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Cross-field plasma acceleration and potential formation induced by electrostatic waves in a relativistic magnetized plasma

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Abstract. Relativistic and non-relativistic particle acceleration along and across a magnetic field, and the generation of an electric field transverse to the magnetic field, both induced by almost perpendicularly propagating electrostatic waves in a relativistic magnetized plasma, are investigated theoretically on the basis of relativistic quasilinear transport equations. The electrostatic waves accelerate particles via Landau or cyclotron damping, and the ratio of parallel and perpendicular drift velocities $v_{s\parallel}/v_d$ can be proved to be proportional to $k_\parallel/k_\perp$. Simultaneously, an intense cross-field electric field $\mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{v}_\mathrm{d}/c$ is generated via the dynamo effect owing to perpendicular particle drift to satisfy the generalized Ohm's law, which means that this cross-field particle drift is identical to $\mathbf{E} \times \mathbf{B}$ drift. The relativistic quasilinear transport equations for relativistic cross-field particle acceleration are derived by Lorentz transformation of the relativistic quasilinear momentum-space diffusion equation in the moving frame of reference without the electric field and the cross-field particle drift. They can be applied to the investigation of the relativistic perpendicular particle acceleration that may possibly occur in space plasmas.

1. Introduction

Transport of energy and particles across the magnetic flux surface in controlled nuclear fusion experiments is anomalously large, and is thought to be driven by electrostatic turbulence. Recent experimental and theoretical research on cross-field transport in tokamak edge plasmas has shown its potential importance for understanding the fluctuation-induced anomalous transport (Antoni et al. 1997; Wesson et al. 1997). In particular, it has also been shown that perpendicular ion acceleration by localized high-frequency waves is an important aspect of both tokamak plasma experiments (Nemoto et al. 1991; Evance et al. 1992) and space plasma experiments (Kintner et al. 1992; Kintner 1992), where the energy of accelerated ions is much larger than the ambient thermal energy.

In recent years, the author has verified theoretically, on the basis of quasilinear transport equations, that particle acceleration and transport along and across a magnetic field and an electric field transverse to the magnetic field can be induced simultaneously by almost perpendicularly propagating

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electrostatic waves in a magnetized plasma (Sugaya 1997, 1998, 1999). The transport equations are given by the quasilinear velocity-space diffusion equation derived from the Vlasov–Maxwell equations (Sugaya 1991a, b, 1994a, b, 1996; Sugaya et al. 1996). This cross-field transport is thought to be capable of explaining the fluctuation-induced anomalous transport and the electric potential formation occurring in sawtooth crashes or edge plasmas in tokamaks and the perpendicular ion acceleration in tokamaks and space plasmas. Moreover, it has been shown by numerical analysis of the transport equations that the parallel and perpendicular transport of plasma electrons induced by localized electrostatic waves can lead to the formation of a strong cross-field electric field and large inhomogeneities in the electron density and temperature (Sugaya et al. 1998). This process can explain the experimental observation of electron transport in an electron-beam–plasma system (Sugaya et al. 1995; Maehara et al. 1996).

In the present work, it is elucidated theoretically on the basis of relativistic quasilinear transport equations that non-relativistic and relativistic particle acceleration along and across a magnetic field and an electric field transverse to the magnetic field can be generated simultaneously by Landau or cyclotron damping of almost perpendicularly propagating electrostatic waves in a relativistic magnetized plasma. The relativistic quasilinear transport equations are obtained by momentum-space integration of the relativistic quasilinear momentum-space diffusion equation derived from the relativistic Vlasov–Maxwell equations. The electrostatic waves accelerate particles and the ratio of parallel and perpendicular drift velocities $v_{s\parallel}/v_d$ can be proved to be proportional to $k_\parallel/k_\perp$. As a result, strong particle acceleration or transport across the magnetic field appears. Simultaneously, an intense cross-field electric field $\mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{v}_\mathrm{d}/c$ is created via the dynamo effect owing to cross-field particle drift to satisfy the generalized Ohm's law; that is, the electrostatic waves can produce a cross-field particle drift that is identical to $\mathbf{E} \times \mathbf{B}$ drift. These results are the same as those for a non-relativistic magnetized plasma that were obtained previously (Sugaya 1997, 1998, 1999). In particular, the relativistic quasilinear transport equations for relativistic cross-field particle acceleration are derived by means of Lorentz transformation of the relativistic quasilinear momentum-space diffusion equation in the frame of reference moving with the $\mathbf{E} \times \mathbf{B}$ drift velocity relative to the laboratory system. In the moving system, there exists no electric field and no $\mathbf{E} \times \mathbf{B}$ drift. It is clarified that the obtained transport equations can be applied to the theoretical investigation of fluctuation-induced particle acceleration or transport and electric potential formation in fusion plasmas and relativistic plasmas. In particular they are of great use in the study of the relativistic perpendicular acceleration that may possibly occur in space plasmas.

In Sec. 2, non-relativistic cross-field particle acceleration in a relativistic magnetized plasma is investigated. In Sec. 3, relativistic cross-field particle acceleration is studied. Finally, a summary is given in Sec. 4.

2. Non-relativistic cross-field particle acceleration

In this section, we consider non-relativistic cross-field particle acceleration in a relativistic magnetized plasma.

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2.1. Relativistic quasilinear momentum-space diffusion equation

In order to derive the relativistic quasilinear momentum-space diffusion equation showing the temporal development of a system where the electrostatic waves propagate almost perpendicularly in a homogeneous relativistic plasma immersed in uniform magnetic and electric fields, we start from the following relativistic Vlasov–Maxwell equations (Sugaya 1994a, b, 1996):

$$\left[\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{v}+e_s\left(\mathbf{E}_0+\frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial}{\partial\mathbf{p}}\right]f_{s\mathbf{k}}+e_s\mathbf{E}_\mathbf{k}\cdot\frac{\partial g_s}{\partial\mathbf{p}}=-e_s\sum_{\mathbf{k}'\neq 0}\mathbf{E}_{\mathbf{k}-\mathbf{k}'}\cdot\frac{\partial f_{s\mathbf{k}'}}{\partial\mathbf{p}}, \quad (1)$$

$$i\mathbf{k}\cdot\mathbf{E}_\mathbf{k}=4\pi\sum_s n_s e_s\int d^3\mathbf{p}f_{s\mathbf{k}}, \quad (2)$$

$$\left(\mathbf{E}_0+\frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial g_s}{\partial\mathbf{p}}=0, \quad (3)$$

where all quantities have been expanded in Fourier series:

$$\mathbf{E}(\mathbf{r},t)=\sum_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{E}_\mathbf{k}(t), \quad (4)$$

$$f_s(\mathbf{r},\mathbf{p},t)=\sum_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}f_{s\mathbf{k}}(\mathbf{p},t). \quad (5)$$

Here $f_{s\mathbf{k}}$ is the $\mathbf{k}$ component of either the ion or the electron momentum distribution function, the subscript s designates the species, $\mathbf{p}=\mathbf{p}'+\mathbf{p}_{ds}$, $\mathbf{p}=\gamma_s' m_s\mathbf{v}$, $\mathbf{p}'=\gamma_s' m_s\mathbf{v}'$, $\mathbf{p}_{ds}=\gamma_s' m_s\mathbf{v}_d$, m_s is the rest mass, and $\gamma_s'=(1-v'^2/c^2)^{-1/2}=(1+p'^2/m_s^2c^2)^{1/2}$. The remaining notation is standard. $g_s(\mathbf{p},t)=f_{s\mathbf{k}=0}(\mathbf{p},t)$ is the background momentum distribution function, which is assumed to vary slowly in time and is governed by the momentum-space diffusion equation. The $\mathbf{k}=0$ terms of $\mathbf{E}$ and $\mathbf{B}_0$ express the background uniform stationary electric and magnetic fields $\mathbf{E}_0=\mathbf{E}_{\mathbf{k}=0}=(0,E_0,0)$ and $\mathbf{B}_0=(0,0,B_0)$, and are assumed to be in the y and z directions respectively. The background momentum distribution function satisfies the unperturbed relativistic Vlasov equation (3), and includes the fluctuation-induced cross-field drift $v_d=\int d^3\mathbf{p}\,v_x g_s$, where $\mathbf{v}_d=c\mathbf{E}_0\times\mathbf{B}_0/B_0^2=(v_d,0,0)$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity. Here a non-relativistic drift velocity ($v_\mathrm{d}/c\ll 1$) is assumed. Accordingly the approximate expression for γ_s' is used because γ_s' *should be defined exactly as* $\gamma_s'=(1-v^2/c^2)^{-1/2}=(1+p^2/m_s^2c^2)^{1/2}$. Further, it is assumed that

$$\mathbf{E}_\mathbf{k}(t)\propto\exp(-i\omega_\mathbf{k}t),\qquad \omega_{-\mathbf{k}}=-\omega_\mathbf{k}^*,\qquad \mathbf{E}_{-\mathbf{k}}=\mathbf{E}_\mathbf{k}^*.$$

