AN ACCURATE METHOD FOR FIRST AND SECOND DERIVATIVES OF DYNAMIC RESPONSES

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ABSTRACT

This paper has developed an accurate method for calculating the first and second derivatives of dynamic responses with respect to the design variables of structures subjected to dynamic loads. An efficient algorithm to calculate the dynamic responses, their first and second derivatives with respect to the design variables is formulated based on the Newmark-β method. The algorithm is achieved by direct differentiation and only a single dynamics analysis is required. An example is demonstrated with the new method proposed in this paper and the analytical method. The comparative numerical results show the new method is highly accurate compared to the analytical method.

Keywords : First derivative of dynamic response, Sensitivity analysis, Second derivative of dynamic response, Hessian matrix.

1. INTRODUCTION

Calculation of the first derivatives of dynamic responses with respect to the design variables is the pre-requisite when the gradient-based methods (also called first order methods) are used in dynamic response optimization design of the structures subjected to transient loads. Many algorithms have been developed to calculate the first derivatives of dynamic responses with respect to the design variables (also called sensitivity analysis) [1-4]. The two primary ways of calculating the design sensitivity of a structure under transient loads are the direct differentiation method (DDM) and the adjoint variable method (AVM) [5,6]. In the direct differentiation method (DDM), the direct integration method [7-10] and the mode superposition method [11-13] are often used to calculate the design sensitivity. The details of the numerical solution of the first derivatives of dynamic responses with the direct integration method can be found in reference [14]. Greene and Haftka [15] reviewed several sensitivity

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analysis methods which are available with the mode superposition method from the viewpoint of numerical method. Paul *et al*. [16] investigated the source of the errors of sensitivity in the mode superposition method. The major errors are from discretization of time, discretization of the continuous body, and truncation of higher modes. To carry out sensitivity analysis by the DDM, we need to solve as many second order differential equations as the number of design variables. Thus, the DDM is not efficient when the number of design variables is large. In the adjoint variable method (AVM), an augmented response function is defined to calculate the sensitivity of a response. The adjoint variable is introduced to define the augmented response function [17-20]. The AVM is efficient compared with the DDM when the number of design variables is large and the number of active constraints is small. The AVM is not very useful when the constraints in a nonintegral form, *i.e.*, pointwise constraints, are involved. In most work on structural optimization, AVM is only applied to the constraints in an integral form [21-24]. The sensitivity of the equivalent function can be calculated by using AVM [25]. The pointwise constraints can be converted into the constraints in integral form by using the Dirac delta function. However, if the number of time grid points to be considered is large, the application of AVM to pointwise constraints is not efficient. Therefore, AVM is generally only applied to the constraints in an integral form. However, AVM has an attractive aspect when it is applied to the transformation method, such as augmented Lagrange function [26] and penalty function [27].

Generally, the second order method (we call it gradient-Hessian matrix-based method) is more efficient than the gradient-based method (also called the first order method) when the Hessian matrix can be calculated efficiently. The gradient-Hessian matrix-based methods, *i.e.*, Newton's Method and Marquardt's method [28], generally require the dynamic responses, their first and second derivatives with respect to the design variables to construct optimal algorithms. Although many researchers have successfully calculated the first derivatives of dynamic responses, there is little work published on the second derivative analysis of dynamic responses. Second derivative analysis is more complicated than first derivative analysis. However, the efficiency of the optimization algorithm using both first and second derivatives can be greatly improved when the gradient and Hessian matrix can be calculated efficiently.

The purpose of this paper is to develop an accurately computational method for the first and second derivatives of dynamic responses with respect to the design variables. We formulate an efficient algorithm to calculate the dynamic responses, their first and second derivatives with respect to the design variables. The algorithm is achieved by direct differentiation and only a single dynamic analysis, based on the Newmark-β method, is required. The detailed computational procedures are developed to calculate accurately dynamic responses, their first and second derivatives with respect to the design variables. In the numerical example, in order to evaluate the accuracy of the new method proposed in this paper, we also formulate an analytical method to compute the dynamic responses, their first and second derivatives with respect to design variables. The results show that the new method presented in this paper is sufficiently accurate compared with the analytical method.

2. PROBLEM PRESENTATION

The structural design variable vector is denoted by $\boldsymbol{d}$, the number of design variable is m. Consider the equations of motion for a linear system subjected to dynamic forces,

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t)+\boldsymbol{C}\dot{\boldsymbol{x}}(t)+\boldsymbol{K}\boldsymbol{x}(t)=\boldsymbol{f}(t) \tag{1}$$

with the following initial conditions:

$$\begin{cases}\boldsymbol{x}(0)=\boldsymbol{x}_0\\ \dot{\boldsymbol{x}}(0)=\dot{\boldsymbol{x}}_0\end{cases},\quad \begin{cases}\dfrac{\partial \boldsymbol{x}(0)}{\partial d_i}=\boldsymbol{x}_{0\,d_i}\\ \dfrac{\partial \dot{\boldsymbol{x}}(0)}{\partial d_i}=\dot{\boldsymbol{x}}_{0\,d_i}\end{cases}\quad (i=1,2,\cdots,m),$$

$$\begin{cases}\dfrac{\partial^2 \boldsymbol{x}(0)}{\partial d_i\,\partial d_j}=\boldsymbol{x}_{0d_i\,d_j}\\ \dfrac{\partial^2 \dot{\boldsymbol{x}}(0)}{\partial d_i\,\partial d_j}=\dot{\boldsymbol{x}}_{0d_i\,d_j}\end{cases}\quad (i,j=1,2,\cdots,m) \tag{2}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ are structural mass matrix, damping matrix and stiffness matrix, respectively. $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$, $\ddot{\boldsymbol{x}}(t)$ and $\boldsymbol{f}(t)$ are unknown nodal displacement, velocity and acceleration vectors and the load vector, respectively.

The problem is to calculate the dynamic responses of the structure, their first and second derivatives with respect to the structural design variables.

