5}$$

holds for all $x, y \in X$. Equation (1) shows that X is balanced if and only if the equation

$$ x \cdot y = (y \cdot x) \cdot y \tag{6}$$

is satisfied. By [28, Proposition 3], every square-free cycle set is balanced.

Proposition 1. *A cycle set X satisfies* ***lri*** *if and only if it is balanced.*

Proof. Equation (5) says that the operation ${}^{x}(\ )$ is inverse to $(\ )^{x}$. Note that by Equation (1), the left-hand equation in **lri** is already equivalent to Equation (5). □

A cycle set X is said to be *non-degenerate* if the corresponding solution $S(x, y) = ({}^{x}y, x^{y})$ is non-degenerate. By [28, Proposition 2], X is non-degenerate if and only if the square map $x \mapsto x \cdot x$ is bijective. Equation (5) shows that balanced cycle sets are non-degenerate. More precisely, we have

Proposition 2. *For a cycle set X, the following are equivalent:*

(a) X is balanced.

(b) X is non-degenerate and satisfies the equation $x \cdot (y \cdot y) = (x \cdot y) \cdot (x \cdot y)$.

(c) The map $X \times X \to X \times X$ with $(x, y) \mapsto (x \cdot y, y \cdot x)$ has an inverse $(u, v) \mapsto (v^{u}, u^{v})$.

Proof. (a) ⇔ (b) follows since $x \cdot (y \cdot y) = (y \cdot x^{y}) \cdot (y \cdot y) = (x^{y} \cdot y) \cdot (x^{y} \cdot y)$. Condition (c) states that $(x \cdot y)^{y \cdot x} = y$ and $v^{u} \cdot u^{v} = u$, where the first equation is equivalent to Equation (6). To obtain the second equation from the first, replace v by v^{u} in $(u \cdot v)^{v \cdot u} \cdot u = v \cdot u$, which gives $v^{v \cdot u} \cdot u = v \cdot u$, and then replace u by u^{v}. □

In other words, a cycle set is balanced if and only if the adjoint action preserves the associated T-structure (see [9], Appendix; [32], Section 7).

An abelian group A with a binary operation $\cdot$ such that the maps $b \mapsto a \cdot b$ are bijective is said to be a *linear cycle set* or a *brace* [29] if it satisfies the equations

$$ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \tag{7}$$

$$ (a + b) \cdot c = (a \cdot b) \cdot (a \cdot c). \tag{8}$$

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Thus, Equation (8) implies that every brace A is a cycle set. The operation

$$a \circ b := a^b + b$$

makes A into a group, the *adjoint group* A° of A. The inverse a' of $a \in A^\circ$ is given by

$$a' = -(a \cdot a), \tag{9}$$

which shows that braces A are non-degenerate as cycle sets. By the commutativity of addition, we have

$$(a \cdot b) \circ a = (b \cdot a) \circ b. \tag{10}$$

The group A° acts on the additive group A via Equation (7) and

$$(a \circ b) \cdot c = a \cdot (b \cdot c). \tag{11}$$

Every radical ring A is a brace with $a \circ b = a + ab + b$ (hence $a^b = a(1 + b)$). Therefore, braces A can be viewed as generalized radical rings with ring multiplication ab given by

$$a \circ b = a + ab + b.$$

Accordingly, a subgroup I of A is said to be a *right ideal* [29] if $a \in I$ and $b \in A$ implies that $ab \in I$. If also $ba \in I$, then I is called an *ideal*. Any right ideal is invariant under the adjoint action, and a subgroup of A°. In addition, an ideal is a normal subgroup of A°. As in ring theory, any ideal I gives rise to a factor brace A/I with a canonical morphism of braces $A \twoheadrightarrow A/I$. An important ideal of any brace A is its *socle*

$$\mathrm{Soc}(A) := \{a \in A \,|\, \forall b \in A\colon a \cdot b = b\}.$$

Iterating this construction, we obtain the *socle series* [29]

$$0 = \mathrm{Soc}_0(A) \subset \mathrm{Soc}_1(A) \subset \mathrm{Soc}_2(A) \subset \mathrm{Soc}_3(A) \subset \cdots$$

of ideals in A with $\mathrm{Soc}_{n+1}(A)/\mathrm{Soc}_n(A) = \mathrm{Soc}(A/\mathrm{Soc}_n(A))$.

By [28, Proposition 6], every non-degenerate cycle set X embeds into a brace A_X with additive group $\mathbb{Z}^{(X)}$. Its adjoint group $G_X := A_X^\circ$ is the *structure group* [9] of X. The subset $X \subset A_X$ is invariant under the adjoint action (11) of the structure group G_X on $(A_X; +)$. The adjoint group $G(X)$ of the factor brace $A(X) := A_X/\mathrm{Soc}(A_X)$ can be identified with the subgroup of the symmetric group $\mathfrak{S}(X)$ generated by the permutations $\sigma(x)$ with $x \in X$. Therefore, it is also called the *permutation group* of X. The image σX of X in $A(X)$ is a sub-cycle set of $A(X)$, the *retraction* of X. (Its corresponding solution to the Yang–Baxter equation coincides with the retraction [9] of S.) By [28, Proposition 10], the retraction σX is again non-degenerate. In particular, $A(X) = A_X/\mathrm{Soc}(A_X)$ is the retraction of A_X as a cycle set.

Definition 1. A subset X of a brace A is said to be a *cycle base* [29] if X is invariant under the adjoint action (11) and generates the additive group of A.