An asterisk indicates the complex conjugate, and $\mathbf{k}=(k_\perp,0,k_\parallel)$ is in the (x,z) plane. Here $f_{s\mathbf{k}}$ can be obtained by orbital integration of (1) along the zeroth-order particle orbit, which is the relativistic Larmor motion, on which the $\mathbf{E}\times\mathbf{B}$ drift is superimposed.

The solution of (3) is given by

$$\begin{aligned} g_s&=a_s\sum_{m=0}^{\infty}\frac{1}{m!}p_x^m\left(-\frac{p_{ds}}{p_\perp}\frac{\partial}{\partial p_\perp}\right)^m g_{s0}(p_\perp,p_\parallel,t)\\ &=a_s g_{s0}((p_\perp^2-2p_x p_{ds})^{1/2},p_\parallel,t), \end{aligned} \quad (6)$$

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with normalization constant a_s, which is determined such that $\int d^3\mathbf{p}\, g_s = \int d^3\mathbf{p}\, g_{s0} = 1$. The momentum distribution function g_s is expressed in displaced cylindrical coordinates in momentum space ($p_x = p_s' + p_{ds} = p_\perp \cos\theta + p_{ds}$, $p_y = p_y' = p_\perp \sin\theta$, $p_z = p_z' = p_\parallel$). The generalized Ohm's law for a collisionless plasma,

$$\mathbf{E}_0 + \frac{1}{c}\mathbf{v}_d \times \mathbf{B}_0 = 0, \tag{7}$$

is provided by the momentum-space integration of (3) multiplied by $\mathbf{p}$. This implies that $\mathbf{v}_d$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity. At the same time, as will be interpreted later, it can be stated that the cross-field electric field $\mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{v}_d/c$ is produced by the dynamo effect of the cross-field particle drift arising from the acceleration due to electrostatic waves. When $\mathbf{v}_d = 0, g_s$ is reduced to g_{s0}, being symmetric with respect to the magnetic field.

The dielectric constant for electrostatic waves propagating obliquely to the magnetic field can be derived from (1) and (2), and is given by

$$\varepsilon_{\mathbf{k}} = \mathbf{1} + \sum_s \varepsilon_{\mathbf{k}}^{(s)}, \tag{8}$$

$$\varepsilon_{\mathbf{k}}^{(s)} = -\frac{\omega_{ps}^2 m_s}{k^2} \sum_{r=-\infty}^{\infty} \int d^3\mathbf{p}\, \frac{J_r^2(\mu_{\mathbf{k}}) U_r(\mathbf{k}) g_{s0}}{k_\parallel v_\parallel + k_\perp v_{\mathrm{d}} - \omega_{\mathbf{k}} + r\omega_{cs}}, \tag{9}$$

$$U_r(\mathbf{k}) = k_\parallel \frac{\partial}{\partial p_\parallel} + \frac{r\omega_{cs}}{v_\perp}\frac{\partial}{\partial p_\perp}, \tag{10}$$

where J_r is the Bessel function of rth order,

$$\mu_{\mathbf{k}} = \frac{k_\perp v_\perp}{\omega_{cs}}, \qquad \omega_{ps}^2 = \frac{4\pi n_s e_s^2}{m_s}, \qquad \omega_{cs} = \frac{|e_s| B_0}{\gamma_s' m_s c},$$

$$p_\perp = \gamma_s' m_s v_\perp, \qquad p_\parallel = \gamma_s' m_s v_\parallel.$$

The momentum-space integration in (9) is performed in cylindrical coordinates $\mathbf{p} = (p_\perp, \theta, p_\parallel)$, which are displaced by p_{ds} in the x direction. Consequently, g_{s0} appears in the integrand. The term $k_\perp v_d$ represents the Doppler shift due to cross-field particle drift. The dielectric constant (8) with $v_d = 0$ is strictly identical to the well-known dielectric constant for obliquely propagating electrostatic Bernstein waves (see e.g. Hasegawa 1975). These waves with or without the cross-field particle drift undergo strong damping when $k_\parallel/k_\perp$ increases, because their Landau and cyclotron damping becomes stronger with increasing $k_\parallel$. For almost-perpendicular propagation with weak damping ($k_\parallel/k_\perp \ll 1$), large-amplitude electrostatic waves can propagate easily to generate strong particle acceleration. On the other hand, for strictly perpendicular propagation ($k_\parallel = 0$), the wave damping vanishes owing to the infinite parallel velocity of resonant particles causing Landau or cyclotron damping ($v_{\parallel r} = (\omega_{\mathbf{k}} - k_\perp v_d - r\omega_{cs})/k_\parallel \to \pm\infty$). Accordingly, strictly perpendicularly propagating electrostatic waves cannot accelerate particles, because no resonant particles exist.

We can derive the following relativistic quasilinear momentum-space

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diffusion equation from (1) in the same manner as in the previous work (Sugaya 1991a, b, 1994a, b, 1996):

$$\frac{\partial g_s}{\partial t} = \sum_{k \neq 0} |E_{\mathbf{k}}|^2 Q_{\mathbf{k}} g_{s0}, \tag{11}$$

with the θ-dependent momentum-space diffusion coefficient

$$Q_{\mathbf{k}} = \mathrm{Im}\left[\frac{e_s^2}{k^2} \sum_{n,r=-\infty}^{\infty} Y_{n,r}(\mathbf{k}) \frac{e^{-i(r-n)\theta} J_r(\mu_{\mathbf{k}})\, U_r(\mathbf{k})}{k_\parallel v_\parallel + k_\perp v_d - \omega_{\mathbf{k}} + r\omega_{cs}}\right], \tag{12}$$

$$Y_{n,r}(\mathbf{k}) = U_n(\mathbf{k})\, J_n(\mu_{\mathbf{k}}) + \frac{k_\perp}{p_\perp}(r-n)\, J'_n(\mu_{\mathbf{k}}), \tag{13}$$

where $Q_{\mathbf{k}}$ is expressed in displaced cylindrical coordinates in momentum space (thereby g_{s0} appears on the right-hand of (11)) and J'_n is the first derivative of J_n with respect to the argument. The azimuthal dependence of $Q_{\mathbf{k}}$ represents the anisotropy of the momentum-space diffusion around the magnetic field and the resulting cross-field particle acceleration or transport in the x direction. In the velocity-space diffusion equation obtained previously (Kennel and Engelmann 1966), the azimuthal dependence disappears because of the assumption of symmetry of the unperturbed velocity distribution function around the magnetic field ($\partial g_s/\partial\theta = 0$). In the present case, the azimuthal dependence ($\partial g_s/\partial\theta \neq 0$) must be retained, since the unperturbed momentum distribution function includes the cross-field particle drift. The quasilinear diffusion coefficient $Q_{\mathbf{k}}$ becomes the familiar delta-function-like diffusion coefficient on setting

$$\mathrm{Im}\left[\frac{1}{k_\parallel v_\parallel + k_\perp v_d - \omega_{\mathbf{k}} + r\omega_{cs}}\right] = \pi\delta(k_\parallel v_\parallel + k_\perp v_d - \omega_{\mathbf{k}} + r\omega_{cs}). \tag{14}$$

