Probability in the Engineering and Informational Sciences, **14**, 2000, 375–383. Printed in the U.S.A.

SOME CONJECTURED UNIFORMLY OPTIMAL RELIABLE NETWORKS

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We consider all-terminal reliability, one of the more popular models in the field of network reliability. A graph with n nodes and e edges, where the nodes are perfectly reliable and the edges survive independently with equal probability p, is said to be a uniformly optimally reliable graph if it has for all values of p ($0 \leq p \leq 1$) an equal or higher reliability among all graphs with the same number of nodes and edges. Boesch et al. [4] verified the existence of uniformly optimally reliable graphs for $e = n - 1$, $e = n$, $e = n + 1$, and $e = n + 2$; he has also given a conjecture for $e = n + 3$. Wang [8] proved this conjecture. In this article, we present four new infinite families of graphs that we conjecture to be uniformly optimal.

1. INTRODUCTION

A network is an undirected simple graph $G = (n, e)$. We model a network whose edges are subject to independent component failure in some fixed time period T by an undirected probabilistic graph where vertices represent communication centers (or telephones) and edges represent communication links between them. Edges are assumed to operate independently, survive with common probability p, and fail with probability $q = 1 - p$; vertices do not fail in this model. The reliability of a graph G, $\mathrm{rel}(G)$, is simply the probability that G remains connected after the fixed time period

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T. Colbourn's monograph [5] is an excellent starting point for a researcher interested in the combinatorics of network reliability. A standard formula for the reliability of a given graph G [denoted by $\mathrm{rel}(G)$ and called the reliability polynomial] is

$$\mathrm{rel}(G) = \sum_{i=0}^{e} N_i(G) p^i (1-p)^{e-i} = 1 - \sum_{i=0}^{e} C_i(G) p^{e-i} (1-p)^i, \tag{1.1}$$

where $N_i(G)$, or N_i, denotes the number of i-edge connected spanning subgraphs of G and the total number of cutsets of size i in G is denoted by $C_i(G)$; note that

$$C_i(G) = \binom{e}{i} - N_{e-i}(G). \tag{1.2}$$

Let $\Omega(n,e)$ denote the set of all connected graphs with n nodes and e edges, with no loops or multiple edges. A graph $G \in \Omega(n,e)$ is uniformly best (or most reliable) if $\mathrm{rel}(G) > \mathrm{rel}(H)$ for all $H \in \Omega(n,e)$ and for all edge-surviving probabilities p ($0 \leq p \leq 1$); here, we are using the symbol "$>$" to indicate a uniform result for all p ($0 \leq p \leq 1$).

The following two theorems are obvious results about uniformly most reliable graphs.

THEOREM 1.1: *If $N_i(G) = N_i(H)$ for $i = 0,1,2,\ldots,k$ and $N_{k+1}(G) > N_{k+1}(H)$, then G is more reliable than H for p close to zero* (*when the edges are each very unreliable*).

THEOREM 1.2: *If $N_i(G) = N_i(H)$ for $i = e, e-1, e-2, \ldots, e-k$, and $N_{e-k-1}(G) > N_{e-k-1}(H)$, then G is more reliable than H for p close to one* (*when the edges are each very reliable*).

The following corollaries of the above theorems were originally proved by Bauer et al. [2] and also presented by Boesch [3].

COROLLARY (of Theorem 1.1): *If G has more spanning trees than H, then G is more reliable than H for p close to zero.*

Define $\lambda(G)$ to be the common size of the minimal cutsets of G, and $C_\lambda(G)$ to be the number of minimal cutsets of G.

COROLLARY (of Theorem 1.2): *If either $[\lambda(G) > \lambda(H)]$ or $[C_\lambda(G) < C_\lambda(H)$ and $\lambda(G) = \lambda(H)]$, then G is more reliable than H for p close to one.*

2. KNOWN RESULTS AND FOUR NEW INFINITE SEQUENCES CONJECTURED TO BE UNIFORMLY OPTIMAL GRAPHS

We use K_n to denote the complete graph with n vertices, in which each vertex is connected to each of the others (with one edge between each pair of vertices). A graph is bipartite if its vertices can be partitioned into two disjoint subsets A and B such that each edge connects a vertex from A to one from B. A bipartite graph is a complete bipartite graph if every vertex in A is connected to every vertex in B. If A

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TABLE 2.1. Graphs Whose Optimal Results Are Known

(n, e)	Best Graphs	Ref.
$(n, n-1)$	Any tree	
(n, n)	Any n-cycle	
$(n, n+1)$	A Theta-graph with path lengths close to equality	9
$(n, n+2)$	A particular subdivision of K_4 (see Fig. 2.1)	7
$(n, n+3)$	A particular subdivision of $K_{3,3}$ (see Fig. 2.2)4	4

has n elements and B has m elements, then we denote the resulting complete bipartite graph by $K_{n,m}$. A Theta-graph $G = \theta(a, b, c)$ has two vertices of degree 3 and paths between these vertices of lengths a, b, and c. Theta-graphs have a, b, and c as close to equal as possible (i.e., the difference between any pair is at most one).