Thus σX is a cycle base in $A(X)$. Any cycle base X of a brace A is a non-degenerate sub-cycle set, and X also generates the adjoint group A°. If the iterated retraction $\sigma^n X$ is

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a singleton for some $n \in \mathbb{N}$, the corresponding solution S is said to be a *multipermutation solution* [9]. Accordingly, we define the *multipermutation level* $\mathrm{mpl}(X) \in \mathbb{N} \cup \{\infty\}$ to be the infimum of all $n \in \mathbb{N}$ for which $\sigma^n X$ is a singleton. Thus, in general,

$$\mathrm{mpl}(X) \leqslant \mathrm{mpl}(A_X). \tag{12}$$

In [14, Theorem 5.15], Gateva-Ivanova proved that equality holds if $|X| > 1$ and X satisfies a condition $(*)$ which states that any $x \in X$ admits an element $y \in X$ with ${}^y x = x$. By Equation (1), the condition translates into

$(*)$ For every $x \in X$ there exists an element $y \in X$ with $y \cdot x = x$.

For elements $x_1, \ldots, x_n$ of a cycle set, we define

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{n+1} := (x_1 \cdot x_2 \cdot \ldots \cdot x_n) \cdot x_{n+1}.$$

A cycle set X is said to be *n-reductive* [21, 27] if the equation $x_0 \cdot x_1 \cdot \ldots \cdot x_n = x_1 \cdot \ldots \cdot x_n$ holds for all $x_0, \ldots, x_n \in X$. (We say that X is *0-reductive* if $|X| = 1$.) To get a precise condition for equality in (12), we modify this concept as follows.

Definition 2. We call a cycle set X *strongly reductive* if $n := \mathrm{mpl}(X)$ is finite and X is n-reductive.

Proposition 3. *Let X be a cycle base of a brace A. Then the retraction σX is a cycle base of $A/\mathrm{Soc}(A) \cong A(X)$.*

Proof. The adjoint action of A° on A is determined by the action on X. Therefore, we have a commutative diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & A/\mathrm{Soc}(A) \\ \uparrow & & \uparrow \\ X & \twoheadrightarrow & \sigma X \end{array}$$

and a natural embedding $A/\mathrm{Soc}(A) \hookrightarrow A(X)$. Since σX generates $A(X)$, the embedding $A/\mathrm{Soc}(A) \hookrightarrow A(X)$ is an isomorphism. □

Corollary. Let X be a cycle base of a brace A. Then $\mathrm{mpl}(X) = \mathrm{mpl}(A) < \infty$ if and only if X is strongly reductive.

Proof. The above diagram shows that $n := \mathrm{mpl}(X) \leqslant \mathrm{mpl}(A)$. Assume without loss of generality that $n < \infty$. Then X is strongly reductive if and only if $\sigma^{n-1}(x)$ is the identity map in $\sigma^{n-1}X$ for all $x \in X$, that is, $\sigma^n A = 0$. □

Remark. A cycle set X satisfies $\mathrm{mpl}(X) \leqslant n$ if and only if the expression $x \cdot x_1 \cdot \ldots \cdot x_n$ does not depend on x. The above condition $(*)$ then implies that X is strongly reductive. A square-free cycle set X is strongly reductive if and only if $\mathrm{mpl}(X) < \infty$.

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Recall that X is said to be *trivial* if $x \cdot y = y$ for all $x, y \in X$. So X is trivial if and only if X is strongly reductive with $\mathrm{mpl}(X) \leqslant 1$.

To state our next result, we translate the concept of strong twisted union into the language of cycle sets. By [15], a cycle set X with a partition $X = \bigsqcup_{i \in I} X_i$ into G_X-invariant subsets is said to be a *strong twisted union* if for distinct $i, j \in I$, the equations

$$ {}^{u^y}x = {}^{u}x, \qquad u^{{}^{v}x} = u^x \tag{13} $$

hold for $x, y \in X_i$ and $u, v \in X_j$.

Proposition 4. *Let X be a cycle set. A partition $X = \bigsqcup_{i \in I} X_i$ into G_X-invariant subsets is a strong twisted union if and only if for distinct $i, j \in I$, the equation*

$$ v \cdot x \cdot u = x \cdot u \tag{14} $$

holds for $x \in X_i$ and $u, v \in X_j$.

Proof. By Equation (1), the second equation in (13) is equivalent to $u^{v^x \cdot x} = u^x$, that is, $u = v^x \cdot x \cdot u^x$. Replacing u by $x \cdot u$ and v by $x \cdot v$, this turns into Equation (14). The first equation in (13) states that $(u^y)^x \cdot x = u^x \cdot x$. By Equation (14), the right-hand side is $u^x \cdot x = x \cdot u^x \cdot x = u \cdot x$. With the same operation applied to the left-hand side, the equation reduces to $u^y \cdot x = u \cdot x$. With i and j interchanged, this follows by Equation (14). □

Proposition 4 suggests the following version of Definition 2.

Definition 3. We call a cycle set X *weakly n-reductive* if for $x_0, \ldots, x_n \in X$, the equation $x_0 \cdot x_1 \cdot \ldots \cdot x_n = x_1 \cdot \ldots \cdot x_n$ holds whenever x_0 and x_n are in the same G_X-orbit of X.

Again, we interpret the case $n = 0$ by saying that X is *weakly 0-reductive* if X is a singleton. Note that a weakly 2-reductive cycle set is balanced. As a consequence of Proposition 4, we have

Corollary. A weakly 2-reductive cycle set X is a strong twisted union of its G_X-orbits.