2.2. Relativistic quasilinear transport equations

The momentum-space integration of the relativistic quasilinear momentum-space diffusion equation multiplied by $w_s = n_s \gamma'_s (1 + v_d v'_x/c^2) m_s c^2$ or $\mathbf{p}_s = n_s \mathbf{p}$ leads to the relativistic quasilinear transport equations, since the following simple relations hold:

$$\int d^3\mathbf{p}\, w_s Q_{\mathbf{k}} g_{s0} = \frac{\omega_{\mathbf{k}}}{4\pi} \mathrm{Im}\, \varepsilon^{(s)}_{\mathbf{k}}. \tag{15}$$

$$\int d^3\mathbf{p}\, \mathbf{p}_s Q_{\mathbf{k}} g_{s0} = \frac{\mathbf{k}}{4\pi} \mathrm{Im}\, \varepsilon^{(s)}_{\mathbf{k}} \tag{16}$$

Thus the relativistic quasilinear transport equations showing the temporal development of the energy and momentum densities of magnetized particles of species s are derived as

$$\frac{\partial U_s}{\partial t} = -\sum_{\mathbf{k} \neq 0} 2\gamma^{(s)}_{\mathbf{k}} U_{\mathbf{k}}, \tag{17}$$

$$\frac{\partial \mathbf{P}_s}{\partial t} = -\sum_{\mathbf{k} \neq 0} \frac{2\gamma^{(s)}_{\mathbf{k}} \mathbf{k}}{\omega_{\mathbf{k}}} U_{\mathbf{k}}, \tag{18}$$

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where

$$U_{\mathbf{k}} = \frac{1}{8\pi}\frac{\partial(\varepsilon'_{\mathbf{k}}\omega_{\mathbf{k}})}{\partial\omega_{\mathbf{k}}}|E_{\mathbf{k}}|^2$$

is the wave energy density, $\mathbf{k}U_{\mathbf{k}}/\omega_{\mathbf{k}}$ is the wave momentum density,

$$U_s = \int d^3\mathbf{p}\, w_s g_s, \qquad \mathbf{P}_s = \int d^3\mathbf{p}\, \mathbf{p}_s g_s$$

are the energy and momentum densities of particles of species s,

$$\gamma_{\mathbf{k}}^{(s)} = -\frac{\varepsilon''^{(s)}_{\mathbf{k}}}{\partial\varepsilon'_{\mathbf{k}}/\partial\omega_{\mathbf{k}}} \qquad (\gamma_{\mathbf{k}} = \sum_s \gamma_{\mathbf{k}}^{(s)}, \quad \varepsilon_{\mathbf{k}} = \varepsilon'_{\mathbf{k}} + i\varepsilon''_{\mathbf{k}}, \quad \varepsilon_{\mathbf{k}}^{(s)} = \varepsilon'^{(s)}_{\mathbf{k}} + i\varepsilon''^{(s)}_{\mathbf{k}})$$

is the linear damping rate ascribed to particles of species s, and

$$\mathbf{P}_s = (P_{s\perp}, 0, P_{s\parallel}), \qquad P_{s\perp} = n_s p_{ds}, \qquad P_{s\parallel} = n_s \gamma'_s m_s v_{s\parallel}.$$

The transport equations (17) and (18) predict clearly that the electrostatic waves can generate strong particle acceleration or transport along and across the magnetic field. The particle acceleration results from Landau or cyclotron damping due to quasilinear wave–particle interaction of the electrostatic waves. Thus electrostatic waves propagating almost perpendicularly or obliquely can cause particle acceleration ($k_\parallel, \gamma_{\mathbf{k}}^{(s)} \neq 0$). In contrast strictly perpendicularly propagating electrostatic waves create no particle acceleration ($k_\parallel = \gamma_{\mathbf{k}}^{(s)} = 0$). The relation $P_{s\parallel}/P_{s\perp} = k_\parallel/k_\perp$ ($\mathbf{P}_s \parallel \mathbf{k}$) can be seen from (18) with $\mathbf{P}_s(0) = 0$, and implies that small parallel and large perpendicular particle acceleration appear simultaneously. That is, the same result is obtained as in the nonrelativistic case (Sugaya 1997, 1998, 1999).

The physical mechanism for generation of the cross-field electric field $\mathbf{E}_0$ can be stated as follows. The Lorentz force $e_s\mathbf{v}_d \times \mathbf{B}_0/c$ due to cross-field particle drift or acceleration causes electrical charge separation and an electric field $\mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{v}_d/c$ that is balanced with the Lorentz force. We can easily understand the physical mechanism of the generation of $\mathbf{E}_0$ when a magnetized plasma of finite size in the y direction is considered. Accumulation of electrical charge at both ends of the plasma in the y direction is caused by the Lorentz force $e_s\mathbf{v}_d \times \mathbf{B}_0/c$, and hence a resulting $\mathbf{E}_0$ is created. This charge accumulation continues until the electrical force is balanced with the Lorentz force. This means that a cross-field electric field is produced by the dynamo effect of the cross-field particle drift or acceleration due to electrostatic waves. When another cross-field electric field already exists, there is initially an $\mathbf{E} \times \mathbf{B}$ drift due to this electric field ($\mathbf{P}_s(0) \neq 0$). Then the cross-field electric field resulting from the dynamo effect of the further particle acceleration due to electrostatic waves is superimposed on the initial electric field. In a collisionless plasma, the created cross-field electric field is determined by the generalized Ohm's law (7) for a collisionless plasma with infinite electrical conductivity. In contrast, in a collisional plasma with finite electrical conductivity, the cross-field electric field is governed by the corresponding generalized Ohm's law. In realistic situations, the cross-field electric field is determined by the magnetohydrodynamic equations, including the cross-field particle drift given by (17) and (18). Thus the real problem becomes perhaps more difficult and complicated to deal with. The cross-field electric field caused by the dynamo effect most likely creates the other electric field to satisfy Poisson's equation. As a result, this excess electric

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field gives another $\mathbf{E}\times\mathbf{B}$ drift. Accordingly, the fluctuation-induced particle drift and the $\mathbf{E}\times\mathbf{B}$ drift due to the excessively generated electric field coexist. Generally speaking, the cross-field drift velocity does not need to be in the x direction (i.e. the $k_\perp$ direction), and hence a cross-field electric field can exist in both x and y directions.

In the absence of nonlinear wave–wave and wave–particle interactions, the kinetic wave equation for the electrostatic waves is

$$\frac{\partial U_{\mathbf{k}}}{\partial t} = 2\gamma_{\mathbf{k}} U_{\mathbf{k}}. \tag{19}$$

From (17)–(19), as has been obtained similarly in the non-relativistic case (Sugaya 1997, 1998, 1999), the conservation laws for the total energy and momentum densities of waves and particles are given by

$$\frac{\partial}{\partial t}\left(\sum_{\mathbf{k}} U_{\mathbf{k}} + \sum_{s} U_s\right) = 0 \tag{20}$$

$$\frac{\partial}{\partial t}\left(\sum_{\mathbf{k}} \frac{\mathbf{k}}{\omega_{\mathbf{k}}} U_{\mathbf{k}} + \sum_{s} \mathbf{P}_s\right) = 0 \tag{21}$$

It can also be verified by adding the appropriate nonlinear terms to (17)–(19) that these conservation laws hold in the presence of nonlinear wave–wave and wave–particle interactions.