3. FIRST AND SECOND DERIVATIVES OF STIFFNESS, MASS, AND DAMPING MATRIX

In this work, the truss element is used to demonstrate the accurate method for calculating the first and second derivatives of dynamic responses. In order to calculate the first and second derivatives of dynamic responses, the first and second derivatives of the total stiffness matrix, total mass matrix, and total damping matrix with respect to the design variables have to be obtained. The first and second derivatives of the truss element stiffness matrix and mass matrix are calculated first. Then the first and second derivatives of the truss element stiffness matrix and mass matrix are assembled to obtain the first and second derivatives of the total stiffness matrix and mass matrix of the truss structure. The assembly process is similar to the one used in the finite element method, *i.e.*, the element stiffness matrix

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and mass matrix are assembled to obtain the total structural stiffness matrix and mass matrix. The first and second derivatives of the truss damping matrix with respect to the design variables are calculated according to Rayleigh damping hypothesis.

We define the variables that will be used in the calculations. Nodal displacement of a truss element is shown in Fig. 1. Element number is denoted by e. Local and global coordinate system are $\bar{x}0\bar{y}$ and $x0y$, respectively. Nodal local and global numbers of element are represented by (1, 2) and (i, j), respectively. $\bar{x}$ is assumed to be directed from node 1 to node 2. Angle between the global and local coordinate axes, x and $\bar{x}$, is φ (clockwise).

The cross-section of the truss element is shown in Fig. 2. d^e and t^e are the outer diameter and wall thickness, respectively.

Nodal displacement vector of element e in a local coordinate is defined as:

$$\bar{\boldsymbol{\delta}}^e = (\bar{u}_i \quad \bar{v}_i \quad \bar{u}_j \quad \bar{v}_j)^{eT} \tag{3}$$

and in a global coordinate is:

$$\boldsymbol{\delta}^e = (u_i \quad v_i \quad u_j \quad v_j)^{eT} \tag{4}$$

where superscript T is a vector transposition and the subscript is node number of element.

3.1 First and Second Derivatives of Stiffness Matrix

Element stiffness matrix in a local coordinate system can be expressed with element design variables (*i.e.*, nodal coordinates (x_i, y_i) and (x_j, y_j), and cross-sectional outer diameter d^e and wall thickness t^e):

$$\bar{\boldsymbol{K}}^e = \frac{EA_e}{l_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

where l_e is the length of element:

$$l_e = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \tag{6}$$

and A_e is the area of element:

$$A_e = \pi\left(\frac{d^e}{2}\right)^2 - \pi\left(\frac{d^e}{2} - t^e\right)^2 \tag{7}$$

The coordinate transformation matrix of element e is defined as:

$$\boldsymbol{T}^e = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \tag{8}$$

where $\sin\varphi = (x_j - x_i) / l_e$ and $\cos\varphi = (y_j - y_i) / l_e$.

Then the stiffness matrix of element e in a global coordinate is:

$$\boldsymbol{K}^e = \boldsymbol{T}^{eT}\, \bar{\boldsymbol{K}}^e\, \boldsymbol{T}^e \tag{9}$$

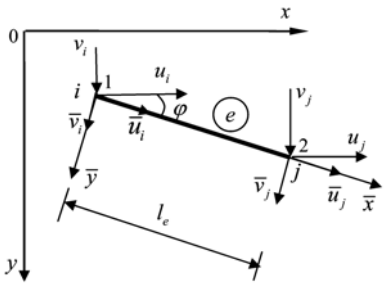


Fig. 1 Truss element

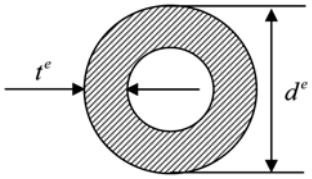


Fig. 2 Element cross-section

The first and second derivatives of the stiffness matrix of element e with respect to the element design variables are obtained by differentiating Eq. (9) with respect to the design variables, *i.e.*, nodal coordinates, cross-sectional outer diameter and wall thickness. Then the element stiffness matrix, its first and second derivatives are assembled into the total stiffness matrix of the truss, its first and second derivatives with respect to the truss design variables.

3.2 First and Second Derivatives of Mass Matrix

The lumped mass matrix of the element e in the local coordinate system expressed by the element design variables is:

$$\bar{\boldsymbol{M}}^e = \frac{\rho\, A_e\, l_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

where ρ is material density of element e.

With $\bar{\boldsymbol{M}}^e$ and $\boldsymbol{T}^e$ from Eq. (10) and Eq. (8), the lumped mass matrix of the element e in the global coordinate system is:

$$\boldsymbol{M}^e = \boldsymbol{T}^{eT}\, \bar{\boldsymbol{M}}^e\, \boldsymbol{T}^e \tag{11}$$

The first and second derivatives of the mass matrix of element e with respect to the element design variables are obtained by differentiating Eq. (11) with respect to the element design variables. Then the element mass matrix, its first and second derivatives are assembled into the total mass matrix of the truss, its first and second derivatives with respect to the truss design variables.

3.3 First and Second Derivatives of Damping Matrix

Rayleigh damping is used in this work and the structural damping matrix is

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$$\boldsymbol{C} = \alpha_1 \boldsymbol{M} + \alpha_2 \boldsymbol{K} \tag{12}$$

where α_1 and α_2 are constants.

The first and second derivatives of the truss damping matrix can be achieved by differentiating Eq. (12) with respect to the truss design variables.

4. FORMULAS FOR THE FIRST AND SECOND DERIVATIVES OF DYNAMIC RESPONSES

Equations (1) and (2) must be satisfied for all of time period $t \in [0, TT]$. TT is the duration of dynamic loads. In practice, the solution of this initial- boundary-value problem (IBVP) requires integration through time. This is achieved numerically by discretising in time the continuous temporal derivatives that appear in the equation. Any one of the time integration procedures can be used for this purpose. The most widely used family of direct time integration methods for solving Eq. (1) is the Newmark family of methods. The Newmark method can be formulated by considering equilibrium at any discrete time $t + \Delta t$, and is given by the following equation:

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t+\Delta t) + \boldsymbol{C}\dot{\boldsymbol{x}}(t+\Delta t) + \boldsymbol{K}\boldsymbol{x}(t+\Delta t) = \boldsymbol{f}(t+\Delta t) \tag{13}$$

The nodal displacements, nodal accelerations and nodal velocities can be achieved at any time $t + \Delta t$ by solving Eq. (13).