For graphs with n nodes and having up to $n + 3$ edges, many results are known. It has been proved that these graphs are uniformly-most reliable by Boesch et al. [4] for up to $n + 2$ edges and by Wang [8] for $n + 3$ edges. These known results on optimality are summarized in Table 2.1.

Although any one of the six indicated nodes in Figure 2.1 can be added for $n = 5$, the order of these additions is not entirely irrelevant for $n \geq 6$. Thus, according to [4], for $n = 6$ the two added nodes can be A and B in Figure 2.1 but not A and C and not A and D. However, the order suggested in Figure 2.1 does satisfy the conditions in [4] for an optimal result for any $n \geq 4$. For $n = 11$, we complete the set of six additions (A, B, C, D, E, F) and add one neighbor to (say) A; for $n = 12$, we add one neighbor to each of A and B, and so forth.

In Figure 2.2 again the order is irrelevant for adding one node (i.e., for $n = 7$) but not for $n \geq 8$. Thus, according to [8], we can add the two nodes (A and C) but not (say) A and D. However, the order suggested in Figure 2.2 does satisfy the conditions

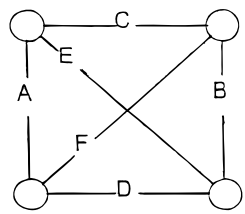


FIGURE 2.1. The optimal graph on $n + 2$ edges ($n = 4,5,6,\ldots$). The increase in n from 4 is carried out by subdividing edges and adding points one by one as indicated by the letters $A, B, C, \ldots$.

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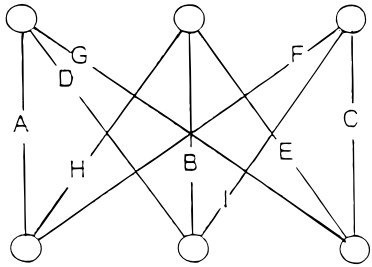


FIGURE 2.2. The optimal graph on $n + 3$ edges ($n = 6,7,8,\ldots$). The increase in n from 6 is carried out by subdividing edges and adding points one by one as indicated by the letters $A, B, C, \ldots$.

in [8] for an optimal result for any $n \geq 6$. After $n = 15$, we add a neighbor to A and continue as in Figure 2.1.

In Figure 2.3, we present four new infinite classes $(n, n + 4)$, $(n, n + 5)$, $(n, n + 6)$, and $(n, n + 7)$. Every member of each of these four classes is conjectured to be uniformly most reliable in the class of graphs with same pair (n, e). The remarks given after both Figures 2.1 and 2.2 about the continuation in $n(n \to \infty)$ also apply to each of our four families below.

It is curious to note that we can also define a "superfamily" of optimal graphs $(2n, 3n), n \geq 2$. Thus, for $n = 2$, we start with (4,6), which is in Figure 2.1, go to (6,9), which is in Figure 2.2, then go to (8,12), which is in the first of the four families in Figure 2.3, and so on through all four families. It is not known whether this phenomenon continues indefinitely.

From (1.1), the smallest number of edges that a spanning subgraph in any G can have is $n - 1$, and these $n - 1$ edges form a spanning tree in G. Hence, N_{n-1} is the number of spanning trees in the graph G, and the reliability polynomial presented at (1.1) can be also written as

$$\sum_{i=n-1}^{e} N_i(G)p^i(1-p)^{e-i} = N_{n-1}(G)p^{n-1}(1-p)^{e-n+1} + \sum_{i=n}^{e} N_i(G)p^i(1-p)^{e-i},$$

$$0 \leq p \leq 1. \tag{2.1}$$

For example, consider the class $\Omega(n, n + 4), n = 8,9,10,\ldots$. From (2.1), we can find the reliability function (in terms of the common probability p of an edge surviving) either by hand calculation or by the Maple V package (or by Myrvold's code) and we obtain the results shown in Table 2.2.