Lemma. *Every brace satisfies the equation*

$$ a \cdot (b \circ c) = (c \cdot a \cdot b) \circ (a \cdot c). \tag{15} $$

Proof. We have $(a \cdot c) \cdot (a \cdot b^c) = (c \cdot a) \cdot (c \cdot b^c) = (c \cdot a) \cdot b$. Hence $a \cdot b^c = (c \cdot a \cdot b)^{a \cdot c}$, and thus $a \cdot (b \circ c) = a \cdot (b^c + c) = a \cdot b^c + a \cdot c = (c \cdot a \cdot b)^{a \cdot c} + a \cdot c = \big(c \cdot a \cdot b\big) \circ (a \cdot c)$. □

Proposition 5. *Every balanced brace satisfies the equation*

$$ (a \cdot b \cdot c) \circ (b \cdot a) = (b \cdot c) \circ (b^c \cdot a). \tag{16} $$

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Proof. Equation (15) gives $(a \cdot b \cdot c) \circ (b \cdot a) = b \cdot (c \circ a) = b^{c \circ a} \cdot (c \circ a) = \big((a \cdot b^{c \circ a}) \cdot c\big) \circ (b^{c \circ a} \cdot a) = \big((a \cdot (b^c)^a) \cdot c\big) \circ \big((b^c)^a \cdot a\big) = (b \cdot c) \circ (b^c \cdot a)$. □

Theorem 1. *For a non-degenerate cycle set X, the following are equivalent:*

(a) The brace A_X is balanced.

(b) X is 2-reductive.

(c) X is a rack.

(d) $mpl(A_X) \leqslant 2$.

(e) X is balanced and the equation ${}^a(b + c) = {}^a b + {}^a c$ holds for all $a, b, c \in A_X$.

Proof. (a) ⇒ (b): By Equation (16), we have $(x \cdot y \cdot z) \circ (y \cdot x) = (y \cdot z) \circ (y^z \cdot x)$. Thus

$$x \cdot y \cdot z = y \cdot z \Longleftrightarrow y \cdot x = y^z \cdot x \tag{17}$$

holds for all $x, y, z \in X$. By Equation (16), we also have

$$(x \cdot y \cdot z)^{y \cdot x} + y \cdot x = (y \cdot z)^{y^z \cdot x} + y^z \cdot x.$$

Assume that $x \cdot y \cdot z \neq y \cdot z$, or equivalently, $y \cdot x \neq y^z \cdot x$. As the additive group of A_X is isomorphic to $\mathbb{Z}^{(X)}$, this implies that $(x \cdot y \cdot z)^{y \cdot x} = y^z \cdot x$ and $y \cdot x = (y \cdot z)^{y^z \cdot x}$. Hence $(y \cdot x) \cdot (y^z \cdot x) = x \cdot y \cdot z$ and $(y^z \cdot x) \cdot (y \cdot x) = y \cdot z$. By Proposition 2, it follows that $y \cdot x = (y \cdot z)^{x \cdot y \cdot z}$. Since X is balanced, we also have $x^y \cdot y \cdot z \neq y \cdot z$. So the same calculation with x^y instead of x yields $x = y \cdot x^y = (y \cdot z)^{x^y \cdot y \cdot z} = (y \cdot z)^{x \cdot y \cdot z} = y \cdot x$. With (17), we obtain

$$x \cdot y \cdot z \neq y \cdot z \Longleftrightarrow y \cdot x \neq y^z \cdot x \Longrightarrow y \cdot x = x. \tag{18}$$

So the inequality $z \cdot y^z \cdot x = y \cdot x \neq y^z \cdot x$ implies $y \cdot z = y^z \cdot z = z$.

Thus, by interchanging x and z, we have proved the implication

$$z \cdot y \cdot x \neq y \cdot x \Longrightarrow y \cdot x = x.$$

Since $z \cdot y^z \cdot x = y \cdot x \neq y^z \cdot x$, this yields $y^z \cdot x = x$, a contradiction. By (17), we thus have shown that $x \cdot y \cdot z = y \cdot z$ holds for all $x, y, z \in X$. Whence X is 2-reductive.

(b) ⇔ (c): See [21], Corollary 3.2. If (b) holds, Equation (2) gives $x \cdot (y \cdot z) = (y \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, which yields (c). Conversely, (c) implies that $(y \cdot x) \cdot z = (y \cdot x) \cdot (y \cdot z^y) = (x \cdot y) \cdot (x \cdot z^y) = x \cdot (y \cdot z^y) = x \cdot z$ holds for $x, y, z \in X$.

(b) ⇒ (d): If $\mathrm{mpl}(X) = 2$, then X is strongly reductive, and (d) follows by the corollary of Proposition 3. Otherwise, $\mathrm{mpl}(A(X)) \leqslant 1$, which yields $\mathrm{mpl}(A_X) \leqslant 2$.

(d) ⇒ (e): This follows by Equations (5) and (7).

(e) ⇒ (a): For $a, b, c \in A_X$ we have ${}^{a \circ b}c = {}^a({}^b c)$.

Therefore, the equation ${}^x y = x \cdot y$ in X extends to $x \in A_X$. By linearity, it extends to all $x, y \in A_X$. □

Corollary. Let X be a cycle set which is a rack. Then X is a strong twisted union of its G_X-orbits, and each G_X-orbit Z satisfies $\mathrm{mpl}(Z) \leqslant 1$.