Perhaps the most widely used direct method for the equation of motion (1) is the Newmark method, an implicit technique, which consists of the following finite difference assumptions with regard to the evolution of the approximate solution:

$$\boldsymbol{x}(t+\Delta t) = \boldsymbol{x}(t) + \Delta t\dot{\boldsymbol{x}}(t) + \Delta t^2\left[\left(\frac{1}{2}-\beta\right)\ddot{\boldsymbol{x}}(t) + \beta\ddot{\boldsymbol{x}}(t+\Delta t)\right] \tag{14}$$

$$\dot{\boldsymbol{x}}(t+\Delta t) = \dot{\boldsymbol{x}}(t) + \Delta t\left[(1-\delta)\,\ddot{\boldsymbol{x}}(t) + \delta\ddot{\boldsymbol{x}}(t+\Delta t)\right] \tag{15}$$

where any particular choice of the parameters β and δ determines the stability and accuracy characteristics of the solution. In this work, parameters $\delta \geq 0.5$ and $\beta = 0.25(0.5 + \delta)^2$. We define the integral constants:

$a_0 = \frac{1}{\beta\Delta t^2}$, $a_1 = \frac{\delta}{\beta\Delta t}$, $a_2 = \frac{1}{\beta\Delta t}$, $a_3 = \frac{1}{2\beta}-1$,

$a_4 = \frac{\delta}{\beta}-1$, $a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\beta}-2\right)$, $a_6 = \Delta t(1-\delta)$, $a_7 = \delta\Delta t$.

The parameters β and δ will be replaced by those constants in the following formulas.

In addition to Eqs. (14) and (15) the equilibrium Eq. (13) at time station $t + \Delta t$ is considered. This way a system of equations is formed for the determination of the three unknowns $\boldsymbol{x}(t+\Delta t)$, $\dot{\boldsymbol{x}}(t+\Delta t)$ and $\ddot{\boldsymbol{x}}(t+\Delta t)$, assuming that the displacement, velocity, and acceleration vectors at the previous time station t have already been computed. Thus, the solution for the displacement vector is

$$\boldsymbol{K}^*\boldsymbol{x}(t+\Delta t) = \boldsymbol{F}^*(t+\Delta t) \tag{16}$$

where

$$\boldsymbol{K}^* = \boldsymbol{K} + a_0\boldsymbol{M} + a_1\boldsymbol{C} \tag{17}$$

and

$$\boldsymbol{F}^*(t+\Delta t) = \boldsymbol{f}(t+\Delta t) + \boldsymbol{M}\left[a_0\boldsymbol{x}(t) + a_2\dot{\boldsymbol{x}}(t) + a_3\ddot{\boldsymbol{x}}(t)\right] + \boldsymbol{C}\left[a_1\boldsymbol{x}(t) + a_4\dot{\boldsymbol{x}}(t) + a_5\ddot{\boldsymbol{x}}(t)\right] \tag{18}$$

The accelerations, $\ddot{\boldsymbol{x}}(t+\Delta t)$, which are required for the computations at the next time station, can be computed as

$$\ddot{\boldsymbol{x}}(t+\Delta t) = a_0\left[\boldsymbol{x}(t+\Delta t) - \boldsymbol{x}(t)\right] - a_2\dot{\boldsymbol{x}}(t) - a_3\ddot{\boldsymbol{x}}(t) \tag{19}$$

while the velocities, $\dot{\boldsymbol{x}}(t+\Delta t)$, can be obtained directly from Eq. (15), as follows:

$$\dot{\boldsymbol{x}}(t+\Delta t) = \dot{\boldsymbol{x}}(t) + a_6\ddot{\boldsymbol{x}}(t) + a_7\ddot{\boldsymbol{x}}(t+\Delta t) \tag{20}$$

Now we will derive the formulas for the first and second derivatives of the dynamic response. Differentiating Eq. (16) with respect to the design variable d_i, we have

$$\boldsymbol{K}^*\frac{\partial\boldsymbol{x}(t+\Delta t)}{\partial d_i} = \frac{\partial\boldsymbol{F}^*(t+\Delta t)}{\partial d_i} - \frac{\partial\boldsymbol{K}^*}{\partial d_i}\boldsymbol{x}(t+\Delta t) \tag{21}$$

where

$$\frac{\partial\boldsymbol{K}^*}{\partial d_i} = \frac{\partial\boldsymbol{K}}{\partial d_i} + a_0\frac{\partial\boldsymbol{M}}{\partial d_i} + a_1\frac{\partial\boldsymbol{C}}{\partial d_i} \tag{22}$$

and

$$\frac{\partial\boldsymbol{F}^*(t+\Delta t)}{\partial d_i} = \frac{\partial\boldsymbol{M}}{\partial d_i}\left[a_0\boldsymbol{x}(t) + a_2\dot{\boldsymbol{x}}(t) + a_3\ddot{\boldsymbol{x}}(t)\right] + \boldsymbol{M}\left[a_0\frac{\partial\boldsymbol{x}(t)}{\partial d_i} + a_2\frac{\partial\dot{\boldsymbol{x}}(t)}{\partial d_i} + a_3\frac{\partial\ddot{\boldsymbol{x}}(t)}{\partial d_i}\right] + \frac{\partial\boldsymbol{C}}{\partial d_i}\left[a_1\boldsymbol{x}(t) + a_4\dot{\boldsymbol{x}}(t) + a_5\ddot{\boldsymbol{x}}(t)\right] + \boldsymbol{C}\left[a_1\frac{\partial\boldsymbol{x}(t)}{\partial d_i} + a_4\frac{\partial\dot{\boldsymbol{x}}(t)}{\partial d_i} + a_5\frac{\partial\ddot{\boldsymbol{x}}(t)}{\partial d_i}\right] \tag{23}$$

Equations (22) and (23) can be obtained by differentiating Eqs. (17) and (18), respectively.

