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Perturbations by operators converging compactly to zero

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Abstract. This paper considers perturbations in locally convex spaces of semi Fredholm operators T by a sequence of operators $\{K_n\}$ converging compactly to zero in a sense extending that from (2). It is shown that, under suitable conditions, $\alpha(T+K_n) \leqslant \alpha(T), \beta(T+K_n) \leqslant \beta(T)$ and $\kappa(T+K_n) = \kappa(T)$ for large n.

1. *Introduction.* Let E and F be locally convex topological vector spaces. A sequence $\{K_n\}$ of continuous everywhere defined linear operators mapping E into F is said to converge compactly to zero, written $K_n \xrightarrow{c} 0$, if there exists a neighbourhood W of zero such that for any sequence $\{x_n\} \subset W$, $\{K_n x_n\}$ is sequentially relatively compact, and if furthermore, K_n converges strongly to zero, i.e. $K_n x \to 0$ for all x in E. This coincides with the definitions of Goldberg (2) if E and F are Banach spaces. The most general results known so far are those for Banach spaces in (2) and it was the intention of the author to show how the existence of a norm in the arguments in that paper could be dropped. We denote the nullity, defect and index of an operator T by $\alpha(T)$, $\beta(T)$ and $\kappa(T)$ respectively. The following result comes from (5), p. 69.

PROPOSITION 1. *Let E be barrelled and $\{A_n\}$ be a sequence of continuous operators from E into F converging strongly to A. Then A is continuous and the convergence is uniform on precompact sets.*

We include the following lemma for later use.

LEMMA 2. *Let $f \in E'$ (the topological dual of E) and W be an absolutely convex neighbourhood of zero in E with gauge p_W. If $f(x) \neq 0$ implies $p_W(x) > 0$ then $f(x)$ is uniformly bounded on W.*

This is easily proved by factoring through the normed space $E/p_W^{-1}(0)$. Let ω be the set of all sequences and ϕ, the subset of ω of all sequences eventually zero. We will consider $\omega[\sigma(\omega, \phi)]$, which is isomorphic to the countable product of the scalar field.

PROPOSITION 3. *Let $E[\tau]$ be a sequentially complete space. Then E contains an isomorphic image of $\omega[\sigma(\omega, \phi)]$ if and only if there exists a non-zero sequence $\{x_n\} \subset E$ such that for arbitrary choice of scalars λ_n, $\lambda_n x_n \xrightarrow{\tau} 0$.*

Proof. We can clearly assume that the x_n are linearly independent (by taking a subsequence if necessary). Let F denote the subspace of E consisting of

$$x = \sum_{k=0}^{\infty} \lambda_k x_k.$$

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It is easily shown that

$$s_n = \sum_{k=0}^{n} \lambda_k x_k$$

is a Cauchy sequence in E and hence that it converges to a limit in E. Clearly F is algebraically isomorphic to ω. It is straightforward to show that the dual of F is isomorphic to ϕ hence the result, since the weak and Mackey topologies on ω with dual ϕ coincide.

If E is a Fréchet space, the above property is related to the existence of a continuous norm on E, see (3), 31·4.

For simplicity we will not give the minimum conditions for which the proofs of the ensuing results are valid. We henceforth assume that E is fully complete (hence barrelled) and F is fully barrelled (1). We assume also that neither E nor F contains an isomorphic image of ω. T will always be a closed operator with closed range from E into F and suppose $K_n \xrightarrow{c} 0$.

The domain of T need not be all of E although, when use is made of its adjoint, we assume it to be dense in E. In view of (4), theorem 4·10, T is open onto its range, and by (1) $R(T')$ is closed in E'.

For simplicity of notation, when no confusion is likely, we will denote a subsequence by the sequence itself.

2. *Main results*

LEMMA 4. *Suppose T is one–one. Then there exists a $p > 0$ such that $T + K_n$ is one–one for $n > p$.*

Proof. Suppose otherwise. Then there exists a subsequence of $\{T + K_n\}$ such that $T + K_n$ is not one–one for each n. Hence there exist x_n in E with $x_n \neq 0$ such that $(T + K_n)x_n = 0$ for all n. We consider two cases.