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Proof. Proposition 4 implies that X is a strong twisted union. By induction, Equation (14) yields $(a \cdot x) \cdot y = x \cdot y$ for $x, y \in X$ and $a \in A_X$. □

Remark. Theorem 1 characterizes a non-degenerate cycle set X which is at the same time a rack. On the other hand, the solution $(S; X)$ corresponding to a cycle set X is a rack solution if and only if it satisfies the equation ${}^x y = y$. By Equation (1), this means that the solution is trivial. This does not contradict Theorem 1. Indeed, the solution $S(x, y) = ({}^x y, x^y)$ given by a 2-reductive cycle set X differs from the rack solution $S(x, y) = (y, x^y)$, unless the cycle set X is trivial.

Example 1. By Equation (5), the equation in Theorem 1 holds in every balanced brace. But there are non-balanced braces satisfying this equation. For example, consider the radical ring $A = \mathbb{F}_2\alpha \oplus \mathbb{F}_2\alpha^2 \oplus \mathbb{F}_2\alpha^3$ over the 2-element field $\mathbb{F}_2$ with $\alpha^4 = 0$. Then $a \cdot b = b + ba + ba^2$. So the socle is $\mathrm{Soc}(A) = \mathbb{F}_2\alpha^3$, and $\mathrm{mpl}(A) = 3$. Furthermore, ${}^a b = a^b \cdot b = (a + ab) \cdot b = b + b(a + ab) + b(a + ab)^2 = b + ba + bab + ba^2$. Hence ${}^a(b + c) = {}^a b + {}^a c$ holds for all $a, b, c \in A$. But A is not balanced. Indeed, $\alpha^\alpha \cdot \alpha = \alpha + \alpha^2 \neq \alpha + \alpha^2 + \alpha^3 = \alpha \cdot \alpha$.

The elements $x_0 := \alpha,\ x_1 := \alpha + \alpha^2,\ x_2 := \alpha + \alpha^3,\ x_3 := \alpha + \alpha^2 + \alpha^3$, form a cycle base $X := \{x_0, x_1, x_2, x_3\}$ of A with a rotation $\sigma(x_1) = \sigma(x_3) = \sigma(x_0)^{-1} = \sigma(x_2)^{-1} = (x_1, x_2, x_3, x_0)$. This cycle set X is not balanced, too.

It is well known that square-free braces are radical rings [4, Theorem 7.4]. For balanced braces, this is not true:

Example 2. The cyclic brace B_1 with additive group $Z/8\mathbb{Z}$ and $a \cdot b := b(-1)^a$ is balanced, but not a radical ring. Its adjoint group is the dihedral group D_4. Among the other braces $B_2, \ldots, B_8$ with adjoint group D_4 (see [2, 33], Example 3), the braces B_2, B_3, B_4 and B_5 are balanced, not square-free, but radical rings.

Example 3. Theorem 1 provides a lot of balanced braces which are not radical rings. Let X be a 2-reductive cycle set. Thus, A_X is a balanced brace. It is a radical ring if and only if it satisfies the equation $x^{y+z} + x = x^y + x^z$. Applying $\sigma(y + z)$, this equation turns into $x + (y + z) \cdot x = (y + z) \cdot x^y + (y + z) \cdot x^z$. Since $(y + z) \cdot x = (z \cdot y) \cdot (z \cdot x) = y \cdot (z \cdot x)$ and $(y + z) \cdot x^y = (y \cdot z) \cdot x = z \cdot x$, the equation becomes $x + y \cdot (z \cdot x) = y \cdot x + z \cdot x$. For $x \in X$, this equation implies that every $x \in X$ is equal to one of any pair of elements in the G_X-orbit of x. In other words, the G_X-orbits in X are of cardinality at most two, which is rarely the case.

3. The Theorems A and B

Using Proposition 1, Gateva-Ivanova's Theorem A follows immediately by Theorem 1. The same holds for the following result which she used for the proof of Theorem A:

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Proposition (Gateva-Ivanova [14, Proposition 8.1]). *Let $(S; X)$ be a square-free involutive solution such that the extension to G_X satisfies* ***lri****. Then*

(a) Each G_X-orbit of X is a trivial solution.

(b) $mpl(S; X) = mpl(S; G_X) \leqslant 2$.

Remark. Note that no 'square-free' hypothesis is assumed in Theorem 1. So it can be dropped in Theorem A if $\mathrm{mpl}(X) \leqslant 2$ is replaced by '2-reductive'. (In (d) we already added the assumption that X is balanced, which follows by the square-freeness of X.) The modification of (c) means that 'strongly reductive' has been added (see the corollary of Proposition 3). With the same adaption, the above proposition holds without assuming 'square-free'.

Theorem 1 does not remain valid if A_X is replaced by $A(X)$ or some brace with X as a cycle base. The case $A(X)$ is addressed in Theorem B. In this section, the structure of cycle bases of balanced braces will be analysed. We will prove the following general

Theorem 2. *Let X be a non-degenerate cycle set with finitely many adjoint orbits. If the brace $A(X)$ is balanced, then $mpl(X) < \infty$.*

By Proposition 2 and Equation (9), we have

Proposition 6. *A brace is balanced if and only if it satisfies the equation*

$$(a \cdot b)' = a \cdot b'.$$

Now we analyse balanced braces with respect to a given cycle base (cf. [14, Proposition 4.7]).