After the first derivatives of the displacement vector at time $t + \Delta t$ is obtained from Eq. (21), through differentiating Eq. (19) with respect to design variables d_i, we have

$$\frac{\partial\ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i} = a_0\left[\frac{\partial\boldsymbol{x}(t+\Delta t)}{\partial d_i} - \frac{\partial\boldsymbol{x}(t)}{\partial d_i}\right] - a_2\frac{\partial\dot{\boldsymbol{x}}(t)}{\partial d_i} - a_3\frac{\partial\ddot{\boldsymbol{x}}(t)}{\partial d_i} \tag{24}$$

Then the first derivatives of acceleration vector at

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time $t + \Delta t$ are obtained from Eq. (24). Differentiating Eq. (20) with respect to the design variable d_i, we have

$$\frac{\partial \dot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i} = \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial d_i} + a_6 \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial d_i} + a_7 \frac{\partial \ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i} \tag{25}$$

Further differentiating Eq. (21) with respect to the design variable d_j, we have:

$$\boldsymbol{K}^* \frac{\partial^2 \boldsymbol{x}(t+\Delta t)}{\partial d_i \, \partial d_j} = \frac{\partial^2 \boldsymbol{F}^*(t+\Delta t)}{\partial d_i \, \partial d_j} - \frac{\partial^2 \boldsymbol{K}^*}{\partial d_i \, \partial d_j} \boldsymbol{x}(t+\Delta t) - \frac{\partial \boldsymbol{K}^*}{\partial d_i} \frac{\partial \boldsymbol{x}(t+\Delta t)}{\partial d_j} - \frac{\partial \boldsymbol{K}^*}{\partial d_j} \frac{\partial \boldsymbol{x}(t+\Delta t)}{\partial d_i} \tag{26}$$

where

$$\frac{\partial^2 \boldsymbol{K}^*}{\partial d_i \, \partial d_j} = \frac{\partial^2 \boldsymbol{K}}{\partial d_i \, \partial d_j} + a_0 \frac{\partial^2 \boldsymbol{M}}{\partial d_i \, \partial d_j} + a_1 \frac{\partial^2 \boldsymbol{C}}{\partial d_i \, \partial d_j} \tag{27}$$

and

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{F}^*(t+\Delta t)}{\partial d_i \, \partial d_j} = {} & \frac{\partial^2 \boldsymbol{M}}{\partial d_i \, \partial d_j} \left[a_0 \boldsymbol{x}(t) + a_2 \dot{\boldsymbol{x}}(t) + a_3 \ddot{\boldsymbol{x}}(t) \right] \\
& + \frac{\partial \boldsymbol{M}}{\partial d_i} \left[a_0 \frac{\partial \boldsymbol{x}(t)}{\partial d_j} + a_2 \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial d_j} + a_3 \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial d_j} \right] \\
& + \frac{\partial \boldsymbol{M}}{\partial d_j} \left[a_0 \frac{\partial \boldsymbol{x}(t)}{\partial d_i} + a_2 \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial d_i} + a_3 \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial d_i} \right] \\
& + \boldsymbol{M} \left[a_0 \frac{\partial^2 \boldsymbol{x}(t)}{\partial d_i \, \partial d_j} + a_2 \frac{\partial^2 \dot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} + a_3 \frac{\partial^2 \ddot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} \right] \\
& + \frac{\partial^2 \boldsymbol{C}}{\partial d_i \, \partial d_j} \left[a_1 \boldsymbol{x}(t) + a_4 \dot{\boldsymbol{x}}(t) + a_5 \ddot{\boldsymbol{x}}(t) \right] \\
& + \frac{\partial \boldsymbol{C}}{\partial d_i} \left[a_1 \frac{\partial \boldsymbol{x}(t)}{\partial d_j} + a_4 \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial d_j} + a_5 \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial d_j} \right] \\
& + \frac{\partial \boldsymbol{C}}{\partial d_j} \left[a_1 \frac{\partial \boldsymbol{x}(t)}{\partial d_i} + a_4 \frac{\partial \dot{\boldsymbol{x}}(t)}{\partial d_i} + a_5 \frac{\partial \ddot{\boldsymbol{x}}(t)}{\partial d_i} \right] \\
& + \boldsymbol{C} \left[a_1 \frac{\partial^2 \boldsymbol{x}(t)}{\partial d_i \, \partial d_j} + a_4 \frac{\partial^2 \dot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} + a_5 \frac{\partial^2 \ddot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} \right]
\end{aligned} \tag{28}$$

Equations (27) and (28) are derived through further differentiating Eqs. (22) and (23), respectively.

After $\partial^2 \boldsymbol{x}(t + \Delta t)/(\partial d_i \, \partial d_j)$ is computed from Eq. (26), with further differentiating Eq. (24) with respect to the design variables d_j, we have

$$\frac{\partial^2 \ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i \, \partial d_j} = a_0 \left[\frac{\partial^2 \boldsymbol{x}(t+\Delta t)}{\partial d_i \, \partial d_j} - \frac{\partial^2 \boldsymbol{x}(t)}{\partial d_i \, \partial d_j} \right] - a_2 \frac{\partial^2 \dot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} - a_3 \frac{\partial^2 \ddot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} \tag{29}$$

Then $\partial^2 \ddot{\boldsymbol{x}}(t+\Delta t)/(\partial d_i \, \partial d_j)$ is obtained from Eq. (29). Further differentiating Eq. (25) with respect to the design variables d_j, we obtain

$$\frac{\partial^2 \dot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i \, \partial d_j} = \frac{\partial^2 \dot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} + a_6 \frac{\partial^2 \ddot{\boldsymbol{x}}(t)}{\partial d_i \, \partial d_j} + a_7 \frac{\partial^2 \ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i \, \partial d_j} \tag{30}$$

5. COMPUTATIONAL PROCEDURE AND FLOW CHART OF THE FIRST AND SECOND DERIVATIVES OF DYNAMIC RESPONSE

Section 4 gave the formulas for calculating the first and second derivatives of the dynamic response. This section provides the detailed computational procedure.

Procedure of calculating the first and second derivatives of dynamic responses:

Step 1.0 Initial calculations:

Step 1.1 $\boldsymbol{x}(0) = x_0, \dot{\boldsymbol{x}}(0) = \dot{\boldsymbol{x}}_0, \quad \ddot{\boldsymbol{x}}(0) = \boldsymbol{M}^{-1} \left[\boldsymbol{f}(0) - \boldsymbol{C}\dot{\boldsymbol{x}}(0) - \boldsymbol{K}\boldsymbol{x}(0) \right]$.