3. Relativistic cross-field particle acceleration

In the next stage, we investigate relativistic cross-field particle acceleration in a relativistic magnetized plasma. When the $\mathbf{E}\times\mathbf{B}$ drift is relativistic, the Larmor orbit with $\mathbf{E}\times\mathbf{B}$ drift is considerably complicated. Accordingly, orbital integration of the relativistic Vlasov equation along the zeroth-order particle orbit is very difficult (indeed virtually impossible!). Then it is tremendously difficult to obtain the relativistic quasilinear transport equations directly from the relativistic Vlasov–Maxwell equations with relativistic $\mathbf{E}\times\mathbf{B}$ drift. However, it becomes easy and simple to derive them when the problem is treated in the frame of reference that moves with the $\mathbf{E}\times\mathbf{B}$ drift velocity and hence contains no electric field and no $\mathbf{E}\times\mathbf{B}$ drift. In the moving frame of reference, the relativistic quasilinear momentum-space diffusion equation and the relativistic quasilinear transport equations can be found directly from the results obtained in the previous sections. Thus the transport equations in the laboratory (stationary) frame of reference with electric field and $\mathbf{E}\times\mathbf{B}$ drift can be obtained by means of Lorentz transformation for the momentum-space integration of the relativistic quasilinear momentum-space diffusion equation in the moving frame of reference.

In Sec. 3.1, we give a detailed description of the Lorentz transformation between the two frames of reference of the stationary magnetic and electric fields, the equations of motion, and the frequencies and wavenumbers. In Sec. 3.2, we describe the relativistic Vlasov–Maxwell equations in the moving system and the relations between the wave electric and magnetic fields in the two systems. The relativistic quasilinear momentum-space diffusion equation

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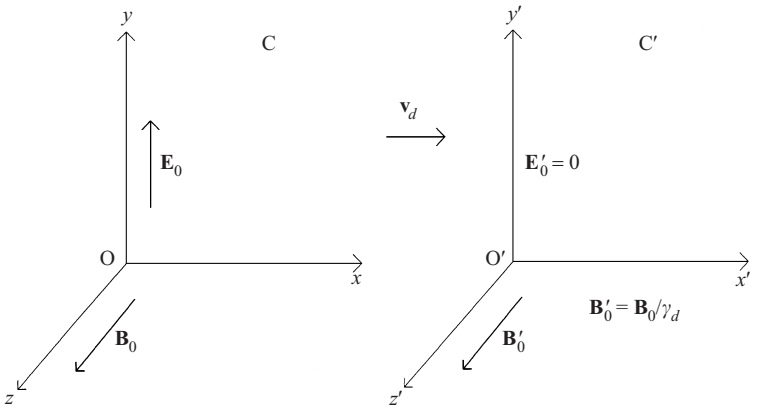


Figure 1. Laboratory coordinate system C and moving coordinate system C′ with $\mathbf{E}\times\mathbf{B}$ drift velocity $\mathbf{v}_d$.

in the moving system, coming from the results in the previous section, is presented in order to derive the relativistic quasilinear transport equations in the laboratory system in Sec. 3.4. It is assumed that the electrostatic waves propagate in the moving system, in order to utilize the results of the previous section. As a result, the electrostatic waves in the moving system become electromagnetic in the laboratory system, as is proved by Lorentz transformation. In Sec. 3.3, we describe the relativistic quasilinear transport equations in the moving system found from the previous section, and explain their meanings. Finally, in Sec. 3.4, we present the relativistic quasilinear transport equations in the laboratory system, which are derived by means of Lorentz transformation for the momentum-space integration of the relativistic quasilinear momentum-space diffusion equation in the moving system.

3.1. Lorentz transformation

In order to investigate relativistic cross-field particle acceleration, we introduce two frames of reference C and C′ as shown in Fig. 1. The frame of reference C is the laboratory system, in which the electromagnetic waves propagate almost perpendicularly in a homogeneous relativistic plasma immersed in a uniform magnetic field $\mathbf{B}_0$ and electric field $\mathbf{E}_0$. The frame of reference C′, in which the electrostatic waves propagate almost perpendicularly in a homogeneous or weakly inhomogeneous relativistic magnetized plasma, is the system moving with velocity $\mathbf{v}_d$ relative to C. Here $\mathbf{v}_d = c\mathbf{E}_0\times\mathbf{B}_0/B_0^2$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity in C, and the electrostatic waves in C′ are shown to be electromagnetic waves in C, as will be described later. The stationary magnetic and electric fields $\mathbf{B}'_0$ and $\mathbf{E}'_0$ in C′ are found by means of Lorentz transformation, and are given as follows:

$$\mathbf{E}'_0 = (1-\gamma_d)(\mathbf{E}_0\cdot\mathbf{v}_d)\frac{\mathbf{v}_d}{v_d^2} + \gamma_d\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}_d\times\mathbf{B}_0\right) = 0, \tag{22}$$

$$\mathbf{B}'_0 = (1-\gamma_d)(\mathbf{B}_0\cdot\mathbf{v}_d)\frac{\mathbf{v}_d}{v_d^2} + \gamma_d\left(\mathbf{B}_0 - \frac{1}{c}\mathbf{v}_d\times\mathbf{E}_0\right) = \frac{\mathbf{B}_0}{\gamma_d}, \tag{23}$$

where $\gamma_d = (1-\beta^2)^{-1/2}$ and $\beta = v_d/c$. In the moving system C′, the stationary

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electric field vanishes and hence no $\mathbf{E}\times\mathbf{B}$ drift exists. The transformation equations connecting the two systems C and C′ are expressed as

$$\mathbf{X}' = \mathbf{S}\cdot\mathbf{X}, \qquad \mathbf{P}' = \mathbf{S}\cdot\mathbf{P}, \tag{24}$$

$$\mathbf{S} = \begin{pmatrix} \gamma_d & 0 & 0 & i\beta\gamma_d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma_d & 0 & 0 & \gamma_d \end{pmatrix}, \tag{25}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}, \qquad \mathbf{X}' = \begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix}, \tag{26}$$

$$\mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ i\gamma_s m_s c \end{pmatrix}, \qquad \mathbf{P}' = \begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ i\gamma'_s m_s c \end{pmatrix}, \tag{26}$$

where

$$\mathbf{p} = \gamma_s m_s \mathbf{v}, \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt}, \qquad \gamma_s = \left(1+\frac{p^2}{m_s^2 c^2}\right)^{1/2} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

are defined in C and

$$\mathbf{p}' = \gamma'_s m_s \mathbf{v}', \qquad \mathbf{v}' = \frac{d\mathbf{r}'}{dt'}, \qquad \gamma'_s = \left(1+\frac{p'^2}{m_s^2 c^2}\right)^{1/2} = \left(1-\frac{v'^2}{c^2}\right)^{-1/2}$$

in C′.

Thus the particle orbits in the stationary magnetic and electric fields in C and C′ are determined by the following two equations of motion:

$$\frac{d\mathbf{p}}{dt} = e_s\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right), \tag{27}$$

$$\frac{d\mathbf{p}'}{dt'} = \frac{e_s}{c}\mathbf{v}'\times\mathbf{B}'_0. \tag{28}$$

The kinetic equation (27) in C is quite difficult to solve, whereas the kinetic equation (28) in C′ can be solved easily, with solutions given by

$$\left.\begin{aligned} x' &= \frac{v'_\perp}{\omega'_{cs}}\sin(\omega'_{cs}t'+\theta') + x'_0, \qquad y' = \frac{\sigma_s v'_\perp}{\omega'_{cs}}\cos(\omega'_{cs}t'+\theta') + y'_0, \\ z' &= v'_\parallel t' + z'_0, \qquad \sigma_s = \frac{e_s}{|e_s|}, \end{aligned}\right\} \tag{29}$$

$$\omega'_{cs} = \frac{|e_s|B'_0}{\gamma'_s m_s c} = \frac{|e_s|B_0}{\gamma_d\gamma'_s m_s c} = \frac{\omega_{cs}}{\gamma_d}. \tag{30}$$

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The particle orbit in C is obtained easily by Lorentz transformation of $\mathbf{X} = \mathbf{S}^{*}\cdot\mathbf{X}'$ (where $\mathbf{S}^{*} = \mathbf{S}^{-1}$ is the complex conjugate of $\mathbf{S}$). We find that the relativistic Vlasov equation in C′ can be solved easily by means of orbital integration along (29).