Without loss of generality, we are using only the coefficients of the reliability polynomials for each member of $G \in \Omega(n, n+4)$; that is, we write $[N_{n-1}, N_n, N_{n+1}, \ldots, N_e]$. Using (1.2), we also obtain the results for the cutset vector for each member of $G \in \Omega(n, n+4)$; that is, we write $[C_{e-(n-1)}, C_{e-n}, C_{e-(n+1)}, \ldots, C_0]$. The corresponding re-

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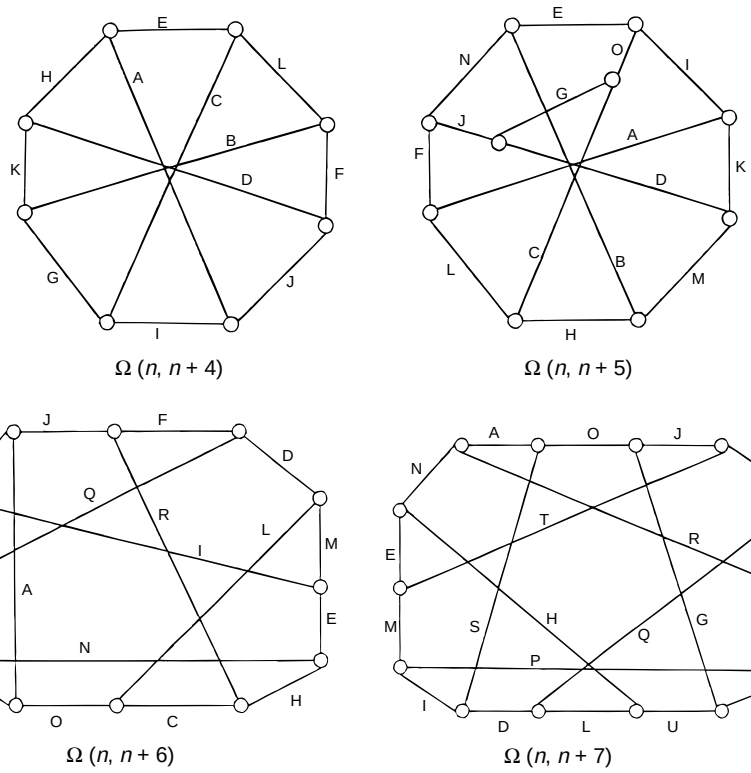


FIGURE 2.3. Four infinite families of optimal graphs. In each of the graphs, for the starting value of n we omit all points corresponding to $A, B, C, \ldots$; for the next n value, we add only A to the indicated unlettered nodes; for the third n value, we include A and B only, and so on, *ad infinitum.*

sults for each member G in the other three classes $\Omega(n, n+5)$, $\Omega(n, n+6)$, and $\Omega(n, n+7)$ are given in Tables 2.3, 2.4, and 2.5, respectively.

3. ADDITIONAL REMARK

The problem of finding the optimally reliable graph for given n and e is highly intractable and is not well known in probabilistic or statistical circles. The literature contains a number of conjectures on optimality (e.g. [1])—some proved and others disproved. It is our hope that future researchers will find our results useful and will try to prove or extend them. Although this field has much literature, the main problems of finding the optimal graph for any given pair (n, e) and proving the optimality have not been solved. These are challenging problems with great potential for applications; we would like to see more probabilists and statisticians interested in

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TABLE 2.2. Members of the Family $\Omega(n, n+4)$ (see Fig. 2.3)

(n = # nodes, e = # edges) (n,e)	Reliability Polynomial (Short Notation) $[N_{n-1}, N_n, N_{n+1}, \ldots, N_e]$	Alternative Cutset Vector Notation $[C_{e-(n-1)}, C_{e-n}, C_{e-(n+1)}, \ldots, C_0]$
(8,12)	[392,409,212,66,12,1]	[400,86,8,0,0,0]
(9,13)	[560,546,265,77,13,1]	[727,169,21,1,0,0]
(10,14)	[800,724,328,89,14,1]	[1202,277,36,2,0,0]
(11,15)	[1120,948,402,102,15,1]	[1883,417,53,3,0,0]
(12,16)	[1568,1232,488,116,16,1]	[2800,588,72,4,0,0]
(13,17)	[2100,1564,585,131,17,1]	[4088,816,95,5,0,0]
(14,18)	[2788,1970,696,147,18,1]	[5780,1090,120,6,0,0]
(15,19)	[3723,2471,822,164,19,1]	[7905,1405,147,7,0,0]
(16,20)	[4838,3049,962,182,20,1]	[10666,1796,178,8,0,0]
(17,21)	[6240,3738,1119,201,21,1]	[14109,2247,211,9,0,0]
(18,22)	[7912,4528,1292,221,22,1]	[18422,2787,248,10,0,0]
(19,23)	[9968,5456,1484,242,23,1]	[23681,3399,287,11,0,0]
(20,24)	[12544,6544,1696,264,24,1]	[29960,4082,328,12,0,0]
⋮	⋮	⋮