Step 1.2 $\dfrac{\partial \boldsymbol{x}(0)}{\partial d_i} = \boldsymbol{x}_{0\,d_i}, \dfrac{\partial \dot{\boldsymbol{x}}(0)}{\partial d_i} = \dot{\boldsymbol{x}}_{0\,d_i}, \dfrac{\partial \ddot{\boldsymbol{x}}(0)}{\partial d_i} = \boldsymbol{M}^{-1} \left\{ \dfrac{\partial \left[\boldsymbol{f}(0) - \boldsymbol{C}\dot{\boldsymbol{x}}(0) - \boldsymbol{K}\boldsymbol{x}(0) \right]}{\partial d_i} - \dfrac{\partial \boldsymbol{M}}{\partial d_i} \ddot{\boldsymbol{x}}(0) \right\}$.

Step 1.3 $\dfrac{\partial^2 \boldsymbol{x}(0)}{\partial d_i \, \partial d_j} = \boldsymbol{x}_{0\,d_i d_j}, \dfrac{\partial^2 \dot{\boldsymbol{x}}(0)}{\partial d_i \, \partial d_j} = \dot{\boldsymbol{x}}_{0\,d_i d_j}$,

$$\frac{\partial^2 \ddot{\boldsymbol{x}}(0)}{\partial d_i \, \partial d_j} = \boldsymbol{M}^{-1} \left\{ \frac{\partial^2 \left[\boldsymbol{f}(0) - \boldsymbol{C}\dot{\boldsymbol{x}}(0) - \boldsymbol{K}\boldsymbol{x}(0) \right]}{\partial d_i \, \partial d_j} - \frac{\partial}{\partial d_j} \left[\frac{\partial \boldsymbol{M}}{\partial d_i} \ddot{\boldsymbol{x}}(0) \right] - \frac{\partial \boldsymbol{M}}{\partial d_j} \frac{\partial \ddot{\boldsymbol{x}}(0)}{\partial d_i} \right\}.$$

Step 1.4 Set Δt.

Step 1.5 Compute integral constants: $a_0 = \dfrac{1}{\beta \Delta t^2}$,

$$a_1 = \frac{\delta}{\beta \Delta t}, \quad a_2 = \frac{1}{\beta \Delta t}, \quad a_3 = \frac{1}{2\beta} - 1,$$

$$a_4 = \frac{\delta}{\beta} - 1, \quad a_5 = \frac{\Delta t}{2} \left(\frac{\delta}{\beta} - 2 \right), \quad a_6 = \Delta t (1 - \delta),$$

$$a_7 = \delta \Delta t .$$

where integral parameters $\delta \geq 0.5$ and $\beta = 0.25(0.5 + \delta)^2$.

Step 1.6 Solve Eq. (17) $\Rightarrow$ $\boldsymbol{K}^*$.

Step 2.0 Calculations for each time step $t + \Delta t$:

Step 2.1 Solve Eq. (16) $\Rightarrow$ $\boldsymbol{x}(t + \Delta t)$.

Step 2.2 Solve Eq. (19) $\Rightarrow$ $\ddot{\boldsymbol{x}}(t + \Delta t)$.

Step 2.3 Solve Eq. (20) $\Rightarrow$ $\dot{\boldsymbol{x}}(t + \Delta t)$.

Step 2.4 Solve Eq. (21) $\Rightarrow$ $\dfrac{\partial \boldsymbol{x}(t+\Delta t)}{\partial d_i}$.

Step 2.5 Solve Eq. (24) $\Rightarrow$ $\dfrac{\partial \ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i}$.

Step 2.6 Solve Eq. (25) $\Rightarrow$ $\dfrac{\partial \dot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i}$.

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Step 2.7 Solve Eq. (26) $\Rightarrow$ $\dfrac{\partial^2 \boldsymbol{x}(t+\Delta t)}{\partial d_i\, \partial d_j}$.

Step 2.8 Solve Eq. (29) $\Rightarrow$ $\dfrac{\partial^2 \ddot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i\, \partial d_j}$.

Step 2.9 Solve Eq. (30) $\Rightarrow$ $\dfrac{\partial^2 \dot{\boldsymbol{x}}(t+\Delta t)}{\partial d_i\, \partial d_j}$.

Step 3 Repetition for the next time step. Replace t by $t + \Delta t$ and implement steps 2.1 to 2.9 for the next time step.

The above detailed computation procedure is expressed as a flow chart shown in Fig. 3.

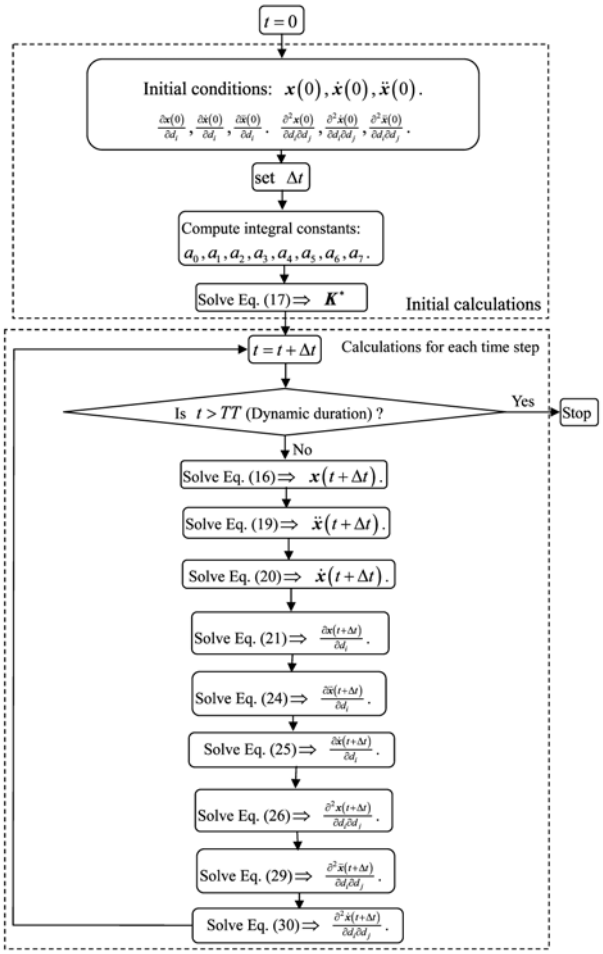


Fig. 3 Flow chart of computing first and second derivatives of dynamic response

6. EXAMPLE

The 2-bar truss to be considered is shown in Fig. 4. The structure is pinned at node 1 and 3. In the examples, we use typical steel materials properties, *i.e.*, elastic modulus E = 210GPa, material density ρ = 7850kg/m^3 and Poisson's ratio μ = 0.3. Node 2 is subjected to both a horizontally dynamic load, $F_h(t) = P_h \sin\omega t$, and a vertically dynamic load, $F_v(t) = P_v \sin\omega t$, where $P_h = P_v$ = 1500N and $\omega = 3\pi/4$. A total time duration of 3 s and an incremental time step of 0.01s are considered in the procedure.