Further, the relation between the frequencies and the wavenumbers in C and C′ is represented similarly by the following transformation equations:

$$\tilde{\mathbf{K}} = \mathbf{S}\cdot\mathbf{K},$$

$$\mathbf{K} = \begin{pmatrix} k_{\perp} \\ 0 \\ k_{\parallel} \\ i\omega_{\mathbf{k}}/c \end{pmatrix}, \qquad \tilde{\mathbf{K}} = \begin{pmatrix} \tilde{k}_{\perp} \\ 0 \\ \tilde{k}_{\parallel} \\ i\tilde{\omega}_{\tilde{\mathbf{k}}}/c \end{pmatrix}. \tag{31}$$

That is, $\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t = \tilde{\mathbf{k}}\cdot\mathbf{r}' - \tilde{\omega}_{\tilde{\mathbf{k}}} t'$ holds, and

$$\tilde{k}_{\perp} = \gamma_d\left(k_{\perp} - \frac{\beta\omega_{\mathbf{k}}}{c}\right), \qquad \tilde{k}_{\parallel} = k_{\parallel}, \qquad \tilde{\omega}_{\tilde{\mathbf{k}}} = \gamma_d(\omega_{\mathbf{k}} - k_{\perp} v_d);$$

or

$$k_{\perp} = \gamma_d\left(\tilde{k}_{\perp} + \frac{\beta\tilde{\omega}_{\tilde{\mathbf{k}}}}{c}\right), \qquad k_{\parallel} = \tilde{k}_{\parallel}, \qquad \omega_{\mathbf{k}} = \gamma_d(\tilde{\omega}_{\tilde{\mathbf{k}}} + \tilde{k}_{\perp} v_d).$$

3.2. Relativistic quasilinear momentum-space diffusion equation in C′

We describe the relativistic quasilinear momentum-space diffusion equation in C′ from which the relativistic quasilinear transport equations in C are derived in Sec. 3.4. The basic equations describing the temporal evolution of the system in C′ are the following relativistic Vlasov–Maxwell equations for electrostatic waves propagating in a uniform relativistic magnetized plasma:

$$\left(\frac{\partial}{\partial t'} + i\tilde{\mathbf{k}}\cdot\mathbf{v}' + \frac{e_s}{c}\mathbf{v}'\times\mathbf{B}_0'\cdot\frac{\partial}{\partial\mathbf{p}'}\right) f_{s\tilde{\mathbf{k}}} + e_s\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}\cdot\frac{\partial g_{s0}}{\partial\mathbf{p}'} = -e_s\sum_{\tilde{\mathbf{k}}'\neq 0}\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}-\tilde{\mathbf{k}}'}\cdot\frac{\partial f_{s\tilde{\mathbf{k}}'}}{\partial\mathbf{p}'}, \tag{32}$$

$$i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}} = 4\pi\sum_s n_s e_s\int d^3\mathbf{p}' f_{s\tilde{\mathbf{k}}}, \tag{33}$$

$$\frac{1}{c}(\mathbf{v}'\times\mathbf{B}_0')\cdot\frac{\partial g_{s0}}{\partial\mathbf{p}'} = 0, \tag{34}$$

where all quantities have been expanded in Fourier series in C′ as follows:

$$\tilde{\mathbf{E}}(\mathbf{r}', t') = \sum_{\tilde{\mathbf{k}}} e^{i\tilde{\mathbf{k}}\cdot\mathbf{r}'}\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}(t'), \tag{35}$$

$$f_s(\mathbf{r}', \mathbf{p}', t') = \sum_{\tilde{\mathbf{k}}} e^{i\tilde{\mathbf{k}}\cdot\mathbf{r}'} f_{s\tilde{\mathbf{k}}}(\mathbf{p}', t'). \tag{36}$$

Here $f_{s\tilde{\mathbf{k}}}$ has the same meaning as in (5), and $g_{s0}(\mathbf{p}', t') = f_{s\tilde{\mathbf{k}}=0}(\mathbf{p}', t')$ is the background momentum distribution function, which varies slowly in time. It is found from (34) that $g_{s0}(\mathbf{p}', t') = g_{s0}(p_{\perp}', p_{\parallel}', t')$ ($p_{x'}' = p_{\perp}'\cos\theta'$, $p_{y'}' = p_{\perp}'\sin\theta'$, $p_{z'}' = p_{\parallel}'$) is symmetric with respect to the magnetic field. Further, it is also assumed that

$$\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}(t') \propto \exp(-i\tilde{\omega}_{\tilde{\mathbf{k}}} t'), \qquad \tilde{\omega}_{-\tilde{\mathbf{k}}} = -\tilde{\omega}_{\tilde{\mathbf{k}}}^{*}, \qquad \tilde{\mathbf{E}}_{-\tilde{\mathbf{k}}} = \tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}^{*}.$$

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The wave electric field and magnetic field $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$ in C are found using the Lorentz transformation. The relation between $\mathbf{E}_{\mathbf{k}}$, $\mathbf{B}_{\mathbf{k}}$ and $\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}$, $\tilde{\mathbf{B}}_{\tilde{\mathbf{k}}}$ is given by

$$\mathbf{E}_{\mathbf{k}} = (1-\gamma_d)\,(\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}\cdot\mathbf{v}_d)\frac{\mathbf{v}_d}{v_d^2}+\gamma_d\left(\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}-\frac{1}{c}\mathbf{v}_d\times\tilde{\mathbf{B}}_{\tilde{\mathbf{k}}}\right), \tag{37}$$

$$\mathbf{B}_{\mathbf{k}} = (1-\gamma_d)\,(\tilde{\mathbf{B}}_{\tilde{\mathbf{k}}}\cdot\mathbf{v}_d)\frac{\mathbf{v}_d}{v_d^2}+\gamma_d\left(\mathbf{B}_{\tilde{\mathbf{k}}}+\frac{1}{c}\mathbf{v}_d\times\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}\right). \tag{38}$$

Taking $\tilde{\mathbf{B}}_{\tilde{\mathbf{k}}}=0$ into account, we get $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$ as follows:

$$\mathbf{E}_{\mathbf{k}} = (\tilde{E}^{x'}_{\tilde{\mathbf{k}}}, 0, \gamma_d\,\tilde{E}^{z'}_{\tilde{\mathbf{k}}}), \tag{39}$$

$$\mathbf{B}_{\mathbf{k}} = (0, -\beta\gamma_d\,\tilde{E}^{z'}_{\tilde{\mathbf{k}}}, 0). \tag{40}$$

The above equations demonstrate that electrostatic waves in C′ become electromagnetic waves in C. It can easily be confirmed that $\mathbf{E}_{\mathbf{k}}$, $\mathbf{B}_{\mathbf{k}}$ and $\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}$ satisfy the Maxwell equations in C and C′ respectively; that is,

$$\boldsymbol{\nabla}\times\mathbf{E}_{\mathbf{k}} = -\frac{\partial\mathbf{B}_{\mathbf{k}}}{\partial t}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B}_{\mathbf{k}} = 0,$$

$$\boldsymbol{\nabla}'\times\tilde{\mathbf{E}}_{\tilde{\mathbf{k}}} = 0,$$

where

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right), \qquad \boldsymbol{\nabla}' = \left(\frac{\partial}{\partial x'},\frac{\partial}{\partial y'},\frac{\partial}{\partial z'}\right).$$