TABLE 2.3. Members of the Family $\Omega(n, n+5)$ (see Fig. 2.3)

(n = # nodes, e = # edges) (n,e)	Reliability Polynomial (Short Notation) $[N_{n-1}, N_n, N_{n+1}, \ldots, N_e]$	Alternative Cutset Vector Notation $[C_{e-(n-1)}, C_{e-n}, C_{e-(n+1)}, \ldots, C_0]$
(10,15)	[2000,2172,1230,445,105,15,1]	[3005,831,135,10,0,0,0]
(11,16)	[2800,2896,1558,534,119,16,1]	[5208,1472,262,26,1,0,0]
(12,17)	[3920,3844,1960,636,134,17,1]	[8456,2344,420,44,2,0,0]
(13,18)	[5432,5059,2446,752,150,18,1]	[13132,3509,614,64,3,0,0]
(14,19)	[7488,6614,3030,883,167,19,1]	[19644,5014,846,86,4,0,0]
(15,20)	[10168,8529,3714,1029,185,20,1]	[28592,6975,1131,111,5,0,0]
(16,21)	[13623,10901,4520,1192,204,21,1]	[40641,9448,1465,138,6,0,0]
(17,22)	[18095,13790,5454,1372,224,22,1]	[56518,12544,1861,168,7,0,0]
(18,23)	[23777,17312,6539,1571,245,23,1]	[77170,16337,2316,200,8,0,0]
(19,24)	[31125,21634,7799,1790,267,24,1]	[103471,20870,2827,234,9,0,0]
(20,25)	[40138,26691,9219,2028,290,25,1]	[136962,26439,3431,272,10,0,0]
(21,26)	[51408,32758,10845,2288,314,26,1]	[178822,33022,4105,312,11,0,0]
(22,27)	[65024,39788,12664,2569,339,27,1]	[230986,40942,4886,356,12,0,0]
(23,28)	[81600,48072,14724,2874,365,28,1]	[295140,50208,5751,402,13,0,0]
⋮	⋮	⋮

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TABLE 2.4. Members of the Family $\Omega(n, n+6)$ (see Fig. 2.3)

(n = # nodes, e = # edges) (n,e)	Reliability Polynomial (Short Notation) $[N_{n-1}, N_n, N_{n+1}, \ldots, N_e]$	Alternative Cutset Vector Notation $[C_{e-(n-1)}, C_{e-n}, C_{e-(n+1)}, \ldots, C_0]$
(12,18)	[9800,11444,7048,2862,804,153,18,1]	[22024,7120,1520,198,12,0,0,0]
(13,19)	[13615,15261,9006,3498,938,170,19,1]	[36773,11871,2622,378,31,1,0,0]
(14,20)	[18963,20272,11441,4249,1088,188,20,1]	[58557,18488,4063,596,52,2,0,0]
(15,21)	[26411,26847,14461,5131,1255,207,21,1]	[89869,27417,5888,854,75,3,0,0]
(16,22)	[36260,35259,18158,6159,1440,227,22,1]	[134284,39354,8176,1156,100,4,0,0]
(17,23)	[49581,46049,22662,7350,1644,248,23,1]	[195576,54898,10987,1505,127,5,0,0]
(18,24)	[66490,59243,27975,8704,1867,270,24,1]	[279614,75353,14529,1922,157,6,0,0]
(19,25)	[88911,75929,34371,10257,2111,293,25,1]	[391789,101171,18759,2393,189,7,0,0]
(20,26)	[116909,96111,41853,12009,2376,317,26,1]	[540891,134119,23927,2941,224,8,0,0]
(21,27)	[153440,121240,50740,13998,2664,342,27,1]	[734590,174770,29990,3552,261,9,0,0]
(22,28)	[199136,151588,61112,16238,2976,368,28,1]	[984904,225152,37168,4237,300,10,0,0]
(23.29)	[256991,188757,73311,18761,3313,395,29,1]	[1303789,286263,45444,4990,341,11,0,0]
(24,30)	[330992,234277,87598,21591,3676,423,30,1]	[1704808,359498,54908,5814,384,12,0,0]
(25,31)	[417136,285746,103411,24665,4062,452,31,1]	[2212439,450535,66500,6800,433,13,0,0]
⋮	⋮	⋮