Proposition 7. *Let X be a cycle base of a balanced brace A. If X is weakly n-reductive and weakly $n-1$-reductive for some $n \geqslant 2$, then X is weakly $n+1$-reductive.*

Proof. Let $x_0, \ldots, x_n \in X$ be arbitrary, and let $x \in X$ be in the A°-orbit of x_0. (Since X is a cycle base, the A°-orbits coincide with the G_X-orbits.) Define $a := x_0 \cdot \ldots \cdot x_{n-2}$ and $d := x_1 \cdot \ldots \cdot x_{n-2}$. The condition that X is weakly $n-1$-reductive means that $a \cdot x = d \cdot x$ holds for a and x as given above. If x is replaced by $d' \cdot x$, which is in the same orbit, the condition becomes $a \cdot (d' \cdot x) = d \cdot (d' \cdot x) = x$, that is, $a' \cdot x = d' \cdot x$. Similarly, with $b := x_{n-1}$, the assumption that X is n-reductive can be expressed as $a \cdot b \cdot x = d \cdot b \cdot x$ or $(a \cdot b)' \cdot x = (d \cdot b)' \cdot x$. Define $c := x_n$. Then Equation (16) gives $x_0 \cdot \ldots \cdot x_n \cdot x = (a \cdot b \cdot c) \cdot x = \big((b \cdot c) \circ (b^c \cdot a) \circ (b \cdot a)'\big) \cdot x$. By Equation (10), $(b^c \cdot a) \circ (b \cdot a)' = (a \cdot b^c) \circ a \circ (b^c)' \circ b \circ a' \circ (a \cdot b)'$. Thus,

$$x_0 \cdot \ldots \cdot x_n \cdot x = ((b \cdot c) \circ (a \cdot b^c) \circ a \circ (b^c)' \circ b \circ a' \circ (a \cdot b)') \cdot x. \tag{19}$$

Now the assumption implies that $(a \cdot b)' \cdot x = (d \cdot b)' \cdot x$. Hence $\big(a' \circ (a \cdot b)'\big) \cdot x = a' \cdot \big((d \cdot b)' \cdot x\big) = d' \cdot \big((d \cdot b)' \cdot x\big) = \big(d' \circ (d \cdot b)'\big) \cdot x$. Proceeding in this way, we replace each a in Equation (3.1) by d. Thus $a \cdot b \cdot c \cdot x = d \cdot b \cdot c \cdot x$, which completes the proof. □

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Corollary. Let X be a square-free cycle base of a balanced brace A. Then X is weakly n-reductive for all $n > 0$.

Proof. Equation (16) with $b = c = x$ implies that $(a \cdot x) \cdot x = x \cdot x = x$. Thus X is weakly 1-reductive. Hence, if $x, y, z \in X$ such that x and z are in the same adjoint orbit, then $x \cdot y \cdot z = (x \cdot y) \cdot (x \cdot z^x) = (y \cdot x) \cdot (y \cdot z^x) = (y \cdot x) \cdot (y \cdot z) = y \cdot z$, which shows that X is weakly 2-reductive. Thus Proposition 7 applies. □

Proof of Theorem B. Assume that X is irretractable. Then X is a cycle base in $A(X)$. Thus Proposition 4 and the above corollary imply that X is a strong twisted union of its G_X-orbits which are trivial cycle sets. If X is finite, a sufficiently long product $x_0 \cdot x_1 \cdot \ldots \cdot x_n$ in X contains two x_i in the same orbit. So the corollary of Proposition 7 implies that the sequence can be shortened. Whence $\mathrm{mpl}(X) < \infty$. □

Remark. We already noted that balanced cycle sets share some cyclicity property with square-free ones. The relationship between both concepts is highlighted by Proposition 2. So it is tempting to ask how far 'square-free' can be removed. Since weakly 1-reductive cycle sets are square-free, the condition is necessary in the preceding corollary. Using a different approach, we will show now that 'square-free' can be dropped in Theorem B.

As a first step, we modify Equation (16).

Proposition 8. *Every balanced brace satisfies the equation*

$$e \circ (c \cdot (a \cdot b) \cdot d) \circ e' = a \cdot (c \cdot b) \cdot d \tag{20}$$

with $e := (b \cdot c) \circ (a \cdot b \cdot c)'$.

Proof. Replacing a by $d \circ a$, Equation (16) turns into $(d \cdot (a \cdot b) \cdot c) \circ \big(b \cdot (d \circ a)\big) = (b \cdot c) \circ \big(b^c \cdot (d \circ a)\big)$. Using Equation (15), this gives $(d \cdot (a \cdot b) \cdot c) \circ (a \cdot b \cdot d) \circ (b \cdot a) = (b \cdot c) \circ (a \cdot b^c \cdot d) \circ (b^c \cdot a)$. By Equation (16), we have $(d \cdot (a \cdot b) \cdot c) \circ \big((a \cdot b) \cdot d\big) = \big((a \cdot b) \cdot c\big) \circ \big((a \cdot b)^c \cdot d\big)$ and $(b^c \cdot a) \circ (b \cdot a)' = (b \cdot c)' \circ (a \cdot b \cdot c)$. So we obtain $\big((a \cdot b) \cdot c\big) \circ \big((a \cdot b)^c \cdot d\big) = (b \cdot c) \circ (a \cdot b^c \cdot d) \circ (b \cdot c)' \circ (a \cdot b \cdot c)$. Replacing c by c' and applying Proposition 6, we obtain $(a \cdot b \cdot c)' \circ (c \cdot (a \cdot b) \cdot d) = (b \cdot c)' \circ (a \cdot (c \cdot b) \cdot d) \circ e$. Rearranging both sides, this yields Equation (3.2). □

As an immediate consequence, Proposition 8 yields

Corollary 1. *Let A be a balanced brace. Then the implication*

$$a \cdot b \cdot c = b \cdot c \implies c \cdot (a \cdot b) \cdot d = a \cdot (c \cdot b) \cdot d$$

holds for $a, b, c, d \in A$.