The design variable vector is $\boldsymbol{d} = [d^1\ d^2\ t^1\ t^2\ x_2\ y_2]^T$ and the current design point is $\boldsymbol{d}_{current}$ = [0.09 0.09 0.012 0.012 0 0]^T(m). Assuming the initial condition is

$$\begin{cases} \boldsymbol{x}(0) = 0 \\ \dot{\boldsymbol{x}}(0) = 0 \end{cases}, \quad \begin{cases} \partial \boldsymbol{x}(0)/\partial d_i = 0 \\ \partial \dot{\boldsymbol{x}}(0)/\partial d_i = 0 \end{cases}, \quad \begin{cases} \partial^2 \boldsymbol{x}(0)/(\partial d_i\, \partial d_j) = 0 \\ \partial^2 \dot{\boldsymbol{x}}(0)/(\partial d_i\, \partial d_j) = 0 \end{cases},$$

(i, j = 1, 2, ⋯, 6). Now we calculate the first and second derivatives of the horizontal and vertical displacements of node 2 with respect to the design variables at the current design point.

6.1 Analytical Method

In order to evaluate the accuracy of the new method proposed in this paper, we formulate an analytical method to calculate the dynamic responses, their first and second derivatives with respect to the design variables for the example.

In this example, the number of dynamic freedom is 2. The horizontal and vertical displacements of node 2 are unknowns. The dynamic load vector is

$$\boldsymbol{f}(t) = \begin{bmatrix} P_h \sin\omega t \\ P_v \sin\omega t \end{bmatrix} = \boldsymbol{P}_0 \sin\omega t \tag{31}$$

Equation (1) becomes

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{P}_0 \sin\omega t \tag{32}$$

Because the initial conditions are $\boldsymbol{x}(0)$ = 0 and $\dot{\boldsymbol{x}}(0) = 0$, the non-steady state response induced by the zero initial condition is zero. Therefore, we only need to calculate the steady state response. Assuming that the steady state response solution is

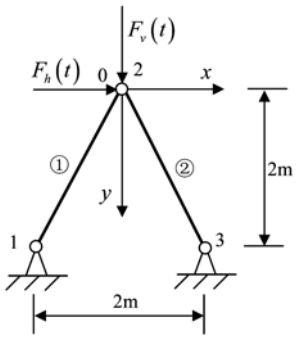


Fig. 4 Two-bar truss

$$\boldsymbol{x} = \begin{bmatrix} U_{21} \\ V_{21} \end{bmatrix} \sin\omega t + \begin{bmatrix} U_{22} \\ V_{22} \end{bmatrix} \cos\omega t = \boldsymbol{B}_1 \sin\omega t + \boldsymbol{B}_2 \cos\omega t \tag{33}$$

Substituting Eq. (33) into Eq. (32), we have

$$\begin{cases} (\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{B}_1 - \omega \boldsymbol{C}\boldsymbol{B}_2 = \boldsymbol{P}_0 \\ \omega \boldsymbol{C}\boldsymbol{B}_1 + (\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{B}_2 = 0 \end{cases} \tag{34}$$

Equation (34) can be written as:

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$$\begin{bmatrix} \boldsymbol{K}-\omega^2\boldsymbol{M} & -\omega\boldsymbol{C} \\ \omega\boldsymbol{C} & \boldsymbol{K}-\omega^2\boldsymbol{M} \end{bmatrix}\begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}=\begin{bmatrix} \boldsymbol{P}_0 \\ 0 \end{bmatrix} \tag{35}$$

$\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ can be obtained by solving Eq. (35). Then the dynamic response $\boldsymbol{x}$ is achieved by substituting $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ into Eq. (33).

Equation (35) can be written as follows:

$$\tilde{\boldsymbol{K}}\tilde{\boldsymbol{B}}=\tilde{\boldsymbol{P}} \tag{36}$$

where $\tilde{\boldsymbol{K}}=\begin{bmatrix} \boldsymbol{K}-\omega^2\boldsymbol{M} & -\omega\boldsymbol{C} \\ \omega\boldsymbol{C} & \boldsymbol{K}-\omega^2\boldsymbol{M} \end{bmatrix}$, $\tilde{\boldsymbol{B}}=\begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}$ and $\tilde{\boldsymbol{P}}=\begin{bmatrix} \boldsymbol{P}_0 \\ 0 \end{bmatrix}$.

Differentiating Eq. (36) with respect to the design variable d_i, we have

$$\tilde{\boldsymbol{K}}\frac{\partial\tilde{\boldsymbol{B}}}{\partial d_i}=-\frac{\partial\tilde{\boldsymbol{K}}}{\partial d_i}\tilde{\boldsymbol{B}} \tag{37}$$

where $\dfrac{\partial\tilde{\boldsymbol{K}}}{\partial d_i}=\begin{bmatrix} \dfrac{\partial\boldsymbol{K}}{\partial d_i}-\omega^2\dfrac{\partial\boldsymbol{M}}{\partial d_i} & -\omega\dfrac{\partial\boldsymbol{C}}{\partial d_i} \\ \omega\dfrac{\partial\boldsymbol{C}}{\partial d_i} & \dfrac{\partial\boldsymbol{K}}{\partial d_i}-\omega^2\dfrac{\partial\boldsymbol{M}}{\partial d_i} \end{bmatrix}$.

By differentiating Eq. (33) with respect to design variable d_i, we have

$$\frac{\partial\boldsymbol{x}}{\partial d_i}=\frac{\partial\boldsymbol{B}_1}{\partial d_i}\sin\omega t+\frac{\partial\boldsymbol{B}_2}{\partial d_i}\cos\omega t \tag{38}$$

where $\partial\boldsymbol{B}_1/\partial d_i$ and $\partial\boldsymbol{B}_2/\partial d_i$ can be obtained by solving Eq. (37). Then the first derivative of the dynamic response, *i.e.*, $\partial\boldsymbol{x}/\partial d_i$, is achieved from substituting $\partial\boldsymbol{B}_1/\partial d_i$ and $\partial\boldsymbol{B}_2/\partial d_i$ into Eq. (38).