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TABLE 2.5. Members of the Family $\Omega(n, n+7)$ (see Fig. 2.3)

(n = # nodes, e = # edges) (n, e)	Reliability Polynomial (Short Notation) $[N_{n-1}, N_n, N_{n+1}, \ldots, N_e]$	Alternative Cutset Vector Notation $[C_{e-(n-1)}, C_{e-n}, C_{e-(n+1)}, \ldots, C_0]$
(14,21)	[50421,61092,40096,17850,5712,1316,210,21,1]	[153069,55188,14168,2499,273,14,0,0,0]
(15,22)	[69629,81456,51552,22100,6800,1504,230,22,1]	[250141,89088,23061,4234,515,36,1,0,0]
(16,23)	[96040,108303,66013,27231,8055,1711,251,23,1]	[394274,136854,34934,6418,800,60,2,0,0]
(17,24)	[132300,143568,84171,33390,9495,1938,273,24,1]	[603171,202536,50425,9114,1131,86,3,0,0]
(18,25)	[180740,188950,106646,40711,11137,2186,296,25,1]	[900835,291750,70454,12419,1513,114,4,0,0]
(19,26)	[246400,247856,134540,49402,13003,2456,320,26,1]	[1315875,409944,95690,16378,1947,144,5,0,0]
(20,27)	[329200,320160,167820,59460,15093,2748,345,27,1]	[1890875,567870,128190,21270,2457,177,6,0,0]
(21,28)	[439200,412272,208476,71254,17447,3064,371,28,1]	[2668905,771768,168264,27026,3028,212,7,0,0]
(22,29)	[581900,527349,257511,84977,20086,3405,398,29,1]	[3710245,1033431,217509,33778,3665,249,8,0,0]
(23,30)	[770552,672970,316966,100944,23037,3772,426,30,1]	[5082373,1362830,276809,41562,4368,288,9,0,0]
(24,31)	[1006554,849675,386963,119171,26301,4165,455,31,1]	[6882171,1779900,349318,50740,5164,330,10,0,0]
(25,32)	[1312740,1069628,470711,140166,29925,4586,485,32,1]	[9205560,2296228,435481,61210,6035,374,11,0,0]
(26,33)	[1690920,1334936,569052,164105,33929,5036,516,33,1]	[12193236,2937112,538516,73231,6991,420,12,0,0]
(27,34)	[2178000,1663008,685940,191502,38349,5516,548,34,1]	[15978204,3716608,658964,86754,8027,468,13,0,0]
⋮	⋮	⋮

them. The report by Myrvold et al. [6] gives a useful recent history of progress in optimal reliability and is a good starting point for research in this area.

Acknowledgments

The authors wish to acknowledge with thanks conversations with Professor Charles Colbourn of the Computer Science Department at the University of Vermont on some open problems in network reliability. We also thank Professor Wendy Myrvold of the Computer Science Department of the University of Victoria, B.C., Canada, who gave us her technical reports and her C programming code (with Maple) to compute reliability polynomials. We also wish to acknowledge conversations about complete graphs and Hamiltonian graphs with Dr. Jim Farned of Santa Barbara.

References

1. Ath, Y. & Sobel, M. (1999). Counterexamples to conjectures for uniformly optimally reliable graphs. Preprint.
2. Bauer, D., Boesch, F., Suffel, C., & Van Slyke, R. (1987). On the validity of a reduction of reliable network design to a graph extremal problem. *IEEE Transactions on Circuit and Systems* CAS-34(12): 1579–1581.
3. Boesch, F.T. (1986). On unreliability polynomials and graph connectivity in reliability network synthesis. *Journal of Graph Theory* 10(3): 339–352.
4. Boesch, F.T., Li, X., & Suffel, C. (1991). On the existence of uniformly optimally reliable networks. *Networks* 21: 181–191.
5. Colbourn, C.J. (1987). *The combinatorics of network reliability.* New York: Oxford University Press.
6. Myrvold, W., Cheung, K.H., Page, L.B., & Perry, J.E. (1991). Uniformly-most reliable networks networks do not always exists. *Networks* 21: 417–419.
7. Tseng, S.S. & Wang, L.R. (1989). Maximizing the number of spanning trees of networks based on cycle basis representation. *International Journal of Computing and Mathematics* 28: 47–56.
8. Wang, G. (1994). A proof of Boesch's conjecture. *Networks* 24: 277–284.
9. Wang, J.F. & Wu, M.H. (1987). Network reliability analysis: on maximizing the number of spanning trees. *Proceedings of the National Science Council of the Republic of China, Part A, Physics and Science Engineering* 11(3): 193–196.

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