Corollary 2. *Let A be a balanced brace, and let $B := A/\mathrm{Soc}(A)$ be its retraction. If $a \cdot b = b$ holds for some $a, b \in B$, then $a \cdot c = c$ holds for all c in the adjoint orbit of b.*

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Proof. Assume $a \cdot b = b$ for some $a, b \in B$. By Corollary 1, this implies that $c \cdot b = c \cdot (a \cdot b) = a \cdot (c \cdot b)$ holds for all $c \in B$. □

Corollary 3. *Let A be a balanced brace. Then $A/Soc(A)$ is weakly 2-reductive.*

Proof. For $b, c, e \in A$, Equation (6) implies that $\big((e \cdot b)^c \cdot e\big) \cdot (e \cdot b)^c = e \cdot (e \cdot b)^c$. With $a := e' \circ \big((e \cdot b)^c \cdot e\big)$, this gives $a \cdot (e \cdot b)^c = (e \cdot b)^c$. So Corollary 1 yields

$$c \cdot (a \cdot (e \cdot b)^c) \cdot d = a \cdot (c \cdot (e \cdot b)^c) \cdot d$$

for all $c, d \in A$. In $A/\mathrm{Soc}(A)$ this leads to the equation $e \cdot b = c \cdot (e \cdot b)^c = c \cdot \big(a \cdot (e \cdot b)^c\big) = a \cdot \big(c \cdot (e \cdot b)^c\big) = a \cdot e \cdot b$. Replacing $e \cdot b$ by b, the equation becomes $b = \big(e' \circ (b^c \cdot e)\big) \cdot b$, that is, $e \cdot b = (b^c \cdot e) \cdot b$. □

Weakly 2-reductive cycle sets are characterized as follows. Recall that a morphism of cycle sets is a map $f\colon X \to Y$ which satisfies $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$.

Proposition 9. *For a cycle set X, the following are equivalent:*

(a) X is weakly 2-reductive.

(b) X is a strong twisted union of its G_X-orbits Z which are 2-reductive.

(c) For each G_X-orbit Z and each $x \in X$, the map $\sigma(x)|_Z$ is an automorphism of Z.

Proof. (a) ⇒ (b): This follows immediately by Proposition 4.
(b) ⇒ (c): Let Z be a G_X-orbit, and $u, v \in Z$. For any $x \in X$, Proposition 4 implies that $x^u \cdot v = x \cdot v$. Hence $x \cdot (u \cdot v) = (u \cdot x^u) \cdot (u \cdot v) = (x^u \cdot u) \cdot (x^u \cdot v) = (x \cdot u) \cdot (x \cdot v)$.
(c) ⇒ (a): For $x \in X$ and a G_X-orbit Z with $u, v \in Z$, we have $v \cdot x \cdot u = (v \cdot x) \cdot (v \cdot u^v) = (x \cdot v) \cdot (x \cdot u^v) = (x \cdot v) \cdot (x \cdot u)^{x \cdot v} = x \cdot u$. □

Our proof of Theorem 2 rests on a careful analysis of weakly 2-reductive braces.

Definition 4. Let A be a weakly 2-reductive brace. For an adjoint orbit Z of A, we write $I_Z(A)$ for the additive subgroup of A generated by Z.

By Proposition 9, $I_Z(A)$ is invariant under the adjoint action, hence a right ideal of A. Note that if A has a cycle base X, then A is generated by the right ideals $I_Z(A)$ if Z runs through the adjoint orbits of X. We hope to say more about the $I_Z(A)$ at another occasion. Here we only show that they are sub-braces of a very special type.

Proposition 10. *Let A be a weakly 2-reductive brace. For each adjoint orbit Z of the twofold retraction $B := A/Soc_2(A)$ of A, the brace $I_Z(B)$ is a commutative radical ring with $I_Z(B)^3 = 0$.*

Proof. For $a, b, c, d \in A$, replace a by $a \cdot c$ in Corollary 1 of Proposition 8. Then $a \cdot c \cdot b \cdot c = b \cdot c$. Hence $c \cdot (a \cdot c \cdot b) \cdot d = (a \cdot c) \cdot (c \cdot b) \cdot d$, and thus $\big(c \circ (a \cdot c)\big) \cdot b \cdot d = \big((a \cdot c) \circ c\big) \cdot b \cdot d$, which shows that c and $a \cdot c$ commute in B. Now if v and $u \cdot v$ belong to some adjoint orbit Z of B, there are $a, c \in A$ with $\sigma^2(a) = u$ and $\sigma^2(c) = v$. Therefore,

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$(u \cdot v) \circ v = v \circ (u \cdot v)$. Since Z is a cycle base of $I_Z(B)$, it generates the adjoint group of $I_Z(B)$ (see [35, Proposition 8]). So $I_Z(B)$ is commutative, hence a radical ring. For $a, b, c \in Z$, Proposition 9 implies that $b^a \cdot c = b \cdot c$. Hence $c = (b \cdot c)^{b^a}$, and thus $c^b = c^{b^a}$. So we obtain $c + cb = c + c(b + ba)$, which yields $cba = 0$. □

Corollary 1. *Let A be a weakly 2-reductive brace. Then the twofold retraction $B := A/Soc_2(A)$ is weakly 3-reductive.*

Proof. With d in the adjoint orbit of c, Proposition 8 gives $(b \cdot c) \circ (a \cdot b \cdot c)' \circ (a \cdot b \cdot d) \circ (a \cdot b \cdot c) \circ (b \cdot c)' = a \cdot (c \cdot b) \cdot d$. By Proposition 10, it follows that $a \cdot b \cdot d = a \cdot (c \cdot b) \cdot d$ holds in B. Replacing a by $c \cdot a$, we obtain $c \cdot a \cdot b \cdot d = (c \cdot a) \cdot (c \cdot b) \cdot d = (a \cdot c) \cdot (a \cdot b) \cdot d = a \cdot b \cdot d$. □