The relativistic Maxwell equations (32)–(34) are identical to (1)–(3) without the cross-field particle drift. Consequently, we can employ (8)–(10) from Sec. 2; that is, it is only necessary to set $\mathbf{v}_d=0$, and replace $\mathbf{p}$, $\mathbf{v}$, $\omega_{\mathbf{k}}$, $\mathbf{k}$ and ω_{cs} by $\mathbf{p}'$, $\mathbf{v}'$, $\tilde{\omega}_{\tilde{\mathbf{k}}}$, $\tilde{\mathbf{k}}$ and ω'_{cs}, respectively. Thus the dielectric constant for the obliquely propagating electrostatic waves in C′ is shown to be given by

$$\tilde{\varepsilon}_{\tilde{\mathbf{k}}} = 1+\sum_s \tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}}, \tag{41}$$

$$\tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}} = -\frac{\omega_{ps}^2\,m_s}{\tilde{k}^2}\sum_{r=-\infty}^{\infty}\int d^3\mathbf{p}'\,\frac{J_r^2(\tilde{\mu}_{\tilde{\mathbf{k}}})\,\tilde{U}_r(\tilde{\mathbf{k}})\,g_{s0}}{\tilde{k}_{\parallel}\,v'_{\parallel}-\tilde{\omega}_{\tilde{\mathbf{k}}}+r\omega'_{\mathrm{cs}}}, \tag{42}$$

$$\tilde{U}_r(\tilde{\mathbf{k}}) = \tilde{k}_{\parallel}\frac{\partial}{\partial p'_{\parallel}}+\frac{r\omega'_{cs}}{v'_{\perp}}\frac{\partial}{\partial p'_{\perp}}, \tag{43}$$

where

$$\tilde{\mu}_{\tilde{\mathbf{k}}} = \frac{\tilde{k}_{\perp}\,v'_{\perp}}{\omega'_{cs}}, \qquad p'_{\perp}=\gamma'_s\,m_s\,v'_{\perp}, \qquad p'_{\parallel}=\gamma'_s\,m_s\,v'_{\parallel}, \qquad d^3\mathbf{p}'=p'_{\perp}\,dp'_{\perp}\,d\theta'\,dp'_{\parallel}.$$

Next, the relativistic quasilinear momentum-space diffusion equation can be obtained similarly from (32)–(34), and is given by

$$\frac{\partial g_{s0}}{\partial t'} = \sum_{\tilde{\mathbf{k}}\neq 0}|\tilde{E}_{\tilde{\mathbf{k}}}|^2\tilde{Q}_{\tilde{\mathbf{k}}}\,g_{s0}, \tag{44}$$

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with the θ'-dependent momentum-space diffusion coefficient

$$\tilde{Q}_{\tilde{\mathbf{k}}} = \mathrm{Im}\left[\frac{e_s^2}{\tilde{k}^2}\sum_{n,r=-\infty}^{\infty}\tilde{Y}_{n,r}(\tilde{\mathbf{k}})\frac{e^{-i(r-n)\theta'}J_r(\tilde{\mu}_{\tilde{\mathbf{k}}})\tilde{U}_r(\tilde{\mathbf{k}})}{\tilde{k}_\parallel v'_\parallel - \tilde{\omega}_{\tilde{\mathbf{k}}} + r\omega'_{cs}}\right], \tag{45}$$

$$\tilde{Y}_{n,r}(\tilde{\mathbf{k}}) = \tilde{U}_n(\tilde{\mathbf{k}})J_n(\tilde{\mu}_{\tilde{\mathbf{k}}}) + \frac{k'_\perp}{p'_\perp}(r-n)\,J'_n(\tilde{\mu}_{\tilde{\mathbf{k}}}). \tag{46}$$

From the momentum-space diffusion equation, it can be seen that there is infinitesimally anisotropic momentum-space diffusion around the magnetic field and a resulting infinitesimal cross-field particle acceleration or transport. That is, g_{s0} evolves temporally toward the infinitesimally anisotropic momentum distribution function.

3.3. Relativistic quasilinear transport equations in C′

A similar procedure as in Sec. 2.2 also gives the relativistic quasilinear transport equations in C′, because of the following simple relations:

$$\int d^3\mathbf{p}'\, w'_s\, \tilde{Q}_{\tilde{\mathbf{k}}}\, g_{s0} = \frac{\tilde{\omega}_{\tilde{\mathbf{k}}}}{4\pi}\,\mathrm{Im}[\tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}}], \tag{47}$$

$$\int d^3\mathbf{p}'\, \mathbf{p}'_s\, \tilde{Q}_{\tilde{\mathbf{k}}}\, g_{s0} = \frac{\tilde{\mathbf{k}}}{4\pi}\,\mathrm{Im}[\tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}}]. \tag{48}$$

Here $w'_s = n_s\gamma'_s m_s c^2$ and $\mathbf{p}'_s = n_s\mathbf{p}'$ are defined in C′. Thus the relativistic quasilinear transport equations in C′ can be derived from (44):

$$\frac{\partial \tilde{U}_s}{\partial t'} = -\sum_{\tilde{\mathbf{k}}\neq 0} 2\tilde{\gamma}^{(s)}_{\tilde{\mathbf{k}}}\,\tilde{U}_{\tilde{\mathbf{k}}}, \tag{49}$$

$$\frac{\partial \tilde{\mathbf{P}}_s}{\partial t'} = -\sum_{\tilde{\mathbf{k}}\neq 0} \frac{2\tilde{\gamma}^{(s)}_{\tilde{\mathbf{k}}}\,\tilde{\mathbf{k}}}{\tilde{\omega}_{\tilde{\mathbf{k}}}}\tilde{U}_{\tilde{\mathbf{k}}}, \tag{50}$$

where

$$\tilde{U}_{\tilde{\mathbf{k}}} = \frac{1}{8\pi}\frac{\partial(\tilde{\varepsilon}'_{\tilde{\mathbf{k}}}\tilde{\omega}_{\tilde{\mathbf{k}}})}{\partial\tilde{\omega}_{\tilde{\mathbf{k}}}}|\tilde{E}_{\tilde{\mathbf{k}}}|^2$$

is the wave energy density, $\tilde{\mathbf{k}}\tilde{U}_{\tilde{\mathbf{k}}}/\tilde{\omega}_{\tilde{\mathbf{k}}}$ is the wave momentum density,

$$\tilde{U}_s = \int d^3\mathbf{p}'\, w'_s\, g_{s0}, \qquad \tilde{\mathbf{P}}_s = \int d^3\mathbf{p}'\, \mathbf{p}'_s\, g_{s0}$$

are the energy and momentum densities of particles of species s,

$$\tilde{\gamma}^{(s)}_{\tilde{\mathbf{k}}} = -\frac{\tilde{\varepsilon}''^{(s)}_{\tilde{\mathbf{k}}}}{\partial\tilde{\varepsilon}'_{\tilde{\mathbf{k}}}/\partial\tilde{\omega}_{\tilde{\mathbf{k}}}} \qquad (\tilde{\gamma}_{\tilde{\mathbf{k}}} = \sum_s \tilde{\gamma}^{(s)}_{\tilde{\mathbf{k}}}, \quad \tilde{\varepsilon}_{\tilde{\mathbf{k}}} = \tilde{\varepsilon}'_{\tilde{\mathbf{k}}} + i\tilde{\varepsilon}''_{\tilde{\mathbf{k}}}, \quad \tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}} = \tilde{\varepsilon}'^{(s)}_{\tilde{\mathbf{k}}} + i\tilde{\varepsilon}''^{(s)}_{\tilde{\mathbf{k}}})$$

is the linear damping rate ascribed to particles of species s, and $\tilde{\mathbf{P}}_s = (\tilde{P}_{s\perp}, 0, \tilde{P}_{s\parallel})$. Here $\tilde{P}_{s\perp}$ is the infinitesimal perpendicular momentum density caused by the infinitesimally anisotropic momentum-space diffusion. This infinitesimal cross-field particle drift generates simultaneously an infinitesimal cross-field electric field given by (7) via its dynamo effect. The above transport equations (49) and (50) are just the same as (17) and (18) in Sec. 2.