(1) Suppose that for a further subsequence, $x_n \in \lambda_n W$ for arbitrary λ_n (W satisfying the definition of K_n), i.e.

$$\{x_n\} \subset \bigcap_{n=1}^{\infty} \frac{1}{n} W = N,$$

the maximal subspace of E contained in W. Then for arbitrary λ_n, $\{K_n \lambda_n x_n\}$ is relatively sequentially compact, thus $K_n \lambda_n x_n \to 0$ for all λ_n, thus $\lambda_n(Tx_n)$ converges to zero for arbitrary λ_n. Since T is open, $\lambda_n x_n$ is convergent for any λ_n hence $x_n = 0$ eventually by Proposition 3. This contradicts our assumption.

(2) For sufficiently large n, there exist scalars μ_n such that $\mu_n x_n \in W - \frac{1}{2}W$. Let $y_n = \mu_n x_n$. Then $(T + K_n)y_n = 0$ with $y_n \in W - \frac{1}{2}W$. Hence $\{K_n y_n\}$ is relatively sequentially compact and thus has a convergent subsequence, hence so has $\{Ty_n\}$. Since T is an isomorphism (onto its range) y_n has a convergent subsequence. Hence by Proposition 1, $\{K_n y_n\}$ and therefore $\{Ty_n\}$ has a subsequence converging to zero. Since T is open, this implies that y_n has a subsequence convergent to zero which is impossible since $y_n \notin \frac{1}{2}W$ for each n. The theorem is thus proved.

THEOREM 5. *If $\alpha(T)$ is finite, there exists a p such that for $n > p$, $\alpha(T + K_n) \leqslant \alpha(T)$.*

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Proof. Let M be a closed topological complement of $\mathscr{N}(T)$, the null space of T, in E. Then $\mathscr{R}(T|_M) = \mathscr{R}(T)$, $\alpha(T|_M) = 0$, and $T|_M$ is clearly closed on the fully complete space M. Thus by Lemma 4, $(T+K_n)_M$ is one–one for large n, i.e. M is contained in a complement of $\mathscr{N}(T+K_n)$ hence $\alpha(T+K_n) \leqslant \alpha(T)$ for large n.

COROLLARY 6. *If $\alpha(T) < \infty$, $\alpha(T+K_n) = \alpha(T)$ for $n > p$ if and only if*

$$E = M \oplus \mathscr{N}(T+K_n) \quad \text{for} \quad n > p.$$

Proof. Clearly $M \oplus \mathscr{N}(T+K_n) \subset E = M \oplus \mathscr{N}(T)$. Since M is finite codimensional the result is trivial.

PROPOSITION 7. *Suppose $\mathscr{R}(T)$ has a topological complement N in F. Then there exists a p such that $n > p$ implies $T' + K_n'$ is one–one on N°.*

Proof. We have $F = \mathscr{R}(T) \oplus N$. Thus $F' = \mathscr{R}(T)^\circ \oplus N^\circ$. Suppose $T' + K_n'$ is not one–one on N° for large n, then by taking a subsequence if necessary, we can find $\{y_n'\} \subset N^\circ$ such that $(T' + K_n')\, y_n' = 0$ for all n, where $y_n' \neq 0$. Let $V = TW$ (the W associated with the K_n). We consider two cases:

(1) Suppose, for some choice of y_n for which $y_n' y_n \neq 0$ we can find a subsequence such that

$$y_n \in \bigcap_p (1/p\, V \oplus N)$$

for each n. If we write $y_n = Tx_n + w_n$ with $x_n \in W$ and $w_n \in N$, this implies that

$$x_n \in \bigcap_p (1/p\, W)$$

for each n. Hence

$$\lambda_n x_n \in W \quad \text{for arbitrary scalars } \lambda_n,$$

hence $K_n \lambda_n x_n = \lambda_n (K_n x_n) \to 0$, thus $K_n x_n = 0$ eventually by Proposition 3. Hence

$$\begin{aligned} y_n' y_n = y_n' T x_n &= (T' + K_n')\, y_n'(x_n) + y_n' K_n x_n \\ &= 0 \quad \text{for large } n. \end{aligned}$$

This contradicts the choice of y_n, hence:

(2) For each choice of y_n such that $y_n' y_n \neq 0$, we have $q(y_n) > 0$, where q denotes the gauge of $V \oplus N$. Hence by taking suitable scalar multiples if necessary we can, in view of Lemma 2, assume that $y \in \sup_{V \oplus N} y_n' y = 1$ for each n. Now choose $y_n \in V \oplus N$ with $y_n' y_n \geqslant \frac{1}{2}$. We can write $y_n = Tx_n + w_n$ with $x_n \in W$ and $w_n \in N$. Now

$$y_n' Tx = (T' + K_n')\, y_n'(x) - y_n' K_n x = -y_n' K_n x \to 0.$$

Since y_n' is zero on N, this means $y_n' v \to 0$ for all $v \in F$. Now

$$\tfrac{1}{2} \leqslant y_n' y_n = (T' + K_n')\, y_n'(x_n) - y_n' K_n x_n. \tag{1}$$

Since $\{x_n\} \subset W$, $\{K_n x_n\}$ is relatively sequentially compact. Hence, since $y_n' v \to 0$, from Proposition 1 we have $\{y_n' K_n x_n\}$ converges to zero. Therefore (1) cannot hold, since $(T' + K_n')\, y_n' = 0$. This completes the proof.

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THEOREM 8. (i) *If $\mathscr{R}(T)$ is complemented in F, there exists a p such that $n > p$ implies $\alpha(T' + K_n') \leqslant \alpha(T')$.*

(ii) *If $\beta(T) < \infty$ there exists a p such that $\beta(T + K_n) \leqslant \beta(T)$ for $n > p$.*

Proof. (i) Choose N and p as in Proposition 7 and $n > p$. Then

$$F' = \mathscr{R}(T)^\circ \oplus N^\circ = \mathscr{N}(T') \oplus N^\circ.$$

Since $T' + K_n'$ is one–one on N°, $F' \supset \mathscr{N}(T' + K_n') \oplus N^\circ$. Thus $\alpha(T' + K_n') \leqslant \alpha(T')$.

(ii) Suppose $\beta(T) < \infty$. Then since $\mathscr{R}(T)$ is closed, it is complemented in F. Thus for large n, $\alpha(T' + K_n') \leqslant \alpha(T')$, hence $\beta(T + K_n) = \alpha(T' + K_n') \leqslant \alpha(T') = \beta(T)$.

COROLLARY 9. *Let $\beta(T) < \infty$, then $\beta(T + K_n) = \beta(T)$ for $n > p$ if and only if*

$$F = \mathscr{R}(T + K_n) \oplus N \quad \text{for} \quad n > p$$

where N is defined as in Proposition 7.

Proof. $\alpha(T' + K_n') = \beta(T + K_n) = \beta(T) = \alpha(T')$. As in the proof of (i) above, $F' = \mathscr{N}(T') \oplus N^\circ \supset \mathscr{N}(T' + K_n') \oplus N^\circ$. The result now follows.

We recall that an operator K from E into F is called bounded if it maps some neighbourhood of E onto a bounded set in F.

THEOREM 10. *Assume that each K_n is a bounded operator. Then if either* (i) *E is a DF-space and $\beta(T) < \infty$, or* (ii) *F is locally complete (Mackey complete) and $\alpha(T) < \infty$, then there exists a p such that for $n > p$, $\kappa(T + K_n) = \kappa(T)$.*

Proof. (i) There exists a p such that for all $\lambda \in [0, 1]$, and $n > p$, $\beta(T + \lambda K_n) < \infty$. If not, there exists a subsequence $\{K_{n'}\}$ and a sequence $\lambda_n \in [0, 1]$ such that

$$\beta(T + \lambda_n K_{n'}) = \infty.$$

This is impossible by Theorem 8 since $\lambda_n K_{n'} \xrightarrow{c} 0$. Given $n > p$, define ψ on $[0, 1]$ with values in the set of extended integers with the discrete topology by $\psi(\lambda) = \kappa(T + \lambda K_n)$. By (7), ψ is continuous and since $[0, 1]$ is connected, ψ is constant. In particular $\kappa(T) = \psi(0) = \psi(1) = \kappa(T + K_n)$.

(ii) Similar to (i) making use of Theorem 5 and (6).

3. *Final remarks.* For Lemma 4, Theorem 5 and Corollary 6 we can weaken the condition on E to assuming only barrelledness and the non-existence of a subspace isomorphic to ω if, instead of closedness of T we require it to be open. In this case F can be an arbitrary space. For Theorem 8 it is sufficient to assume E being barrelled and that T is weakly continuous with closed complemented range in an arbitrary space F. The rather restrictive conditions on E and F are mainly required to prove the last three results, as some form of the open mapping theorem seems impractical to do without.

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