Corollary 2. *Let A be a balanced brace. Then the threefold retraction $A/Soc_3(A)$ is weakly n-reductive for all $n \geqslant 2$.*

Proof. This follows by Proposition 7 and Corollary 3 of Proposition 8. □

Proof of Theorem 2. Assume that X has m adjoint orbits. By Proposition 3, σX is a cycle base of the brace $A(X)$. Hence $\sigma^4 X$ is a cycle base in $B := A(X)/\mathrm{Soc}_3(A(X))$. By Corollary 2 of Proposition 10, B is weakly n-reductive for all $n \geqslant 2$. Consider a product $x_0 \cdot x_1 \cdot \ldots \cdot x_n$ with $x_i \in \sigma^3 X$ and $n > m$. Since $\sigma^4 X$ has at most m adjoint orbits, there are integers i, j with $0 \leqslant i < j \leqslant n$ such that x_i and x_j are in the same B°-orbit and $i + 2 \leqslant j$. Thus $(x_0 \cdot \ldots \cdot x_i) \cdot x_{i+1} \cdot \ldots \cdot x_j = x_{i+1} \cdot \ldots \cdot x_j$. If $i > 0$, then $(x_1 \cdot \ldots \cdot x_i) \cdot x_{i+1} \cdot \ldots \cdot x_j = x_{i+1} \cdot \ldots \cdot x_j$. Hence, in any case, $x_0 \cdot x_1 \cdot \ldots \cdot x_n = x_1 \cdot \ldots \cdot x_n$, which shows that $\mathrm{mpl}(\sigma^4 X) \leqslant n$. Thus $\mathrm{mpl}(X) \leqslant n + 4$. □

Example 4. A radical ring A is balanced as a cycle set if and only if it satisfies the equation $bab = 0$ for all $a, b \in A$. Indeed, Equation (5) is equivalent to $a^b \cdot b = a \cdot b$, that is, $b = a^{b^a} \cdot b^a$. With $a^b = a + ab$, this gives $b = \big(a + a(b + ba)\big) \cdot (b + ba)$, or equivalently, $b + ba = b + b\big(a + a(b + ba)\big)$. So the equation reduces to $(bab)^a = 0$, that is, $bab = 0$. For comparison, A is square-free if and only if it satisfies the equation $a^2 = 0$.

A similar calculation shows that more generally, $b^c \cdot a \cdot b = a \cdot b$ holds in a balanced radical ring. (Replace a by a^{b^c} and proceed as before!) Thus, every balanced radical ring is weakly 2-reductive, which simplifies Corollary 3 of Proposition 8 in this case.

Example 5. Let V be a vector space over a finite field F, and let A be the Jacobson radical of the exterior algebra $\bigwedge(V)$ over V. Thus, A is a square-free brace with $n := \mathrm{mpl}(A) = \dim V$. Let $Y \subset V$ be a basis of V. For any $x \in Y$, the adjoint orbit consists of the elements $x + x \wedge a$ with $a \in A$, and the union X of these orbits is a square-free cycle base X of A. So there are just n orbits. However, if $n > 1$, the retraction of X has the same number of orbits. So there is no way to prove Theorem B by induction on the number of orbits.

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Example 6. Let A be the free non-commutative algebra over the 2-element Galois field $\mathbb{F}_2$, with generators x, y and the relation $bab = 0$. Then

$$A = \mathbb{F}_2 x \oplus \mathbb{F}_2 y \oplus \mathbb{F}_2 xy \oplus \mathbb{F}_2 yx \oplus Z$$

with centre $Z := \mathbb{F}_2 x^2 \oplus \mathbb{F}_2 y^2 \oplus \mathbb{F}_2 x^2 y \oplus \mathbb{F}_2 y^2 x$. The brace A is balanced, but not square-free. So A is not weakly 1-reductive. The eight basis elements form a cycle base X of A with two adjoint orbits. If Z denotes the orbit of x in the retraction of A, then $I_Z(A/\mathrm{Rad}(A))$ is non-commutative, which shows that a single retraction in Proposition 10 does not suffice. Note that in constrast to the exterior algebra $\bigwedge(V)$, it is not enough to assume the relation $bab = 0$ for the elements of the subspace $V := \mathbb{F}_2 x \oplus \mathbb{F}_2 y$ of A, which would lead to the *meson algebra* $B(V)$, an algebra [20, 24, 26] with an additional basis element x^2y^2. In A, this element vanishes since $(x + y)x(x + y)y = 0$.

Remarks. 1. If a cycle base X of a brace A is balanced, then A is usually far from being balanced. By induction, it can be shown that A is balanced if Equation (6) holds for $x \in X$ and $y \in A$. If the variable y in Equation (6) is replaced by a product of elements in X, new equations arise. Equation (3.2) enfolds the essence of these higher equations.

2. A construction of all cycle sets X with $\mathrm{mpl}(X) = 2$ is given in [21]. Such a cycle set need not be balanced (see, e.g. the cycle set X in Example 1). By [21], Lemma 6.7, it is balanced if and only if it is 2-reductive. Then Theorem 1 shows that X is a rack, and that the brace A_X is balanced with multipermutation level $\leqslant 2$. In a forthcoming paper, the cycle sets of multipermutation level 2 will be classified.