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In the absence of nonlinear wave–wave and wave–particle interactions, the kinetic wave equation in C′ becomes

$$\frac{\partial \tilde{U}_{\tilde{\mathbf{k}}}}{\partial t'} = 2\tilde{\gamma}_{\tilde{\mathbf{k}}}\,\tilde{U}_{\tilde{\mathbf{k}}}. \tag{51}$$

It is proved easily from (49)–(51) that the same conservation laws as (20) and (21) hold also in C′:

$$\frac{\partial}{\partial t'}\left(\sum_{\tilde{\mathbf{k}}}\tilde{U}_{\tilde{\mathbf{k}}}+\sum_{s}\tilde{U}_{s}\right)=0. \tag{52}$$

$$\frac{\partial}{\partial t'}\left(\sum_{\tilde{\mathbf{k}}}\frac{\tilde{\mathbf{k}}}{\tilde{\omega}_{\tilde{\mathbf{k}}}}\tilde{U}_{\tilde{\mathbf{k}}}+\sum_{s}\tilde{\mathbf{P}}_{s}\right)=0. \tag{53}$$

3.4. Relativistic quasilinear transport equations in C

Finally, we derive the relativistic quasilinear transport equations in C based on the relativistic quasilinear momentum-space diffusion equation in C′. The momentum-space integration of (44) multiplied by the energy and momentum densities in C,

$$w_s = n_s\gamma_s m_s c^2 = n_s\gamma_d\gamma_s'\left(1+\frac{\beta v_{x'}'}{c}\right)m_s c^2, \qquad \mathbf{p}_s = n_s\mathbf{p}$$

($p_{xs} = n_s\gamma_d(p_{x'}'+\beta\gamma_s' m_s c)$, $p_{ys} = n_s p_{y'}'$, $p_{zs} = n_s p_{z'}'$), yields the following simple relations:

$$\int d^3\mathbf{p}'\, w_s\,\tilde{Q}_{\tilde{\mathbf{k}}}\,g_{s0} = \frac{\omega_{\mathbf{k}}}{4\pi}\,\mathrm{Im}[\tilde{\varepsilon}_{\tilde{\mathbf{k}}}^{(s)}]. \tag{54}$$

$$\int d^3\mathbf{p}'\,\mathbf{p}_s\,\tilde{Q}_{\tilde{\mathbf{k}}}\,g_{s0} = \frac{\mathbf{k}}{4\pi}\,\mathrm{Im}[\tilde{\varepsilon}_{\tilde{\mathbf{k}}}^{(s)}]. \tag{55}$$

Here the relation (31) has been used. Since $p_{x'}' = \gamma_d(p_x-\beta\gamma_s m_s c)$, $p_{y'}' = p_y$, $p_{z'}' = p_\parallel$ and $t = \gamma_d(t'+\beta x'/c)$, the spatially uniform momentum distribution function g_s in C is given by $g_s(\mathbf{p},t) = g_{s0}(p_\perp', p_\parallel', \gamma_d(t'+\beta x'/c))$ and is expressed as

$$g_s(\mathbf{p},t) = g_{s0}((\gamma_d^2(p_x-\beta\gamma_s m_s c)^2+p_y^2)^{1/2}, p_\parallel, t). \tag{56}$$

Thus we see that the spatially uniform distribution function g_s in C becomes weakly inhomogeneous in C′. Then we should use the relativistic quasilinear momentum-space diffusion equation where $\partial/\partial t'$ is replaced by $\partial/\partial t'+v_{x'}'\,\partial/\partial x'$ in (44), because g_{s0} is non-uniform in the x' direction. Accordingly, the modified relativistic quasilinear momentum-space diffusion equation in the spatially non-uniform system C′ should be expressed as

$$\left(\frac{\partial t}{\partial t'}+v_{x'}'\frac{\partial}{\partial x'}\right)g_{s0} = \sum_{\tilde{\mathbf{k}}\neq\mathbf{0}}|\tilde{E}_{\tilde{\mathbf{k}}}|^2\,\tilde{Q}_{\tilde{\mathbf{k}}}\,g_{s0}. \tag{57}$$

In addition, it is easily seen that the following relation holds:

$$\left(\frac{\partial}{\partial t'}+v_{x'}'\frac{\partial}{\partial x'}\right)g_{s0} = \left(\frac{\partial t}{\partial t'}+v_{x'}'\frac{\partial t}{\partial x'}\right)\frac{\partial}{\partial t}g_{s0}(p_\perp', p_\parallel', t)$$

$$= \gamma_d\left(1+\frac{\beta v_{x'}'}{c}\right)\frac{\partial}{\partial t}g_s(\mathbf{p},t).$$

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Further, with respect to the volume elements in the momentum spaces of the two systems, there exists the following relation between $d^3\mathbf{p}$ and $d^3\mathbf{p}'$:

$$d^3\mathbf{p} = \left|\frac{\partial \mathbf{p}}{\partial \mathbf{p}'}\right| d^3\mathbf{p}' = \gamma_d\left(1+\frac{\beta v'_{x'}}{c}\right) d^3\mathbf{p}'.$$

As a result, the following simple relation holds between C and C′.

$$d^3\mathbf{p}\,\frac{\partial g_s}{\partial t} = d^3\mathbf{p}'\left(\frac{\partial}{\partial t'}+v'_{x'}\frac{\partial}{\partial x'}\right) g_{s0} \tag{58}$$

Connecting (57), (58) with (54), (55), we can obtain the relativistic quasilinear transport equations for the energy and momentum densities of particles of species s, $U_{Rs} = \int d^3\mathbf{p}\, w_s g_s$ and $\mathbf{P}_{Rs} = \int d^3\mathbf{p}\,\mathbf{p}_s g_s$, in C:

$$\frac{\partial U_{Rs}}{\partial t} = -\sum_{\mathbf{k}\neq 0} 2\gamma^{(s)}_{R\mathbf{k}} U_{R\mathbf{k}}, \tag{59}$$

$$\frac{\partial \mathbf{P}_{Rs}}{\partial t} = -\sum_{\mathbf{k}\neq 0} \frac{2\gamma^{(s)}_{R\mathbf{k}}\mathbf{k}}{\omega_{\mathbf{k}}} U_{R\mathbf{k}}, \tag{60}$$

where

$$U_{R\mathbf{k}} = \frac{1}{8\pi}\frac{\partial(\varepsilon'_{R\mathbf{k}}\omega_{\mathbf{k}})}{\partial\omega_{\mathbf{k}}}|\tilde{E}_{\tilde{\mathbf{k}}}|^2$$

and $\mathbf{k}U_{R\mathbf{k}}/\omega_{\mathbf{k}}$ are respectively the effective wave energy and momentum densities in C, and $\varepsilon_{R\mathbf{k}}$ and $\varepsilon^{(s)}_{R\mathbf{k}}$ are obtained from $\tilde{\varepsilon}_{\tilde{\mathbf{k}}}$ and $\tilde{\varepsilon}^{(s)}_{\tilde{\mathbf{k}}}$ substituted with $\tilde{\omega}_{\tilde{\mathbf{k}}} = \gamma_d(\omega_{\mathbf{k}} - k_\perp v_d)$, $\tilde{k}_\perp = \gamma_d(k_\perp - \beta\omega_{\mathbf{k}}/c)$ and $\tilde{k}_\parallel = k_\parallel$. The linear damping rate in C is given similarly by

$$\gamma^{(s)}_{R\mathbf{k}} = -\frac{\varepsilon''^{(s)}_{R\mathbf{k}}}{\partial\varepsilon'_{R\mathbf{k}}/\partial\omega_{\mathbf{k}}},$$

with

$$\gamma_{R\mathbf{k}} = \sum_s \gamma^{(s)}_{R\mathbf{k}}, \qquad \varepsilon_{R\mathbf{k}} = \varepsilon'_{R\mathbf{k}} + i\varepsilon''_{R\mathbf{k}}, \qquad \varepsilon^{(s)}_{R\mathbf{k}} = \varepsilon'^{(s)}_{R\mathbf{k}} + i\varepsilon''^{(s)}_{R\mathbf{k}}.$$

The linear kinetic wave equation in C can be obtained similarly, and is given by the usual form:

$$\frac{\partial U_{R\mathbf{k}}}{\partial t} = 2\gamma_{R\mathbf{k}} U_{R\mathbf{k}}. \tag{61}$$

Immediately, we find the same conservation laws as (20) and (21) from (59)–(61):

$$\frac{\partial}{\partial t}\left(\sum_{\mathbf{k}} U_{R\mathbf{k}} + \sum_s U_{Rs}\right) = 0, \tag{62}$$

$$\frac{\partial}{\partial t}\left(\sum_{\mathbf{k}} \frac{\mathbf{k}}{\omega_{\mathbf{k}}} U_{R\mathbf{k}} + \sum_s \mathbf{P}_{Rs}\right) = 0, \tag{63}$$

The transport equations (59) and (60) demonstrate that the electrostatic waves in C′ can create a strong relativistic particle acceleration along and across the magnetic field in C. Simultaneously, a cross-field electric field $\mathbf{E}_0 = \mathbf{B}_0 \times \mathbf{v}_d/c$ is generated in C via the dynamo effect due to relativistic cross-field particle drift.