Further differentiating Eq. (37) with respect to the design variable d_j, we have:

$$\tilde{\boldsymbol{K}}\frac{\partial^2\tilde{\boldsymbol{B}}}{\partial d_i\,\partial d_j}=-\frac{\partial^2\tilde{\boldsymbol{K}}}{\partial d_i\,\partial d_j}\tilde{\boldsymbol{B}}-\frac{\partial\tilde{\boldsymbol{K}}}{\partial d_i}\frac{\partial\tilde{\boldsymbol{B}}}{\partial d_j}-\frac{\partial\tilde{\boldsymbol{K}}}{\partial d_j}\frac{\partial\tilde{\boldsymbol{B}}}{\partial d_i} \tag{39}$$

where

$$\frac{\partial^2\tilde{\boldsymbol{K}}}{\partial d_i\,\partial d_j}=\begin{bmatrix} \dfrac{\partial^2\boldsymbol{K}}{\partial d_i\,\partial d_j}-\omega^2\dfrac{\partial^2\boldsymbol{M}}{\partial d_i\,\partial d_j} & -\omega\dfrac{\partial^2\boldsymbol{C}}{\partial d_i\,\partial d_j} \\ \omega\dfrac{\partial^2\boldsymbol{C}}{\partial d_i\,\partial d_j} & \dfrac{\partial^2\boldsymbol{K}}{\partial d_i\,\partial d_j}-\omega^2\dfrac{\partial^2\boldsymbol{M}}{\partial d_i\,\partial d_j} \end{bmatrix}.$$

Further differentiating Eq. (38) with respect to the design variable d_j, we have:

$$\frac{\partial^2\boldsymbol{x}}{\partial d_i\,\partial d_j}=\frac{\partial^2\boldsymbol{B}_1}{\partial d_i\,\partial d_j}\sin\omega t+\frac{\partial^2\boldsymbol{B}_2}{\partial d_i\,\partial d_j}\cos\omega t \tag{40}$$

$\partial^2\boldsymbol{B}_1/(\partial d_i\,\partial d_j)$ and $\partial^2\boldsymbol{B}_2/(\partial d_i\,\partial d_j)$ can be obtained by solving Eq. (39). Then the second derivative $\partial^2\boldsymbol{x}/(\partial d_i\partial d_j)$ is achieved from substituting $\partial^2\boldsymbol{B}_1/(\partial d_i\,\partial d_j)$ and $\partial^2\boldsymbol{B}_2/(\partial d_i\,\partial d_j)$ into Eq. (40).

6.2 Results and Accuracy

All of the computations are performed on a notebook PC with Window 7 operating system. The computer has an Intel Core i5 CPU, and the frequency of the CPU is 2.27GHz. The computer also has 2GB RAM. The computational time of solving the problem is about 0.94s with the new method proposed in this paper, and about 0.08s with the analytical method. The computational results of the two methods are shown in the same figure to evaluate the accuracy of the new method proposed in this paper. The first derivative time histories of dynamic responses with respect to the design variable are shown in Fig. 5 ~ Fig. 8. The second derivative time histories of dynamic responses with respect to design variable are shown in Fig. 9 ~ Fig. 12. One of the relative error-time curves of the first derivative between the new method and analytical method is shown in Fig. 13. The maximum relative error of the numerical results achieved by the new method is less than 2×10^{-2}. One of the relative error-time curves of the second derivative between the new method and analytical method is also shown in Fig. 14. The maximum of the relative error achieved by the new method is ~ 2×10^{-3}. The relative error decreases rapidly and does not propagate in the time step integration procedure. Therefore, the new method has almost the same accuracy as the analytical method. The important reason of high accuracy is that the formulas are derived by direct differentiations.

6.3 Efficiency

The most popular method to obtain the first and second derivatives of dynamic responses with respect to the design variables is the central difference method. In this example, the computational time of a single dynamic analysis is about 0.025s with the Newmark-β method. For such a problem with 6 design variables, central-difference first and second derivative calculations require total repetition of the dynamic response analysis for total $4\times(6^2-2\times6+1)$ different design points [29]. The central difference method requires more than 2.5 s, *i.e.*, $4\times(6^2-2\times6+1)\times0.025$s, to obtain all of the first and second derivatives of dynamic responses with respect to the design variables. The efficiency is ranked thus: The analytical method (0.08s) > the new method (0.94s) > the central difference method (more than 2.5s). Therefore, the new method is more efficient than the central difference method. The important reason of high efficiency is that the dynamic responses and their first and second derivatives can be calculated simultaneously in only a single dynamics analysis.

6.4 Limitations

The new method is based on the Newmark-β method. Therefore, the accuracy and stability of algorithm depend on the parameters δ and β. If δ = 0.5 and β = 0.25, the algorithm is unconditionally stable and second-order accurate. The smaller time step Δt, the more accurate the computational results, and the longer the computational time. Additionally, the new method is only applicable to the linear dynamic systems.

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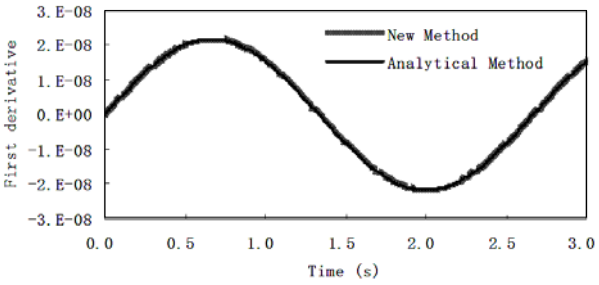