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References

1. I. Angiono, C. Galindo and L. Vendramin, Hopf braces and Yang-Baxter operators, *Proc. Amer. Math. Soc.* **145**(5) (2017), 1981–1995.
2. D. Bachiller, Classification of braces of order p^3, *J. Pure Appl. Algebra* **219**(8) (2015), 3568–3603.
3. D. Bachiller, Counterexample to a conjecture about braces, *J. Algebra* **453** (2016), 160–176.
4. F. Cedó, E. Jespers and J. Okniński, Braces and the Yang-Baxter equation, *Comm. Math. Phys.* **327**(1) (2014), 101–116.
5. L. N. Childs, Fixed-point free endomorphisms and Hopf Galois structures, *Proc. Amer. Math. Soc.* **141**(4) (2013), 1255–1265.
6. F. Chouraqui and E. Godelle, Finite quotients of groups of I-type, *Adv. Math.* **258** (2014), 46–68.
7. P. Dehornoy, Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs, *Adv. Math.* **282** (2015), 93–127.
8. V. G. Drinfeld, On some unsolved problems in quantum group theory, in *Quantum Groups (Leningrad, 1990)*, Lecture Notes in Mathematics, Volume 1510, pp. 1–8 (Springer-Verlag, Berlin, 1992).

https://doi.org/10.1017/S0013091521000134 Published online by Cambridge University Press

9. P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, *Duke Math. J.* **100** (1999), 169–209.
10. R. Fenn and C. Rourke, Racks and links in codimension two, *J. Knot Theory Ramifications* **1**(4) (1992), 343–406.
11. T. Gateva-Ivanova, Regularity of skew-polynomial rings with binomial relations, in *Talk at the International Algebra Conference*, Miskolc, Hungary, 1996.
12. T. Gateva-Ivanova, Noetherian properties of skew-polynomial rings with binomial relations, *Trans. Amer. Math. Soc.* **343** (1994), 203–219.
13. T. Gateva-Ivanova, Quadratic algebras, Yang-Baxter equation, and Artin-Schelter regularity, *Adv. Math.* **230**(4–6) (2012), 2152–2175.
14. T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups, *Adv. Math.* **338** (2018), 649–701.
15. T. Gateva-Ivanova and P. Cameron, Multipermutation solutions of the Yang-Baxter equation, *Comm. Math. Phys.* **309**(3) (2012), 583–621.
16. T. Gateva-Ivanova and S. Majid, Set-theoretic solutions of the Yang-Baxter equation, graphs and computations, *J. Symbolic Comput.* **42**(11–12) (2007), 1079–1112.
17. T. Gateva-Ivanova and S. Majid, Matched pairs approach to set theoretic solutions of the Yang-Baxter equation, *J. Algebra* **319**(4) (2008), 1462–1529.
18. T. Gateva-Ivanova and S. Majid, Quantum spaces associated to multipermutation solutions of level two, *Algebr. Represent. Theory* **14**(2) (2011), 341–376.
19. T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, *J. Algebra* **206** (1998), 97–112.
20. J. Helmstetter and A. Micali, About the structure of meson algebras, *Adv. Appl. Clifford Algebr.* **20**(3–4) (2010), 617–629.
21. P. Jedlička, A. Pilitowska and A. Zamojska-Dzienio, The construction of multipermutation solutions of the Yang-Baxter equation of level 2, *J. Comb. Theory* **176** (2020), 105295.
22. A. Joyal and R. Street, Braided tensor categories, *Adv. Math.* **102**(1) (1993), 20–78.
23. D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23**(1) (1982), 37–65.
24. N. Kemmer, The algebra of meson matrices, *Proc. Cambridge Philos. Soc.* **39** (1943), 189–196.
25. J.-H. Lu, M. Yan and Y.-C. Zhu, On the set-theoretical Yang-Baxter equation, *Duke Math. J.* **104** (2000), 1–18.
26. A. Micali and M. Rachidi, On meson algebras, *Adv. Appl. Clifford Algebr.* **18**(3–4) (2008), 875–889.
27. A. Pilitowska and A. Romanowska, Reductive modes, *Period. Math. Hungar.* **36**(1) (1998), 67–78.
28. W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, *Adv. Math.* **193** (2005), 40–55.
29. W. Rump, Braces, radical rings, and the quantum Yang-Baxter equation, *J. Algebra* **307**(1) (2007), 153–170.
30. W. Rump, The brace of a classical group, *Note Mat.* **34**(1) (2014), 115–144.
31. W. Rump, Right l-groups, geometric garside groups, and solutions of the quantum Yang-Baxter equation, *J. Algebra* **439** (2015), 470–510.
32. W. Rump, Classification of cyclic braces, II, *Trans. Amer. Math. Soc.* **372**(1) (2019), 305–328.
33. W. Rump, Construction of finite braces, *Ann. Comb.* **23**(2) (2019), 391–416.
34. W. Rump, A covering theory for non-involutive set-theoretic solutions to the Yang-Baxter equation, *J. Algebra* **520** (2019), 136–170.

https://doi.org/10.1017/S0013091521000134 Published online by Cambridge University Press

35. W. Rump, One generator braces and indecomposable set-theoretic solutions to the Yang-Baxter equation, *Proc. Edinb. Math. Soc* **63**(3) (2020), 676–696. https://doi.org/610.1017/S0013091520000073.
36. J. Tate and M. Van den Bergh, Homological properties of Sklyanin Algebras, *Invent. Math.* **124** (1996), 619–647.
37. A. Weinstein and P. Xu, Classical solutions of the quantum Yang-Baxter equation, *Comm. Math. Phys.* **148**(2) (1992), 309–343.

https://doi.org/10.1017/S0013091521000134 Published online by Cambridge University Press