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Here it is assumed that the waves in C′ are electrostatic, although the drift velocity $\mathbf{v}_d$ varies. However, when $\mathbf{v}_d$ becomes relativistic, there is a possibility that the electrostatic waves become electromagnetic in C′ owing to the change in the dispersion relation. Then a rigorous theoretical study is needed in this case. Accordingly, a further theoretical investigation of relativistic cross-field particle acceleration due to electromagnetic waves should be performed. Moreover, this would enable us to study relativistic acceleration due to electrostatic waves in a laboratory system when we treat the case in which the electrostatic waves propagate in C and the electromagnetic waves in C′.

In the non-relativistic limit of the cross-field particle acceleration ($\beta \ll 1$, $\gamma_d \approx 1$), the momentum distribution function g_s of (56) is reduced to $g_{s\sim}$ of (6) in Sec. 2. The electrostatic waves in C′ are also electrostatic in C ($\mathbf{E}_{\mathbf{k}} = \tilde{\mathbf{E}}_{\tilde{\mathbf{k}}}$ and $\mathbf{B}_{\mathbf{k}} = 0$), and the dielectric constant $\tilde{\varepsilon}_{\tilde{\mathbf{k}}}$ of (41) and (42) becomes identical to $\varepsilon_{\mathbf{k}}$ of (8) and (9) in Sec. 2, because $\tilde{\omega}_{\tilde{\mathbf{k}}} = \omega_{\mathbf{k}} - k_\perp v_d$, $\tilde{\mathbf{k}} = \mathbf{k}$ and $\omega'_{cs} = \omega_{cs}$ hold. Furthermore, the momentum-space diffusion equation (44) and the transport equations (59) and (60) are also reduced to (11), (17) and (18) in Sec. 2 respectively.

4. Summary

A relativistic quasilinear momentum-space diffusion equation showing non-relativistic and relativistic particle acceleration and electric potential formation in a relativistic magnetized plasma has been derived from the relativistic Vlasov–Maxwell equations. In the case of non-relativistic cross-field particle acceleration, the relativistic quasilinear momentum-space diffusion equation was given directly. Momentum-space integration of this equation yields relativistic quasilinear transport equations that show that particle acceleration along and across the magnetic field together with an electric field transverse to the magnetic field can be generated simultaneously by Landau or cyclotron damping of almost perpendicularly propagating electrostatic waves. The particle acceleration occurs in the $\mathbf{k}$ direction. The cross-field electric field arises via the dynamo effect due to cross-field particle acceleration to satisfy the generalized Ohm's law.

In the case of relativistic cross-field particle acceleration, the relativistic quasilinear transport equations were obtained by means of Lorentz transformation of the relativistic quasilinear momentum-space diffusion equation in the moving frame of reference. The velocity of the moving system relative to the laboratory system is the $\mathbf{E} \times \mathbf{B}$ drift velocity in the laboratory system, and hence no stationary electric field and no $\mathbf{E} \times \mathbf{B}$ drift exist in the moving system. Further, the electrostatic waves in the moving system become electromagnetic in the laboratory system. Relativistic cross-field particle acceleration in the $\mathbf{k}$ direction can arise via Landau or cyclotron damping of electrostatic waves in the frame of reference moving with the cross-field drift velocity of the accelerated particles. Simultaneously, a transverse electric field can be created in the laboratory system via the dynamo effect due to cross-field particle acceleration. The relativistic quasilinear transport equations obtained can be applied usefully to the investigation of high-energy or relativistic perpendicular particle acceleration and the creation of a transverse electric field, which may possibly occur in space plasmas. Because the wave modes are different in the

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moving and laboratory systems, further theoretical work should be performed to investigate relativistic cross-field particle acceleration due to electromagnetic waves in the moving system.

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References

Antoni, A., Desideri, D., Martines, E., Serianni, G. and Tramontin, L. 1997 *Phys. Rev. Lett.* **79**, 4814.

Evance, J. D., Morales, G. J. and Taylor, R. J. 1992 *Phys. Rev. Lett.* **69**, 1528.

Hasegawa, A. 1975 *Plasma Instabilities and Nonlinear Effects*, pp. 27–51. Springer–Verlag, Berlin.

Kennel, C. F. and Engelmann, F. 1966 *Phys. Fluids* **9**, 2377.

Kintner, P. M., Vago, J., Chesny, S., Arnoldy, R. L., Lynch, K. A., Pollock, C. J. and Moore, T. E. 1992 *Phys. Rev. Lett.* **68**, 2448.

Kintner, P. M. 1992 *Phys. Fluids* **B4**, 2264.

Maehara, T., Hase, M., Ikimatsu, M., Misou, T., Sugawa, M. and Sugaya, R. 1996 In: *Proceedings of 1996 International Conference on Plasma Physics*, *Nagoya*, Vol. 1, p. 426. The Japan Society of Plasma Science and Nuclear Fusion Research.

Nemoto, M., Uesugi, K., Imai, T., Ikeda, Y., Naito, O., Kusama, Y., Tobita, K., Takeuchi, H. and JT-60 Team 1991 *Phys. Rev. Lett.* **67**, 70.

Sugaya, R. 1991a *J. Phys. Soc. Jpn* **60**, 518.

Sugaya, R. 1991b *J. Phys. Soc. Jpn* **60**, 3367.

Sugaya, R. 1994a *J. Phys. Soc. Jpn* **63**, 1775.

Sugaya, R. 1994b *Phys. Plasmas* **1**, 2768.

Sugaya, R. 1996 *J. Plasma Phys.* **56**, 193.

Sugaya, R. 1997 In: *Proceedings of 7th European Fusion Theory Conference*, *Jülich*, Vol. 1, p. 207. Forschungszentrum Jülich GmbH, Institute for Plasma Physics, Jülich.

Sugaya, R. 1998 In: *Proceedings of 1998 International Congress on Plasma Physics Combined with 25th European Physical Society Conference on Controlled Fusion and Plasma Physics*, *Praha*, *ECA* Vol. 22C, p. 209. European Physical Society.

Sugaya, R. 1999 In: *Proceedings of 26th European Physical Society Conference on Controlled Fusion and Plasma Physics*, *Maastricht*, ECA Vol. 23J, p. 497. European Physical Society.

Sugaya, R., Tachibana, H., Yamashita, H., Miyake, K., Ue, A., Sugawa, M. and Dodo, T. 1995 *J. Phys. Soc. Jpn* **64**, 2018.

Sugaya, R., Ue, A., Maehara, T. and Sugawa, M. 1996 *Phys. Plasmas* **3**, 3485.

Sugaya, R., Maehara, T. and Sugawa, M. 1998 In: *Proceedings of 1998 International Congress on Plasma Physics Combined with 25th European Physical Society Conference on Controlled Fusion and Plasma Physics*, *Praha*, ECA Vol. 22C, p. 141. European Physical Society.

Wesson, J. A., Alper, B., Edwards, A. W. and Gill, R. D. 1997 *Phys. Rev. Lett.* **79**, 5018.

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