Fig. 5 First derivative, horizontal displacement of node 2, with respect to d^1

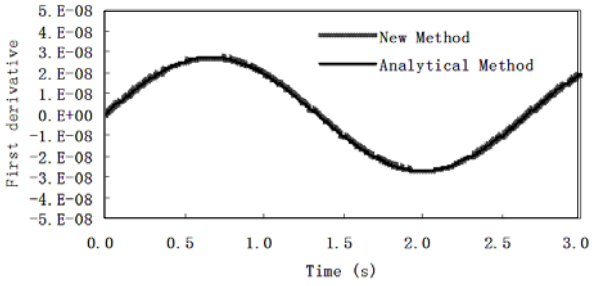


Fig. 6 First derivative, horizontal displacement of node 2, with respect to x_2

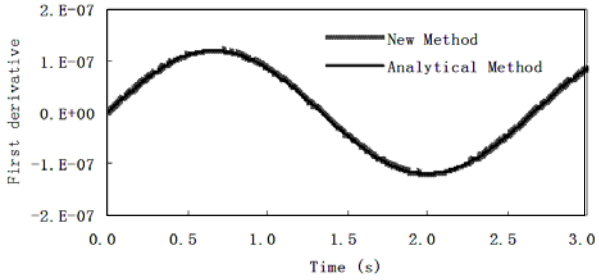


Fig. 7 First derivative, horizontal displacement of node 2, with respect to t^1

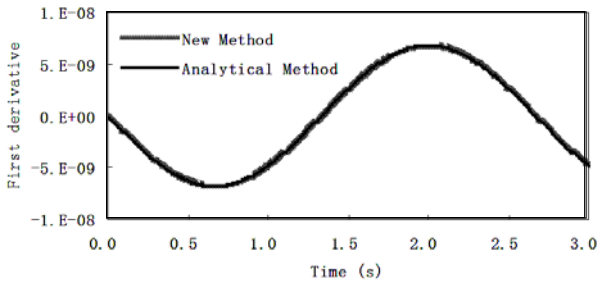


Fig. 8 First derivative, horizontal displacement of node 2, with respect to y_2

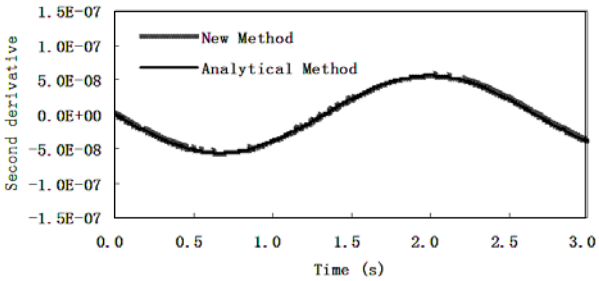


Fig. 9 Second derivative, horizontal displacement of node 2, with respect to d^1, d^2

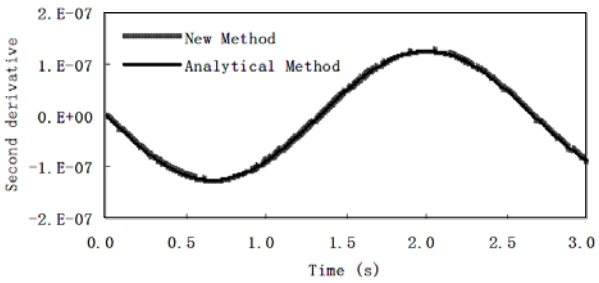


Fig. 10 Second derivative, horizontal displacement of node 2, with respect to d^1, t^1

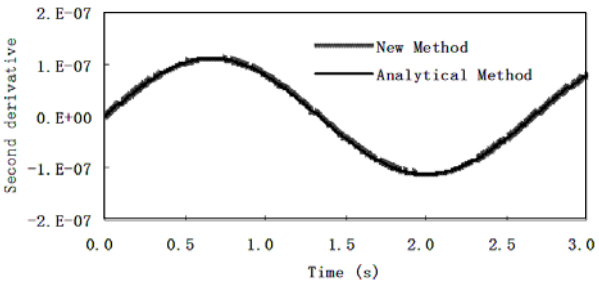


Fig. 11 Second derivative, vertical displacement of node 2, with respect to d^1, d^1

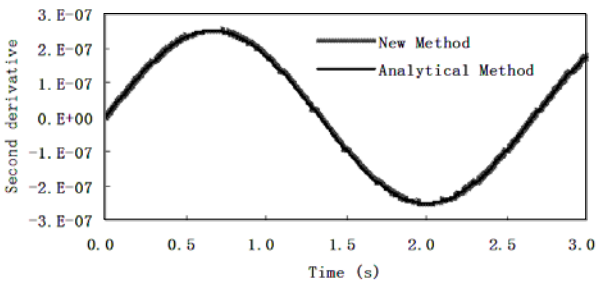


Fig. 12 Second derivative, vertical displacement of node 2, with respect to d^1, t^1

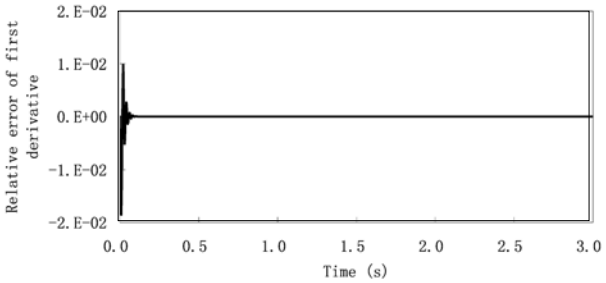


Fig. 13 Relative error of first derivative, horizontal displacement of node 2, with respect to y_2. (Between new method and analytical method)

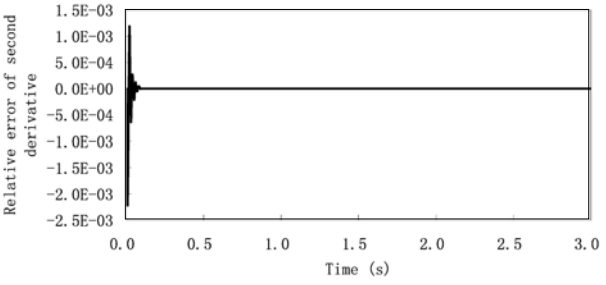


Fig. 14 Relative error of second derivative, horizontal displacement of node 2, with respect to d^1, t^1. (Between new method and analytical method)

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7. CONCLUSIONS

In this paper, a new method for the first and second derivatives of structural dynamic responses with respect to the design variables was developed. The conclusions are summarized as follows:

1. An efficient algorithm is formulated to calculate the structural dynamic responses, their first and second derivatives with respect to the design variables. The algorithm is achieved by direct differentiation and only a single dynamics analysis based on Newmark-β method is required.
2. The comparative numerical results show that the new method presented in this paper is highly accurate compared with the analytical method. The important reason of high accuracy is that the formulas are derived by direct differentiations.

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